Heavy-quark mass effects in the energy-energy correlation in the back-to-back region

Giancarlo Ferrera

Milan University & INFN, Milan

Based:

U.G. Aglietti & G.F.

Phys.Rev.D 110 (2024) 11, 114004, e-Print: 2403.04077

Eur.Phys.J.C 85 (2025) 3, 272, e-Print: 2412.02629

IV Sardinian Worskshop on Spin

Pula – 12/6/2025

Heavy-quark mass effects in the Sudakov form factor

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Energy-Energy Correlation (EEC) function

$$\mathrm{e^+ + e^- \;\to\; h_i + h_j + X}$$

$$\frac{d\Sigma}{d\cos\chi} = \sum_{i,j=1}^{n} \int \frac{E_i}{Q}\frac{E_j}{Q}\,\delta(\cos\chi - \cos\theta_{ij})\, d\sigma_{e^+e^-\to h_i h_j + X}$$

where $Q = \sqrt{s}$ and θ_{ij} is the angle between momenta $\vec{p}_i$ and $\vec{p}_j$ [Basham,Brown, Ellis,Love('78)].

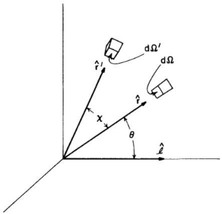

FIG. 2. Geometry for the experiment.

- EEC is IRC safe ($E_iE_j/Q^2 \overset{E_i\to 0}{\to} 0$, $xE_i + (1-x)E_i = E_i$). While $d\sigma$ depends on parton fragmentation functions $D_{h,q}$, EEC does not: $\sum_h \int_0^1 dx\, x\, D_{h,q}(x,\mu_F^2) = 1$.
  EEC calculable in pure pQCD.
- Normalization gives

$$\int_{-1}^{+1} \frac{d\Sigma}{d\cos\chi}\, d\cos\chi = \int \left(\sum_{i=1}^{n} \frac{E_i}{Q}\right)^2 d\sigma = \sigma_{tot}.$$

- In the CoM frame at $\mathcal{O}(\alpha_S^0)$ we have a back-to-back $q\bar{q}$ pair:

$$\frac{1}{\sigma_{tot}}\frac{d\Sigma}{d\cos\chi} = \frac{1}{2}\delta(1-\cos\chi) + \frac{1}{2}\delta(1+\cos\chi) + \mathcal{O}(\alpha_S).$$

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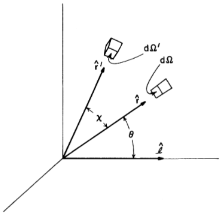

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EEC in fixed-order pQCD

At higher orders in QCD we have (we use $z = 1 - y = (1 - \cos\chi)/2 = \sin^2(\chi/2)$):

$$\frac{1}{\sigma_{tot}}\frac{d\Sigma}{dz} = \frac{1}{2}\left(\delta(1-z) + \delta(z)\right) + \frac{\alpha_S}{\pi}\mathcal{A}(z) + \left(\frac{\alpha_S}{\pi}\right)^2\mathcal{B}(z) + \left(\frac{\alpha_S}{\pi}\right)^3\mathcal{C}(z) + \mathcal{O}(\alpha_S^4)\,,$$

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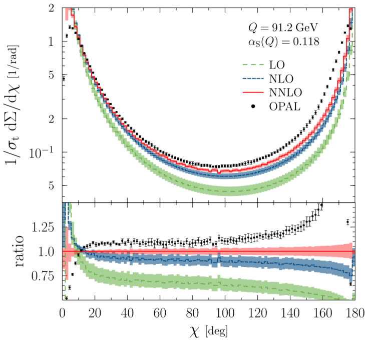
- The $\mathcal{O}(\alpha_S)$ function $\mathcal{A}(z)$ is known analytically from [Basham et al.('78)].
- At $\mathcal{O}(\alpha_S^2)$ function $\mathcal{B}(z)$ known analytically by [Dixon et al.('18)] (numerically by [Richards et al.('82,83)]).
- The $\mathcal{O}(\alpha_S^3)$ function $\mathcal{C}(z)$ known numerically by [Del Duca et al.('16)],[Tulipant,Kardos, Somogyi('17)] from (fully differential) NNLO calculation of 3-jets cross-section in e^+e^- ann. using CoLoRFulNNLO subtraction method [Del Duca et al.('16)].



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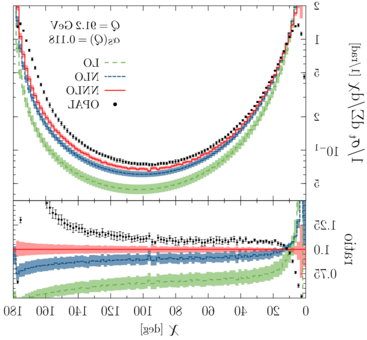

EEC in the back-to-back limit

In the back-to-back limit $z \to 1$ ($y \to 0$, $\chi \to \pi$) we have

$$\mathcal{A}(z) = C_F \left\{ -\frac{1}{2}\left[\frac{\ln(1-z)}{1-z}\right]_+ - \frac{3}{4}\left[\frac{1}{1-z}\right]_+ - \left(\frac{\pi^2}{12} - \frac{11}{8}\right)\delta(1-z) + \cdots . \right\}$$

- In general at any order α_S^n large infrared (Sudakov) logarithms appears

$$\alpha_S^n \left[\frac{\ln^k(1-z)}{1-z}\right]_+ , \quad 0 \le k \le 2n-1$$

- Large logs spoils the convergence of fixed-order perturbative expansion. Reliable QCD predictions requires all order Sudakov resummation.
- In the back-to-back region the q_T between 2 hadrons is

$$y = 1 - z = \cos^2(\chi/2) \simeq q_T^2/Q^2 \to 0$$

  and EEC is closely related to Drell–Yan process at small-q_T.
- EEC function also contains large (single) logarithmic corrections of hard-collinear nature in the forward region $z \to 0$ (or $\chi \to 0$), $\ln^{n-1}(z)/z$, where hadrons have small angular separations [Dixon et al.('19)].

EEC in the back-to-back limit

ECC can be written in terms of the unitegrated parton fragmentation functions $D_{h,q}(x, p_T, Q)$ [Bassetto,Ciafaloni,Marchesini('80)].

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CAN SOFT GLUON EFFECTS BE MEASURED IN ELECTRON-POSITRON ANNIHILATION? *

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ABSTRACT

The energy-energy correlation at large angles in e^+e^- annihilation is calculated by resumming soft gluon contributions through two-loop level. The result is compared with experimental data. No agreement is obtained using a purely perturbative analysis. The relevance of nonperturbative effects at present energies is emphasized.

Using the unintegrated parton densities[4,8] $D(Q^2, p_T, x)$ the energy-energy correlation is written as[9] (see Fig. 1)

$$\frac{1}{\sigma_{TOT}} \frac{d\Sigma}{d^2Q_T} = \frac{1}{\sigma_{TOT}} \frac{1}{2} \sum_{A,B} \int x_A dx_A \int x_B dx_B \sum_{q,\bar{q}} \int d^2p_T^A \, d^2p_T^B \, d^2p_T^S \times \delta^2\left(Q_T - \frac{p_T^A}{x_A} - \frac{p_T^B}{x_B} - p_T^S\right) D_q^A\left(Q^2, p_T^A, x_A\right) D_{\bar{q}}^B\left(Q^2, p_T^B, x_B\right) S\left(Q^2, p_T^S\right) \quad (1)$$

$$Q^2 \frac{\partial}{\partial Q^2} D_q\left(Q^2, b_T, x\right) = \int \frac{dz}{z} \int dq_T^2 \left[\frac{\alpha_s(q_T^2)}{2\pi} + K\left(\frac{\alpha_s(q_T^2)}{2\pi}\right)^2\right] \times C_F\left(\frac{1+z^2}{1-z}\right)_+ \delta\left[z(1-z)Q^2 - q_T^2\right] J_0\left(\frac{bq}{z}\right) D_q\left(Q^2, \frac{b_T}{z}, \frac{x}{z}\right) , \quad (2)$$

Soft gluon exponentiation

Sudakov resummation feasible when:
dynamics AND kinematics factorize
$\Rightarrow$ exponentiation.

- Dynamics factorization: general property of matrix element for soft emissions based on colour coherence. QCD analogous [Gatheral('83)],[Frenkel,Taylor('84)], [Catani,Ciafaloni('84,'85)] of eikonal approximation in QED [Bloch,Nordsieck('37)],[Yennie,Frautschi,Suura('61)]

$$dw_n(q_1,\dots,q_n) \simeq \frac{1}{n!}\prod_{i=1}^{n} dw_1(q_i)$$

- Kinematics factorization: not valid in general. For q_T distribution it holds in the impact parameter space (Fourier transform)[Parisi,Petronzio('79)]

$$\int d^2\mathbf{q_T}\,\exp(-i\mathbf{b}\cdot\mathbf{q_T})\,\delta^{(2)}\left(\mathbf{q_T}-\sum_{j=1}^{n}\mathbf{q_{T_j}}\right) = \exp(-i\mathbf{b}\cdot\sum_{j=1}^{n}\mathbf{q_{T_j}}) = \prod_{j=1}^{n}\exp(-i\mathbf{b}\cdot\mathbf{q_{T_j}})\,.$$

- Exponentiation holds in the impact parameter space. Results have then to be transformed back to the physical space: $q_T \ll M \Leftrightarrow Mb \gg 1$, $\log M/q_T \gg 1 \Leftrightarrow \log Mb \gg 1$.

Sudakov resummation for EEC

[Kodaira,Trentadue('81)],
[Collins,Soper('83),de Florian,Grazzini('04),Tulipant et al.('17),Kardos et al.('18)]

$$\frac{1}{\sigma_{tot}}\frac{d\Sigma}{dz} = \frac{1}{\sigma_{tot}}\frac{d\Sigma_{res.}}{dz} + \frac{1}{\sigma_{tot}}\frac{d\Sigma_{fin.}}{dz}\ ;$$

In the impact parameter (b) space: $1-z \ll 1 \Leftrightarrow Qb \gg 1$, $\ln(1-z) \gg 1 \Leftrightarrow \ln(Qb) \gg 1$

$$\frac{1}{\sigma_{tot}}\frac{d\Sigma_{res.}}{dz} = \frac{1}{4} H(\alpha_S) \int_0^\infty db\, Q^2\, b\, J_0(\sqrt{1-z}\,Qb)\, S(Q,b),$$

$$S(Q,b) \ = \ e^{-w(Q,b)} = \exp\left\{-\int_{b_0^2/b^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2}\left[A(\alpha_S(k_\perp^2))\ln\frac{Q^2}{k_\perp^2} + B(\alpha_S(k_\perp^2))\right]\right\}$$

$$H(\alpha_S) = \sum_{n=1}^{\infty} H_n\,\alpha_S^n, \quad A(\alpha_S) = \sum_{n=1}^{\infty} A_n\,\alpha_S^n, \quad B(\alpha_S) = \sum_{n=1}^{\infty} B_n\,\alpha_S^n\ .$$

- Analogous results have been obtained, more recently, in the framework of SCET framework [Moult,Zhu('18)],[Ebert et al.('20)],[Duhr et al.('22)]

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Sudakov resummation in pQCD

Distinctive features of the formalism [Catani,de Florian,Grazzini('01)], [Bozzi,Catani,de Florian,Grazzini('03)],[de Florian,Grazzini('04)]:

$$S(Q,b) \;=\; \exp\left\{ L\,g_1(\lambda) + g_2(\lambda) + \frac{\alpha_S}{\pi} g_3(\lambda) + \left(\frac{\alpha_S}{\pi}\right)^2 g_4(\lambda) + \cdots \right\}$$

$$\text{with} \qquad L = \ln(Q^2 b^2/b_0^2), \quad \lambda = \beta_0 \alpha_S L/\pi \lesssim 1, \quad b_0 = 2e^{-\gamma_E} \simeq 1.123$$

LL ($\sim \alpha_S^n L^{n+1}$): g_1; NLL ($\sim \alpha_S^n L^n$): g_2, H_1; $\cdots$ N^{k}LL ($\sim \alpha_S^n L^{n+k-1}$): g_{k+1}, H_k;

- Introduction of resummation scale $\mu_Q \sim Q$ [Bozzi at al. ('03)]: variations give an estimate of the uncertainty from uncalculated logarithmic corrections.

$$\ln(Q^2 b^2) \to \ln(\mu_Q^2 b^2) + \ln(Q^2/\mu_Q^2)$$

- Perturbative unitarity constraint: recover *exactly* the fixed-order total cross-section (upon integration on z)

$$\ln(Q^2 b^2/b_0^2) \to \widetilde{L} \equiv \ln(Q^2 b^2/b_0^2 + 1) \;\Rightarrow\; \exp\{\alpha_S^n \widetilde{L}^k\}\big|_{b=0} = 1 \;\Rightarrow\; \int_0^1 dz \left(\frac{d\sigma}{dz}\right) = \sigma_{tot};$$

- No explicit NP models: Landau singularity of α_S regularized using a *Minimal Prescription* without power-suppressed corrections [Catani et al.('96)],[Laenen et al.('00)].

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Function-series representation

The explicit functions up to NNLL are:

$$
\begin{aligned}
g_1(\lambda) &= \frac{A_1}{\beta_0} \frac{\lambda + \ln(1-\lambda)}{\lambda}; \\
g_2(\lambda) &= -\frac{A_2}{\beta_0^2} \left[\frac{\lambda}{1-\lambda} + \ln(1-\lambda) \right] + \frac{A_1 \beta_1}{\beta_0^3} \left[\frac{\ln(1-\lambda) + \lambda}{1-\lambda} + \frac{1}{2} \ln^2(1-\lambda) \right] \\
&\quad + \frac{B_1}{\beta_0} \ln(1-\lambda); \\
g_3(\lambda) &= \frac{A_3}{2\beta_0^2} \frac{\lambda^2}{(1-\lambda)^2} - \frac{B_2}{\beta_0} \frac{\lambda}{1-\lambda} - \frac{A_2 \beta_1}{2\beta_0^3} \frac{\lambda(2-3\lambda) + 2(1-2\lambda)\ln(1-\lambda)}{(1-\lambda)^2} \\
&\quad + \frac{A_1 \beta_2}{2\beta_0^3} \left[\frac{\lambda(2-3\lambda)}{(1-\lambda)^2} + 2\ln(1-\lambda) \right] + \frac{B_1 \beta_1}{\beta_0^2} \frac{\lambda + \ln(1-\lambda)}{1-\lambda} \\
&\quad + \frac{A_1 \beta_1^2}{2\beta_0^4} \frac{\lambda^2 + (1-2\lambda)\ln^2(1-\lambda) + 2\lambda(1-\lambda)\ln(1-\lambda)}{(1-\lambda)^2}.
\end{aligned}
$$

$$
\text{with} \quad A_1 = C_F \,, \quad B_1 = -\frac{3}{2} C_F \,, \cdots .
$$

Remark: $g_k(\lambda)$ are singular at $\lambda = 1$ (i.e. $b \simeq 1/\Lambda_{QCD}$).

EEC resummation: perturbative accuracy

- Coefficients A_1, A_2, A_3, A_4, B_1, B_2 and H_1 already known [Basham et al.('78)],Kodaira,Trentadue('82),de Florian,Grazzini('04),Becher,Neubert('11). We have determined the new coefficients B_3 and H_2 and H_3 in full QCD from results in SCET [Ebert,Mistlberger,Vita('20)].
- We thus performed all-order resummation up to N^{3}LL logarithmic accuracy all orders (i.e. up to $\exp(\sim\alpha_S^n L^{n-2})$) including hard-virtual contribution up to factor N^{3}LO.
- Matching with NNLO corrections (i.e. up to $\mathcal{O}(\alpha_S^3)$) from results in [Del Duca et al.('16)],[Tulipant,Kardos,Somogyi('17)];
- Results up to N^{3}LO (i.e. up to $\mathcal{O}(\alpha_S^3)$) recovered for the total cross section (from unitarity).
- N^{3}LO results also includes four-loop QCD coupling $(\beta_0, \beta_1, \beta_2, \beta_3)$.
- pQCD prediction fully determined from the knowledge of $\alpha_S(m_Z^2)$.

Finite (remainder) function

Remainder function obtained subtracting the asymptotic expansion from the f.o. result:

$$\frac{1}{\sigma_{\rm tot}}\frac{d\Sigma_{\rm (fin.)}}{dz} = \frac{1}{\sigma_{\rm tot}}\frac{d\Sigma}{dz} - \frac{1}{\sigma_{\rm tot}}\frac{d\Sigma_{\rm (res.)}}{dz}\bigg|_{\rm f.o.} = \mathcal{A}_{\rm (fin.)}(z)\,\frac{\alpha_S}{\pi} + \mathcal{B}_{\rm (fin.)}(z)\left(\frac{\alpha_S}{\pi}\right)^2 + \mathcal{C}_{\rm (fin.)}(z)\left(\frac{\alpha_S}{\pi}\right)^3 + \cdots$$

$$\mathcal{A}_{\rm (fin.)}(z) = -\frac{2}{3\,z^5}\left(z^4 + z^3 - 3z^2 + 15z - 9\right)\ln(1-z) - \frac{z^3 + z^2 + 7z - 6}{z^4}\,,$$

$$\mathcal{B}_{\rm (fin.)}(z) = \frac{1080z^6 - 3240z^5 + 4164z^4 - 2924z^3 + 1134z^2 - 229z + 1}{9z(1-z)}\,z_3 + \cdots + \cdots$$

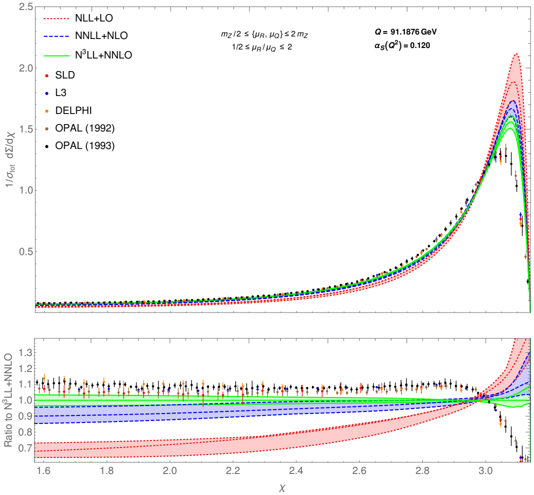
Third-order remainder function fitted with the following function:

$$\mathcal{C}_{\rm (fin.)}(z) \approx 15\ln^5(1-z) + 130\ln^4(1-z) + 408\ln^3(1-z) + 544\ln^2(1-z) + 308\ln(1-z) + 226 + 0.70545\,\frac{\ln^2(z)}{z} - 15.494\,\frac{\ln(z)}{z} + 39.568\,\frac{1}{z}\,,$$

where the terms enhanced for $z \to 0$ known from analysis in [Dixon et al.('19)].

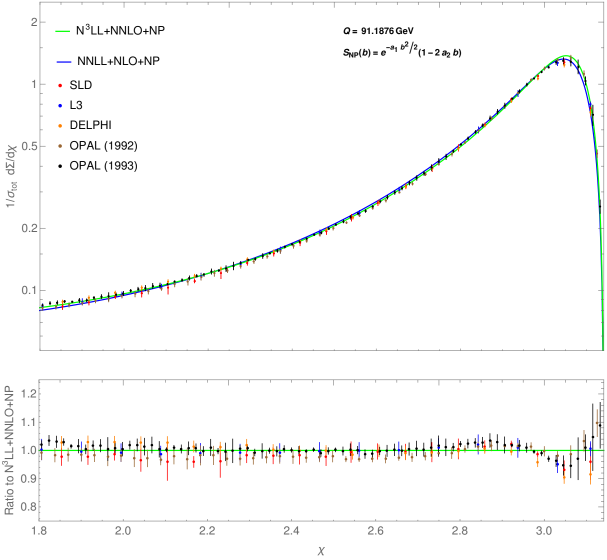
Similar results obtained with the unitarity constraint.

Numerical results: perturbative results



Resummed EEC at $\sqrt{s} = 91.1876$ GeV at various perturbative orders in QCD.

Numerical results: non perturbative effects



Comparison at N^{3}LL+NNLO and NNLL+NLO with NP effects parameterized by a form factor

$$S(Q,b) \to S(Q,b) S_{NP}(b)$$

$$S_{NP} = \exp\{-a_1 b^2\}(1 - a_2 b)$$

[Dokshitzer,Marchesini, Webber('99)]

Fit results

NNLL+NLO:
$\alpha_S(m_Z) = 0.121 \pm 0.002$,
$a_1 = 1.9 \pm 1.4\ GeV^2$,
$a_2 = 0.4 \pm 0.1\ GeV$

N^{3}LL+NNLO:
$\alpha_S(m_Z) = 0.120 \pm 0.002$,
$a_1 = 1.8 \pm 1.4\ GeV^2$,
$a_2 = 0.3 \pm 0.1\ GeV$

Mass effects in EEC

[Csikor('84)],[Ali,Barreiro('84)]

- at $Q = m_Z$, $\left(\frac{2m_b}{Q}\right)^2 \sim 0.01 \sim (\alpha_S(m_Z^2))^2$;
- at $Q = 30\,\text{GeV}$, $\left(\frac{2m_b}{Q}\right)^2 \sim 0.1 \sim \alpha_S(Q^2)$.

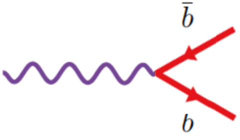
Two distinct effects:

- **primary** decay into massive quarks $e^+e^- \to b\bar{b}$:

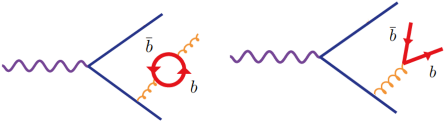
  Suppression factors for flavor-inclusive:

  $\frac{e_b^2}{\sum_{f=1}^5 e_f^2} \sim 0.09$, for γ^*-exchange

  $\frac{V_d^2+A_d^2}{2(V_u^2+A_u^2)+3(V_d^2+A_d^2)} \sim 0.22$, for Z-exchange



- **secondary** splittings of (real or virtual) massive quark pair $g \to b\bar{b}$ (affecting both $e^+e^- \to b\bar{b}$ and $e^+e^- \to q\bar{q}$).



Mass effects in EEC at back-to-back

Starting from the (effective) one-soft gluon emission rate from a massive quark ($m \ll Q$) in the quasi-collinear limit ($k_\perp \to 0$, $k_\perp/m \to$ const.):

$$\frac{C_F}{\pi}\int_0^{Q^2} dk_\perp^2 \alpha_S(k_\perp^2) \int_{k_\perp/Q}^{1} \frac{dz}{k_\perp^2 + z^2 m^2}\left[\frac{2}{z} - 2 + z - \frac{2m^2 z(1-z)}{k_\perp^2 + z^2 m^2}\right]\left[1 - J_0\left(bk_\perp\right)\right]$$

$$\to \quad w_m(Q,b) = \int_{m^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2}\left\{\ln\left(\frac{Q^2}{k_\perp^2}\right) A\left[\alpha_S(k_\perp^2)\right] + B\left[\alpha_S\left(k_\perp^2\right)\right]\right\}\left[1 - J_0\left(bk_\perp\right)\right]$$

$$+ \quad \int_0^{m^2} \frac{dk_\perp^2}{k_\perp^2}\left\{\ln\left(\frac{Q^2}{m^2}\right) A\left[\alpha_S(k_\perp^2)\right] + D\left[\alpha_S(k_\perp^2)\right]\right\}\left[1 - J_0\left(bk_\perp\right)\right]$$

where $\quad 1 - J_0\left(b\,k_\perp\right) \simeq \theta\left(k_\perp^2 - \frac{b_0^2}{b^2}\right)$ and

$$D(\alpha_S) = \sum_{n=1}^{\infty} D_n\,\alpha_S^n \qquad \text{produced by soft, large-angle emissions}$$

Massive Sudakov form factor

$$S_m(Q,b) = \exp\{-w_m(Q,b)\} = S(Q,b)\,\theta(b \le b_{cr}) \; + \; S_L(Q,b)\,\theta(b > b_{cr})\,,$$

where $b_{cr} = b_0/m\,,$

$$\begin{aligned} S_L(Q,b) \quad &= \quad S(Q,b_{cr})\,\exp\left\{\ln\left(\frac{Q^2}{m^2}\right)\left(f_1(\rho) + \frac{\alpha_S(m^2)}{\pi} f_2(\rho) + \cdots\right)\right. \\ &+ \quad \left.\left(h_2(\rho) + \frac{\alpha_S(m^2)}{\pi} h_3(\rho) + \cdots\right)\right\} \end{aligned}$$

where $\rho = \beta_0 \alpha_S(m^2)\ln(m^2 b^2/b_0^2)/\pi$ (with $\ln(m^2 b^2/b_0^2) = L - \ln(Q^2/m^2)$)

$$\lim_{b \to b_{cr}} S_L(Q,b) = S(Q,b_{cr}) \text{ (continuity)};$$

$$\lim_{m \to 0} S_m(Q,b) = S(Q,b) \text{ (massless limit)}$$

Secondary mass effects included in $S_L(Q,b)$ by removing a massless quark for $k_\perp < m$ i.e. reducing the number of active flavors n_f.

Function series representation

The explicit functions up to NNLL$_m$ are:

$$f_1(\rho) = \frac{A_1}{\beta_0} \ln(1-\rho)\,;$$

$$f_2(\rho) = -\frac{A_2}{\beta_0}\frac{\rho}{1-\rho} + \frac{A_1\beta_1}{\beta_0^2}\frac{\ln(1-\rho)+\rho}{1-\rho}\,;$$

$$f_3(\rho) = \frac{-A_3}{2\beta_0}\frac{\rho(2-\rho)}{(1-\rho)^2} + \frac{A_2\beta_1}{2\beta_0^2}\frac{2\ln(1-\rho)+\rho(2-\rho)}{(1-\rho)^2} - \frac{A_1\beta_2}{2\beta_0^2}\frac{\rho^2}{(1-\rho)^2} + \frac{A_1\beta_1^2}{2\beta_0^3}\frac{\rho^2-\ln^2(1-\rho)}{(1-\rho)^2}\,;$$

$$h_2(\rho) = +\frac{D_1}{\beta_0}\ln(1-\rho)\,;$$

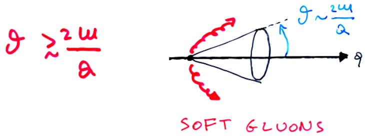
$$h_3(\rho) = -\frac{D_2}{\beta_0}\frac{\rho}{1-\rho} + \frac{D_1\beta_1}{\beta_0^2}\frac{\ln(1-\rho)+\rho}{1-\rho}\,;$$

$$\text{with} \quad D_1 = -C_F\,, \quad D_2 = \frac{C_F}{2}\left[C_A\left(z_2 - z_3 - \frac{49}{18}\right) + \frac{5}{9}n_f\right].$$

Remark: $f_k(\rho)$ and $h_{k+1}(\rho)$ have same functional form.

Relation with the dead cone

Collinear mass effects ($f_k(\rho)$ functions) can be obtained by simply imposing the (classical!) dead-cone effect. Collinear radiation screened inside a cone $\theta_{DC} = 2m/Q$ by heavy-quark mass ($\ln Q^2/m^2$ replace $\ln Q^2 b^2/b_0^2$).



Soft large-angle functions $h_k(\rho)$ (depending on $D(\alpha_S)$ coefficients) modify the dead cone [Dokshitzer,Khoze,Troian,('91,'96)]:

$$\text{At NLL}_m: \ln(Q^2/m^2)\, f_1(\rho) + h_2(\rho) = \ln(e^{D_1/A_1} Q^2/m^2)\, f_1(\rho) = \ln(Q^2/(\sqrt{e}m)^2)\, f_1(\rho)\,,$$

$$\text{and} \quad \theta_{DC} = 2m/Q \to \sqrt{e}\, 2m/Q \simeq 1.65\, \theta_{DC}$$

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$$\simeq \ln(Q^2/(1.65m)^2)\, f_1(\rho)|_{A_1} + \ln(Q^2/(1.8m)^2)\, f_1(\rho)|_{A_2}\,,$$

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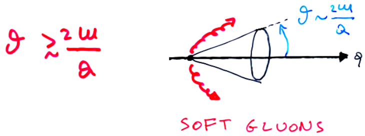

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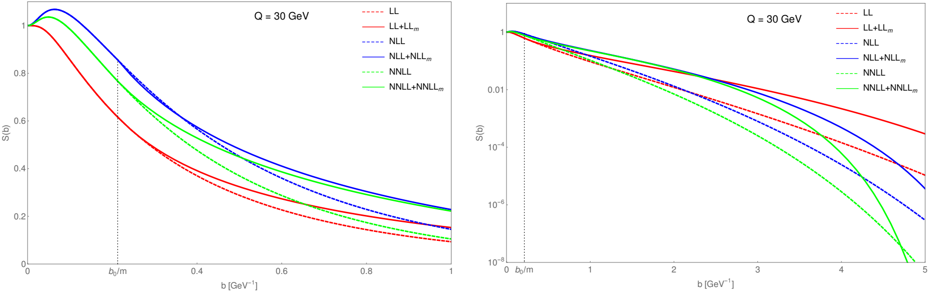
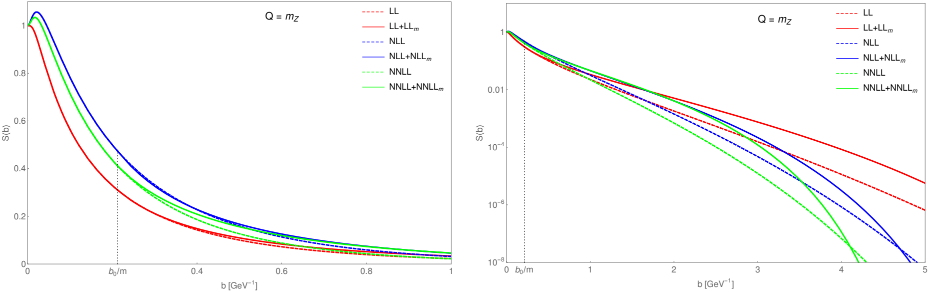
$$\simeq \ln(Q^2/(1.65m)^2)\, f_1(\rho)|_{A_1} + \ln(Q^2/(1.8m)^2)\, f_1(\rho)|_{A_2}\,,$$

Numerical results



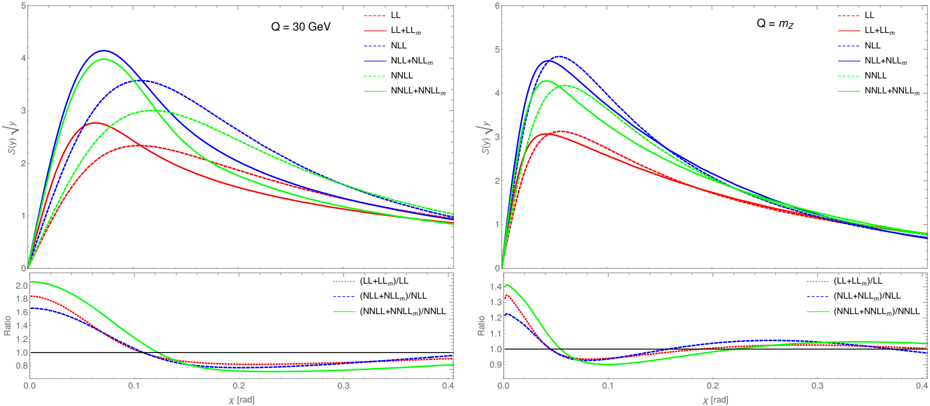
The massive Sudakov form factor $S_m(Q, b)$ in b-space at a hard scale $Q = 30$ GeV both in linear (left panel) and logarithmic (right panel) scales, at various logarithmic orders. The solid lines represent the massive case with $m = m_b$, while the dashed lines the massless case.

Numerical results



The massive Sudakov form factor $S_m(Q,b)$ in b-space at a hard scale $Q = m_Z$ both in linear (left panel) and logarithmic (right panel) scales, at various logarithmic orders. The solid lines represent the massive case with $m = m_b$, while the dashed lines the massless case.

Numerical results



The massive Sudakov form factor $\mathcal{S}_{\Updownarrow}(Q, y)$ in physical (angle-χ) space for a hard scale $Q = 30$ GeV (left) and $Q = m_Z$ (right) in the massive case with $m = m_b$ (solid lines) and massless case (dashed lines). Lower panel shows the ratio of the massive form factor over the massless one.

Conclusions

- Presented resummation for energy-energy-correlation in e^+e^- in the back-to-back region at full N^{3}LL accuracy (including N^{3}LO hard-virtual effects) matched with the known NNLO results (important away the back-to-back region).
- Very precise pQCD: percent level perturbative uncertainty.
- Inclusion of NP QCD effects allows us to provide a very good description of precise experimental data from LEP and SLD at $\sqrt{s} = m_Z$ (extraction of $\alpha_S(m_Z)$ consistent with the world average).
- Inclusion of heavy-quark mass effects ($m \ll Q$) in the Sudakov form factor through a simple generalization of the massless formula.
- Full perturbative analysis which does not involve model dependence.
- Our general formula can be easily implemented to improve analysis in regions where Sudakov resummation is relevant.