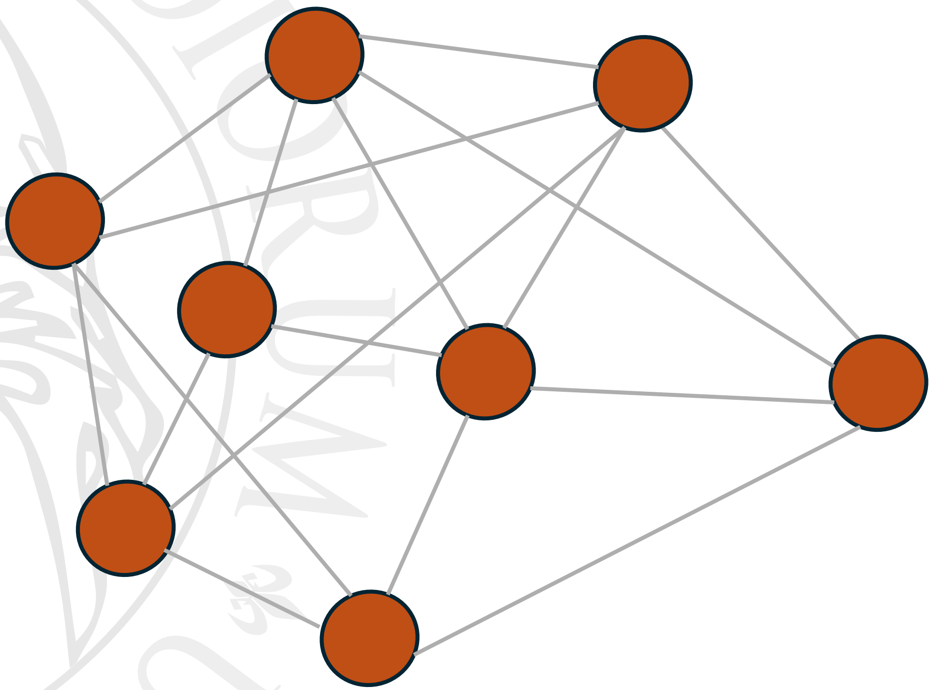
UNIVERSITÀ DEGLI STUDI FIRENZE

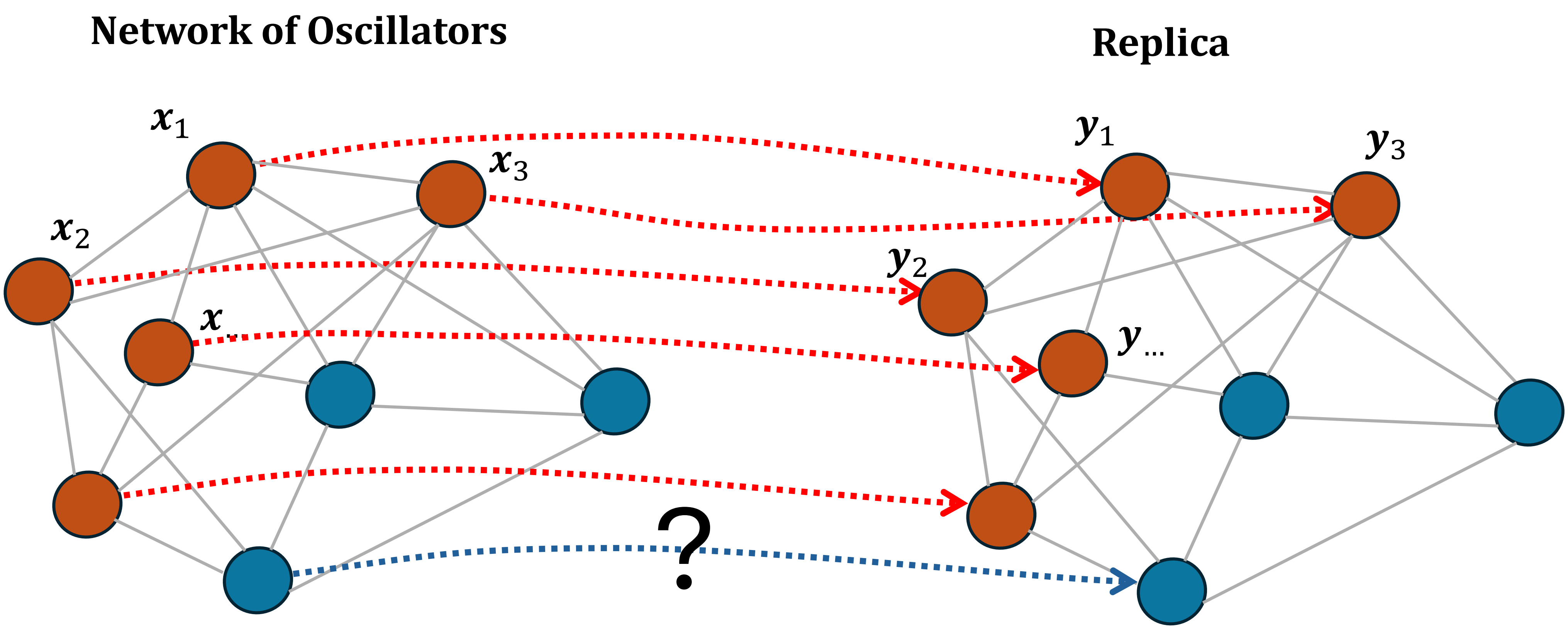
Internal reliability and anti-reliability in oscillator networks



Tommaso Matteuzzi[1],
Franco Bagnoli[1],
Michele Baia[1],
Stefano Iubini[3]
Arkady Pikovsky[2]

1) Department of Physics and Astronomy, University of Firenze and CSDC and INFN,
2) Department of Physics and Astronomy, University of Potsdam,
3) CNR-ISC, Institute of Complex Systems, Firenze

Syncronization as a measure method



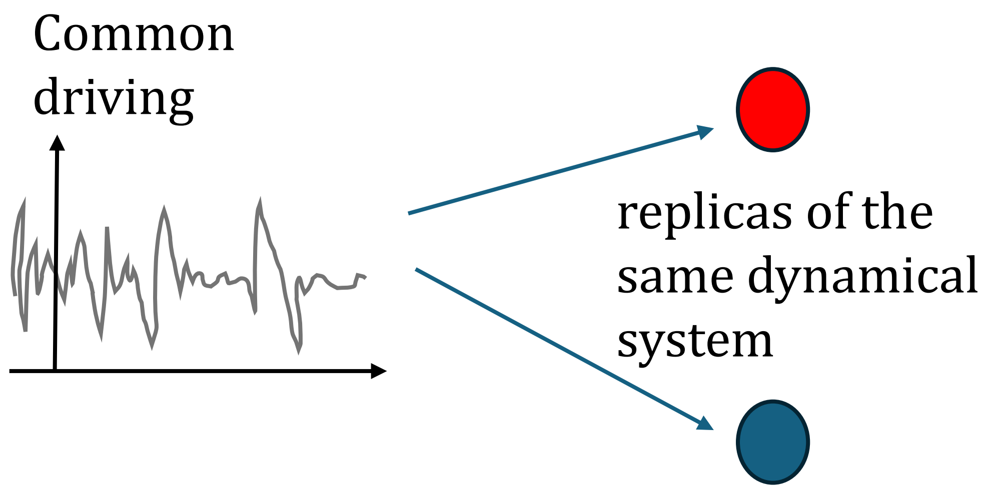
For example, a network of neurons

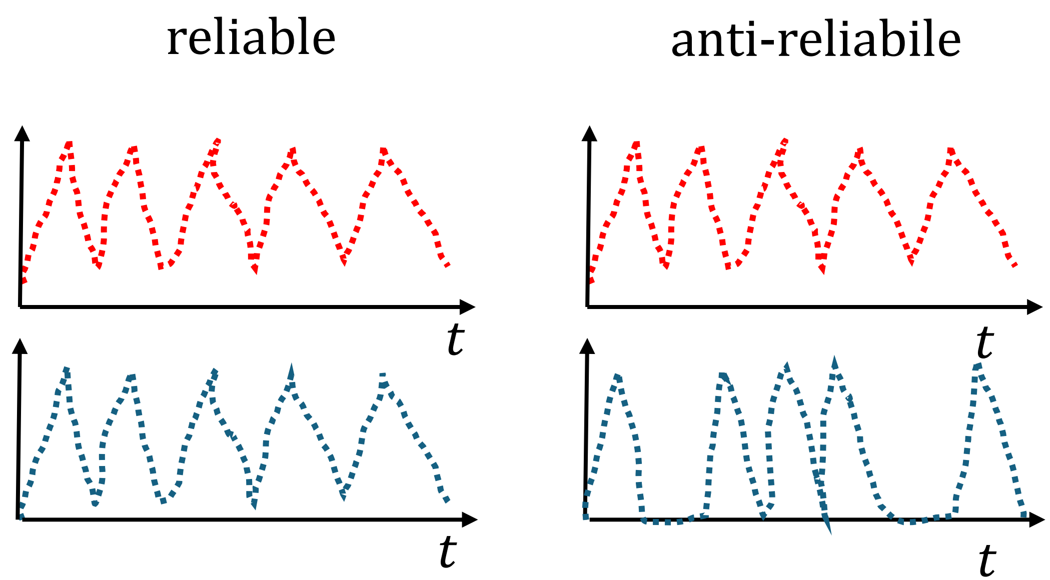
e.g. simulated model

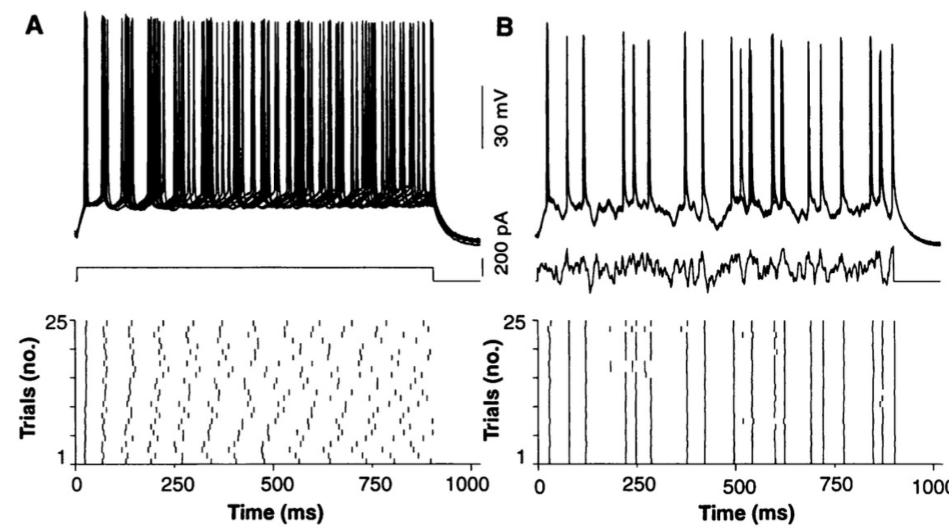
Assume that the replica has the same structure of the original and that we know the dynamics of units. We have access to a subset of the system: we can measure their state and impose them to the corresponding units of the replica.
Do the other units of the replica follow the original system?

Reliability and anti-reliability under external forces

The stability of synchronized state is related to the response to a common noise



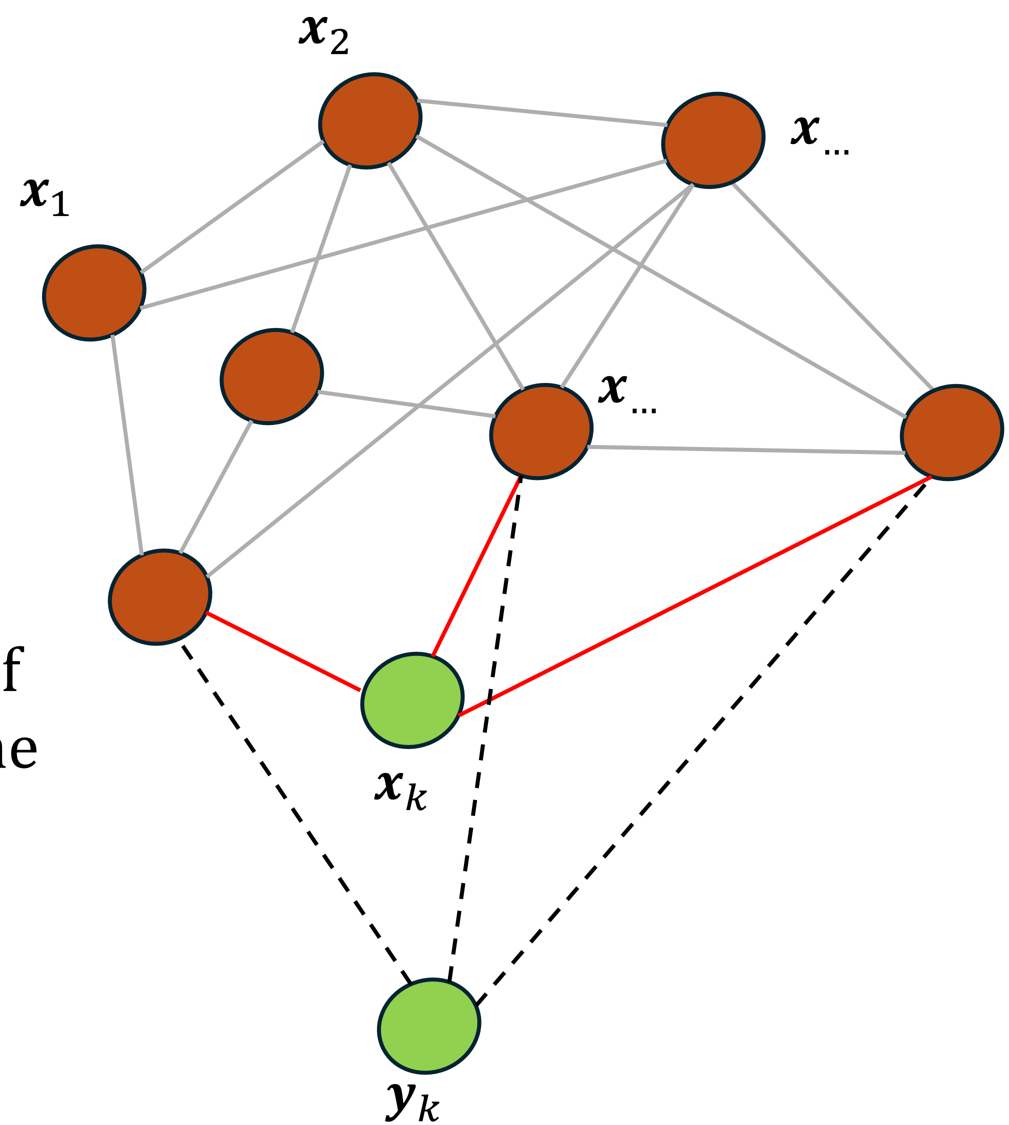




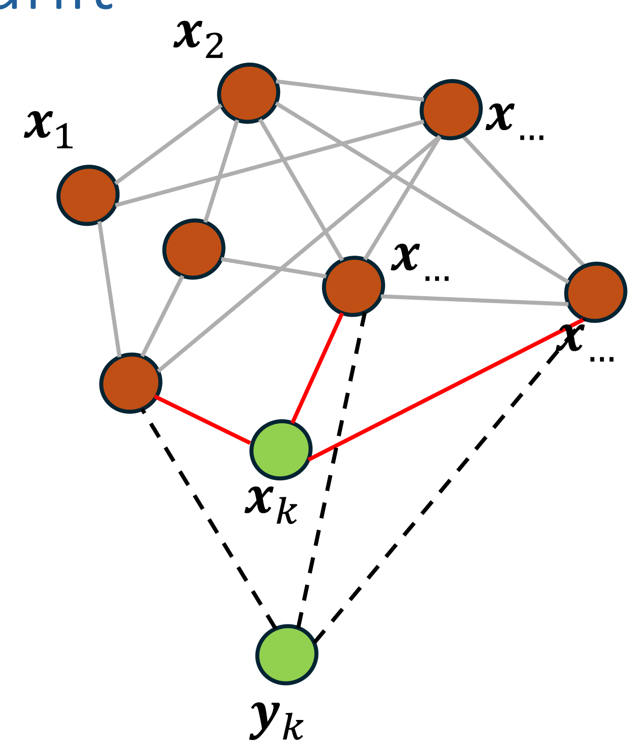
The notions of reliability and anti-reliability are used to describe a response of replicas to nontrivial external forces, in particular to external noise.
If two replicas driven by the same noisy force show exactly then same behaviour they are **reliable**, otherwise they are **anti-reliable**

Internal reliability and anti-reliability

We apply concepts of reliability and anti-reliability to the internal dynamics of oscillator networks.
Complex networks often demonstrate chaotic behaviour. In such a regime, does a particular unit (or, more generally, a subgroup of the units) reliably follow the driving from the other units?
We make a replica of one unit and look if this replica has the same dynamics as the original unit.



Internal reliability and anti-reliability of **one** unit



Consider a set $k = 1,2,\dots,N$ of units described by variables, $\boldsymbol{x}_k(t)$. Each unit follows its own dynamics $(\boldsymbol{F}_k)$ plus a coupling:

$$\dot{\boldsymbol{x}}_k(t) = \boldsymbol{F}_k(\boldsymbol{x}_k) + \boldsymbol{H}_k(\boldsymbol{x}_k; \boldsymbol{x}_1,\dots,\boldsymbol{x}_{k-1},\boldsymbol{x}_{k+1},\boldsymbol{x}_N)$$

We prepare a replica, $\boldsymbol{y}_k$, of the unit $\boldsymbol{x}_k$, that receives the same inputs from other elements as the unit $\boldsymbol{x}_k(t)$,

$$\dot{\boldsymbol{y}}_k(t) = \boldsymbol{F}_k(\boldsymbol{y}_k) + \boldsymbol{H}_k(\boldsymbol{y}_k; \boldsymbol{x}_1,\dots,\boldsymbol{x}_{k-1},\boldsymbol{x}_{k+1},\boldsymbol{x}_N)$$

Of course, if $\dot{\boldsymbol{y}}_k(0) = \dot{\boldsymbol{x}}_k(0)$ then the states of the prototype and its replica coincide for all future times.

For a small deviation $\boldsymbol{v}_k = \boldsymbol{y}_k - \boldsymbol{x}_k$ we obtain, by subtracting and linearizing, the linear system

$$\dot{\boldsymbol{v}}_k(t) = \left(\frac{\partial \boldsymbol{F}_k}{\partial \boldsymbol{x}_k} + \frac{\partial \boldsymbol{H}_k}{\partial \boldsymbol{x}_k}\right)\boldsymbol{v}_k.$$

This linear equation gives an asymptotic grows $|\boldsymbol{v}_k| \sim e^{\lambda_k t}$, where $\lambda_k$is the maximal transversal Lyapunov exponent (TLE).
Positive λ_k means that the replica **diverges** and thus the unit k is **anti-reliable**;
negative λ_k means **reliability**: the replica converges and eventually

$$\boldsymbol{y}_k(t) \xrightarrow[t\to\infty]{} \boldsymbol{x}_k(t).$$

Internal reliability of recurrent neural network

First (trivial) example: a recurrent network of neural network units (like in the Hopfield network with parallel dynamics)

$$\dot{x}_k = -x_k + \sum_{j \neq k} J_{jk} \tanh(x_j)$$

$(F_k(x_k) = -x_k;\ H_k(x_k) = \sum_{j \neq k} \tanh(x_j))$

The growth of a small deviation between the prototype and the replica is:

$$\dot{v}_k = -v_k$$
$$\lambda_k = -1,$$

Therefore, all units are reliable, although the system may be chaotic.

[H. Sompolinsky, A. Crisanti, and H. J. Sommers, *Chaos in random neural networks*, Phys. Rev. Lett. 61, 259 (1988)]

Kuramoto model

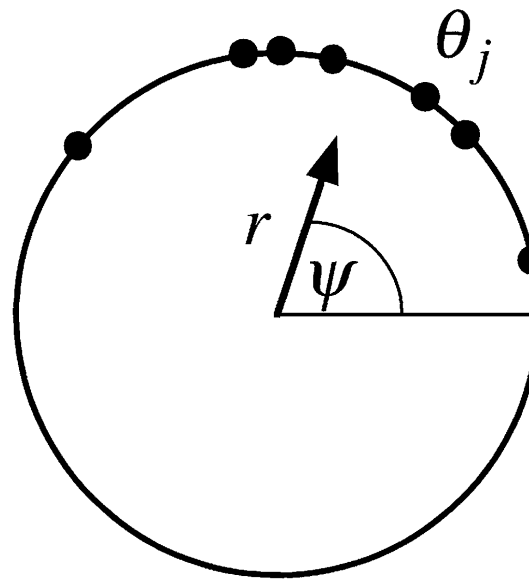
The Kuramoto model is formed by N all-to-all coupled oscillators different natural frequencies ω_k..

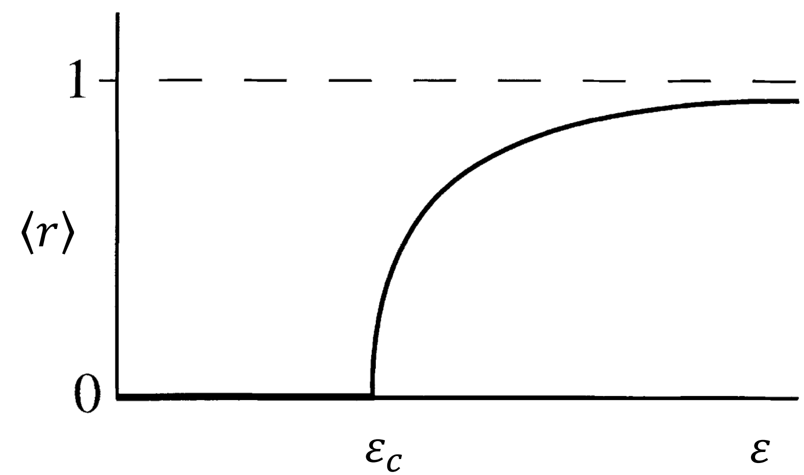
$$\dot{x}_k = \omega_k + \frac{\varepsilon}{N}\sum_{j\neq k} \sin(x_j - x_k)$$

We can define an order parameter (radial position of the center of mass of oscillators)

$$r = \frac{1}{N}\left(\left(\sum_k \cos(x_k)\right)^2 + \left(\sum_k \sin(x_k)\right)^2\right)$$

Which shows a transition from $\langle r\rangle = 0$ to $\langle r\rangle > 0$.
Above the transition all units are trivially reliable.
We investigate the system below (near) the critical point.
It is possible to compute analytically ε_c for $N \to \infty$.





Internal reliability of Kuramoto units

Let us consider now a Kuramoto model of N coupled oscillators with natural frequencies, ω_k, distributed following the $Beta$ distribution (in order to have them bounded). The frequences are also "optimally" spaced to avoid a source of variability.

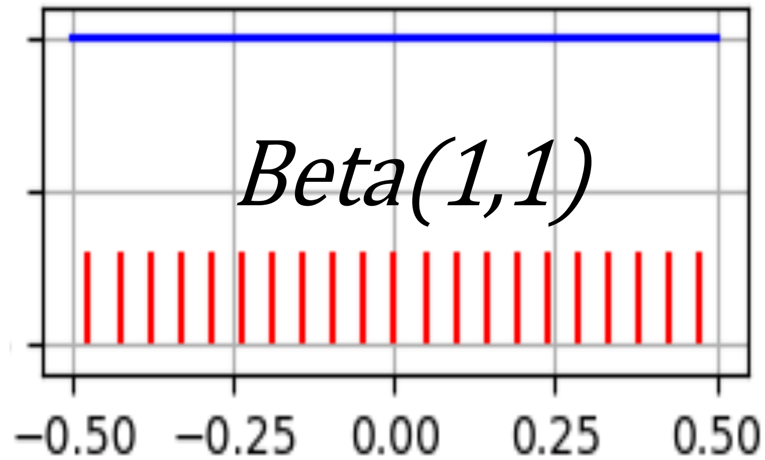
$$\dot{x}_k = \omega_k + \frac{\varepsilon}{N}\sum_{j\neq k} \sin(x_j - x_k)$$

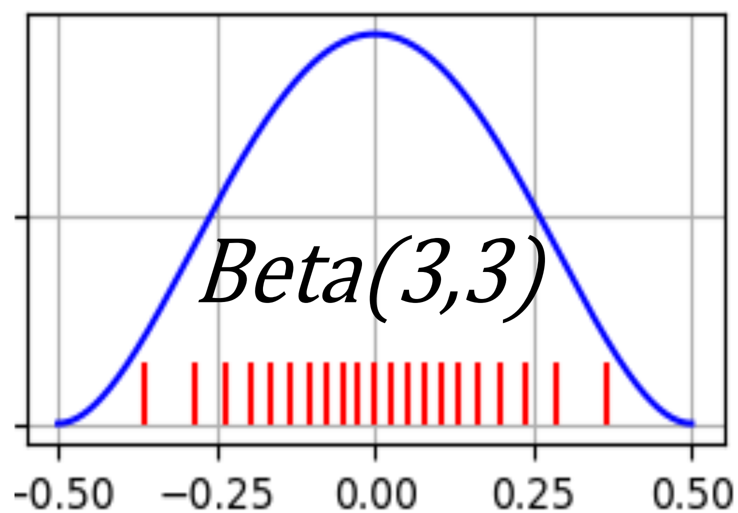
$(F_k(x_k) = \omega_k; H_k(x_k) = \frac{\varepsilon}{N}\sum_{j\neq k} \sin(x_j - x_k))$

The growth of a small deviation between the prototype and the replica is:

$$\dot{v}_k = -\frac{\varepsilon}{N}\sum_{j\neq k} \cos(x_j - x_k)\, v_k$$

$$\lambda_k = -\frac{\varepsilon}{N}\left\langle \sum_{j\neq k} \cos(x_j - x_k) \right\rangle$$

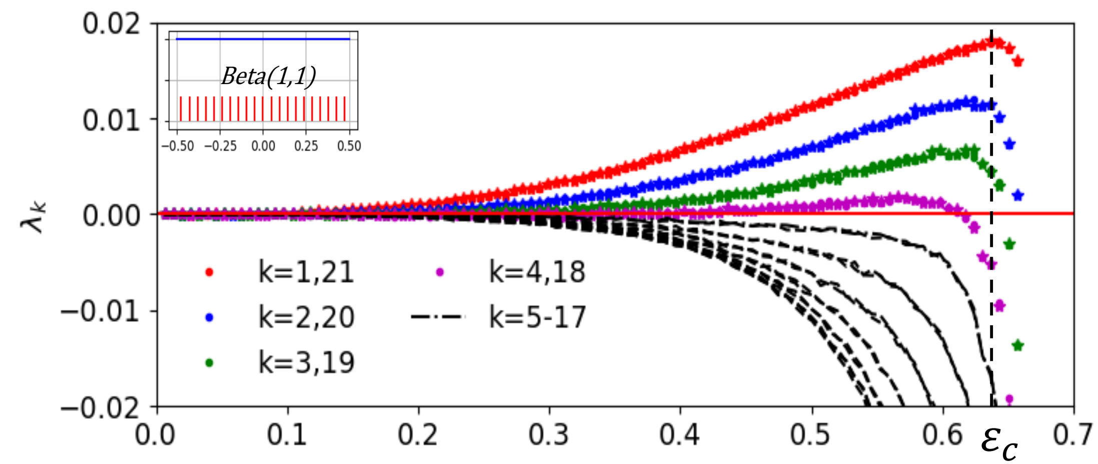


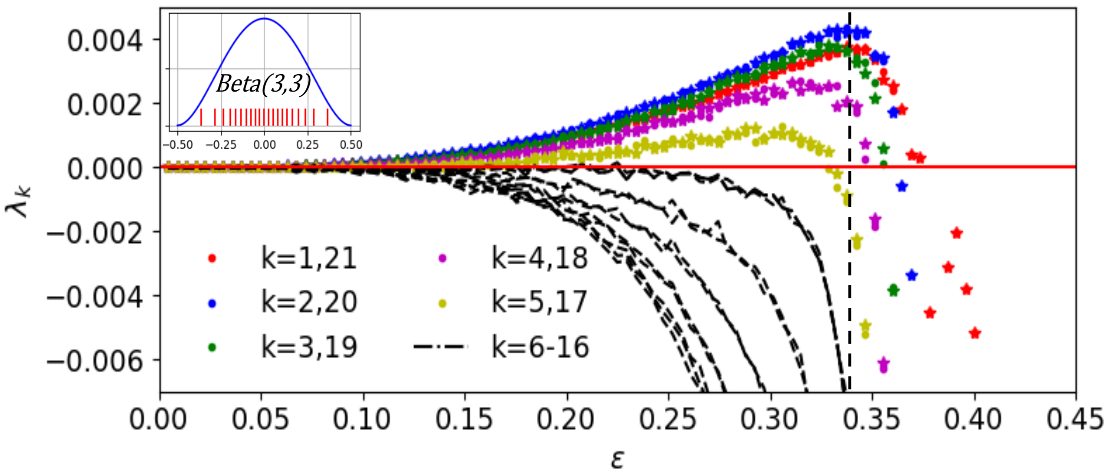


Internal reliability of Kuramoto units

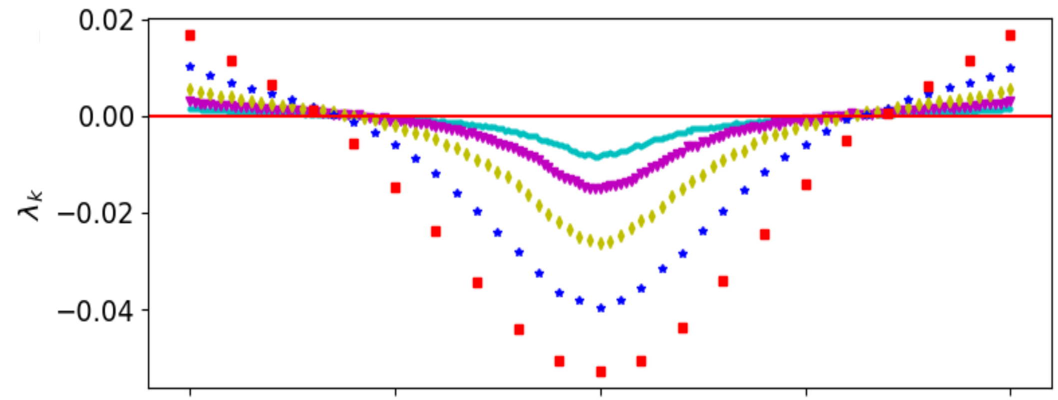
In this case for $N \to \infty$ the TLE is zero or negative, but for finite N the reliability is related to coupling and natural frequency (although the coupling is all to all).

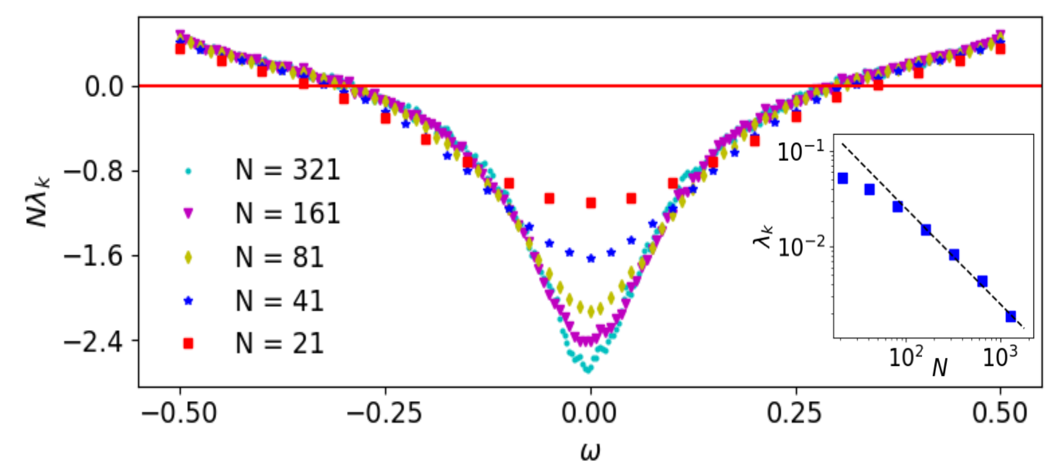
Reliability vs coupling strength, $N = 21$





Reliability vs oscillator natural freq.





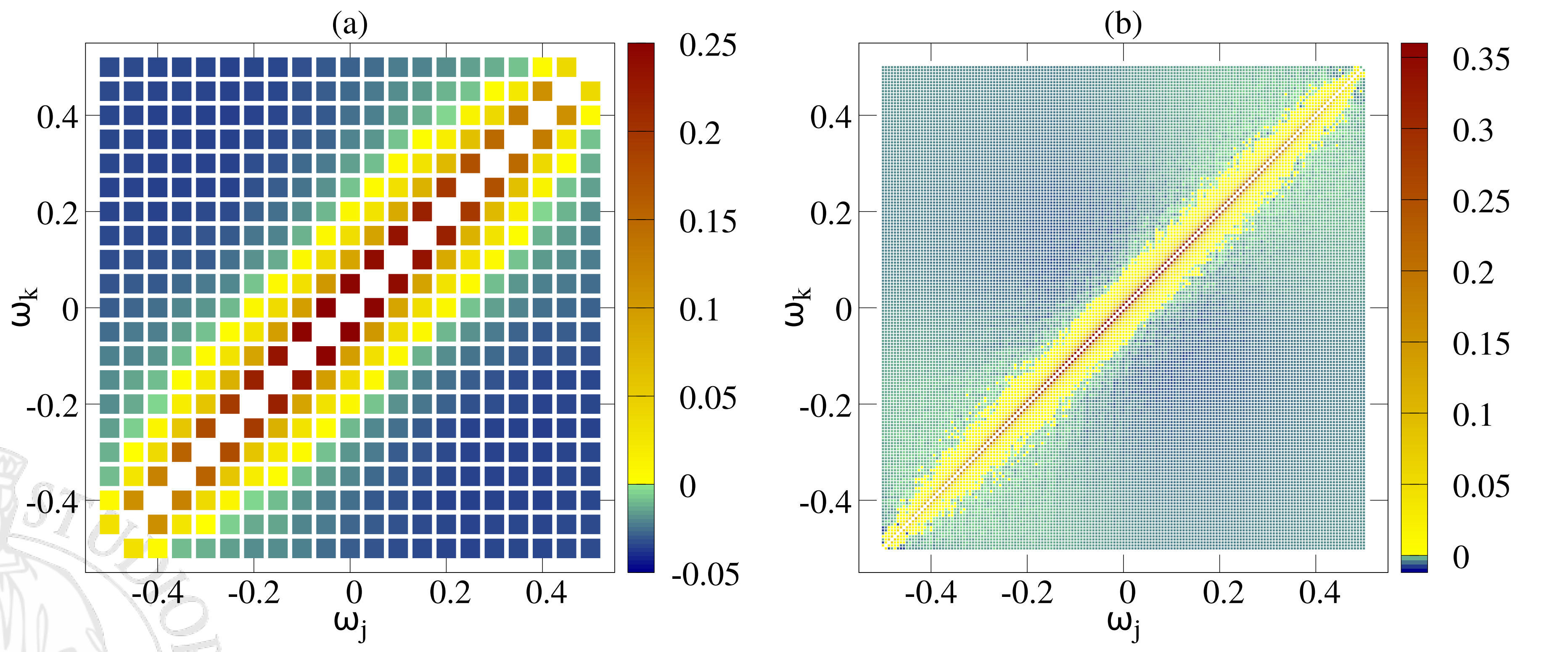
Correlations matrices

Representing correlations among two units as $c_{jk} = c_{kj} = \langle \cos(x_j - x_k) \rangle$ we have

$$\lambda_k = -\frac{\varepsilon}{N} \sum_{j \neq k} c_{jk}$$

And indeed we have positive correlations (negative TLE) for central units.

$N = 21$ $N = 161$



Fluctuation-dissipation relations

The relationship

$$\lambda_k = -\frac{\varepsilon}{N}\sum_{j\neq k} c_{jk}$$

can be considered a kind of fluctuation-dissipation relation.

The variance of the Kuramoto order parameter

$$r = |Z| = \frac{1}{N}\left|\sum_k \exp(ix_k)\right|$$

i.e.,

$$\langle |Z|^2\rangle = \frac{1}{N^2}\sum_{jk}\left\langle e^{i(x_j - x_k)}\right\rangle = \frac{1}{N^2}\left(N + \sum_{j\neq k} c_{jk}\right) = \frac{1}{N}\left(1 - \frac{1}{\varepsilon\sum_k \lambda_k}\right),$$

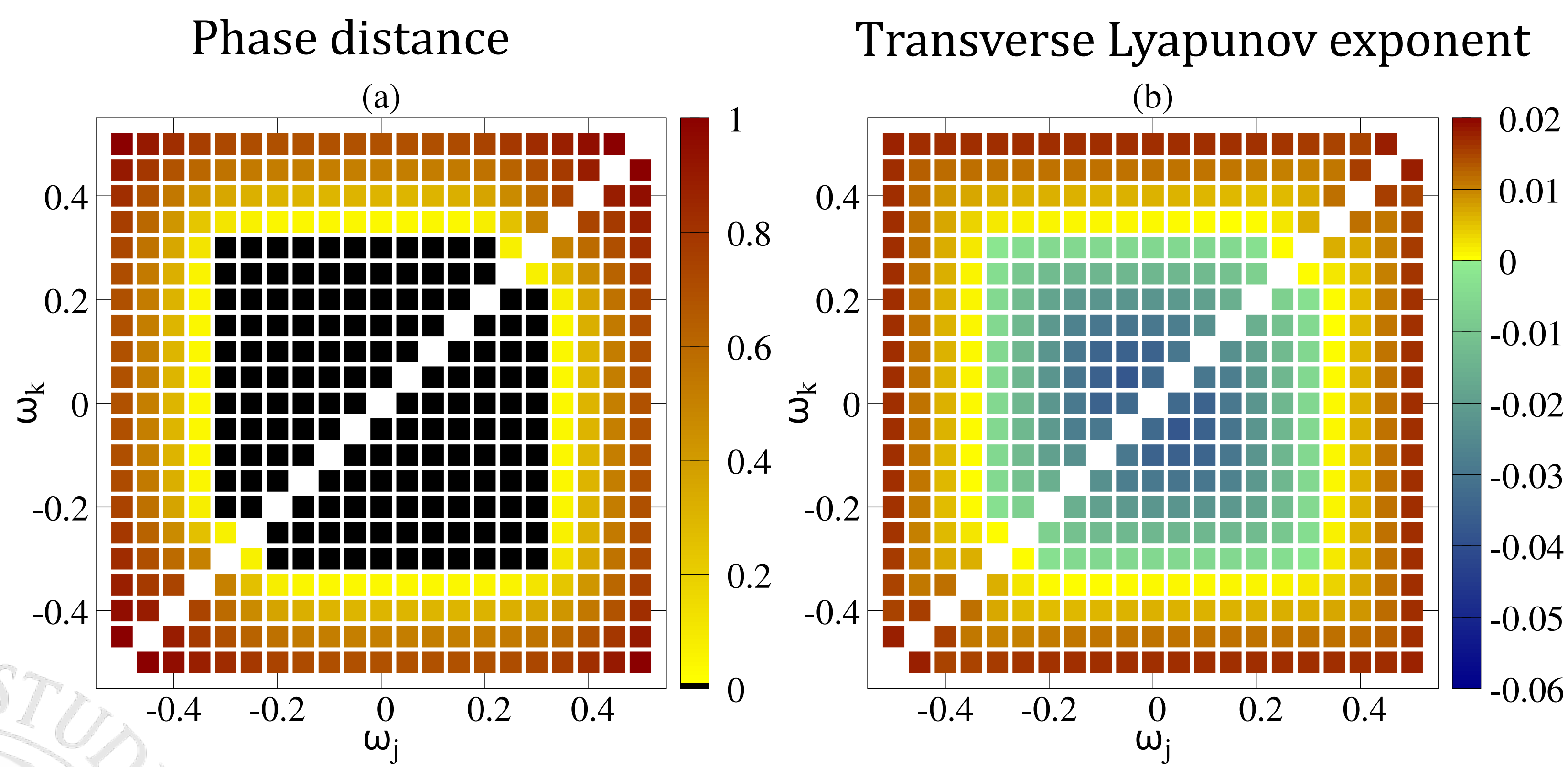
can be considered as another fluctuation-dissipation relation.

Reliability of pairs

Representing correlations among two units as $c_{jk} = c_{kj} = \langle\cos(x_j - x_k)\rangle$ we have

$$\lambda_k = -\frac{\varepsilon}{N}\sum_{j\neq k} c_{jk}$$

$N = 21,$
$\varepsilon = 0.6$



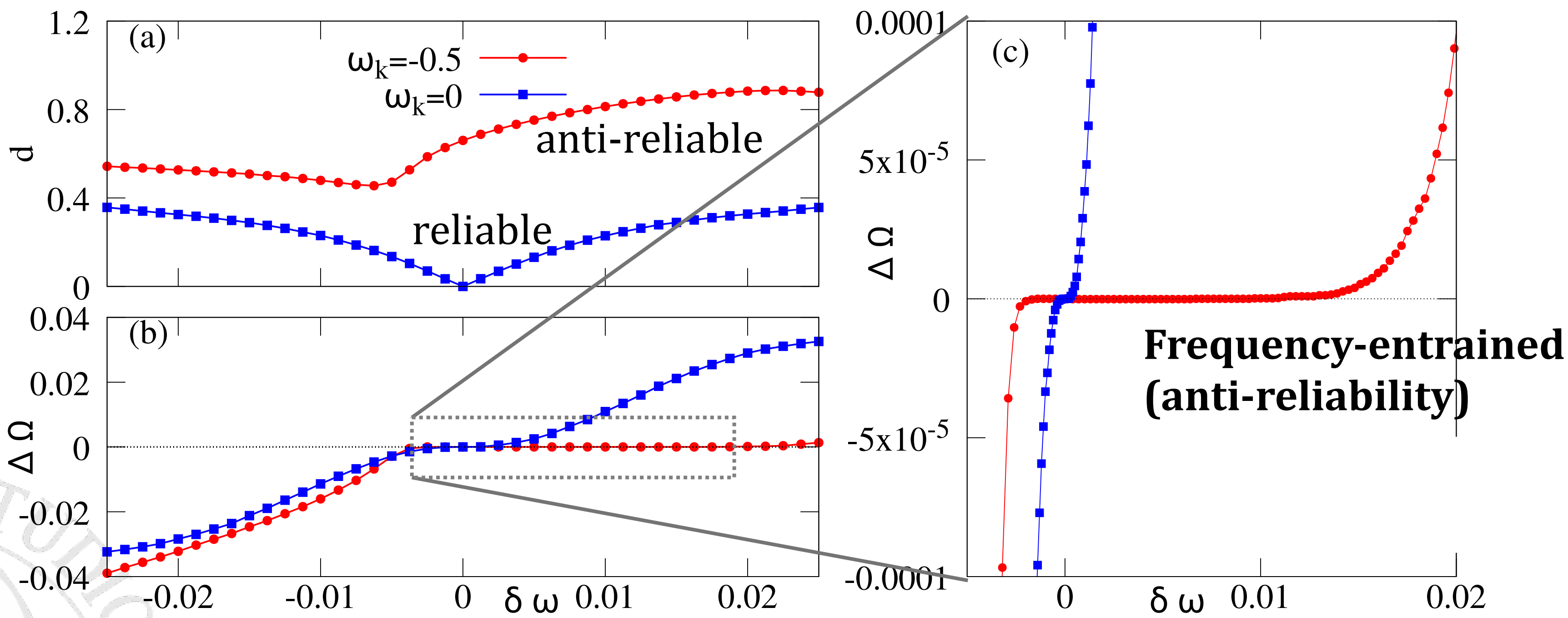
A pair is ***reliable*** if both units are individually reliable

Imperfect replicas

We insert a detuning $\boldsymbol{\delta\omega}$ of the natural frequency of the replica

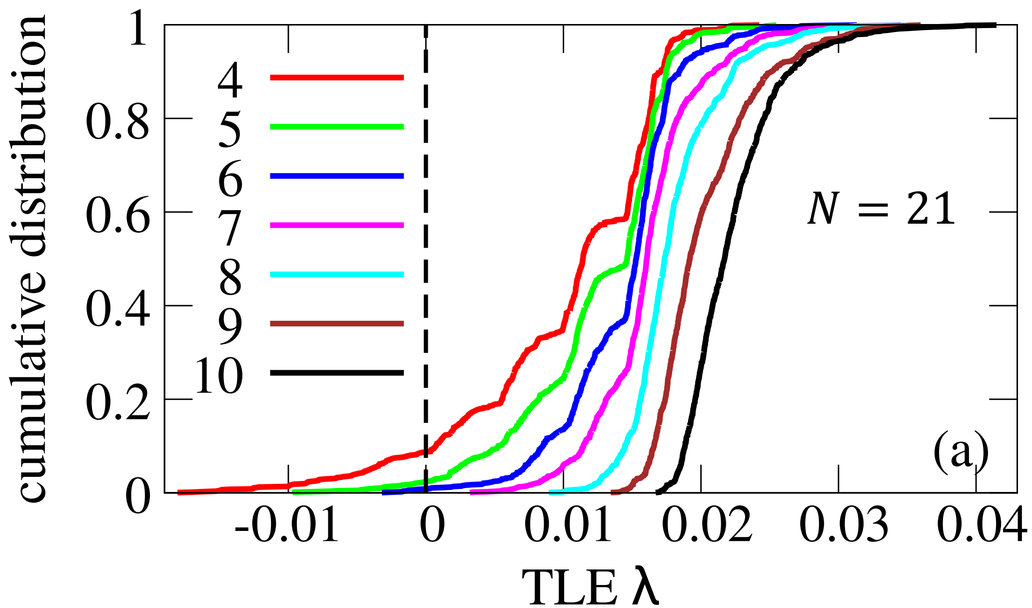
$$\dot{y}_k = \omega_k + \boldsymbol{\delta\omega_k} + \frac{\epsilon}{N}\sum_{j\neq k} \sin(x_j - y_k)$$

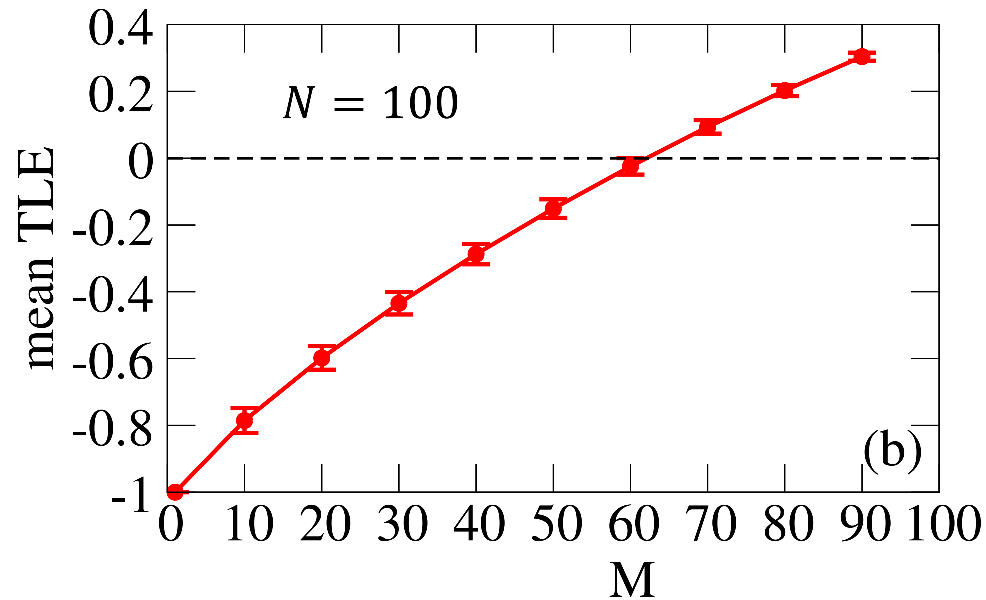
and compute the phase distance $d = \left\langle \left| \sin\left(\frac{x_k - y_k}{2}\right) \right| \right\rangle$ and the observed frequency distance $\Delta\Omega = \langle \dot{y}_k - \dot{x}_k \rangle$.



Reliability of subnetworks

For larger networks the number of possible sub-networks is too large to explore all of them.





We computed the TLEs for a random sample of 1000 sub-networks of sizes $4, \ldots, 10$ out of $N = 21$. Already for subnetworks of size 4 only 10% of all cases are reliable, and starting from 7 all tested subnetworks are anti-reliable.

Mean values and standard deviations (depicted as error bars) of the TLEs for randomly sampled sub-networks of the recurrent neural network.
For sufficiently large subnetwork the system becomes anti-reliable.

Conclusions

We explored phase oscillators coupled via Winfree-type coupling terms; Stuart-Landau oscillators, coupled rotators ("oscillators with inertia") described by second-order phase equations, and all of them demonstrated qualitatively the same properties: units with peripheral frequencies are anti-reliable, while those with central frequencies are reliable.

Furthermore, we explored several examples of random Kuramoto networks

$$\dot{x}_k = \omega_k + \varepsilon \sum_{j \neq k} A_{jk} \sin(x_j - x_k)$$

with a uniform distribution of natural frequencies and randomly sampled A_{jk}.

For both symmetric and asymmetric random matrices, the typical picture is that peripheral units are anti-reliable, and the central units are reliable. However, some central units become anti-reliable in a certain range of coupling strengths ε.

In the reliable regime, we can exploit the fact that the synchronized state is stable only if all parameters are the same to estimate them.

[Bagnoli, F., & Baia, M. (2023). *Synchronization, control and data assimilation of the Lorenz system.* Algorithms, 16(4), 213]