First insight into TMD fragmentation physics at photon-photon colliders

Simone Anedda
In collaboration with: F. Murgia, C. Pisano
Based on : S. Anedda, F. Murgia, C. Pisano, 2504.12802; **Accepted for publication at Phys. Rev. D**

INFN CAGLIARI
Istituto Nazionale di Fisica Nucleare
Sezione di Cagliari

UNICA
UNIVERSITÀ DEGLI STUDI DI CAGLIARI

Index

Probes for TMDs FF

↓

Photon-Photon collider

↓

$$\ell^+\ell^- \rightarrow \gamma^*\gamma \rightarrow q\bar{q} \rightarrow h_1 h_2 + X$$

↓

Differential cross section

↓

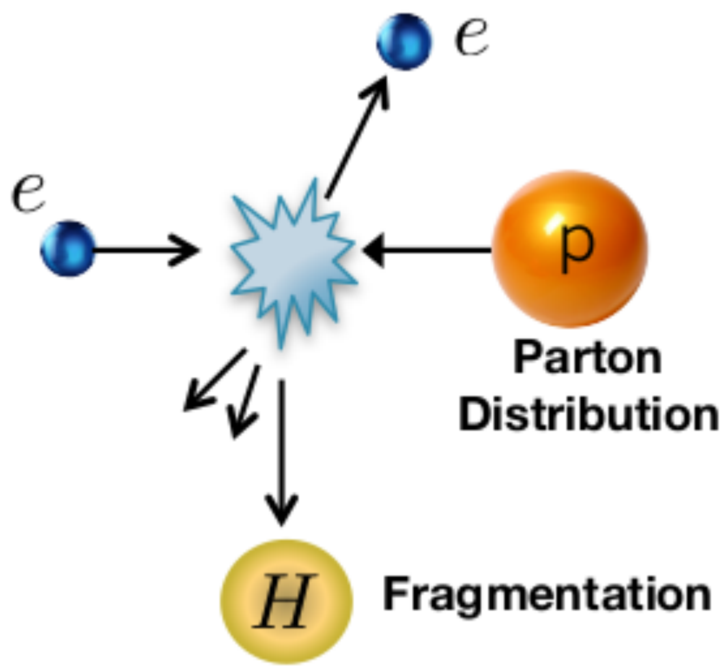
Azimuthal moments

↓

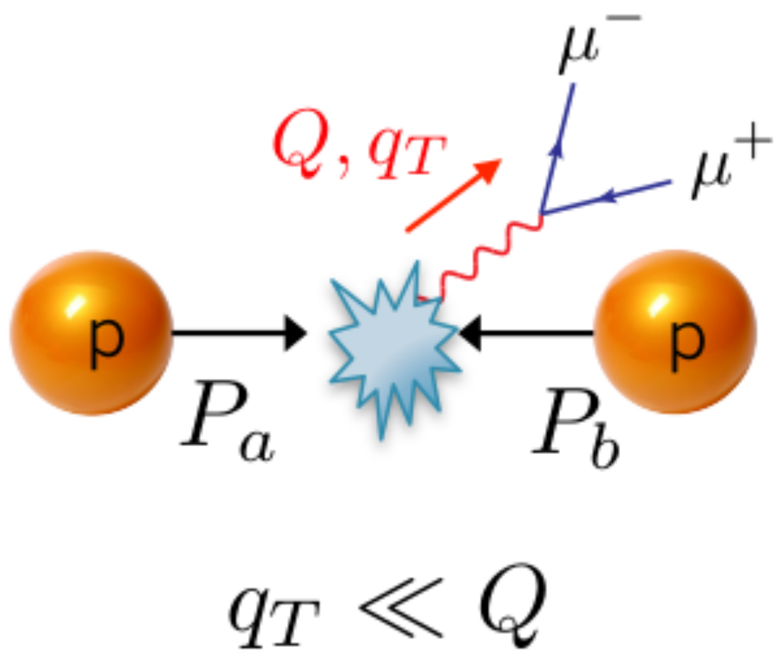
Summary and outlook

Probe for TMDs

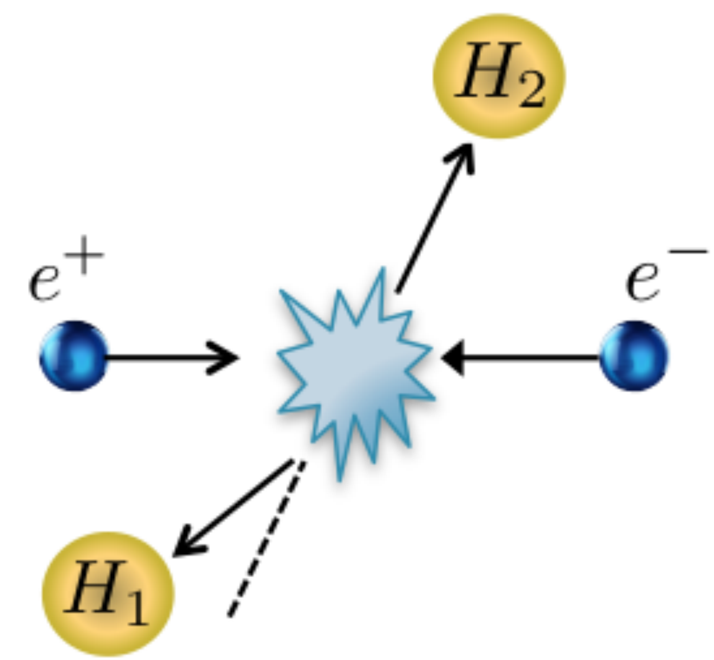
Semi-Inclusive DIS



Drell-Yan



Dihadron in e+e-



R. Boussarie et al., "TMD Handbook," arXiv:2304.03302 [hep-ph]

QCD factorization theorems within TMD approach have been proven in:

- X.-d. Ji, J.-P. Ma, and F. Yuan, Phys. Lett. B 597 (2004) 299–308.
- X.-d. Ji, J.-p. Ma, and F. Yuan, Phys. Rev. D 71 (2005) 034005.
- J. Collins, vol. 32. Cambridge University Press, 2011.
- M. G. Echevarria, A. Idilbi, and I. Scimemi, JHEP 07 (2012) 002.

Photon - Photon collider

$$\gamma_1^*(q_1)\,+\,\gamma_2(q_2)\to q(K_q)\,+\bar{q}(K_{\bar{q}})\to h_1(P_1)\,+\,h_2(P_2)+X$$

- The lepton-antilepton colliders can be used as effective two photons collider, with the initial state of the process relatively simple;

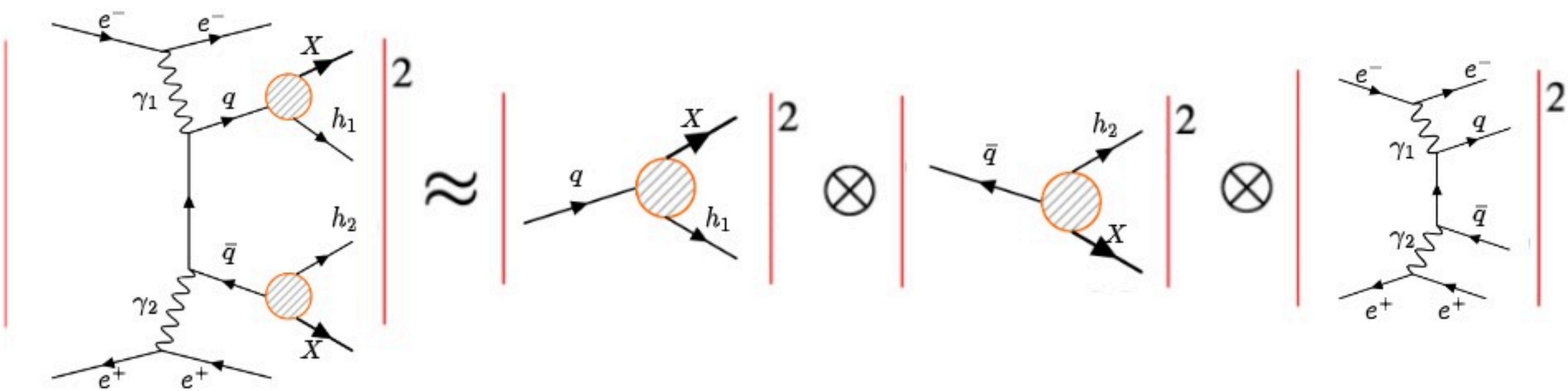
- Its possible to make similar studies using the intense photon beams available in hadronic colliders;

- Can provide unique and complementary information on the flavor structure of the quark TMD fragmentation functions involved;

- While lepton colliders operate at some fixed cm energy, photon photon collisions allow to vary the perturbative energy scale.

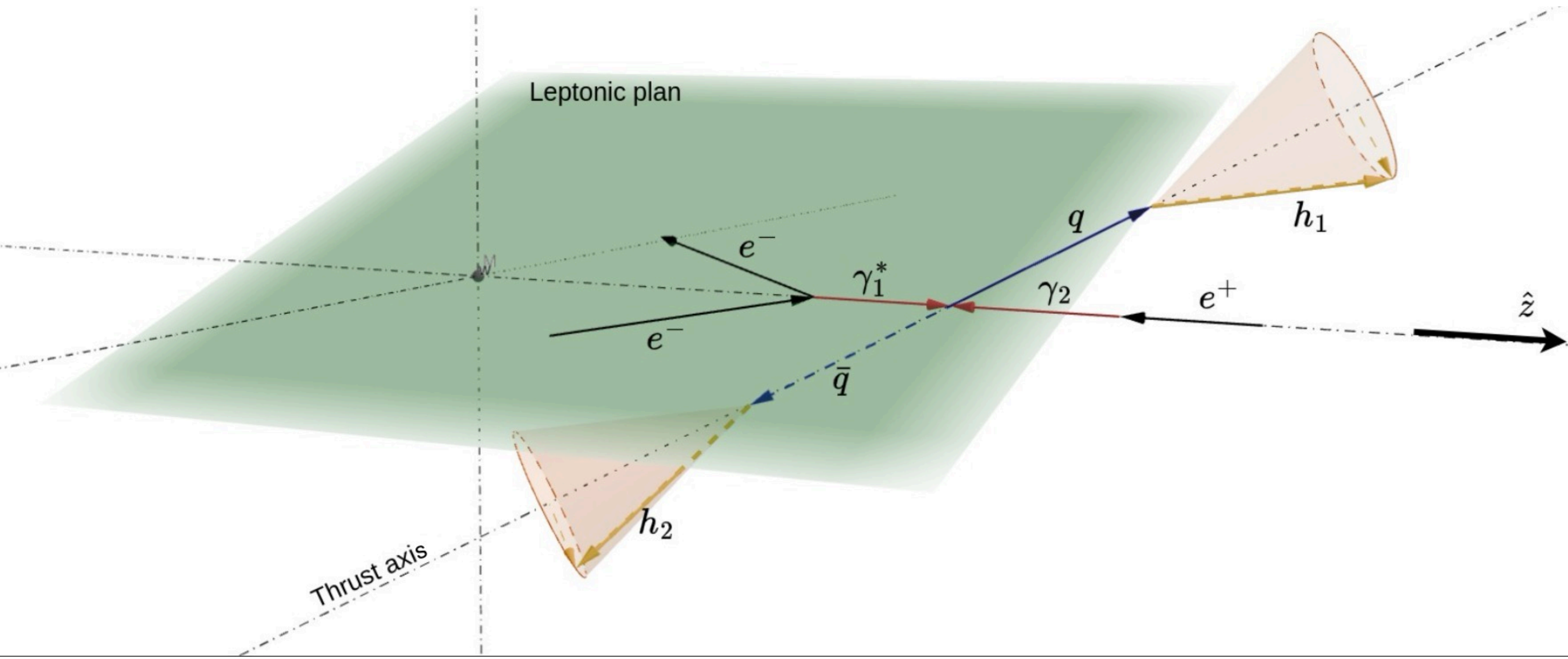
Lepton collisions as effective photon collider

$$\ell^+(l_+) + \ell^-(l_-) \to \ell^+(l'_+) + h_1(P_1) + h_2(P_2) + X$$



Photon-photon center of mass frame

$$\ell^+(l_+) + \ell^-(l_-) \to \ell^+(l'_+) + h_1(P_1) + h_2(P_2) + X$$



Further consideration

- Validity of TMD factorization within helicity formalism as in SIA processes. TMD FF universality and process independent proven in:
    - J. C. Collins and A. Metz, Phys. Rev. Lett. 93 (2004) 252001.
    - F. Yuan, Phys. Rev. Lett. 100 (2008) 032003.
    - L. P. Gamberg, A. Mukherjee, and P. J. Mulders, Phys. Rev. D 83 (2011) 071503.
- Azimuthal distribution around thrust axis is cleanest theoretically, but experimentally more challenging;
- Simplified framework at a fixed energy scale, full TMD evolution available in
    - U. D'Alesio, F. Murgia, and M. Zaccheddu, JHEP 10 (2021) 078.
- Possible competing gluon contribution inside the quasi-real lepton should be suppressed

Differential cross section

$$
\mathrm{d}\sigma^{\ell^+\ell^-\to\ell'^+\,h_1\,h_2\,X} = \frac{1}{4\,(l_+\cdot l_-)}\,\frac{\mathrm{d}^3\boldsymbol{l}'_+}{2(2\pi)^3\,l'^0_+}\,\frac{\mathrm{d}^3\boldsymbol{K}_q}{2(2\pi)^3K_q^0}\,\frac{\mathrm{d}^3\boldsymbol{K}_{\bar q}}{2(2\pi)^3K_{\bar q}^0}\,(2\pi)^4\,\delta^{(4)}(q_1+q_2-K_q-K_{\bar q})
$$

$$
\times\sum_q\sum_{\{\lambda_i\}}\tilde\rho_{\lambda_1,\lambda_1'}(\gamma_1^*)\,\rho_{\lambda_2,\lambda_2'}(\gamma_2)\,f_{\gamma/\ell^-,\mathcal{P}^{\ell^-}_{\hat z_-}}(\xi)\,\frac{\mathrm{d}\xi}{\xi}\,\hat H_{\lambda_q,\lambda_{\bar q};\lambda_1,\lambda_2}\,\hat H^*_{\lambda_q',\lambda_{\bar q}';\lambda_1',\lambda_2'}
$$

$$
\times\,\hat D^{h_1}_{\lambda_q,\lambda_q'}(z_1,\boldsymbol{p}_{\perp 1})\,\mathrm{d}z_1\,\mathrm{d}^2\boldsymbol{p}_{\perp 1}\,\hat D^{h_2}_{\lambda_{\bar q},\lambda_{\bar q}'}(z_2,\boldsymbol{p}_{\perp 2})\,\mathrm{d}z_2\,\mathrm{d}^2\boldsymbol{p}_{\perp 2}
$$

Differential cross section

$$
\mathrm{d}\sigma^{\ell^+\ell^-\to\ell'^+\,h_1\,h_2\,X} = \boxed{\frac{1}{4\,(l_+\cdot l_-)}\,\frac{\mathrm{d}^3\boldsymbol{l}'_+}{2(2\pi)^3\,l'^0_+}\,\frac{\mathrm{d}^3\boldsymbol{K}_q}{2(2\pi)^3 K^0_q}\,\frac{\mathrm{d}^3\boldsymbol{K}_{\bar q}}{2(2\pi)^3 K^0_{\bar q}}\,(2\pi)^4\,\delta^{(4)}(q_1+q_2-K_q-K_{\bar q})}
$$

$$
\times\sum_q\sum_{\{\lambda_i\}}\tilde\rho_{\lambda_1,\lambda'_1}(\gamma_1^*)\,\rho_{\lambda_2,\lambda'_2}(\gamma_2)\,f_{\gamma/\ell^-,\mathcal{P}^{\ell^-}_{\hat z_-}}(\xi)\,\frac{\mathrm{d}\xi}{\xi}\,\hat H_{\lambda_q,\lambda_{\bar q};\lambda_1,\lambda_2}\,\hat H^*_{\lambda'_q,\lambda'_{\bar q};\lambda'_1,\lambda'_2}
$$

$$
\times\,\hat D^{h_1}_{\lambda_q,\lambda'_q}(z_1,\boldsymbol{p}_{\perp 1})\,\mathrm{d}z_1\,\mathrm{d}^2\boldsymbol{p}_{\perp 1}\,\hat D^{h_2}_{\lambda_{\bar q},\lambda'_{\bar q}}(z_2,\boldsymbol{p}_{\perp 2})\,\mathrm{d}z_2\,\mathrm{d}^2\boldsymbol{p}_{\perp 2}
$$

- Lorentz invariant phase space factor

Differential cross section

$$\mathrm{d}\sigma^{\ell^+\ell^-\to\ell'^+\,h_1\,h_2\,X} = \frac{1}{4\,(l_+\cdot l_-)}\,\frac{\mathrm{d}^3\boldsymbol{l}'_+}{2(2\pi)^3\,l'^0_+}\,\frac{\mathrm{d}^3\boldsymbol{K}_q}{2(2\pi)^3K^0_q}\,\frac{\mathrm{d}^3\boldsymbol{K}_{\bar q}}{2(2\pi)^3K^0_{\bar q}}\,(2\pi)^4\,\delta^{(4)}(q_1+q_2-K_q-K_{\bar q})$$
$$\times\sum_q\sum_{\{\lambda_i\}}\tilde\rho_{\lambda_1,\lambda'_1}(\gamma^*_1)\,\rho_{\lambda_2,\lambda'_2}(\gamma_2)\,f_{\gamma/\ell^-,\mathcal{P}^{\ell^-}_{\hat z_-}}(\xi)\,\frac{\mathrm{d}\xi}{\xi}\,\hat H_{\lambda_q,\lambda_{\bar q};\lambda_1,\lambda_2}\,\hat H^*_{\lambda'_q,\lambda'_{\bar q};\lambda'_1,\lambda'_2}$$
$$\times\,\hat D^{h_1}_{\lambda_q,\lambda'_q}(z_1,\boldsymbol{p}_{\perp1})\,\mathrm{d}z_1\,\mathrm{d}^2\boldsymbol{p}_{\perp1}\,\hat D^{h_2}_{\lambda_{\bar q},\lambda'_{\bar q}}(z_2,\boldsymbol{p}_{\perp2})\,\mathrm{d}z_2\,\mathrm{d}^2\boldsymbol{p}_{\perp2}$$

- Lorentz invariant phase space factor
- Dynamical kernel

Differential cross section

$$
\begin{aligned}
d\sigma^{\ell^+\ell^-\to\ell'^+\,h_1\,h_2\,X} = & \frac{1}{4\,(l_+\cdot l_-)}\,\frac{d^3\boldsymbol{l}'_+}{2(2\pi)^3\,l'^0_+}\,\frac{d^3\boldsymbol{K}_q}{2(2\pi)^3 K_q^0}\,\frac{d^3\boldsymbol{K}_{\bar q}}{2(2\pi)^3 K_{\bar q}^0}\,(2\pi)^4\,\delta^{(4)}(q_1+q_2-K_q-K_{\bar q}) \\
& \times \sum_q \sum_{\{\lambda_i\}} \tilde\rho_{\lambda_1,\lambda'_1}(\gamma_1^*)\,\rho_{\lambda_2,\lambda'_2}(\gamma_2)\,f_{\gamma/\ell^-,\mathcal{P}_{\hat z_-}^{\ell^-}}(\xi)\,\frac{d\xi}{\xi}\,\hat H_{\lambda_q,\lambda_{\bar q};\lambda_1,\lambda_2}\,\hat H^*_{\lambda'_q,\lambda'_{\bar q};\lambda'_1,\lambda'_2} \\
& \times \hat D^{h_1}_{\lambda_q,\lambda'_q}(z_1,\boldsymbol{p}_{\perp 1})\,dz_1\,d^2\boldsymbol{p}_{\perp 1}\,\hat D^{h_2}_{\lambda_{\bar q},\lambda'_{\bar q}}(z_2,\boldsymbol{p}_{\perp 2})\,dz_2\,d^2\boldsymbol{p}_{\perp 2}
\end{aligned}
$$

- Lorentz invariant phase space factor
- Dynamical kernel
- TMD fragmentation function

Helicity density matrix for the virtual photon

- The helicity density matrix for the tagged (virtual) photon can be written as:

$$\rho(\gamma_1^*) = \frac{1}{2(2-y)^2}$$

$$\times \begin{pmatrix} 1+(1-y)^2+\mathcal{P}^{\ell^+}_{\hat{z}_+}\,y(2-y) & -e^{-i\phi_\ell}\sqrt{2(1-y)}\left[(2-y)+\mathcal{P}^{\ell^+}_{\hat{z}_+}\,y\right] & -e^{-i2\phi_\ell}\,2(1-y) \\ -e^{i\phi_\ell}\sqrt{2(1-y)}\left[(2-y)+\mathcal{P}^{\ell^+}_{\hat{z}_+}\,y\right] & 4(1-y) & e^{-i\phi_\ell}\sqrt{2(1-y)}\left[(2-y)-\mathcal{P}^{\ell^+}_{\hat{z}_+}\,y\right] \\ -e^{i2\phi_\ell}\,2(1-y) & e^{i\phi_\ell}\sqrt{2(1-y)}\left[(2-y)-\mathcal{P}^{\ell^+}_{\hat{z}_+}\,y\right] & 1+(1-y)^2-\mathcal{P}^{\ell^+}_{\hat{z}_+}\,y(2-y) \end{pmatrix}$$

$$\tilde{\rho} = \rho\,\mathrm{Tr}[\tilde{\rho}] \qquad \mathrm{Tr}[\tilde{\rho}] = \frac{2\,e^2\,(2-y)^2}{Q^2\,y^2} \equiv \frac{2\,e^2\,(2-y)^2}{x_B\,y^3\,s}$$

Helicity density matrix for the quasi real photon and the Weizäsker-Williams function

- Instead, for the untagged (quasi-real) photon:

$$\rho(\gamma_2) = \frac{1}{2}\begin{pmatrix} 1+\mathcal{P}^{\gamma_2}_{\hat{z}_2} & 0 \\ 0 & 1-\mathcal{P}^{\gamma_2}_{\hat{z}_2} \end{pmatrix}$$

$$[\,\rho_{++}(\gamma_2)+\rho_{--}(\gamma_2)\,]\; f_{\gamma/\ell^-,\mathcal{P}^{\ell^-}_{\hat{z}_-}}(\xi) \;=\; f_{\gamma,+/\ell^-,\mathcal{P}^{\ell^-}_{\hat{z}_-}}(\xi)+f_{\gamma,-/\ell^-,\mathcal{P}^{\ell^-}_{\hat{z}_-}}(\xi) \;=\; f_{\gamma/\ell}(\xi)\,,$$

$$[\,\rho_{++}(\gamma_2)-\rho_{--}(\gamma_2)\,]\; f_{\gamma/\ell^-,\mathcal{P}^{\ell^-}_{\hat{z}_-}}(\xi) \;=\; \mathcal{P}^{\gamma_2}_{\hat{z}_2}\, f_{\gamma/\ell^-,\mathcal{P}^{\ell^-}_{\hat{z}_-}}(\xi) \;=\; f_{\gamma,+/\ell^-,\mathcal{P}^{\ell^-}_{\hat{z}_-}}(\xi) \;-\; f_{\gamma,-/\ell^-,\mathcal{P}^{\ell^-}_{\hat{z}_-}}(\xi) \;=\; \mathcal{P}^{\ell^-}_{\hat{z}_-}\,\Delta_L f_{\gamma/\ell}(\xi)$$

Helicity scattering amplitude

$$\hat{H}_{+-;1,1} = -\hat{H}_{-+;-1,-1} = -2\sqrt{3}\,e^2\,e_q^2\,\frac{Q^2}{\hat{s}+Q^2}\sqrt{\frac{\hat{u}}{\hat{t}}} = -2\sqrt{3}\,e^2\,e_q^2\,\frac{x_B}{\xi}\sqrt{\frac{1-\zeta}{\zeta}}\,,$$

$$\hat{H}_{+-;1,-1} = -\hat{H}^*_{-+;-1,1} = -2\sqrt{3}\,e^2\,e_q^2\,e^{i2\phi_q}\,\frac{\hat{s}}{\hat{s}+Q^2}\sqrt{\frac{\hat{u}}{\hat{t}}} = -2\sqrt{3}\,e^2\,e_q^2\,e^{i2\phi_q}\,\frac{\xi-x_B}{\xi}\sqrt{\frac{1-\zeta}{\zeta}}\,,$$

$$\hat{H}_{+-;-1,1} = -\hat{H}^*_{-+;1,-1} = 2\sqrt{3}\,e^2\,e_q^2\,e^{-i2\phi_q}\,\frac{\hat{s}}{\hat{s}+Q^2}\sqrt{\frac{\hat{t}}{\hat{u}}} = \sqrt{3}\,e^2\,e_q^2\,e^{-i2\phi_q}\,\frac{\xi-x_B}{\xi}\sqrt{\frac{\zeta}{1-\zeta}}\,,$$

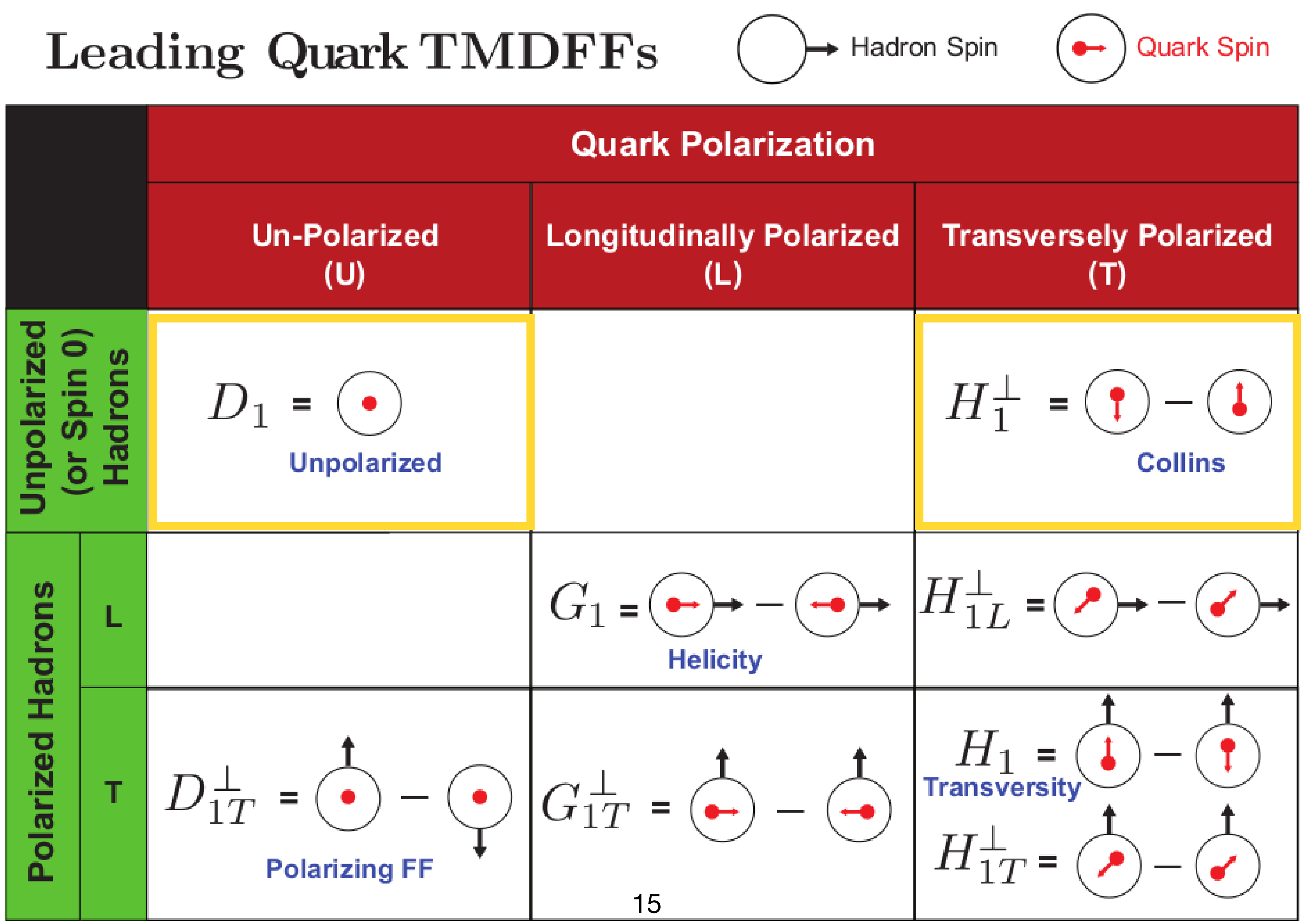
$$\hat{H}_{+-;-1,-1} = -\hat{H}_{-+;1,1} = 2\sqrt{3}\,e^2\,e_q^2\,\frac{Q^2}{\hat{s}+Q^2}\sqrt{\frac{\hat{t}}{\hat{u}}} = 2\sqrt{3}\,e^2\,e_q^2\,\frac{x_B}{\xi}\sqrt{\frac{\zeta}{1-\zeta}}\,,$$

$$\hat{H}_{+-;0,\pm1} = -\hat{H}_{-+;0,\pm1} = \pm 2\sqrt{6}\,e^2\,e_q^2\,e^{\mp i\phi_q}\,\frac{\sqrt{\hat{s}}Q}{\hat{s}+Q^2} = \pm 2\sqrt{6}\,e^2\,e_q^2\,e^{\mp i\phi_q}\,\frac{\sqrt{x_B(\xi-x_B)}}{\xi}\,.$$

TMD Fragmentation Function

- The non perturbative process is embodied into the transverse momentum dependent functions for q->h_1 + X

$$\hat{D}^{h/a}_{\lambda_a,\lambda'_a}(z,\boldsymbol{p}_\perp) = \sum_{\lambda_h} \sum\!\!\!\!\!\!\!\int_{X,\lambda_X} \hat{\mathcal{D}}_{\lambda_h,\lambda_X;\lambda_a}(z,\boldsymbol{p}_\perp)\hat{\mathcal{D}}^*_{\lambda_h,\lambda_X;\lambda'_a}(z,\boldsymbol{p}_\perp)$$



- Unpolarized FF

$$\hat{D}^{h/a}_{++}(z,\boldsymbol{p}_\perp) = \hat{D}^{h/a}_{--}(z,\boldsymbol{p}_\perp) = D^h_a(z,p_\perp)$$

- Collins FF

$$\hat{D}^{h/a}_{+-}(z,\boldsymbol{p}_\perp) = D^{h/a}_{+-}(z,p_\perp)e^{i\phi^h_a}$$

$$\Delta^N D^h_{a\uparrow}(z,p_\perp) = \frac{2p_\perp}{zm_h}\, H_1^{\perp,a}(z,p_\perp) = -i2D^{h/a}_{+-}(z,p_\perp)$$

- For future use:

$$\int \mathrm{d}^2\boldsymbol{p}_\perp\, D^h_a(z,p_\perp) = D^h_a(z)$$

$$\int \mathrm{d}^2\boldsymbol{p}_\perp\, \Delta^N D^h_{a\uparrow}(z,p_\perp) \equiv \int \mathrm{d}^2\boldsymbol{p}_\perp\, \frac{2p_\perp}{zm_h}\, H_1^{\perp,a}(z,p_\perp) = 2\pi \int \mathrm{d}p_\perp\, p_\perp\, \Delta^N D^h_{a\uparrow}(z,p_\perp)$$
$$= \Delta^N D^h_{a\uparrow}(z) = 4\, H_1^{\perp(1/2)a}(z)$$

Differential cross section

$$\frac{d\sigma^{\ell^+\ell^-\to\ell'^+\,h_1\,h_2\,X}\;(\mathcal{P}_+,\,\mathcal{P}_-)}{dx_B\,dy\,d\zeta\,d\phi_q\,d\xi\,dz_1\,d^2\boldsymbol{p}_{\perp 1}\,dz_2\,d^2\boldsymbol{p}_{\perp 2}} = \frac{3\,\alpha^3}{4\,\pi}\,\frac{1}{x_B\,y^2\,\xi^3\,s}\sum_q e_q^4$$

$$\times\Big\{\Big[A_U + \mathcal{P}_+\mathcal{P}_-\,A_L + \Big(A_U^{\cos\phi_q} + \mathcal{P}_+\mathcal{P}_-\,A_L^{\cos\phi_q}\Big)\cos\phi_q + A_U^{\cos 2\phi_q}\cos 2\phi_q\Big]\,D_q^{h_1}(z_1,p_{\perp 1})D_{\bar q}^{h_2}(z_2,p_{\perp 2})$$

$$+\Big[\Big(B_U^{\cos\phi_{12}} + \mathcal{P}_+\mathcal{P}_-\,B_L^{\cos\phi_{12}}\Big)\cos\phi_{12} + \Big(B_U^{\cos(\phi_q-\phi_{12})} + \mathcal{P}_+\mathcal{P}_-\,B_L^{\cos(\phi_q-\phi_{12})}\Big)\cos(\phi_q-\phi_{12})$$

$$+\Big(B_U^{\cos(\phi_q+\phi_{12})} + \mathcal{P}_+\mathcal{P}_-\,B_L^{\cos(\phi_q+\phi_{12})}\Big)\cos(\phi_q+\phi_{12}) + B_U^{\cos(2\phi_q-\phi_{12})}\cos(2\phi_q-\phi_{12})$$

$$+\,B_U^{\cos(2\phi_q+\phi_{12})}\cos(2\phi_q+\phi_{12})\Big]\,\Delta^N D_{q^\uparrow}^{h_1}(z_1,p_{\perp 1})\Delta^N D_{\bar q^\uparrow}^{h_2}(z_2,p_{\perp 2})\Big\}$$

- Azimuthal angle of the quark-antiquark direction relative to the z axis -> ϕ_q
- Azimuthal angle of the hadron relative to the thrust axis -> $\phi_q^{h_1}, \phi_{\bar q}^{h_2}$, $\phi_q^{h_1} - \phi_{\bar q}^{h_2} \equiv \phi_{12}$

$$
\begin{aligned}
A_U &= 2\,\Big\{ [\,1+(1-y)^2\,]\,[\,x_B^2+(\xi-x_B)^2\,]\,\frac{1-2\zeta(1-\zeta)}{\zeta(1-\zeta)} \;+\; 16\,(1-y)\,x_B(\xi-x_B)\Big\}\, f_{\gamma/\ell}(\xi)\,,\\
A_U^{\cos\phi_q} &= -\,8\,(2-y)\,\sqrt{1-y}\,(\xi-2x_B)\,\sqrt{x_B(\xi-x_B)}\;\frac{1-2\zeta}{\sqrt{\zeta(1-\zeta)}}\, f_{\gamma/\ell}(\xi)\,,\\
A_U^{\cos 2\phi_q} &= 16\,(1-y)\,x_B(\xi-x_B)\, f_{\gamma/\ell}(\xi)\,,\\
A_L &= -\,2y(2-y)\,\xi(\xi-2x_B)\,\frac{1-2\zeta(1-\zeta)}{\zeta(1-\zeta)}\,\Delta_L f_{\gamma/\ell}(\xi)\,,\\
A_L^{\cos\phi_q} &= 8\,y\sqrt{1-y}\,\xi\,\sqrt{x_B(\xi-x_B)}\,\frac{1-2\zeta}{\sqrt{\zeta(1-\zeta)}}\,\Delta_L f_{\gamma/\ell}(\xi)\,.
\end{aligned}
$$

$$B_U^{\cos\phi_{12}} = \left\{ [1+(1-y)^2]\,[\,x_B^2 + (\xi - x_B)^2\,] \;-\; 8\,(1-y)\,x_B(\xi - x_B) \right\}\; f_{\gamma/\ell}(\xi)\,,$$

$$B_U^{\cos(\phi_q-\phi_{12})} = -\,2\,(2-y)\,\sqrt{1-y}\,(\xi - 2x_B)\,\sqrt{x_B(\xi - x_B)}\,\sqrt{\frac{\zeta}{1-\zeta}}\; f_{\gamma/\ell}(\xi)\,,$$

$$B_U^{\cos(\phi_q+\phi_{12})} = 2\,(2-y)\,\sqrt{1-y}\,(\xi - 2x_B)\,\sqrt{x_B(\xi - x_B)}\,\sqrt{\frac{1-\zeta}{\zeta}}\; f_{\gamma/\ell}(\xi)\,,$$

$$B_U^{\cos(2\phi_q-\phi_{12})} = 2\,(1-y)\,x_B(\xi - x_B)\,\frac{\zeta}{1-\zeta}\; f_{\gamma/\ell}(\xi)$$

$$B_U^{\cos(2\phi_q+\phi_{12})} = 2\,(1-y)\,x_B(\xi - x_B)\,\frac{1-\zeta}{\zeta}\; f_{\gamma/\ell}(\xi)\,,$$

$$B_L^{\cos\phi_{12}} = -\,y\,(2-y)\,\xi(\xi - 2x_B)\;\Delta_L f_{\gamma/\ell}(\xi)\,,$$

$$B_L^{\cos(\phi_q-\phi_{12})} = 2\,y\,\sqrt{1-y}\,\xi\,\sqrt{x_B(\xi - x_B)}\,\sqrt{\frac{\zeta}{1-\zeta}}\;\Delta_L f_{\gamma/\ell}(\xi)\,,$$

$$B_L^{\cos(\phi_q+\phi_{12})} = -\,2\,y\,\sqrt{1-y}\,\xi\,\sqrt{x_B(\xi - x_B)}\,\sqrt{\frac{1-\zeta}{\zeta}}\;\Delta_L f_{\gamma/\ell}(\xi)\,.$$

Isolating the azimuthal modulations

$$\frac{\mathrm{d}\sigma^{\ell^+\ell^-\to\ell'^+\,h_1\,h_2\,X}(\mathcal{P}_+,\,\mathcal{P}_-)}{\mathrm{d}x_B\,\mathrm{d}y\,\mathrm{d}\zeta\,\mathrm{d}\phi_q\,\mathrm{d}\xi\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}\phi_{12}} = \frac{3\,\alpha^3}{8\,\pi^2}\,\frac{1}{x_B\,y^2\,\xi^3\,s}\sum_q e_q^4$$
$$\times\Big\{\Big[A_U + \mathcal{P}_+\mathcal{P}_-\,A_L + \Big(A_U^{\cos\phi_q} + \mathcal{P}_+\mathcal{P}_-\,A_L^{\cos\phi_q}\Big)\cos\phi_q + A_U^{\cos 2\phi_q}\cos 2\phi_q\Big]\,D_q^{h_1}(z_1)\,D_{\bar q}^{h_2}(z_2)$$
$$+\Big[\Big(B_U^{\cos\phi_{12}} + \mathcal{P}_+\mathcal{P}_-\,B_L^{\cos\phi_{12}}\Big)\cos\phi_{12} + \Big(B_U^{\cos(\phi_q-\phi_{12})} + \mathcal{P}_+\mathcal{P}_-\,B_L^{\cos(\phi_q-\phi_{12})}\Big)\cos(\phi_q-\phi_{12})$$
$$+\Big(B_U^{\cos(\phi_q+\phi_{12})} + \mathcal{P}_+\mathcal{P}_-\,B_L^{\cos(\phi_q+\phi_{12})}\Big)\cos(\phi_q+\phi_{12}) + B_U^{\cos(2\phi_q-\phi_{12})}\cos(2\phi_q-\phi_{12})$$
$$+B_U^{\cos(2\phi_q+\phi_{12})}\cos(2\phi_q+\phi_{12})\Big]\,\Delta^N D_{q^\uparrow}^{h_1}(z_1)\,\Delta^N D_{\bar q^\uparrow}^{h_2}(z_2)\Big\}$$

$$\int \mathrm{d}^2\boldsymbol{p}_\perp\,D_a^h(z,p_\perp) = D_a^h(z)$$

$$\int \mathrm{d}^2\boldsymbol{p}_\perp\,\Delta^N D_{a^\uparrow}^h(z,p_\perp) \equiv \int \mathrm{d}^2\boldsymbol{p}_\perp\,\frac{2p_\perp}{zm_h}\,H_1^{\perp,a}(z,p_\perp) = 2\pi\int \mathrm{d}p_\perp\,p_\perp\,\Delta^N D_{a^\uparrow}^h(z,p_\perp)$$
$$= \Delta^N D_{a^\uparrow}^h(z) = 4\,H_1^{\perp(1/2)a}(z)$$

Azimuthal moments

$$\begin{aligned} d\sigma^{\text{unp}} &= \frac{1}{4}\left[d\sigma(1,1) + d\sigma(1,-1) + d\sigma(-1,1) + d\sigma(-1,-1)\right] \\ &= \frac{1}{2}\left[d\sigma(1,1) + d\sigma(1,-1)\right], \\ \Delta_L\sigma &= d\sigma(1,1) - d\sigma(1,-1) = d\sigma(-1,-1) - d\sigma(-1,+1)\,. \end{aligned}$$

$$A_{LL} = \frac{d\sigma(1,1) - d\sigma(1,-1)}{d\sigma(1,1) + d\sigma(1,-1)} = \frac{\Delta_L\sigma}{2\,d\sigma^{\text{unp}}}\,.$$

$$\langle\, d\sigma^{\text{unp}} |\, n_q; m_{12}\,\rangle = 2\,\frac{\int d\phi_q\, d\phi_{12}\, d\sigma^{\text{unp}}(\phi_q,\phi_{12})\,\cos[n_q\phi_q + m_{12}\phi_{12}]}{\int d\phi_q\, d\phi_{12}\, d\sigma^{\text{unp}}(\phi_q,\phi_{12})}$$

$$\langle\, A_{LL} |\, n_q; m_{12}\,\rangle = 2\,\frac{\int d\phi_q\, d\phi_{12}\, A_{LL}\, d\sigma^{\text{unp}}(\phi_q,\phi_{12})\,\cos[n_q\phi_q + m_{12}\phi_{12}]}{\int d\phi_q\, d\phi_{12}\, d\sigma^{\text{unp}}(\phi_q,\phi_{12})}$$

$$n_q = 0,1,2 \qquad m_{12} = 0,\pm 1$$

n_q	m_{12}	$\langle\, \mathrm{d}\sigma^{\mathrm{unp}} \vert n_q; m_{12}\,\rangle$	$\langle\, A_{LL} \vert n_q; m_{12}\,\rangle$
0	0	-	$\dfrac{A_L}{A_U}$
± 1	0	$\dfrac{A_U^{\cos\phi_q}}{A_U}$	$\dfrac{A_L^{\cos\phi_q}}{A_U}$
± 2	0	$\dfrac{A_U^{\cos 2\phi_q}}{A_U}$	0
0	± 1	$\dfrac{B_U^{\cos\phi_{12}}}{A_U}\, \dfrac{\sum_q\, e_q^4\, \Delta^N D_{q\uparrow}^{h_1}\, \Delta^N D_{\bar{q}\uparrow}^{h_2}}{\sum_q\, e_q^4\, D_q^{h_1}\, D_{\bar{q}}^{h_2}}$	$\dfrac{B_L^{\cos\phi_{12}}}{A_U}\, \dfrac{\sum_q\, e_q^4\, \Delta^N D_{q\uparrow}^{h_1}\, \Delta^N D_{\bar{q}\uparrow}^{h_2}}{\sum_q\, e_q^4\, D_q^{h_1}\, D_{\bar{q}}^{h_2}}$
1	± 1	$\dfrac{B_U^{\cos(\phi_q \pm \phi_{12})}}{A_U}\, \dfrac{\sum_q\, e_q^4\, \Delta^N D_{q\uparrow}^{h_1}\, \Delta^N D_{\bar{q}\uparrow}^{h_2}}{\sum_q\, e_q^4\, D_q^{h_1}\, D_{\bar{q}}^{h_2}}$	$\dfrac{B_L^{\cos(\phi_q \pm \phi_{12})}}{A_U}\, \dfrac{\sum_q\, e_q^4\, \Delta^N D_{q\uparrow}^{h_1}\, \Delta^N D_{\bar{q}\uparrow}^{h_2}}{\sum_q\, e_q^4\, D_q^{h_1}\, D_{\bar{q}}^{h_2}}$
-1	∓ 1	//	//
2	± 1	$\dfrac{B_U^{\cos(2\phi_q \pm \phi_{12})}}{A_U}\, \dfrac{\sum_q\, e_q^4\, \Delta^N D_{q\uparrow}^{h_1}\, \Delta^N D_{\bar{q}\uparrow}^{h_2}}{\sum_q\, e_q^4\, D_q^{h_1}\, D_{\bar{q}}^{h_2}}$	0
-2	∓ 1	//	//

Summary and outlook

- Photon-photon scattering provides a clean environment for studying quark TMD fragmentation functions
    - Complementing the flavor separation
    - The variable energy scale
- Circular and linear lepton colliders or ultraperipheral collisions at the LHC or RHIC as effective photon colliders
- Further studies could investigate hadron pair production involving either two spin-1/2 particles or a spin-0 and a spin-1/2 particle