Numerical Methods for Holography

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Contents

Numerical methods and Holography

Holography often appears in talks' titles.
Set it aside for now.

A couple of crowd questions about numerical methods:

Crowd question:

Who enjoys doing numerical tasks?

Crowd question:

Who, in the past few months, had to use numerical methods?

Crowd question:

Who would have preferred not to do so?

Numerical methods and Holography

Claim of the talk:

Numerical methods are often **necessary**, or at least useful.

They can sometimes be as effective[1] as analytical ones, and provide what we want from a proper theoretical result.

(often with a clearer visualization!)

[1] Obvious exaggeration.

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Strongly coupled theories

Research area of good use for numerical methods

What: **QCD**

QCD-like theories are interesting and immediate examples of strong coupling

How: **HOLOGRAPHY**

Duality used as a tool to probe the strong-coupling regime of the field theory

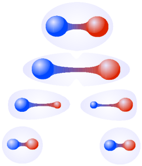
QCD-like theories

SM's QCD is state-of-the-art for strong interaction, but it is not perfect

Examples:

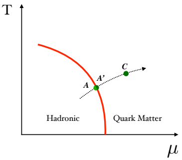
Confinement

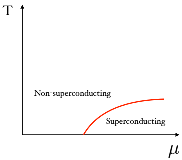
It is phenomenologically known that quarks and gluons are confined within hadrons. What is the mechanism that gives rise to this?

[a]

Phase diagram

A rich phase structure is expected. Is there a critical point at finite density? How does the phase transition happen?





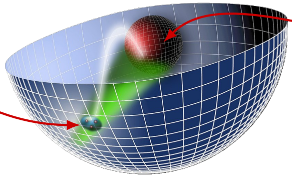
[a] Courtesy of nLab

Holography

*Conjectured equivalence between **strong** gauge and **weak** gravity*

Holographic principle: duality between

nonperturbative QFT in D dimensions $\Leftrightarrow$ classical gravity in $D+1$ dimensions



As of today, there is **no QCD holographic dual**, but known theories mimic *aspects* of QCD (confinement, phase structure) $\Rightarrow$ useful tool to study strong interaction

Numerical methods

What are they good for?

Analysis: numerical integration and differentiation, even via standard techniques (quadrature, finite differences) often allow *arbitrarily high precision*

Optimization: numerical solution of EoM via action minimization, allows for *arbitrary number of boundary conditions* to be imposed

Simulation: solving (linearized) EoM on a spacetime grid to study evolution phenomena like phase transitions

Modeling: well-built numerical approximations of exact objects (functions, surfaces) are *easily controllable*

Numerical methods

How to use them properly

Error analysis: check convergence of numerical approximation error wrt numerical parameters

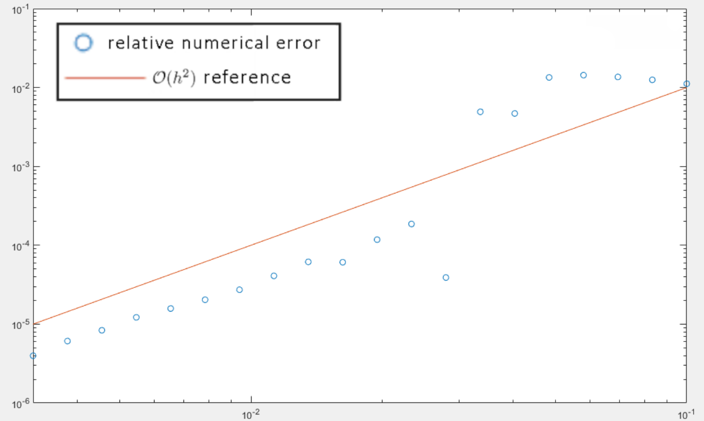
Toy models: test algorithms on problems with a similar structure but known (or computable) solution

Data-aware programming: different languages use memory differently, and some programs might require a lot of data to be used/stored

Algorithmic efficiency: exploit the available resources, by vectorization, parallelization, usage of GPUs

Numerical methods

How to use them properly (example)

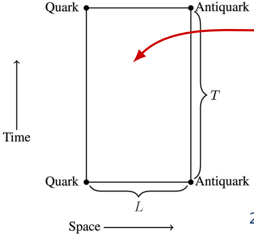


Confinement and screening

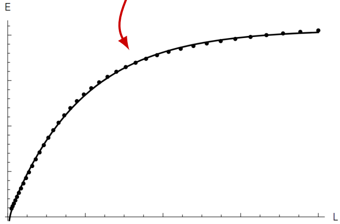
Aim: study **confinement** and **screening** in a QCD-like theory

$\rightarrow$ study how the **Wilson loop VEV** (i.e. quark-antiquark potential) scales with quark-antiquark separation

$$\langle W_l \rangle = \mathrm{Tr}\, P \exp\left(-i \oint_l A \right) \sim \exp(-T \cdot E(L))$$



[2]

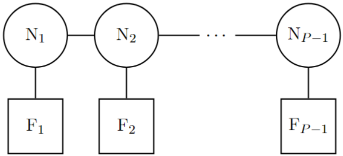


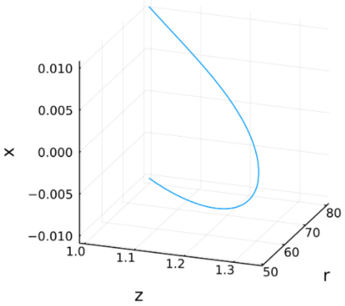
[2] courtesy of S. Chatterje

Holographic Wilson loop

Holographically dual to a **quiver gauge theory** [Cremonesi, Tomasiello 2015] with supersymmetry

$\longrightarrow$ calculate Wilson loop VEV by **hanging a fundamental string probe** with endpoints separated in the boundary QFT; string dives into the gravity bulk, minimizing its action S_{NG} in a nontrivial way [Maldacena 1998]





Numerical solution

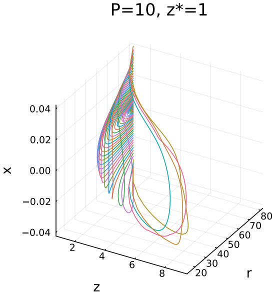
EoM for probe string is highly *non-linear*, *coupled*, *boundary value* problem. Daunting to tackle analytically.
But hanging string can be modeled numerically (e.g. via *splines*, i.e. piecewise polynomials on a subdivided interval)!

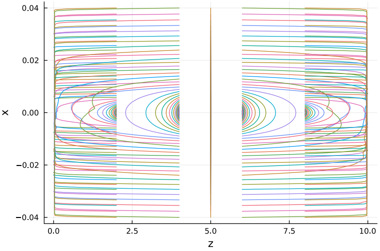
Interest: L dependancy in $E(L)$, a (good) numerical approximation would suffice

$\Rightarrow$ find solution via ***numerical optimization*** (i.e. minimization) of NG action, computed via simple trapezoidal quadrature

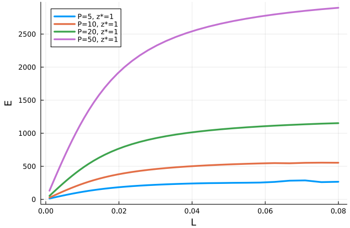
Julia module `RobinHood.jl`, publicly available on github [M.G., Fatemiabhari, Nunez 2024]

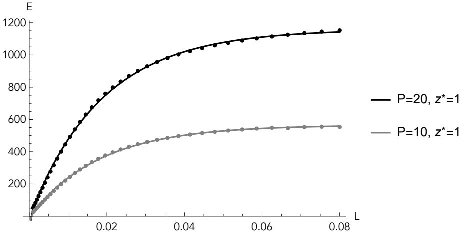
Numerical solutions





Fit of $E(L)$





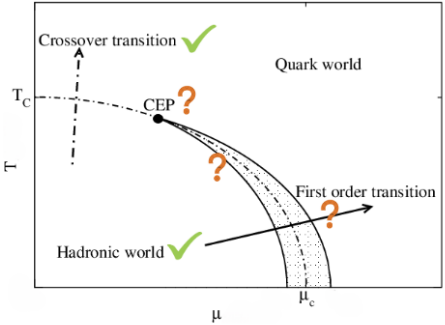
$$E(L) = -\frac{a}{L} + \gamma \frac{1 - e^{-bL}}{b}$$

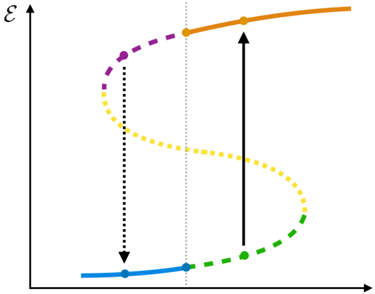
For small L conformal-to-confining: $E(L \to 0) = -\frac{a}{L} + \gamma L$

For large L screening $E(L \to \infty) = \frac{\gamma}{b}$

Bubbles in HQCD

Aim: simulate **bubbles** in a QCD-like theory
$\longrightarrow$ study *bubble wall velocity* (nonequilibrium parameter)





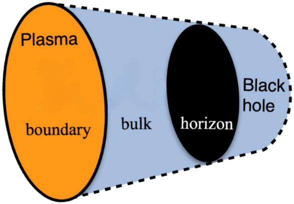
Holographic setup

Gravity bulk theory: Einstein-Maxwell-Dilaton theory [DeWolfe, Gubser, Rosen 2010]

$$S = \frac{2}{\kappa_5^2} \int d^5x \sqrt{-g} \left(\frac{1}{4}\mathcal{R} - \frac{1}{2}(\partial\phi)^2 - V(\phi) - \frac{f(\phi)}{8} F^2 \right)$$

on $5D$ black hole metric

$$\mathrm{d}s^2 = \mathrm{e}^{2A(r)}(-h(r)\mathrm{d}t^2 + \mathrm{d}\vec{x}^2) + \frac{\mathrm{e}^{2B(r)}}{h(r)}\mathrm{d}r^2$$



Solutions of gravity theory used to compute (meta-)stability

Time evolution

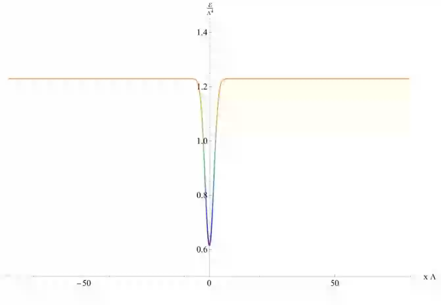
Evolving perturbation in metastable states, we observe bubbles

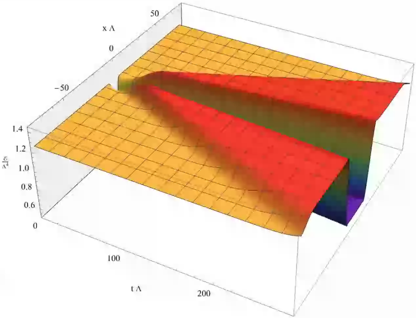
Using Julia module `Jecco.jl`, simulate time evolution in the bulk of expanding planar configurations [M.G., Mateos *et al.* 2024]

Stress tensor on the boundary $\rightarrow$ relate to thermodynamical quantities of QFT, like **energy density evolution during the phase transition**

Bubble wall velocity

Nonequilibrium parameter





Conclusions

- In settings like holographic dual of QCD-like theories, using numerical methods can yield interesting results

1. In a top-down model, we found evidence of a **screening mechanism**
   - *Setup*: quiver field theory
   - *Tool*: holographic Wilson loop
   - *Numerics*: action minimization on splines
2. In a phenomenological model, we computed **bubble wall velocity**
   - *Setup*: Einstein-Maxwell-Dilaton
   - *Tool*: bubble nucleation
   - *Numerics*: time-evolution simulation

- Moreover, the numerical tools employed are easily generalizable to other, similar, problems

Thank you!

Extra slides

Numerical methods

Programming languages used:

- MATLAB
  - → Standard for numerical modeling and analysis
  - → Intuitive vectorization and parallelization
  - → Availability of libraries, easiness to write new ones
- **julia**
  - → Compiled-level performance, fastest among high-level languages
  - → Optimized for numerical tasks
  - → Open-source and free

Holographic Wilson loop

Probing the bulk to study the boundary

Wilson loop expectation value in QFT $\Leftrightarrow$ embedding probe string in gravity dual $\rightarrow$ study **Nambu-Goto action**, will be minimized

$$S_{NG} = T_{F1} \int \mathrm{d}\sigma \mathrm{d}\tau \sqrt{-g}$$

Different submanifold embedding possible:

usual $\rightarrow x$ quark separation, r holographical direction
ours $\rightarrow x,\ r$ and z *quiver direction*

Common choice: $x(\sigma) = \sigma$

$$\Rightarrow \quad S_{NG} = T_{F1} T \int \mathrm{d}x \sqrt{F^2 + G^2 r'^2 + S^2 z'^2}$$

where $F^2,\ G^2,\ S^2$ depend on r and $\alpha(z)$, only further specification to be made

Holographic Wilson loop

Open string with endpoints on D-brane at $r \to \infty$

Distance L between endpoints $\Leftrightarrow$ separation non-dynamical $q\bar{q}$ in QFT

After regularization (two static strings, rest mass of $q\bar{q}$) can study $E(L)$

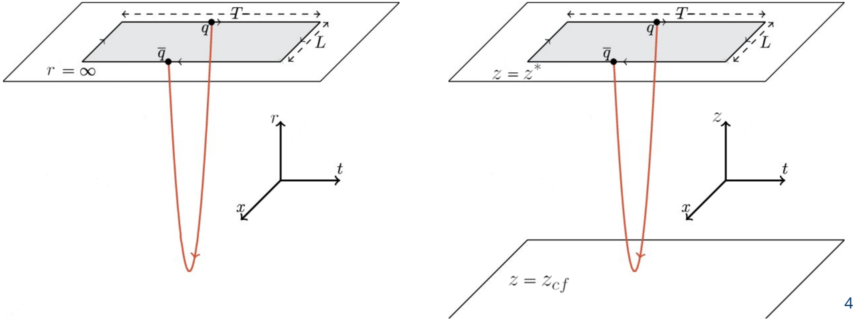
Specify $\alpha(z)$: our QFT has *flavor*, $\forall$ kink in $R(z)$ there is a flavor group (stack of localized D-branes). Endpoints on D-brane at $z = z^* \in \mathbb{N}$

$\to$ **screening**[3], creation of dynamical $Q\bar{Q}$ on flavor brane, disrupts the $q\bar{q}$ string. Even if $q\bar{q}$ sits at different z, there is interaction "pulling" the string

$$q\bar{q} \longrightarrow q\bar{Q} + Q\bar{q}$$

[3] or something closely related

Holographic Wilson loop



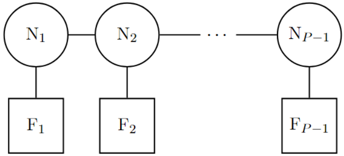
[4]

[4] courtesy of S. Chakraborty

Quiver gauge theory

Diagrammatic representation of groups and matter

Circular node N_i = gauge group $U(N_i)$
Square node F_i = flavor group $U(F_i)$
Link = matter in bifundamental



In Holography, the dual of quiver theories have **localized *D*-branes** at quiver nodes along quiver direction [Cremonesi, Tomasiello 2015]

Supergravity background: step 1

Construct the dual to the desired QFT

massive IIA $\Leftrightarrow$ $6D$ $\mathcal{N} = (1,0)$ **quiver SCFT**

$\mathrm{d}s^2_{10} = \ldots \mathrm{d}s^2_{AdS_7} + \ldots \mathrm{d}z^2 + \ldots \mathrm{d}\Omega^2$

with NS forms, Ramond field, dilaton

dependent on $\alpha(z, P)$ $\rightarrow$ specify geometry solution of mIIA EoM

linear quiver, $P - 1$ gauge nodes, anomaly-free, with flavor

described by rank function $R(z, P) \propto \alpha''(z, P)$

color and flavor group ranks are found at integer z in R and its derivatives

Supergravity background: step 2

Construct the dual to the desired QFT

compactification to AdS_5 $\Leftrightarrow$ **flow to $4D$ $\mathcal{N} = 1$ SCFT**

compactify on H_2 to preserve SUSY

flow metric, NS forms, Ramond fields, dilaton (still dependent on α) solve mIIA EoM and BPS eq

$r \to \infty$ goes to gauge transformation of the AdS_7 background

$r \to -\infty$ goes to fixed point dual to

symmmetry group $SO(2,4) \times U(1)_R \times SU(N)_{P-1}$ with flavor

still preserves SUSY (good)

still conformal (bad)

Supergravity background: step 3

Construct the dual to the desired QFT

AdS_5 **compactified on** S^1_ϕ $\Leftrightarrow$ **(2+1)D gapped QFT**

compactify on S^1_ϕ with 1-form $\mathcal{A} \propto \mathrm{d}\phi$ to preserve SUSY

$\mathrm{d}s^2_{10} = \ldots \mathrm{d}s^2_5 + \ldots \mathrm{d}z^2 + \ldots$

$\mathrm{d}s^2_5 =$

$r^2(-\mathrm{d}t^2 + \mathrm{d}\vec{x}^2 + f(r)\mathrm{d}\phi^2) + \frac{\mathrm{d}r^2}{r^2 f(r)}$

$f(r) = 1 - \frac{\mu}{r^4} - \frac{1}{r^6}$

metric and fields solve Einstein, Maxwell, Bianchi EoM

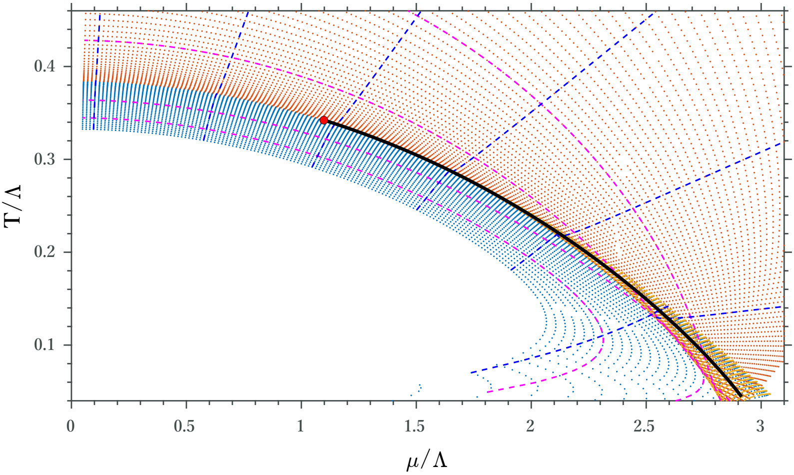
Bianchi indicates sources, $D8$

admits possibility of "screening"

can still preserves SUSY (very good)

good field theory to probe with Wilson loop

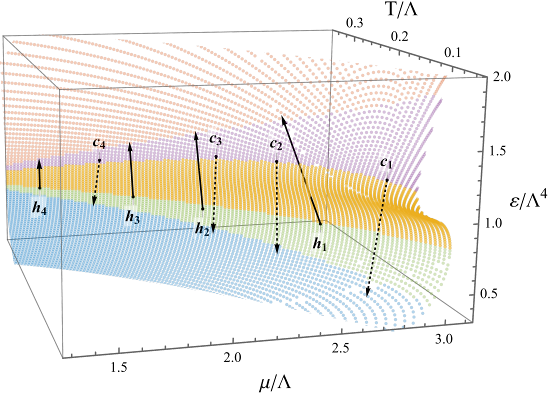
Phase diagram



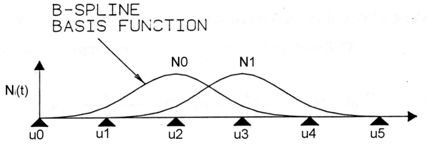
Bubble dynamics

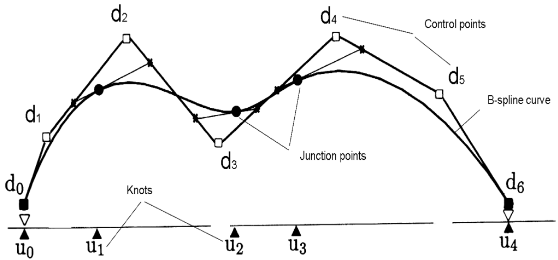
Initial state: metastable homogeneous (superheated/supercooled) with large locali-zed perturbation

After FOPT: stable state inside the bubble (hot/cold)

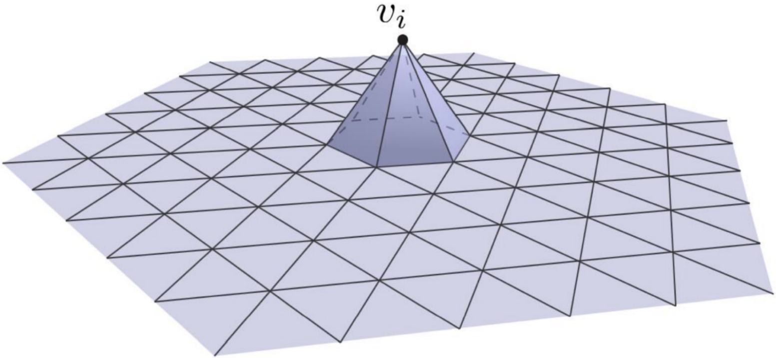


B-splines

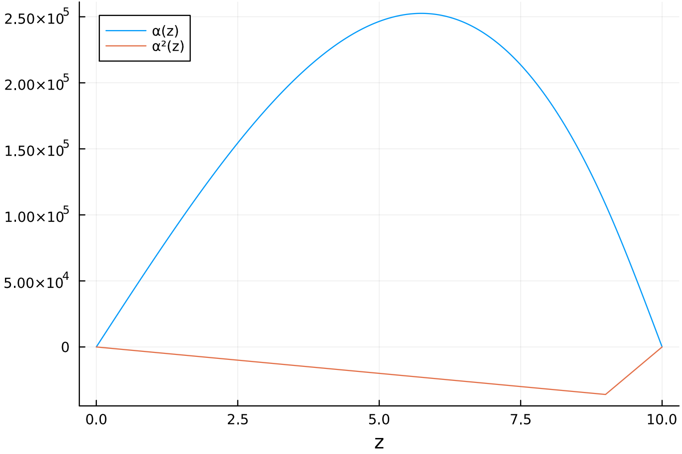

B-SPLINE
BASIS FUNCTION
N0
N1
Ni(t)
u0
u1
u2
u3
u4
u5


d2
d4
Control points
d5
d1
B-spline curve
d3
Junction points
d0
Knots
d6
u0
u1
u2
u3
u4

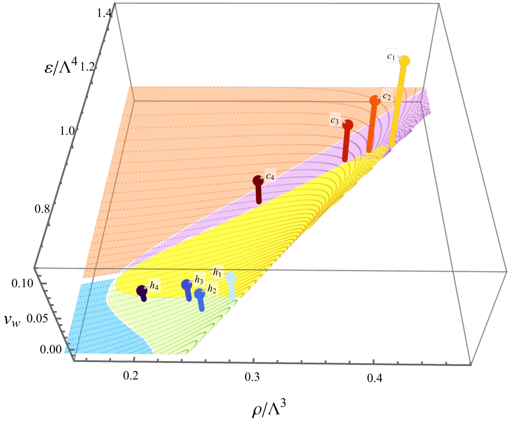
Surface hat function



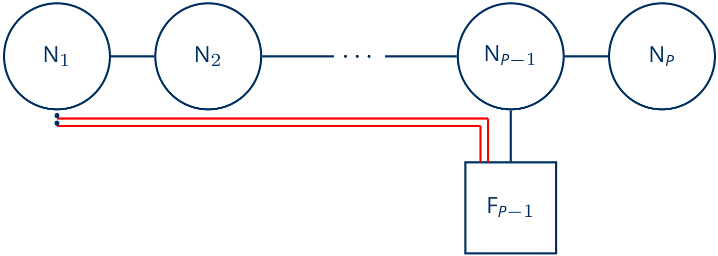
$\alpha(z)$



Wall velocity on phase diagram



Quiver pulling string



Full supergravity metric

$$ds^2 = f_1(z) ds^2_{AdS_7} + f_2(z) dz^2 + f_3(z) d\Omega^2(\theta_2, \phi_2),$$

$$B_2 = f_4(z) \mathrm{Vol}(S^2), \quad F_2 = f_5(z) \mathrm{Vol}(S^2), \quad e^{\Psi} = f_6(z).$$

$$f_1(z) = 8\sqrt{2}\pi \sqrt{-\frac{\alpha}{\alpha''}}, \quad f_2(z) = \sqrt{2}\pi \sqrt{-\frac{\alpha''}{\alpha}},$$

$$f_3(z) = \sqrt{2}\pi \sqrt{-\frac{\alpha''}{\alpha}} \left(\frac{\alpha^2}{\alpha'^2 - 2\alpha\alpha''} \right),$$

$$f_4(z) = \pi \left(-z + \frac{\alpha\alpha'}{\alpha'^2 - 2\alpha\alpha''} \right), \quad f_5(z) = \left(\frac{\alpha''}{162\pi^2} + \frac{\pi F_0 \alpha\alpha'}{\alpha'^2 - 2\alpha\alpha''} \right),$$

$$f_6(z) = 2^{\frac{5}{4}} \pi^{\frac{5}{2}} 3^4 \frac{(-\alpha/\alpha'')^{\frac{3}{4}}}{\sqrt{\alpha'^2 - 2\alpha\alpha''}}.$$

Free energy phase diagram