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Number density interpretation of dihadron fragmentation functions

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We present a new quantum field-theoretic definition of fully unintegrated dihadron fragmentation functions (DiFFs) as well as a generalized version for n-hadron fragmentation functions. We demonstrate that this definition allows certain sum rules to be satisfied, making it consistent with a number density interpretation. Moreover, we show how our corresponding so-called extended DiFFs that enter existing phenomenological studies are number densities and also derive their evolution equations. Within this new framework, DiFFs extracted from experimental measurements will have a clear physical meaning.

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\overline{OCD} factorization with multi QCD factorization with multihadron fragmentation functions

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arXiv:2305.11995v2 [hep-ph] 28 Dec 2023 1 (x, m)
T (x, m) \cdot PR D62(2000)034008 hen-n $k = 1.112$ DOL (2000) 00 1000, 110 p annihilation $\overline{52}$, semi-inelastic scatter-inelastic sc BBJR: P.R. D**62** (2000) 034008, hep-ph/9907475 with momentum *k* into two hadrons *P*¹ ,*P*² #see Fig. 3\$: BBJR: P.R. D62 (2000) 034008, hep-ph/9907475 00) 034008. hep-ph/9907475 rence and properties of ''*T* odd'' FF is to look at residual $\sum_{i=1}^n \sum_{i=1}^n \sum_{i$ %(*zh* ,+,*Ph* \$,, *^h* ,-*^h* ,*Mh* ² ,-*d*) after integrating over the #hardscale suppressed\$ light-cone component *k*# and, conse- $\mathcal{L}_{\mathcal{A}}$, the light-like $\mathcal{L}_{\mathcal{A}}$, i.e., i.e. \overline{D} , and the pair, namely on the pair, namely of the pair, namely on the relative momentum of the two hadrons (*R* \overline{D} \overline **BB**the BB **BCC** (0000) 00 1000 had the (0007175 BBJR: P.R. D62 (2000) 034008, hep-ph/9907475 which means in that case *B*5!*B*6!*B*7!*B*8!0, i.e., terms $\mathcal{L}(\mathcal{L})$ of $\mathcal{L}(\mathcal{L})$ that ensures color gauge color g invariance [2, 59]. A sum over color indices in Eq. (2) is \overline{OOZ} *inter*

\n $P_1 \quad P_2 \quad P_3 \quad P_1$ \n	\n $\text{quark-quark correlator}$ \n	\n $\Delta_{ij}(k; P_1, P_2) = \sum_{x} \int \frac{d^4x}{(2\pi)^4} e^{ikx} \langle 0 \psi_i(x) a_2^{\dagger}(P_2) a_1^{\dagger}(P_1) X \rangle \langle X a_1(P_1) a_2(P_2) \overline{\psi}_j(0) 0 \rangle$ \n	
\n $\Delta^{[\Gamma]} = \frac{1}{4z_h} \int dk^+ \int dk^- \delta \left(k^- - \frac{P_h^-}{z_h} \right) \text{Tr}[\Delta \Gamma].$ \n	\n $P_h = P_1 + P_2$ \n		
\n $\Delta (k; P_1, P_2)$ \n	\n unpolarized DIFF \n	\n $\Delta^{[\gamma^-]}(z_h, \zeta, \vec{k}_T^2, \vec{R}_T^2, \vec{k}_T \cdot \vec{R}_T) \equiv D_1(z_h, \zeta, \vec{k}_T^2, \vec{R}_T^2, \vec{k}_T \cdot \vec{R}_T)$ \n	\n $\epsilon_h = P_h^- / k^- = z_1 + z_2$ \n

$$
\text{new definition} \quad \tfrac{1}{64\pi^3z_1z_2} \text{Tr}\Big[\Delta^{h_1h_2/q}(z_1,z_2,\vec{P}_{1\perp},\vec{P}_{2\perp})\gamma^-\Big] = D_1^{h_1h_2/q}(z_1,z_2,\vec{P}_{1\perp}^2,\vec{P}_{2\perp}^2,\vec{P}_{1\perp}\cdot\vec{P}_{2\perp})
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 $\sum_{i=1}^{n}$, $\sum_{i=1}^{n}$

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annihilation [52], semi-inclusive deep-inelastic scatter-with momentum *k* into two hadrons *P*¹ ,*P*² #see Fig. 3\$:

arXiv:2305.11995v2 [hep-ph] 28 Dec 2023 gives the number density in the momentum fraction x and transverse momentum ⃗ k^T of a parton i = q or g in a nucleon N [1, 2]. Similarly, the unpolarized TMD FF Dh/i ¹ (z, ^P⃗ ² [⊥]) gives the number density in the momentum fraction z and transverse momentum P⃗[⊥] of a hadron h fragmenting from a parton i [1, 2]. Since PDFs and FFs are number densities, one can also use them to calculate expectation values (see, e.g., Refs. [6–9]). The information contained in sum rules and expectation values are important pieces to understanding hadronic structure as well as constraining or cross-checking phenomenological extractions and model calculations of PDFs and FFs. case of two hadrons h1, h² being detected from the same parton-initiated jet, i → (h1h2)X, where dihadron FFs (DiFFs) become relevant [10–50]. The quantum fieldtheoretic definition of DiFFs at the fully unintegrated level (what we will call uDiFFs) was first written down almost 25 years ago in the pioneering paper of Bianconi, Boffi, Jakob, and Radici (BBJR) [20]. This work has been the basis for all subsequent dihadron-related research for observables sensitive to the relative transverse momentum of the two hadrons [21–25, 27, 31, 32, 35– 39, 41–48]. Unfortunately, the BBJR definition does ing [53, 54], and proton-proton collisions [55, 56], and cross section and SIDIS dihadron multiplicities, one eventually will be able to perform rigorous fits of extDiFFs within QCD global analyses. These studies must tracted extDiFFs to have a clear physical meaning – see Refs. [57, 58]. briefly discussing two different reference frames that will be relevant for our analysis: the "parton frame" (p), where the fragmenting parton has no transverse momentum, transformation (see, e.g., Ref. [2] Sec. 12.4.1): V [−] ^p = V [−] ^h [≡] ^V [−]; ^V ⁺ ^p = (⃗ k^T /k−)² V [−]/2 + V ⁺ ^h [−] ⃗ ^k^T · ^V⃗^T /k−; is more practical for proofs of factorization needed for phenomenological applications. The quantum field-theoretic correlator for the fragmentation of a parton i into two hadrons h1, h2, after ing the remnant of the jet as a spectator and summing over all its possible configurations. Therefore, in the following the In the field-theoretical description of hard processes the soft parts connecting quark and gluon lines to hadrons are pair momentum carried by each individual hadron, + !*z*¹ /*zh*!1\$*z*² /*zh* , and of the four independent invariants that can be formed by means of the momenta *k*,*P*¹ ,*P*² at , *^h*!*k*2, -*h*!2*k*•#*P*1#*P*2\$.2*k*•*Ph* , !#*P*1#*P*2\$ ².*Ph* where we define the vector *R*!(*P*1\$*P*2)/2 for later use. scale suppressed\$ light-cone component *k*# and, consequently, taking & as light-like !2", i.e., ! *dk*#Tr!%/""&\$!⁰ ! *dk*# ! *dk*\$⁰ # *^k*\$\$ *Ph* The function %[/] now depends on five variables, apart from the Lorentz structure of the Dirac matrix /. In order to make this more explicit and to reexpress the set of variables in a \$!*d*-*^h dk*# , 2*k*#! *^d*, *^h dk*\$, #12\$ A. BIANCONI, S. BOFFI, R. JAKOB, AND M. RADICI PHYSICAL REVIEW D **62** 034008 quark-quark correlator namely, %*i j*#*k*;*P*¹ ,*P*2\$!X ! *^d*4& expressed in terms of specific Dirac projections of rence and properties of ''*T* odd'' FF is to look at residual %(*zh* ,+,*Ph* \$,, *^h* ,-*^h* ,*Mh* ² ,-*d*) after integrating over the #hard-%*i j*#*k*;*P*¹ ,*P*2\$!X *X* ! *^d*4& #2'\$ 4 "*eik*•& (0")*i*#&\$*a*² #*P*2\$*a*¹ #*P*1\$"*X** "(*X*"*a*1#*P*1\$*a*2#*P*2\$*¯*) *^j*#0\$"0*, #9\$ where the sum runs over all the possible intermediate states involving the two final hadrons *P*¹ ,*P*2. For the Fourier transquark momentum *k*. We choose for convenience the frame where the total pair momentum *Ph*!*P*1#*P*² has no transverse component. The constraint to reproduce on-shell hadrons with fixed mass !*M*¹ dent degrees of freedom. As shown in Appendix A #where also the light-cone components of a 4-vector are defined\$, they can conveniently be reexpressed in terms of the lightcone component of the hadron pair momentum, *Ph* light-cone fraction of the quark momentum carried by the \$/*k*\$!*z*1#*z*2, of the fraction of hadron 2*k*# ⁰ # *^k*\$\$ *Ph zh* \$!⁰ # ²*k*#*k*\$\$ ²*k*#*Ph* !⁰ # , *^h*#*k*! *^T* which leads to the result #*zh* ,+,*k*! *^T* ² ,*Mh* ² ,-*d*\$ " Tr!%#*zh* ,+,*Ph* function of *zh* ,+,*k*! *^T* transverse momentum between the two hadrons in the con-034008-4 #2'\$ "*eik*•& (0")*i*#&\$*a*² † #*P*2\$*a*¹ † #*P*1\$"*X** "(*X*"*a*1#*P*1\$*a*2#*P*2\$*¯* where the sum runs over all the possible intermediate states involving the two final hadrons *P*¹ ,*P*2. For the Fourier transform only the two space-time points 0 and & matter, i.e., the positions of quark creation and annihilation, respectively. constraint to reproduce on-shell hadrons with fixed mass (*P*¹ 2 !*M*¹ ² ,*P*² 2 !*M*² 2) reduces to seven the number of independent degrees of freedom. As shown in Appendix A #where light-cone fraction of the quark momentum carried by the hadron pair, *zh*!*Ph* \$/*k*\$!*z*1#*z*2, of the fraction of hadron which leads to the result %[/] #*zh* ,+,*k*! *^T* ! ! *^d*-*^h ^d*, *^h*⁰ # , *^h*#*k*! *^T* "(*X*"*a*1#*P*1\$*a*2#*P*2\$*¯*) *^j*#0\$"0*, #9\$ where the sum runs over all the possible intermediate states involving the two final hadrons *P*¹ ,*P*2. For the Fourier transform only the two space-time points 0 and & matter, i.e., the positions of quark creation and annihilation, respectively. Their relative distance & is the conjugate variable to the quark momentum *k*. (*P*¹ !*M*¹ ² ,*P*² !*M*²) reduces to seven the number of independent degrees of freedom. As shown in Appendix A #where hadron pair, *zh*!*Ph* task suggests that a more convenient way to model occurrence and properties of ''*T* odd'' FF is to look at residual interactions between two hadrons in the same jet, considering the remnant of the jet as a spectator and summing over all its possible configurations. Therefore, in the following the formalism for two-hadron semi-inclusive production and FF **III. QUARK-QUARK CORRELATION FUNCTION FOR** defined as certain matrix elements of non-local operators involving the quark and gluon fields themselves !17–19". In with momentum *k* into two hadrons *P*¹ ,*P*² #see Fig. 3\$: %(*zh* ,+,*Ph* \$,, *^h* ,-*^h* ,*Mh* ² ,-*d*) after integrating over the #hardscale suppressed\$ light-cone component *k*# and, consequently, taking& as light-like !2", i.e., %[/] ! ¹ 4*zh* ! *dk*#Tr!%/""&\$!⁰ ! ¹ 4*zh* ! *dk*# ! *dk*\$⁰ # *^k*\$\$ *Ph zh* \$ Tr!%/". #11\$ this more explicit and to reexpress the set of variables in a more convenient way, let us rewrite the integrations in Eq. #11\$ in a covariant way using and the relation 2*k*# ⁰ # *^k*\$\$ *Ph zh* \$!⁰ # ²*k*#*k*\$\$ ²*k*#*Ph* \$ *zh* \$ interactions between two hadrons in the same jet, considering the remnant of the jet as a spectator and summing over all its possible configurations. Therefore, in the following the formalism for two-hadron semi-inclusive production and FF **III. QUARK-QUARK CORRELATION FUNCTION FOR TWO-HADRON PRODUCTION** In the field-theoretical description of hard processes the soft parts connecting quark and gluon lines to hadrons are quark correlation function describing the decay of a quark with momentum *k* into two hadrons *P*¹ ,*P*² #see Fig. 3\$: "*eik*•& (0")*i*#&\$*a*² † #*P*2\$*a*¹ #*P*1\$"*X** where the sum runs over all the possible intermediate states scale suppressed\$ light-cone component *k*# and, consequently, taking & as light-like !2", i.e., %[/] ! ¹ 4*zh* ! *dk*#Tr!%/""&\$!⁰ ! ¹ 4*zh* ! *dk*# ! *dk*\$⁰ # *^k*\$\$ *Ph* \$ *zh* \$ Tr!%/". #11\$ The function %[/] now depends on five variables, apart from the Lorentz structure of the Dirac matrix /. In order to make this more explicit and to reexpress the set of variables in a 2*Ph* \$!*d*-*^h dk*# , 2*k*#! *^d*, *^h dk*\$, #12\$ and the relation !⁰ # , *^h*#*k*! *^T* ²\$ -*^h zh* # *zh* which leads to the result tive momentum of the two hadrons (*R*! *^T* ²) and on the relative orientation between the pair plane and the quark jet axis ² , *k*! *^T*•*R*! *^T* , see also Fig. 4\$. **IV. ANALYSIS OF INTERFERENCE FRAGMENTATION FUNCTIONS** If the polarizations of the two final hadrons are not ob-!%*k*;*P*¹ ,*P*2\$!*B*1%*M*1"*M*2\$"*B*2*P*" ¹"*B*3*P*" ²"*B*4*k*" &'(*P*1'*P*2()5*'(+&) '*P*1(*P*2+*k*& . %15\$ *Bi* *!*Bi* for *i*!1, . . . ,4, *Bi* tive momentum of the two hadrons (*R*! *^T* ²) and on the relative which means in that case *B*5!*B*6!*B*7!*B*8!0, i.e., terms involving *B*⁵ ,...,*B*⁸ are naive ''*T* odd.'' Inserting the ansatz %15\$ in Eq. %14\$ and reparametrizing the momenta *k*,*P*¹ ,*P*² with the indicated new set of variables, we get the following Dirac projections: ![)#] %*zh* ,#,*k*! *^T* ² ,*R*! *^T* ² ,*k*! *^T*•*R*! *^T*\$! ¹ 2*zh* ! -*d*&*hd*. *^h*/" *^B*2#"*B*3%1##\$"*B*⁴ *zh* # , %19\$![)#) 5] , *M*1*M*² *G*1 !%*zh* ,#,*k*! *^T* ² ,*R*! *^T* ² ,*k*! *^T*•*R*! *^T*\$! * *^T i jRTikT j M*1*M*² ! -*d*&*hd*. *^h*/-#*B*8/, %20\$ orientation between the pair plane and the quark jet axis ² , *k*! *^T*•*R*! *^T* , see also Fig. 4\$. **IV. ANALYSIS OF INTERFERENCE FRAGMENTATION** If the polarizations of the two final hadrons are not observed, the quark-quark correlation !(*k*;*P*¹ ,*P*2) of Eq. %9\$ can be generally expanded, according to Hermiticity and par-!%*k*;*P*¹ ,*P*2\$!*B*1%*M*1"*M*2\$"*B*2*P*" ¹"*B*3*P*" ²"*B*4*k*" " *B*5 &'(*P*1'*k*(" *B*6 *M*² &'(*P*2'*k*(&'(*P*1'*P*2(" *B*8)5*'(+&) '*P*1(*P*2+*k*& . %15\$ Symmetry constraints are obtained in the form which means in that case *B*5!*B*6!*B*7!*B*8!0, i.e., terms involving *B*⁵ ,...,*B*⁸ are naive ''*T* odd.'' Inserting the ansatz %15\$ in Eq. %14\$ and reparametrizing ables, we get the following Dirac projections: ![)#] %*zh* ,#,*k*! *^T* ,*D*1%*zh* ,#,*k*! *^T* ² ,*R*! *^T* ² ,*k*! *^T*•*R*! *^T*\$![)#) 5] %*zh* ,#,*k*! *^T* ² ,*R*! *^T* ² ,*k*! *^T*•*R*! *^T*\$! * *^T i jRTikT j M*1*M*² ![*i*&*i*#) 5] %*zh* ,#,*k*! *^T* ² ,*R*! *^T* ² ,*k*! *^T*•*R*! *^T*\$ BBJR: P.R. D**62** (2000) 034008, hep-ph/9907475 unpolarized DiFF positions of quark creation and annihilation, respectively. Their relative distance & is the conjugate variable to the momentum *Ph*!*P*1#*P*² has no transverse component. The constraint to reproduce on-shell hadrons with fixed mass ² ,*P*² 2 !*M*² 2) reduces to seven the number of independent degrees of freedom. As shown in Appendix A #where also the light-cone components of a 4-vector are defined\$, they can conveniently be reexpressed in terms of the lightall its possible configurations. Therefore, in the following the formalism for two-hadron semi-inclusive production and FF will be addressed. **III. QUARK-QUARK CORRELATION FUNCTION FOR** In the field-theoretical description of hard processes the defined as certain matrix elements of non-local operators involving the quark and gluon fields themselves !17–19". In analogy with semi-inclusive hard processes involving one detected hadron in the final state !2", the simplest matrix element for the hadronization into two hadrons is the quarkquark correlation function describing the decay of a quark fixed masses *M*¹ ,*M*2, i.e., , *^h*!*k*2, -*h*!2*k*•#*P*1#*P*2\$.2*k*•*Ph* , where we define the vector *R*!(*P*1\$*P*2)/2 for later use. By generalizing the Collins-Soper light-cone formalism !18" for fragmentation into multiple hadrons !12,11", the cross section for two-hadron semi-inclusive emission can be expressed in terms of specific Dirac projections of %[/] ! ¹ 4*zh* ! *dk*#Tr!%/""&\$!⁰ #11\$ in a covariant way using 2*Ph* \$!*d*-*^h dk*# , 2*k*#! *^d*, *^h dk*\$, #12\$ FIG. 3. Quark-quark correlation function for the fragmentation of a quark into a pair of hadrons. (*P*¹ 2 !*M*¹ ² ,*P*² 2 !*M*² 2) reduces to seven the number of independent degrees of freedom. As shown in Appendix A #where they can conveniently be reexpressed in terms of the lightcone component of the hadron pair momentum, *Ph* light-cone fraction of the quark momentum carried by the hadron pair, *zh*!*Ph* \$/*k*\$!*z*1#*z*2, of the fraction of hadron new definition mental representation of SU(3) that ensures color gauge implied. For a gluon, Nⁱ = N² invariance [2, 59]. A sum over color indices in Eq. (2) is implied. For a gluon, Nⁱ = N² ^c − 1, and Oh1h2/g αβ (ξ) = ⟨0|Wba(∞, ^ξ)F^a ⁺α(ξ⁺, ⁰−, ⃗ξ⊥)|P1, P2; ^X⟩ × ⟨P1, P2; ^X|F^c ⁺β(0⁺, ⁰−,⃗0⊥)Wcb(0, [∞])|0⟩, (3) ^µ^ν = ∂µA^a ^ν [−] [∂]νA^a ^µ + gf abcA^b µA^c strength tensor involving the gluon field A, and the Wilson lines are now in the adjoint representation of SU(3). Throughout this Letter we focus on the production of 1 64π³z1z² Tr\$ ∆h1h2/q(z1, z2, P⃗¹⊥, P⃗²⊥)γ[−] % (4) 32π3z1z2P [−] = D^h1h2/g ¹ (z1, z2, ^P⃗ ² ¹⊥, ^P⃗ ² ²⊥, ^P⃗1[⊥] ·P⃗2⊥), where z = z¹ + z² is the total momentum fraction of the dihadron and P^h = P¹ + P2. As we will show in the next section, the prefactor of 1/(64π3z1z2) in Eq. (4) ^Nˆh^j [≡] (2π)³ 2P [−] j aˆ† h^j aˆh^j = by having the specific prefactors on the l.h.s. of Eqs. (4), (5). Indeed, a derivation is not possible if a prefactor of 1/(4z)=1/(4(z¹ + z2)) is used on the l.h.s. of Eq. (4). The result in Eq. (6) gives a clear interpretation for the uDiFFs we defined in Eqs. (4), (5): they are fragmentation process for an unpolarized quark (γ[−] projection of the correlator). The number density inter-Material. We can also derive a momentum sum rule involving uDiFFs and TMD FFs, O^h1h2/g αβ (ξ) = ⟨0|Wba(∞, ^ξ)F^a ⁺α(ξ⁺, ⁰[−], ⃗ξ⊥)|P1, P2; ^X⟩ × ⟨P1, P2; ^X|F^c ⁺β(0⁺, ⁰[−],⃗0⊥)Wcb(0, [∞])|0⟩, (3) where F^a ^µ^ν = ∂µA^a ^ν [−] [∂]νA^a ^ν is the field strength tensor involving the gluon field A, and the Wilson lines are now in the adjoint representation of SU(3). Throughout this Letter we focus on the production of unpolarized hadrons. For the fragmentation of an unpolarized parton, we parameterize the correlator in Eq. (1) = D^h1h2/q ¹ (z1, z2, ^P⃗ ² ¹⊥, ^P⃗ ² ²⊥, ^P⃗¹[⊥] ·P⃗²⊥), where z = z¹ + z² is the total momentum fraction of the dihadron and P^h = P¹ + P2. As we will show in the next section, the prefactor of 1/(64π3z1z2) in Eq. (4) is crucial to justifying the number density interpretation of the quark uDiFFs (and similarly for the gluon case for each hadron (j = 1 or 2). This can only be achieved for the uDiFFs we defined in Eqs. (4), (5): they are densities in the momentum fractions z1, z² and transverse momenta P⃗¹⊥, P⃗²[⊥] for the number of hadron (2π)³ ^eik·^ξ ^Oh1h2/i αβ (ξ) # # # ^ξ−=0 , momentum carried by each hadron, and P⃗¹⊥, P⃗²[⊥] are the parton model framework. The proofs of the sum rules in this section are left for Supplemental Material. We focus first on the number sum rule, " ^dPS ^Dh1h2/i ¹ (z1, z2, ^P⃗ ² ¹⊥, ^P⃗ ² ²⊥, ^P⃗1⊥·P⃗2⊥) = ⟨^N (^N [−] 1)⟩, (6) where & dPS = ' ' h² & 1 ⁰ dz² & ¹−z² ⁰ dz¹ d2P⃗¹[⊥] & d2P⃗²⊥, parton model framework. The proofs of the sum rules in this section are left for Supplemental Material. We focus " ^dPS ^Dh1h2/i ¹ (z1, z2, ^P⃗ ² ¹⊥, ^P⃗ ² ²⊥, ^P⃗1⊥·P⃗2⊥) = ⟨^N (^N [−] 1)⟩, where & dPS = ' h¹ ' h² & 1 ⁰ dz² & ¹−z² ⁰ dz¹ & d2P⃗1[⊥] & d2P⃗2⊥, and N is the total number of hadrons produced when the total # of hadron pairs sum rule recover number density interpretation i with momentum (z2, P⃗2⊥). Further integrating Eq. (8) over P⃗2[⊥] yields The momentum sum rule Eq. (9) was first put forth in Refs. [11, 14]. We also mention that the study of DiFFs has a close connection to double PDFs (DPDFs), where two partons emerge from a single nucleon. Indeed, an analogous sum rule to Eq. (9) exists for DPDFs, as was derived in Refs. [62, 63]. The quantum field-theoretic derivation of the sum rule Eq. (8) at the unintegrated (transverse momentum dependent) operator level (from which Eq. (9) follows immediately) is a new aspect pre-One can readily generalize to n-hadron (n ≥ 1) fragmentation in a way that retains a number density inter-^T ⁼ ¹−ζ² ⁴ M² ^h [−] ¹−^ζ ² M² ¹ [−] 1+^ζ change of reference frames, one naturally thinks of uDiFFs as now depending on (z, ζ,⃗ sider making a change of variables from (z1, z2, P⃗1⊥, P⃗2⊥) to (w, x, Y , ⃗ Z⃗), where we understand w, x to be scalars is a number density in (w, x, Y , [|]∂(z1, z2, ^P⃗1⊥, ^P⃗2⊥)/∂(w, x, Y , ⃗ ^Z⃗)[|] is the Jacobian for the change of variables from (z1, z2, P⃗1⊥, P⃗2⊥) to (w, x, Y , *ζ ζ ^ζ* ⁼ (*z*¹ [−] *^z*2)/*zh*

1 (k)
k2
k2

transverse momenta of the hadrons relative to the parton.

with momentum *k* into two hadrons *P*¹ ,*P*² #see Fig. 3\$: 1eralizable to n-hadron ca $\frac{1}{4(16\pi^3)^{n-1}z_1\cdots z_n} \text{Tr}\Big[$ form only the two space-time points 0 and & matter, i.e., the generalizable to n-nadron case: $\frac{1}{\sqrt{1-\frac{1$ $\mathcal{L}^{\text{max}}(\mathcal{L}^{\text{max}}(\mathcal{L}^{\text{max}}(\mathcal{L}^{\text{max}}(\mathcal{L}^{\text{max}}(\mathcal{L}^{\text{max}}(\mathcal{L}^{\text{max}}(\mathcal{L}^{\text{max}}(\mathcal{L}^{\text{max}}(\mathcal{L}^{\text{max}}(\mathcal{L}^{\text{max}}(\mathcal{L}^{\text{max}}(\mathcal{L}^{\text{max}}(\mathcal{L}^{\text{max}}(\mathcal{L}^{\text{max}}(\mathcal{L}^{\text{max}}(\mathcal{L}^{\text{max}}(\mathcal{L}^{\text{max}}(\mathcal{$ %*k*;*P*¹ ,*P*2\$)0!!%*k*;*P*¹ ,*P*2\$ from Hermiticity, $\binom{n-1}{2}, \cdots, \binom{n}{n}$ of the quark unit up the gluon case \mathbf{a} $\frac{1}{\sqrt{(12.2 \cdot 2) \cdot 1}}$ Tr $\left| \frac{1}{\sqrt{11}} \right|$ $4(10\pi^3)^{n+1}z_1\cdots z_n$ L tation \mathcal{I}_1 , \mathcal{I}_2 , \mathcal{I}_3 would not retain a set of retain and retain a set of retain and retain a set parton i fragments. Thus, ⟨N (N −1)⟩ is the expectation case: $\frac{1}{\sqrt{(16-3)x-1}}$ Tr $\mathcal{H}(10\pi^{\circ})^{\alpha-1}\mathcal{Z}_1\cdots\mathcal{Z}_n$ L ϵ -hadron case in the ϵ in ϵ the fraction of $\frac{1}{4(16\pi^3)n-1}$ over $\frac{1}{x}$ $i(10n) \approx 1 \approx n$ generalizable to n-hadron case: 1 $4(16\pi^3)^{n-1}z_1\cdots z_n$ $\text{Tr}\left[\ldots \right]$

$$
E_{\rm F}(s, \zeta_{F}) = \int d^2 \mathbf{q}_{T} K(\mathbf{q}_{T}; \mu) F_{f/P_{1}}(x_{1}, \mathbf{k}_{T} - \mathbf{q}_{T}; \mu, \zeta_{F})
$$

BBJR definition of correlator compatible with factorization theorem $\alpha, \zeta_F) = \gamma_F(g(\mu); \zeta_F/\mu^2)$ *d*q*^T ···* $\partial \left({\bm k}_{T};\mu \right) = - \gamma_K(g(\mu)) \, \delta({\bf k}_{T}) \, ,$ $q \neq k$ and $\left\langle k \right\rangle$ $\not k_2$ $J_2(\tilde k_2)$ $J_1(\tilde{k}, \{p_h\})$ $f(\mu,\zeta_F) = \gamma_F(g(\mu); \zeta_F \sqrt{\mu \mu^2})^{\chi} F_f/\c{P}_1}(x_1,\mathbf k_T;\mu,\zeta_F).$ ${\bf k}_*({\bf k}_T)\,\overleftarrow{\equiv}\, {\bf \hat k}_{\overline{k}_c}$ $\overline{}$ $k_{min}^2+k_{\vert T}^2$ $\mu_*(k_T) \equiv C_1 k_*$ other familiar forms. To simplify the discussion we will continue to assume that each hadron in a dihadron pair has $2E_{p_{h_1}}(2\pi)^3 2E_{p_{h_2}}(2\pi)$ $\frac{3}{\pi}$ d σ $\mathrm{d}^3\bm{p}_{h_1}$ $\mathrm{d}^3\bm{p}_{h_2}$ = \int_0^1 *z* $\mathrm{d}\xi$ ξ^2 $\sqrt{ }$ $2E_{\hat k}(2\pi)$ $_3$ $\mathrm{d}\hat{\sigma}$ $\mathrm{d}^3\bm{\hat{k}}$ ◆ $d(\xi, \{p_h\}) + \text{p.s.}$ with the fragmentation defined as in Eq. (81). For factorization to the kinematics need to be restricted as in \mathbb{R}^n to *M*² *^h/z* ⌧ *^Q*². A standard way to characterize the dihadron momentum that is common to many treatments is with the variables ζ_i k^+ p_h^+ k^+ z^0 $\xi_i =$ $p_{h_i}^+$ *k*+ = $p_{h_i}^+$ *p*+ *h* $ξ = ξ₁ + ξ₂$

 p_h^+

k+

=

zi

z ξ

$$
\kappa_s(\mu_*(k_T)) \stackrel{k_T\to 0}{=} \alpha_s(C_1 k_{min})
$$

$$
\mathbf{b}_*(\mathbf{b}_T) \equiv \frac{\mathbf{b}_T}{\sqrt{1+b_T^2/b_{\max}^2}}
$$

$$
\mu_*(b_T) = C_1/b_*
$$

 $\alpha_s(\mu_*(b_T))$ ^{*b*} $T \equiv \infty$ </sub> $\alpha_s(C_1/b_{max})$

$$
\frac{d\sigma}{d\mathbf{q}_T\,\cdots}
$$

 P_1 P_2 *P*¹ *P*²

$$
,\zeta_F)=\int d^2\mathbf{q}_T\,K(\mathbf{q}_T;\mu)\,F_{f/P_1}(x_1,\mathbf{k}_T-\mathbf{q}_T;\mu,\zeta_F)
$$

BBJR definition of correlator compatible with factorization theorem $\alpha, \zeta_F) = \gamma_F(g(\mu); \zeta_F/\mu^2)$. *d*q*^T ···* $\beta(k_{T};\mu)=-\gamma_{K}(g(\mu))\,\delta(\mathbf{k}_{T})$ $q \neq k$ and $\left\langle k \right\rangle$ $\not k_{2}$ $J_2(\tilde k_2)$ $J_1(\tilde{k}, \{p_h\})$ $\mu, \zeta_F) = \gamma_F(g(\mu); \zeta_F \sqrt{\mu \mu^2})^2 F_f / P_1(x_1, \mathbf{k}_T; \mu, \zeta_F)$ $F_n(\epsilon \sim 1.11)^{1/2}$ $E_{1}(s,s,\mathbf{w}_{T};m_{h}) = \frac{1}{4\xi} \int$ $\int_s^s (\mu_*(k) \cdot k) \cdot k \cdot T \to 0$ $\alpha_s(C_1 k_{min})$ $h_{\rm T}$ \rightarrow $\frac{1}{2}$ $\frac{1}{2}$ $\mathbf{b}_*(\mathbf{b}_T) \equiv \frac{\mathbf{b}_T}{\mathbf{b}_T}$ $\mathbf{b}_*(\mathbf{b}_T) \equiv \frac{\Sigma_1}{\sqrt{1+b_T^2/b_{\max}^2}}$ $\mathbf{v} = \mathbf{1}$, max $\mathbf{r} = \mathbf{1}$ 4ξ) is insensitive to infrared contributions. Here the demonstrated by can be defined by calculating its lowest order of η μ (b_{as}) $\equiv C_1/b$ $\frac{d}{dx}$ $\overline{d\mathbf{q}_T\,\cdot\cdot\cdot}$ */* ˆ *k^µ/* $\mathbf{k}_*(\mathbf{k}_T) \stackrel{\vee}{\equiv} \mathbf{\hat{k}}_{\overline{k}}$ $\overline{}$ $k_{min}^2+k_{\vert T}^2$ $\mu_*(k_T) \equiv C_1 k_*$ \mathbf{b}_T $\sqrt{1+b_T^2/b_{\max}^2}$ $\mu_*(b_T) = C_1/b_*$ $(\mu_*(b_T))$ ^{*b*} $T \rightrightarrows \infty$ </sub> $\alpha_s(C_1/b_{max})$ $d\sigma$ d **q***T* \cdots other familiar forms. To simplify the discussion we will continue to assume that each hadron in a dihadron pair has $2E_{p_{h_1}}(2\pi)^3 2E_{p_{h_2}}(2\pi)$ $\frac{3}{\pi}$ d σ $\mathrm{d}^3\bm{p}_{h_1}$ $\mathrm{d}^3\bm{p}_{h_2}$ = \int_0^1 *z* $\mathrm{d}\xi$ ξ^2 $\sqrt{ }$ $2E_{\hat k}(2\pi)$ $_3$ $\mathrm{d}\hat{\sigma}$ $\mathrm{d}^3\bm{\hat{k}}$ ◆ $d(\xi, \{p_h\}) + \text{p.s.}$ with the fragmentation defined as in Eq. (81). For factorization to the kinematics need to be restricted as in \mathbb{R}^n to *M*² *^h/z* ⌧ *^Q*². $\frac{1}{T}$ standard wave that is momental treatments is $\frac{1}{T}$, we can rewrite the dependence of the $\frac{1}{T}$ ζ_i k^+ p_h^+ k^+ z^0 p_{h} , M_{h}^{2}) = $\frac{1}{4\xi} \int dk^{+}$ *I***r** $[\Delta(k, P_{h}, R) \gamma^{-}]|_{k=-h}$ $\int d^2k + \int d^2k + \int d^2k$ *, z*¹ ⌘ *^Q*² *, z*² ⌘ \mathbf{r} \mathbf{y} \mathbf{z} $\mathbf{z$ \mathbf{r} *p*+ *h*1*,H* $\frac{1}{\sqrt{5}}$ dk_1 *p*+ *h*1*,H* $\frac{1}{4}$ Tr γ $+\tilde{k}$ γ $\frac{p_h}{\gamma}$ $\Big\vert$ $\Big\vert$ *k*+ *H* $\sum_{i=1}^{n}$ *p*+ *h*2*,H p*+ *h,H* $\xi_i =$ $p_{h_i}^+$ *k*+ = $p_{h_i}^+$ p_h^+ *h* p_h^+ *k*+ = *zi z ξ* $ξ = ξ_1 + ξ_2$ ition of correlator compatible with factorization theorem restricts to the analysis to the parties with redeematical model so that the can be identified with the kinematical α \mathcal{T} $\mathcal{L}_{\mathcal{A}}$ would be to translate the equation in the following way: $(x_1, \mathbf{k}_T; \mu, \zeta_F)$ $_{2E_{p_h}(2\pi)^3 2E_{p_h}(2\pi)^3 \frac{1}{\pi^2}}$ correlator as (⇠*,* ⇣*, k^T , M*² $p_{h_i}^+$ is $p_{h_i}^+$ in $p_{h_i}^+$ in $p_{h_i}^+$ is the aximuthal and p_{h_i} of *R^T* with respect to the reaction plane. κ can be extracted from the p₁ can be extracted from the following through the following through the following the following through the following through the following through the following through the following th $D_1(\xi, \zeta, \bm{k}_T, M_h^2) = \frac{1}{4\kappa}$ 4ξ dk^+ Tr $\left[\Delta(k, P_h, R) \, \gamma^- \right] \left. \right|_{k^- = P_h^- / \xi}$ $=\frac{1}{4}$ 4ξ $\int dk^+$ $\frac{d\mathcal{K}}{(2\pi)^4}$ Tr $[J_1(k,\{p_h\})\,\gamma^{-}]|_{k^-={P_h^-}/{\xi}}$ The factor of 1*/*(4⇠) follows from the same factorization-based reasoning as in Sec. IV and Sec. V of this paper. On $\begin{bmatrix} \n\text{max} \\ \n\text{max} \n\end{bmatrix}$ the notation in Eq. (B2), which shows that $\begin{bmatrix} \n\text{max} \\ \n\text{max} \n\end{bmatrix}$ $\begin{bmatrix} \n\text{max} \\ \n\text{max} \n\end{bmatrix}$ $t = \frac{1}{4\pi} \int \frac{d\kappa_H}{(2\pi)^4} \text{Tr} \left[\gamma^+ \tilde{k} \right]$ d²*k*^T *D*1(⇠*,* ⇣*, k^T , M*² *X ^d* (⇠*,* ⇠*kH*T*, {ph}*)h*k*1*|k*2ⁱ ⁼ ¹ 2⇠(2⇡)³ The d*Y* has just been generalized to a multihadron phase space. Using the expressions for *b † ^k* and *b^k* in terms of the quark field operator (*x*) (and retracing the steps in Sec. (6.7.3) of Ref. [22]) puts the definition in the more familiar $\forall NN$ and \forall in the final state with *n* not necessarily equal to 1, \int_0^1 \int_0^1 $\mathrm{d}k^+$ \int_0^1 $\mathrm{d}k^+$ \int_0^1 $f(x) = 4\xi \int (2\pi)^{4}$ into Feynman rules, the Feynman rules, the function $f(x)$ $=\frac{1}{4\xi}$ $\int \frac{\mathrm{d}k_{H}^{-}}{(2\pi)^{4}} \mathrm{Tr}% \left\vert \mathcal{H}\right\vert ^{2}$ $\sqrt{2}$ $\overline{1}$ $\overline{1}$ $\overline{1}$ $\overline{1}$ 4 γ^+ \tilde{k} $\overbrace{\qquad \qquad \qquad }^{p_h}$ $\qquad \qquad \ddots$ ˜ *k* 3 \mathcal{L} $\overline{1}$ $\overline{1}$ \mathcal{L} $\overline{1}$ k_H^+ $_{H}^{+}$ = p_{h}^{+} $_{h,H}^{+}/\xi$ $\frac{p_h}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\right]$

˜ *k*

 P_1 P_2 P_1 P_2

$$
,\zeta_F)=\int d^2\mathbf{q}_T\,K(\mathbf{q}_T;\mu)\,F_{f/P_1}(x_1,\mathbf{k}_T-\mathbf{q}_T;\mu,\zeta_F)
$$

BBJR definition of correlator compatible with factorization theorem $\alpha, \zeta_F) = \gamma_F(g(\mu); \zeta_F/\mu^2)$. *d*q*^T ···* $\beta(k_{T};\mu)=-\gamma_{K}(g(\mu))\,\delta(\mathbf{k}_{T})$ $q \neq k$ and $\left\langle k \right\rangle$ $\not k_{2}$ $J_2(\tilde k_2)$ $J_1(\tilde{k}, \{p_h\})$ $\mu, \zeta_F) = \gamma_F(g(\mu); \zeta_F \sqrt{\mu \mu^2})^2 F_f / P_1(x_1, \mathbf{k}_T; \mu, \zeta_F)$ $F_n(\epsilon \sim 1.11)^{1/2}$ $E_{1}(s,s,\mathbf{w}_{T};m_{h}) = \frac{1}{4\xi} \int$ $\int_s^s (\mu_*(k) \cdot k) \cdot k \cdot T \to 0$ $\alpha_s(C_1 k_{min})$ h_{th} and h_{th} is put the overall factor into a more recognizable form. $\mathbf{b}_*(\mathbf{b}_T) \equiv \frac{\mathbf{b}_T}{\mathbf{b}_T}$ $\mathbf{b}_*(\mathbf{b}_T) \equiv \frac{\Sigma_1}{\sqrt{1+b_{\rm m}^2/b_{\rm max}^2}}$ $\mathbf{v} = \mathbf{1}$, max $\mathbf{r} = \mathbf{1}$ same factor as in single-hadron \overline{a} \overline{a} k ² was sepin-summed in the spin-summer subset of the spin-supplier subgraph is $\frac{1}{2}$ \overline{a} o‹ $\frac{3}{2}$ *n*e irres $\overline{}$ Ξ Z ve of th 2 */* ˜ *k*2 $\overline{1}$ same irrespective of the kind of final state phase same irrespective of the kind of final $\frac{d\mathbf{x}}{d\mathbf{x}}$ */* ˆ *k^µ/* ˆ *k*2⌫ that $M_h^2 \ll Q^2$ and *z* is fixed $\frac{d\sigma}{dt}$ ² *^J*1(˜ $-\zeta \frac{d\mathbf{q}}{\mathrm{max}}\zeta =$ $\frac{1}{\epsilon} \sqrt{1 - \frac{(m_{h_1} + m_{h_2})^2}{M^2}} \le \zeta_{\text{max}} \Rightarrow \zeta =$ $=\frac{z_1 - z_2}{z_1 - z_2}$ limited Here, we have also replaced *z*ⁿ in the expressions for ˜ P_1 P_2 $\mathbf{k}_*(\mathbf{k}_T) \stackrel{\vee}{\equiv} \mathbf{\hat{k}}_{\overline{k}}$ $\overline{}$ $k_{min}^2+k_{\vert T}^2$ $\mu_*(k_T) \equiv C_1 k_*$ \mathbf{b}_T $\sqrt{1+b_T^2/b_{\max}^2}$ *µ*⇤(*b^T*) = *C*1*/b*⇤ *D*1(⇠*,* ⇣*, M*² fragmentation: factorization works the $(\mu_*(b\phi)$ adromi ω state $(1,2)$... n hadrons) provided $d\sigma$ d **q** \mathcal{I} **z** ζ \div P_1 P_2 other familiar forms. To simplify the discussion we will continue to assume that each hadron in a dihadron pair has $2E_{p_{h_1}}(2\pi)^3 2E_{p_{h_2}}(2\pi)$ $\frac{3}{\pi}$ d σ $\mathrm{d}^3\bm{p}_{h_1}$ $\mathrm{d}^3\bm{p}_{h_2}$ = \int_0^1 *z* $\mathrm{d}\xi$ ξ^2 $\sqrt{ }$ $2E_{\hat k}(2\pi)$ $_3$ $\mathrm{d}\hat{\sigma}$ $\mathrm{d}^3\bm{\hat{k}}$ ◆ $d(\xi, \{p_h\}) + \text{p.s.}$ with the fragmentation defined as in Eq. (81). For factorization to the kinematics need to be restricted as in \mathbb{R}^n to *M*² *^h/z* ⌧ *^Q*². $\frac{1}{T}$ standard wave that is momental treatments is $\frac{1}{T}$, we can rewrite the dependence of the $\frac{1}{T}$ ζ_i k^+ p_h^+ k^+ z^0 p_{h} , M_{h}^{2}) = $\frac{1}{4\xi} \int dk^{+}$ *I***r** $[\Delta(k, P_{h}, R) \gamma^{-}]|_{k=-h}$ $\int d^2k + \int d^2k + \int d^2k$ *, z*¹ ⌘ *^Q*² *, z*² ⌘ $\mathcal{L}_{\mathcal{S}}$ \mathcal{J} $(2n)$ \mathbf{r} *p*+ *h*1*,H* $\frac{1}{\sqrt{5}}$ dk_1 *p*+ *h*1*,H* $\frac{1}{4}$ Tr γ $+\tilde{k}$ γ $\frac{p_h}{\gamma}$ $\Big\vert$ $\Big\vert$ *k*+ *H* $\sum_{i=1}^{n}$ *p*+ *h*2*,H p*+ *h,H* $\xi_i =$ $p_{h_i}^+$ *k*+ = $p_{h_i}^+$ p_h^+ *h* p_h^+ *k*+ = *zi z ξ* $ξ = ξ_1 + ξ_2$ ition of correlator compatible with factorization theorem restricts to the analysis to the parties with redeematical model so that the can be identified with the kinematical α \mathcal{T} $\mathcal{L}_{\mathcal{A}}$ would be to translate the equation in the following way: $(x_1, \mathbf{k}_T; \mu, \zeta_F)$ $_{2E_{p_h}(2\pi)^3 2E_{p_h}(2\pi)^3 \frac{1}{\pi^2}}$ correlator as (⇠*,* ⇣*, k^T , M*² $p_{h_i}^+$ is $p_{h_i}^+$ in $p_{h_i}^+$ in $p_{h_i}^+$ is the aximuthal and p_{h_i} of *R^T* with respect to the reaction plane. κ can be extracted from the p₁ can be extracted from the following through the following through the following the following through the following through the following through the following through the following th $D_1(\xi, \zeta, \bm{k}_T, M_h^2) = \frac{1}{4\kappa}$ 4ξ dk^+ Tr $\left[\Delta(k, P_h, R) \, \gamma^- \right] \left. \right|_{k^- = P_h^- / \xi}$ $=\frac{1}{4}$ 4ξ $\int dk^+$ $\frac{d\mathcal{K}}{(2\pi)^4}$ Tr $[J_1(k,\{p_h\})\,\gamma^{-}]|_{k^-={P_h^-}/{\xi}}$ Γ *i* \sim 1^{*/(400)* for the same factorization-based reasoning as in Sec. II and Sec. V of this paper. On this} $\begin{bmatrix} \n\text{max} \\ \n\text{max} \n\end{bmatrix}$ the notation in Eq. (B2), which shows that $\begin{bmatrix} \n\text{p}_h \\ \n\text{p}_h \n\end{bmatrix}$ is a shows that $\begin{bmatrix} \n\text{p}_h \\ \n\text{p}_h \n\end{bmatrix}$ is a shows that $\begin{bmatrix} \n\text{p}_h \\ \n\text{p}_h \n\end{bmatrix}$ is a shown t $\begin{aligned} \mathcal{L}_{\mathbf{A}}^{\mathbf{A}}(\mathbf{A}) = \frac{1}{4\pi} \int \frac{\mathrm{d}\kappa_{H}}{(2\pi)^{4}} \mathrm{Tr} \left[\gamma^{+} \tilde{k} \right] \end{aligned}$ d²*k*^T *D*1(⇠*,* ⇣*, k^T , M*² *X ^d* (⇠*,* ⇠*kH*T*, {ph}*)h*k*1*|k*2ⁱ ⁼ ¹ 2⇠(2⇡)³ The d*Y* has just been generalized to a multihadron phase space. Using the expressions for *b † ^k* and *b^k* in terms of the quark field operator (*x*) (and retracing the steps in Sec. (6.7.3) of Ref. [22]) puts the definition in the more familiar $\forall NN$ and \forall in the final state with *n* not necessarily equal to 1, \int_0^1 \int_0^1 $\mathrm{d}k^+$ \int_0^1 $\mathrm{d}k^+$ \int_0^1 $\int_{-\infty}^{\infty} 4\xi \int_{-\infty}^{\infty} (2\pi)^{4}$ into Feynman rules, the function $\int_{-\infty}^{\infty} 4\xi \int_{-\infty}^{\infty} (2\pi)^{4}$ is the function function is the function function function function is the function of the function of the fun $=\frac{1}{4\xi}$ $\int \frac{\mathrm{d}k_{H}^{-}}{(2\pi)^{4}} \mathrm{Tr}$ $\sqrt{2}$ $\overline{1}$ $\overline{1}$ $\overline{1}$ $\overline{1}$ 4 γ^+ \tilde{k} $\overbrace{\qquad \qquad \qquad }^{p_h}$ $\qquad \qquad \ddots$ ˜ *k* 3 \mathcal{L} $\overline{1}$ \mathcal{L} \mathcal{L} $\overline{1}$ k_H^+ $_{H}^{+}$ = p_{h}^{+} $_{h,H}^{+}/\xi$ $-\zeta \frac{d\mathbf{q}}{dx} \zeta = \sqrt{1 - \frac{(m_{h_1} + m_{h_2})^2}{M_h^2}} \le \zeta_{\text{max}} \Rightarrow \zeta = \frac{z_1 - z_2}{z}$ limited $\frac{p_h}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\right]$ ˜ *k* M_h^2 $\leq \zeta_{\text{max}} \Rightarrow \zeta =$ *z*₁ − *z*₂ *z* limited

critique of JAM: 1. z_1 , z_2 , are external kinematical variables, not to be confused with parton momentum fractions $ξ_1$, $ξ_2$:

$$
z_i = \frac{2p_{h_i} \cdot q}{q^2} \qquad \qquad \overrightarrow{\text{only in parton model approx.}} \qquad \xi_i = \frac{p_{h_i}^+}{k^+} = \frac{p_{h_i}^+}{p_h^+} \frac{p_h^+}{k^+} = \frac{z_i}{z} \xi
$$

parton momentum fractions ξi must be used in definition of correlators

We define *p*+ *h*1*,H* 2*E^h*¹ (2⇡) ³2*E^h*² (2⇡) where *dependent indenandant on datails of nonnarturhative final state* by an extra 1 - hard part not independent on details of nonperturbative final state - changes evolution kernel

critique : critique :
(continued) 3. generalization to n hadrons $\frac{1}{4(16\pi^3)^{n-1}}$ 4(16*π*3)*n*−1*ξ*1*ξ*² . . *ξⁿ* $\text{Tr}[\ldots]$ α the standard evolution equation equation equation given in Eq. (132), but we are are definition given in Eq. (132), but we are definition given in Eq. (132), but we are definition given in Eq. (132), but we are defin unable to retrace the steps based on the information provided there. If we use Eq. (132), then we instead find that at led) of gonoral extra factor of the hard part of the state for the model in the model of the model in the model $f(10\mu)$ $5152 \cdot Sn$ $2E_{p_{h_1}}(2\pi)^3 2E_{p_{h_2}}(2\pi)$ $\frac{3}{\pi}$ d σ = \int_1^1 $\mathrm{d}\xi$ $\sqrt{ }$ $2E_{\hat k} (2\pi)^3 \!\!\!\! \int\limits_{\rm d3i}^{\rm d\hat \sigma}$ ◆ $d_{\text{mod}}(\xi, \{p_h\}) + \text{p.s.}$ *ξn*−¹

 $\mathrm{d}^3\bm{p}_{h_1} \ \mathrm{d}^3\bm{p}_{h_2}$

z

 ξ^2

waxaan by aling of factorization worsen breaking of factorization..

 $\mathrm{d}^3\hat{k}$

4. if we have really to interpret JAM formula as $\frac{1}{\sqrt{1-\lambda}}$ Tr[...]

4. if we have really to interpret JAM formula as
$$
\frac{1}{64\pi^3 z_1 z_2} \text{Tr}[\ldots]
$$

From Eq. (137), retracing the steps from Eqs. (103)–(110)–(110)–(110)–(110)–(109) α kinematic factors

2*Eh*(2⇡) d³*p^h* d*M^h* = on would de d³*k*ˆ *d*mod,red,1(⇠*, Mh*) + p.s. *.* (139) depend on q => breaking universality lid Z p formul *zd*^ˆ *zd*^ˆ ⌦ *^d*mod*,*red*,*1(⇠*, Mh*) + p.s. + *^O* (↵*s*) then operator definition would depend on process because z_1 , z_2 (and still factorization formula would be broken by a 1/ξ term…)