Quantum correlations in neutrino oscillations

Massimo Blasone

Dipartimento di Fisica, Università di Salerno and INFN, Salerno, Italy

Summary

1. Entanglement in neutrino mixing and oscillations
2. Quantum Field Theory of neutrino mixing and oscillations
3. Quantum correlations & nonlocality in neutrino oscillations
4. Chiral oscillations

Motivations

- Understanding entanglement generation in fundamental processes;
- Quantum correlations in relativistic systems as a possible resource for quantum information;
- Investigation of fundamental properties of (elementary) particles via quantum correlations;
- Entanglement in high energy processes as a probe for new physics beyond Standard Model.

Entanglement in particle physics[¶]

- Entanglement, decoherence, Bell inequalities for the $K^0\bar{K}^0$ (or $B^0\bar{B}^0$) system[*];

- Quantumness in neutrino oscillations (entanglement[†], Leggett-Garg inequalities[‡], quantum correlations);

- Entanglement in scattering processes[§].

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[*] R.A.Bertlmann and B.Hiesmayr, Phys.Rev. A (2001); A.Di Domenico (Ed.) Frascati Physics Series (2007).
[†] M.B. et al., EPL (2009).
[‡] J.A. Formaggio et al., Phys. Rev. Lett. (2016).
[§] G. Aad et al. [ATLAS], Nature (2024)
M.B. et al., Phys.Rev. D (2024), (2025).

[¶] R.A.Bertlmann, *Entanglement, Bell inequalities and decoherence in particle physics*, Lect. Not. Phys. (2006);
Y.Shi, *Historical origins of quantum entanglement in particle physics*, arXiv:2507.13582.

Entanglement in neutrino mixing and oscillations

Neutrino oscillations in QM *

Pontecorvo mixing relations

$$|\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle$$

$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle$$

– Time evolution:

$$|\nu_e(t)\rangle = \cos\theta\, e^{-iE_1 t}\,|\nu_1\rangle + \sin\theta\, e^{-iE_2 t}\,|\nu_2\rangle$$

– Flavor oscillations:

$$P_{\nu_e\to\nu_e}(t) \;=\; |\langle\nu_e|\nu_e(t)\rangle|^2 = 1 - \sin^2 2\theta\,\sin^2\left(\frac{\Delta E}{2}t\right) = 1 - P_{\nu_e\to\nu_\mu}(t)$$

– Flavor conservation:

$$|\langle\nu_e|\nu_e(t)\rangle|^2 + |\langle\nu_\mu|\nu_e(t)\rangle|^2 = 1$$

*S.M.Bilenky and B.Pontecorvo, Phys. Rep. (1978)

Entanglement in neutrino mixing[†]

– Flavor mixing (neutrinos)

$$|\nu_e\rangle = \quad \cos\theta\,|\nu_1\rangle \,+\, \sin\theta\,|\nu_2\rangle$$

$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle \,+\, \cos\theta\,|\nu_2\rangle$$

• Correspondence with two-qubit states:

$$|\nu_1\rangle \equiv |1\rangle_1|0\rangle_2 \equiv |10\rangle, \qquad |\nu_2\rangle \equiv |0\rangle_1|1\rangle_2 \equiv |01\rangle,$$

$|\rangle_i$ denotes states in the Hilbert space for neutrinos with mass m_i.

⇒ flavor states are entangled superpositions of the mass eigenstates:

$$|\nu_e\rangle = \quad \cos\theta\,|10\rangle \,+\, \sin\theta\,|01\rangle.$$

[†]M.B., F.Dell'Anno, S.De Siena, M.Di Mauro and F.Illuminati, PRD (2008).

Composite structure of Hilbert space for neutrinos

• Necessity of tensor-product structure of Hilbert space for two generations:

Orthogonality of Hilbert spaces for fields with different masses[‡]

Example: two scalar fields with different masses

$$(\Box + \mu_1^2)\phi_1(x) = 0 \quad , \qquad (\Box + \mu_2^2)\phi_2(x) = 0$$

with boundary conditions $\phi_1(0, \mathbf{x}) = \phi_2(0, \mathbf{x})$ and $\dot{\phi}_1(0, \mathbf{x}) = \dot{\phi}_2(0, \mathbf{x})$

One obtains

$$_1\langle 0|0\rangle_2 \simeq \exp\left\{-\frac{V}{64\pi^2}\int_0^\infty dk \frac{(\mu_1^2 - \mu_2^2)^2}{k^2}\right\}$$

which vanishes in the infinite volume limit.

[‡]G.Barton, Introduction to Advanced Field Theory, Intersc. Publ. (1963)

Entanglement - mathematical definition

- Given a bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, a system is entangled, iff

$$\rho_{AB} \neq \sum_k p_k \, \rho_k^{(A)} \otimes \rho_k^{(B)}$$

with $0 \leq p_i \leq 1, \quad \sum_k p_k = 1.$

- For a generic pure state of the form:

$$|\psi\rangle_{AB} = \sum_{ij} c_{ij} |i\rangle_A \otimes |j\rangle_B$$

the condition for entanglement reads

$$|\psi\rangle_{AB} \neq |\phi\rangle_A \otimes |\chi\rangle_B$$

Single-particle entanglement*

- A state like $|\psi\rangle_{A,B} = |0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B$ is entangled;

– entanglement among field modes, rather than particles;

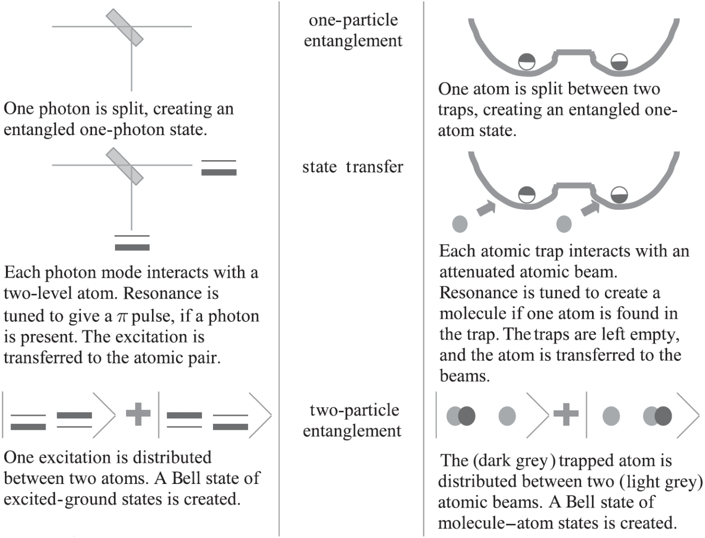
– entanglement is a property of composite systems, rather than of many-particle systems;

– entanglement and non-locality are not synonyms;

– single-particle entanglement is as good as two-particle entanglement for applications (quantum cryptography, teleportation, violation of Bell inequalities, etc..).

* J.van Enk, Phys. Rev. A (2005), (2006);
J.A.Dunningham and V.Vedral, Phys. Rev. Lett. (2007).

Protocols for extraction of single-particle entanglement †



† M.O.Terra Cunha, J.A.Dunningham and V.Vedral, Proc. Royal Soc. A (2007)

Multipartite entanglement in neutrino mixing[‡]

– Neutrino mixing (three flavors):

$$|\underline{\nu}_f\rangle \;=\; U(\tilde{\theta},\delta)\,|\underline{\nu}_m\rangle$$

with $|\underline{\nu}_f\rangle = (|\nu_e\rangle, |\nu_\mu\rangle, |\nu_\tau\rangle)^T$ and $|\underline{\nu}_m\rangle = (|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle)^T$.

– Mixing matrix (PMNS)

$$U(\tilde{\theta},\delta) \;=\; \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix},$$

where $(\tilde{\theta},\delta) \equiv (\theta_{12},\theta_{13},\theta_{23};\delta)$, $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$.

- Correspondence with three-qubit states:

$$|\nu_1\rangle \equiv |1\rangle_1|0\rangle_2|0\rangle_3 \equiv |100\rangle, \qquad |\nu_2\rangle \equiv |0\rangle_1|1\rangle_2|0\rangle_3 \equiv |010\rangle,$$

$$|\nu_3\rangle \equiv |0\rangle_1|0\rangle_2|1\rangle_3 \equiv |001\rangle$$

[‡] M.B., F.Dell'Anno, S.De Siena, M.Di Mauro and F.Illuminati, PRD (2008).

(Flavor) Entanglement in neutrino oscillations[§]

– Two-flavor neutrino states

$$|\underline{\nu}^{(f)}\rangle \;=\; \mathbf{U}(\theta,\delta)\,|\underline{\nu}^{(m)}\rangle$$

where $|\underline{\nu}^{(f)}\rangle \;=\; (|\nu_e\rangle, |\nu_\mu\rangle)^T$ and $|\underline{\nu}^{(m)}\rangle \;=\; (|\nu_1\rangle, |\nu_2\rangle)^T$ and
$\mathbf{U}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

– Flavor states at time t:

$$|\underline{\nu}^{(f)}(t)\rangle \;=\; \mathbf{U}(\theta,\delta)\,\mathbf{U}_0(t)\,\mathbf{U}(\theta,\delta)^{-1}\,|\underline{\nu}^{(f)}\rangle \equiv\; \widetilde{\mathbf{U}}(t)|\underline{\nu}^{(f)}\rangle\,,$$

with $\mathbf{U}_0(t) = \begin{pmatrix} e^{-iE_1 t} & 0 \\ 0 & e^{-iE_2 t} \end{pmatrix}$.

[§] M.B., F.Dell'Anno, S.De Siena and F.Illuminati, EPL (2009).

– Transition probability for $\nu_\alpha \to \nu_\beta$

$$P_{\nu_\alpha \to \nu_\beta}(t) = |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 = |\widetilde{\mathbf{U}}_{\alpha\beta}(t)|^2 \,.$$

• We now take the flavor states at initial time as our qubits:

$$|\nu_e\rangle \equiv |1\rangle_e |0\rangle_\mu \equiv |10\rangle_f, \quad |\nu_\mu\rangle \equiv |0\rangle_e |1\rangle_\mu \equiv |01\rangle_f,$$

• Starting from $|10\rangle_f$ or $|01\rangle_f$, time evolution generates the (entangled) Bell-like states:

$$|\nu_\alpha(t)\rangle = \widetilde{\mathbf{U}}_{\alpha e}(t) |1\rangle_e |0\rangle_\mu + \widetilde{\mathbf{U}}_{\alpha\mu}(t) |0\rangle_e |1\rangle_\mu, \quad \alpha = e, \mu.$$

Entanglement measures

It is necessary to distinguish the various entanglement measures for pure and mixed states (which may contain classical correlations).

Measures for pure states:

- von Neumann entropy
- Geometric Entanglement

Measures for mixed states:

- Entanglement of Formation and Concurrence
- Logarithmic negativity
- Relative Entropy of Entanglement

Entanglement in neutrino oscillations: two-flavors

Consider the density matrix for the electron neutrino state $\rho^{(e)} = |\nu_e(t)\rangle\langle\nu_e(t)|$, and trace over mode $\mu \;\Rightarrow\; \rho_e^{(e)}$.

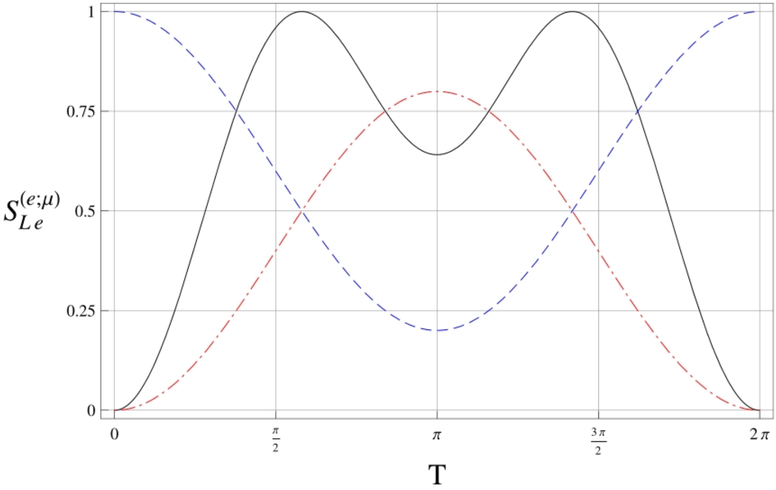
- The associated linear entropy is :

$$S_L^{(e;\mu)}(\rho^{(e)}) \,=\, 4\,|\widetilde{\mathbf{U}}_{e\mu}(t)|^2\,|\widetilde{\mathbf{U}}_{ee}(t)|^2 \,=\, 4\,P_{\nu_e\to\nu_e}(t)\,P_{\nu_e\to\nu_\mu}(t)$$

The linear entropy for the state $\rho^{(\alpha)}$ is:

$$\begin{aligned} S_{L\alpha}^{(e;\mu)} = S_{L\alpha}^{(\mu;e)} &= 4\,|\widetilde{\mathbf{U}}_{\alpha\mu}(t)|^2\,|\widetilde{\mathbf{U}}_{\alpha e}(t)|^2 \\ &= 4|\widetilde{\mathbf{U}}_{\alpha e}(t)|^2\,(1-|\widetilde{\mathbf{U}}_{\alpha e}(t)|^2) \\ &= 4|\widetilde{\mathbf{U}}_{\alpha\mu}(t)|^2\,(1-|\widetilde{\mathbf{U}}_{\alpha\mu}(t)|^2)\,. \end{aligned}$$

- Linear entropy given by product of transition probabilities



Linear entropy $S_{Le}^{(e;\mu)}$ (full) as a function of the scaled time $T = \frac{2Et}{\Delta m_{12}^2}$, with $\sin^2\theta = 0.314$. Transition probabilities $P_{\nu_e\to\nu_e}$ (dashed) and $P_{\nu_e\to\nu_\mu}$ (dot-dashed) are reported for comparison.

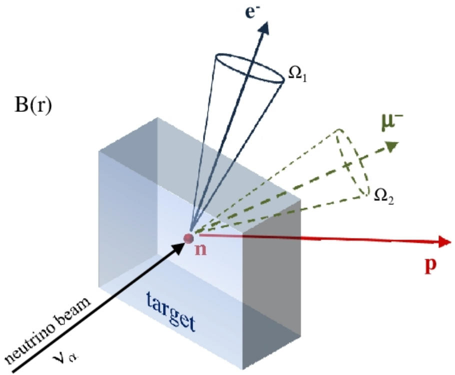
ν–oscillations as a resource for quantum information

• Single-particle entanglement encoded in flavor states $|\underline{\nu}^{(f)}(t)\rangle$ is a real physical resource that can be used, at least in principle, for protocols of quantum information.

– Experimental scheme for the transfer of the flavor entanglement of a neutrino beam into a single-particle system with *spatially separated modes.*

Charged-current interaction between a neutrino ν_α with flavor α and a nucleon N gives a lepton α^- and a baryon X:

$$\nu_\alpha + N \longrightarrow \alpha^- + X\,.$$



Generation of a single-particle entangled lepton state (two flavors):

In the target the charged-current interaction occurs: $\nu_\alpha + n \longrightarrow \alpha^- + p$ with $\alpha = e\,, \mu$. A spatially nonuniform magnetic field $\mathbf{B}(\mathbf{r})$ constraints the momentum of the outgoing lepton within a solid angle Ω_i, and ensures spatial separation between lepton paths. The reaction produces a superposition of electronic and muonic spatially separated states.

• Given the initial Bell-like superposition $|\nu_\alpha(t)\rangle$ the unitary process associated with the weak interaction leads to the superposition

$$|\alpha(t)\rangle \,=\, \Lambda_e |1\rangle_e |0\rangle_\mu + \Lambda_\mu |0\rangle_e |1\rangle_\mu \,,$$

where $|\Lambda_e|^2 + |\Lambda_\mu|^2 = 1$, and $|k\rangle_\alpha$, with $k = 0, 1$, is the lepton qubit.

The coefficients Λ_α are proportional to $\widetilde{\mathbf{U}}_{\alpha\beta}(t)$ and to the cross sections associated with the creation of an electron or a muon.

• Analogy with single-photon system: quantum uncertainty on the "*which path*" of the photon at the output of an unbalanced beam splitter ⇔ uncertainty on the "*which flavor*" of the produced lepton.

The coefficients Λ_α plays the role of the transmissivity and of the reflectivity of the beam splitter.

Other results

- Generalization to three flavors. Extension to wave packets;*
- Flavor entanglement in Quantum Field Theory.†

*M.B., F.Dell'Anno, S.De Siena and F.Illuminati, EPL (2015).
†M.B., F.Dell'Anno, S.De Siena and F.Illuminati, EPL (2014).

Quantum Field Theory of neutrino mixing and oscillations

Neutrino mixing in QFT

- Mixing relations for two Dirac fields

$$\nu_e(x) = \cos\theta\,\nu_1(x) + \sin\theta\,\nu_2(x)$$

$$\nu_\mu(x) = -\sin\theta\,\nu_1(x) + \cos\theta\,\nu_2(x)$$

can be written as*

$$\nu_e^\alpha(x) = G_\theta^{-1}(t)\,\nu_1^\alpha(x)\,G_\theta(t)$$

$$\nu_\mu^\alpha(x) = G_\theta^{-1}(t)\,\nu_2^\alpha(x)\,G_\theta(t)$$

– Mixing generator:

$$G_\theta(t) = \exp\left[\theta\int d^3\mathbf{x}\,\left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)\right)\right]$$

For ν_e, we get $\frac{d^2}{d\theta^2}\,\nu_e^\alpha = -\nu_e^\alpha$ with i.c. $\nu_e^\alpha\big|_{\theta=0} = \nu_1^\alpha$, $\frac{d}{d\theta}\nu_e^\alpha\big|_{\theta=0} = \nu_2^\alpha$.

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*M.B. and G.Vitiello, Annals Phys. (1995)

• The vacuum $|0\rangle_{1,2}$ is not invariant under the action of $G_\theta(t)$:

$$|0(t)\rangle_{e,\mu} \equiv G^{-1}_\theta(t)\;|0\rangle_{1,2}$$

• Relation between $|0\rangle_{1,2}$ and $|0(t)\rangle_{e,\mu}$: orthogonality! (for $V \to \infty$)

$$\lim_{V\to\infty}\ {}_{1,2}\langle 0|0(t)\rangle_{e,\mu} = \lim_{V\to\infty} e^{V\int \frac{d^3\mathbf{k}}{(2\pi)^3}\ \ln\left(1-\sin^2\theta\,|V_\mathbf{k}|^2\right)^2} = 0$$

with

$$|V_\mathbf{k}|^2 \ \equiv\ \sum_{r,s} |\ v^{r\dagger}_{-\mathbf{k},1} u^s_{\mathbf{k},2}\ |^2 \neq 0 \quad for \quad m_1 \neq m_2$$

.

Quantum Field Theory vs. Quantum Mechanics

- Quantum Mechanics:

- finite ♯ of degrees of freedom.

- unitary equivalence of the representations of the canonical commutation relations (von Neumann theorem).

- Quantum Field Theory:

- infinite ♯ of degrees of freedom.

- ∞ many unitarily inequivalent representations of the field algebra $\Leftrightarrow$ many vacua .

- The mapping between interacting and free fields is "weak", i.e. representation dependent (LSZ formalism)[*]. Example: theories with spontaneous symmetry breaking.

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[*] F. Strocchi, *Elements of Quantum Mechanics of Infinite Systems* (W. Sc., 1985).
H. Umezawa, *Advanced Field Theory* (AIP, 1993).

- The "flavor vacuum" $|0(t)\rangle_{e,\mu}$ is a $SU(2)$ generalized coherent state[†]:

$$|0\rangle_{e,\mu} = \prod_{\mathbf{k},r} \Big[(1 - \sin^2\theta\,|V_{\mathbf{k}}|^2) - \epsilon^r \sin\theta\,\cos\theta\,|V_{\mathbf{k}}|\,(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1})$$

$$+\,\epsilon^r \sin^2\theta\,|V_{\mathbf{k}}||U_{\mathbf{k}}|\,(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}) + \sin^2\theta\,|V_{\mathbf{k}}|^2\,\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big]\,|0\rangle_{1,2}$$

- Condensation density:

$$_{e,\mu}\langle 0(t)|\alpha^{r\dagger}_{\mathbf{k},i}\alpha^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} \;=\; _{e,\mu}\langle 0(t)|\beta^{r\dagger}_{\mathbf{k},i}\beta^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} \;=\; \sin^2\theta\,|V_{\mathbf{k}}|^2$$

vanishing for $m_1 = m_2$ and/or $\theta = 0$ (in both cases no mixing).

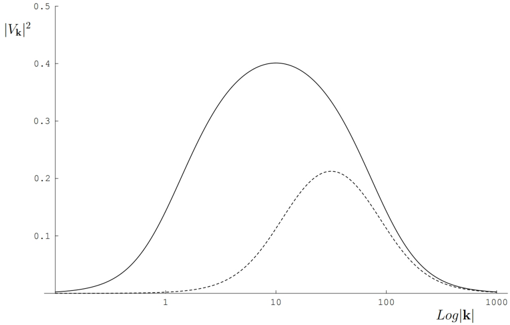
– Condensate structure as in systems with SSB (e.g. superconductors)

– Exotic condensate: mixed pairs

– Note that $|0\rangle_{e\,\mu} \neq |a\rangle_1 \otimes |b\rangle_2 \Rightarrow$ entanglement.

[†] A. Perelomov, *Generalized Coherent States,* (Springer V., 1986)

Condensation density for mixed fermions



Solid line: $m_1 = 1$, $m_2 = 100$; Dashed line: $m_1 = 10$, $m_2 = 100$.

- $V_{\mathbf{k}} = 0$ when $m_1 = m_2$ and/or $\theta = 0$.
- Max. at $k = \sqrt{m_1 m_2}$ with $V_{max} \to \frac{1}{2}$ for $\frac{(m_2 - m_1)^2}{m_1 m_2} \to \infty$.
- $|V_{\mathbf{k}}|^2 \simeq \frac{(m_2 - m_1)^2}{4k^2}$ for $k \gg \sqrt{m_1 m_2}$.

• Structure of the annihilation operators for $|0(t)\rangle_{e,\mu}$:

$$\alpha^r_{\mathbf{k},e}(t) = \cos\theta\,\alpha^r_{\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},2} + \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},2}\right)$$

$$\alpha^r_{\mathbf{k},\mu}(t) = \cos\theta\,\alpha^r_{\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},1} - \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},1}\right)$$

$$\beta^r_{-\mathbf{k},e}(t) = \cos\theta\,\beta^r_{-\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},2} - \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},2}\right)$$

$$\beta^r_{-\mathbf{k},\mu}(t) = \cos\theta\,\beta^r_{-\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},1} + \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},1}\right)$$

• Mixing transformation = Rotation + Bogoliubov transformation .

– Bogoliubov coefficients:

$$U_{\mathbf{k}}(t) \;=\; u^{r\dagger}_{\mathbf{k},2} u^r_{\mathbf{k},1}\, e^{i(\omega_{k,2}-\omega_{k,1})t} \quad ; \qquad V_{\mathbf{k}}(t) = \epsilon^r\, u^{r\dagger}_{\mathbf{k},1} v^r_{-\mathbf{k},2}\, e^{i(\omega_{k,2}+\omega_{k,1})t}$$

$$|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1$$

Decomposition of mixing generator *

The mixing generator can be expressed in terms of a rotation and a Bogoliubov transformation. Define:

$$R(\theta) \equiv \exp\left\{\theta \sum_{\mathbf{k},r} \left[\left(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r}_{\mathbf{k},2} + \beta^{r\dagger}_{\mathbf{k},1}\beta^{r}_{\mathbf{k},2}\right)e^{i\psi_k} - \left(\alpha^{r\dagger}_{\mathbf{k},2}\alpha^{r}_{\mathbf{k},1} + \beta^{r\dagger}_{\mathbf{k},2}\beta^{r}_{\mathbf{k},1}\right)e^{-i\psi_k}\right]\right\},$$

$$B_i(\Theta_i) \equiv \exp\left\{\sum_{\mathbf{k},r}\Theta_{\mathbf{k},i}\ \epsilon^r\left[\alpha^{r}_{\mathbf{k},i}\beta^{r}_{-\mathbf{k},i}e^{-i\phi_{k,i}} - \beta^{r\dagger}_{-\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i}e^{i\phi_{k,i}}\right]\right\}, \qquad i = 1,2$$

Since $[B_1, B_2] = 0$ we put $B(\Theta_1,\Theta_2) \equiv B_1(\Theta_1)\, B_2(\Theta_2)$.

- We find:

$$G_\theta \ = \ B(\Theta_1,\Theta_2)\ \ R(\theta)\ \ B^{-1}(\Theta_1,\Theta_2)$$

which is realized when the $\Theta_{\mathbf{k},i}$ are chosen as:

$$U_{\mathbf{k}} = e^{-i\psi_k}\ \cos(\Theta_{\mathbf{k},1} - \Theta_{\mathbf{k},2})\,; \qquad V_{\mathbf{k}} = e^{\frac{(\phi_{k,1}+\phi_{k,2})}{2}}\ \sin(\Theta_{\mathbf{k},1} - \Theta_{\mathbf{k},2})$$

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*M. B., M.V. Gargiulo and G. Vitiello, Phys. Lett. B (2017)

Bogoliubov vs Pontecorvo

- Bogoliubov and Pontecorvo do not commute!

[ , ] ≠ 0

As a result, flavor vacuum gets a non-trivial term:

$$|0\rangle_{e,\mu} \equiv G_\theta^{-1}|0\rangle_{1,2} \,=\, |0\rangle_{1,2} \,+\, \Big[B(m_1,m_2)\,,\; R^{-1}(\theta)\Big]\;|\widetilde{0}\rangle_{1,2}$$

with $|\widetilde{0}\rangle_{1,2} \equiv B^{-1}(\Theta_1,\Theta_2)|0\rangle_{1,2}$.

- Non-diagonal Bogoliubov transformation

$$|0\rangle_{e,\mu} \cong \left[\mathbb{1} + \theta\, a \int \frac{d^3\boldsymbol{k}}{(2\pi)^{\frac{3}{2}}}\; \tilde{V}_{\mathbf{k}} \sum_r \epsilon^r \Big(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big)\right]|0\rangle_{1,2}\,,$$

with $a \equiv \frac{(m_2-m_1)^2}{m_1 m_2}$.

Currents and charges for mixed fermions *

- Lagrangian in the mass basis:

$$\mathcal{L} = \bar{\Psi}_m \left(i \not\partial - M_d\right) \Psi_m$$

where $\Psi_m^T = (\nu_1, \nu_2)$ and $M_d = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$.

– Lagrangian in the flavor basis:

$$\mathcal{L} = \bar{\Psi}_f \left(i \not\partial - M\right) \Psi_f$$

where $\Psi_f^T = (\nu_e, \nu_\mu)$ and $M = \begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix}$.

* M. B., P. Jizba and G. Vitiello, Phys. Lett. B (2001)

– Two sets of charges:

$$Q_i = \int d^3\mathbf{x}\, \nu_i^\dagger(x)\, \nu_i(x)\,; \qquad i = 1, 2$$

$$Q_\sigma(t) = \int d^3\mathbf{x}\, \nu_\sigma^\dagger(x)\, \nu_\sigma(x)\,; \qquad \sigma = e, \mu$$

– In presence of mixing, neutrino flavor charges not conserved charges $\Rightarrow$ flavor oscillations.

– They are still (approximately) conserved in the vertex $\Rightarrow$ define flavor neutrinos as their eigenstates.

- Problem: find the eigenstates of the above charges.

- Flavor charge operators are diagonal in the flavor ladder operators:

$$:: Q_\sigma(t) :: \;\equiv\; \int d^3\mathbf{x} \; :: \nu_\sigma^\dagger(x)\, \nu_\sigma(x) ::$$

$$= \sum_r \int d^3\mathbf{k} \left(\alpha^{r\dagger}_{\mathbf{k},\sigma}(t) \alpha^{r}_{\mathbf{k},\sigma}(t) \,-\, \beta^{r\dagger}_{-\mathbf{k},\sigma}(t) \beta^{r}_{-\mathbf{k},\sigma}(t) \right) , \quad \sigma = e, \mu.$$

Here :: ... :: denotes normal ordering w.r.t. flavor vacuum:

$$:: A :: \equiv A \,-\, {}_{e,\mu}\langle 0|A|0\rangle_{e,\mu}$$

- Define flavor neutrino states with definite momentum and helicity:

$$|\nu^r_{\mathbf{k},\sigma}\rangle \,\equiv\, \alpha^{r\dagger}_{\mathbf{k},\sigma}(0)\, |0\rangle_{e,\mu}$$

– Such states are eigenstates of the flavor charges (at t=0):

$$:: Q_\sigma :: |\nu^r_{\mathbf{k},\sigma}\rangle \,=\, |\nu^r_{\mathbf{k},\sigma}\rangle$$

Neutrino oscillation formula (QFT)

– We have, for an electron neutrino state:

$$\mathcal{Q}_{\mathbf{k},\sigma}(t) \equiv \langle \nu^r_{\mathbf{k},e}| :: Q_\sigma(t) :: |\nu^r_{\mathbf{k},e}\rangle$$

$$= \left|\left\{\alpha^r_{\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 + \left|\left\{\beta^{r\dagger}_{-\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2$$

with $Q_\sigma(t) \equiv \int d^3\mathbf{x}\, \nu^\dagger_\sigma(x)\, \nu_\sigma(x)$.

- Neutrino oscillation formula (exact result)*:

$$\mathcal{Q}_{\mathbf{k},e}(t) = 1 - |U_\mathbf{k}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2} - \omega_{k,1}}{2}\, t\right) - |V_\mathbf{k}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2} + \omega_{k,1}}{2}\, t\right)$$

- For $k \gg \sqrt{m_1 m_2}$, $|U_\mathbf{k}|^2 \to 1$ and $|V_\mathbf{k}|^2 \to 0 \Rightarrow$ Pontecorvo formula is recovered.

* M.B., P.Henning and G.Vitiello, Phys. Lett. **B** (1999).

Lepton charge violation for Pontecorvo states[†]

– Pontecorvo states:

$$|\nu^r_{\mathbf{k},e}\rangle_P \;=\; \cos\theta\,|\nu^r_{\mathbf{k},1}\rangle \;+\; \sin\theta\,|\nu^r_{\mathbf{k},2}\rangle$$

$$|\nu^r_{\mathbf{k},\mu}\rangle_P \;=\; -\sin\theta\,|\nu^r_{\mathbf{k},1}\rangle \;+\; \cos\theta\,|\nu^r_{\mathbf{k},2}\rangle\;,$$

are *not* eigenstates of the flavor charges.

⇒ *violation of lepton charge conservation* in the production/detection vertices, at tree level:

$${}_P\langle\nu^r_{\mathbf{k},e}|:Q_e(0):|\nu^r_{\mathbf{k},e}\rangle_P \;=\; \cos^4\theta+\sin^4\theta+2|U_{\mathbf{k}}|\,\sin^2\theta\cos^2\theta \;<\; 1,$$

for any $\theta\neq 0$, $\mathbf{k}\neq 0$ and for $m_1\neq m_2$.

[†] M. B., A. Capolupo, F. Terranova and G. Vitiello, Phys. Rev. **D** (2005)
C. C. Nishi, Phys. Rev. **D** (2008).

Entanglement for flavor neutrino states in QFT

– Entanglement for flavor neutrino states in QFT can be expressed by means of the variances* of the neutrino charges:† Q_i, $Q_\sigma(t)$

– Variance of $Q_i \to$ static entanglement:

$$\begin{aligned} \Delta Q_i(\nu_e) &= \langle \nu^r_{\mathbf{k},e}|Q_i^2(t)|\nu^r_{\mathbf{k},e}\rangle - \langle \nu^r_{\mathbf{k},e}|Q_i|\nu^r_{\mathbf{k},e}\rangle^2 \\ &= \cos^2\theta \sin^2\theta \end{aligned}$$

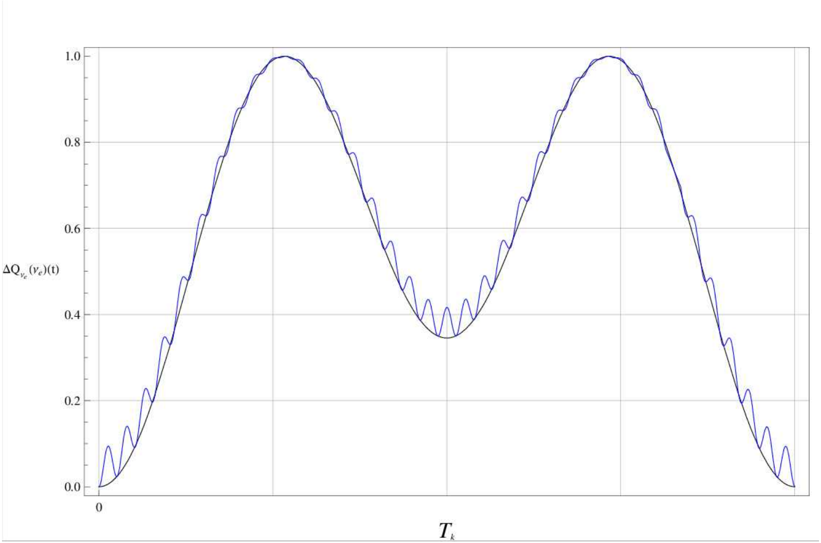
– Variance of $Q_\sigma \to$ flavor (dynamical) entanglement:

$$\begin{aligned} \Delta Q_\sigma(\nu_e)(t) &= \langle \nu^r_{\mathbf{k},e}|Q_\sigma^2(t)|\nu^r_{\mathbf{k},e}\rangle - \langle \nu^r_{\mathbf{k},e}|Q_\sigma|\nu^r_{\mathbf{k},e}\rangle^2 \\ &= \mathcal{Q}^{\mathbf{k}}_{e\to e}(t)\, \mathcal{Q}^{\mathbf{k}}_{e\to\mu}(t) \end{aligned}$$

in formal agreement with results obtained in QM.

* A. A. Klyachko, B. Öztop, and A. S. Shumovsky, *Phys. Rev.* **A** (2007).
† M.Blasone, F. Dell'Anno, S.De Siena and F. Illuminati, EPL (2014)

QFT flavor entanglement



QM vs. QFT flavor entanglement for $|\nu_e(t)\rangle$.

Neutrino ontology: flavor or mass?

- In view of the unitary inequivalence of mass and flavor representations, we have the problem of the fundamental (ontological) nature of neutrino.

Flavor or mass, that is the question...

Neutrino ontology: research directions

- How to verify the fundamental nature of neutrino states?

Two directions:

- Investigate the phenomenology of flavor neutrinos, with corrections expected in the non-relativistic regime: oscillations, beta decay endpoint, quantum correlations, ...
- Use the formal consistency of QFT, by comparing neutrino processes in two different frames (inertial and comoving) for accelerated particle: Unruh effect.*

*M. B., G. Lambiase, G. Luciano and L.Petruzziello, Phys. Rev. D (2018); G.Cozzella, S.Fulling, A.Landulfo, G.Matsas and D.Vanzella, Phys.Rev.(2018) M. B., G.Lambiase, G. Luciano and L.Petruzziello, Phys. Lett. B (2020)

- Dynamical generation of fermion mixing*.
- Flavor-energy uncertainty relations for mixed states†.
- Poincaré invariance for flavor neutrinos‡.
- Violation of equivalence principle§.

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*M.B., P.Jizba, N.E.Mavromatos and L.Smaldone, Phys. Rev. D (2019)
†M. B., P. Jizba and L. Smaldone, Phys. Rev. D (2019)
‡M.B., P.Jizba, N.E.Mavromatos and L.Smaldone, Phys. Rev. D (2020) ; A. E. Lobanov, Ann. Phys. (2019)
§M.B., P.Jizba, G.Lambiase and L.Petruzziello, Phys. Lett. B (2020)

Flavor neutrino as unstable particles

- Time-energy uncertainty relations (TEUR) in the Mandelstam–Tamm form, furnish lower-bounds on neutrino energy uncertainty compatible with flavor oscillations*.

- QFT formulation of neutrino oscillations suggests that these bounds can be read as flavor-energy uncertainty relations (FEUR)†. Energy uncertainty is connected with the intrinsic unstable nature of flavor neutrinos.

*S.M Bilenky, F. von Feilitzsch and W. Potzel, J. Phys. G (2008)

†M. B., P. Jizba and L. Smaldone, Phys. Rev. D (2019)

Time-energy uncertainty relations

- Mandelstam–Tamm TEUR is*:

$$\Delta E \, \Delta t \, \geq \frac{1}{2}$$

where

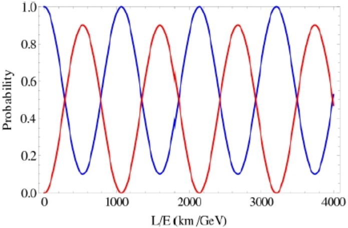
$$\Delta E \equiv \sigma_H \qquad \Delta t \equiv \sigma_O / \left| \frac{\mathrm{d}\langle O(t) \rangle}{\mathrm{d}t} \right|$$

Here $\langle \ldots \rangle \equiv \langle \psi | \ldots | \psi \rangle$ and $O(t)$ represents the "clock observable" whose dynamics quantifies temporal changes in a system.

– The above inequality is obtained by means of the Cauchy-Schwarz inequality and using the fact that $[\hat{O}, \hat{H}] \neq 0$.

*L. Mandelstam and I.G. Tamm, J. Phys. USSR (1945)

Clock observables



Flavor-energy uncertainty relations

• Choose flavor charges as clock observables. Then $[Q_{\nu_\sigma}(t)\,,\,H] \neq 0 \Rightarrow$ flavor-energy uncertainty relation[†]:

$$\langle \Delta H \rangle \, \langle \Delta Q_\sigma(t) \rangle \;\geq\; \frac{1}{2} \left| \frac{\mathrm{d}\langle Q_\sigma(t)\rangle}{\mathrm{d}t} \right|$$

Taking the state $|\psi\rangle \;=\; |\nu^r_{\mathbf{k},\sigma}\rangle$ we have $\langle Q_{si}(t)\rangle = \mathcal{Q}_{\sigma\to\sigma}(t)$ and

$$\langle \Delta Q_{\nu_\sigma}(t) \rangle \;=\; \sqrt{\mathcal{Q}_{\sigma\to\sigma}(t)(1-\mathcal{Q}_{\sigma\to\sigma}(t))} \leq \frac{1}{2}.$$

Integrating over time from 0 to T, and using the triangular inequality, we obtain:

$$\Delta E \, T \;\geq\; \mathcal{Q}_{\sigma\to\rho}(T)\,, \quad \sigma \neq \rho$$

[†] M. B., P. Jizba and L.Smaldone, Phys. Rev. D (2019)

Neutrino oscillation condition

When $m_i/|\mathbf{k}| \to 0$:

$$\Delta E \ \geq \ \frac{2\,\sin^2 2\theta}{L_{osc}}$$

This relation is usually interpreted as neutrino oscillation condition[‡].

The situation is similar to that of unstable particles:

$$\Delta E \ \approx \ \frac{1}{2\tau}$$

where the τ is the particle life-time.

– As for unstable particles only energy distribution are meaningful.
The width of the distribution is related to the oscillation length.

[‡] S.M Bilenky, F. von Feilitzsch and W. Potzel, J. Phys. G (2008)

- Electron neutrino mass

$$\langle H \rangle|_{k=0} \;=\; m_1 \cos^2\theta \;+\; m_2 \sin^2\theta$$

$$\Delta E|_{k=0} \;\geq\; \frac{\sin^2 2\theta\,(m_2 - m_1)}{4\pi}$$

- Note that

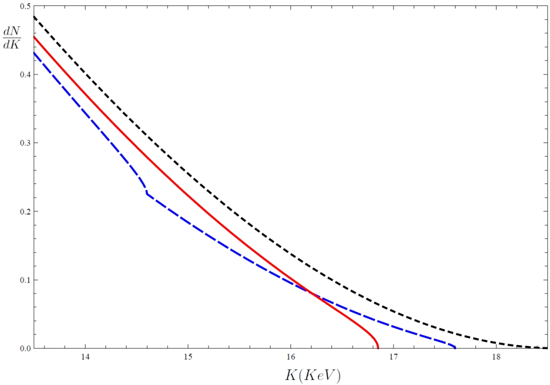
$$\begin{aligned}\sigma_Q^2 &= \langle Q_\sigma^2(t)\rangle_\sigma \;-\; \langle Q_\sigma(t)\rangle_\sigma^2 \\ &= \mathcal{Q}_{\sigma\to\sigma}(t)\,(1-\mathcal{Q}_{\sigma\to\sigma}(t))\;.\end{aligned}$$

quantifies dynamical (flavor) entanglement for neutrino states[§] since it concides with the linear entropy in terms of the flavor qubits:

$$|\nu_e\rangle \equiv |1\rangle_e|0\rangle_\mu \equiv |10\rangle_f\,, \quad |\nu_\mu\rangle \equiv |0\rangle_e|1\rangle_\mu \equiv |01\rangle_f\,,$$

[§] M.B., F.Dell'Anno, S.De Siena and F.Illuminati, EPL (2009)

Phenomenological consequences



The tail of the tritium β spectrum for:

- a massless neutrino (dotted line);
- fundamental flavor states (continuous line);
- superimposed prediction for 2 mass states (short-dashed line):

We used $m_e = 1.75$ KeV, $m_1 = 1$ KeV, $m_2 = 4$ KeV, $\theta = \pi/6$.

Neutrino oscillations in the interaction picture

- Analogy with unstable particles suggests alternative approach: treat the mixing term as a perturbation and compute oscillation formula from QFT at finite time*.

- Decompose neutrino Lagrangian as $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$ with

$$\mathcal{L}_0 = \sum_{\sigma=e,\mu} \overline{\nu}_\sigma \left(i \not\partial - m_\sigma\right) \nu_\sigma$$

$$\mathcal{L}_{int} = -m_{e\mu} \left(\overline{\nu}_e \nu_\mu + \overline{\nu}_\mu \nu_e\right)$$

Time-evolution operator ($\mathcal{H}_{int} = -\mathcal{L}_{int}$):

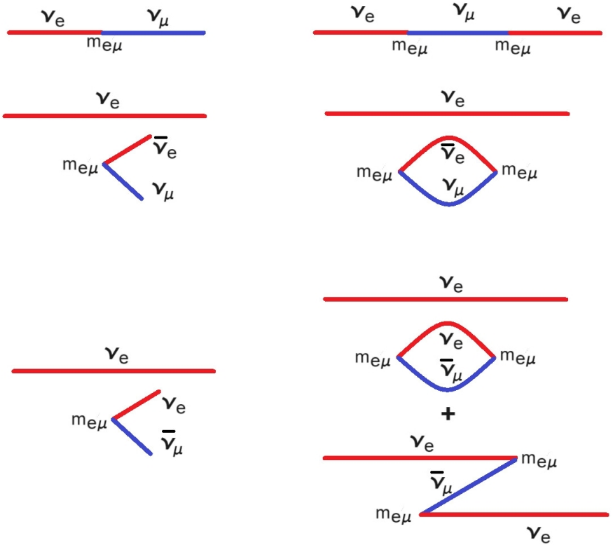
$$U(t_i, t_f) = \mathcal{T} \exp\left[-i \int_{t_i}^{t_f} \mathrm{d}^4 x \, :\mathcal{H}_{int}(x):\right].$$

*M. B., F. Giacosa, L. Smaldone and G.Torrieri, EPJC (2023)

C. Bernardini, L. Maiani and M. Testa, Phys. Rev. Lett. (1993).

P. Facchi and S. Pascazio, La regola d'oro di Fermi, (Bibliopolis, 1999).

Diagrams for neutrino oscillations



Neutrino oscillation formula

Total flavor transition probability

$$\mathcal{P}^e_D(\mathbf{p};\Delta t) \;=\; 4m^2_{e\mu}\left[\frac{W^2_{\mathbf{p}}}{\left(\omega^-_{\mathbf{p}}\right)^2}\sin^2\left(\frac{\omega^-_{\mathbf{p}}\Delta t}{2}\right)+\frac{Y^2_{\mathbf{p}}}{\left(\omega^+_{\mathbf{p}}\right)^2}\sin^2\left(\frac{\omega^+_{\mathbf{p}}\Delta t}{2}\right)\right]$$

with $\omega^{\pm}_{\mathbf{p}} \equiv \omega_{\mathbf{p},e} \pm \omega_{\mathbf{p},\mu}$. Note that

$$|U_{\mathbf{p}}| \;=\; W_{\mathbf{p}}\frac{m_\mu - m_e}{\omega^-_{\mathbf{p}}}\,, \qquad |V_{\mathbf{p}}| \;=\; Y_{\mathbf{p}}\frac{m_\mu - m_e}{\omega^+_{\mathbf{p}}}$$

when $m_1 \approx m_e$, $m_2 \approx m_\mu$. Then

$$\mathcal{P}^e_D(\mathbf{p};\Delta t) \;=\; \sin^2(2\theta)\left[|U_{\mathbf{p}}|^2\sin^2\left(\frac{\omega^-_{\mathbf{p}}\Delta t}{2}\right)+|V_{\mathbf{p}}|^2\sin^2\left(\frac{\omega^+_{\mathbf{p}}\Delta t}{2}\right)\right]$$

with $\theta = m_{e\mu}/(m_\mu - m_e) \approx \sin\theta$. Oscillation formula of the flavor Fock-space approach!!

Results

• The interaction picture approach[†] matches results of the flavor Fock space approach, at the lowest order in $m_{e\mu}$

• It should be possible to sum up the perturbative series and recover the flavor space (nonperturbative) result.

• Similar results for chiral oscillations (see below).

[†] M.B., F.Giacosa, L.Smaldone and G.Torrieri, EPJC (2023)

Quantum correlations & nonlocality in neutrino oscillations

Quantum Resource Theory

Resource theories are a versatile set of tools developed in quantum information theory‡.

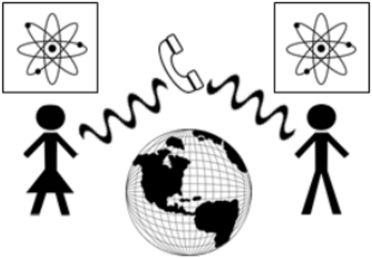
The basic idea of a quantum resource theory is to study quantum information processing under a restricted set of physical operations, called *free* operations.

These allow as to prepare only certain physical states, called *free* states. The other are called **resource** states.

‡E. Chitambar, G. Gour, Rev. Mod. Phys. (2019)
A. Streltsov, G. Adesso and M. B. Plenio, Rev. Mod. Phys. (2017),

Entanglement in QRT

Alice and Bob work in their laboratory separated by a large distance. They can communicate only by telephone.



The free operations consist in local operations and classical communication (LOCC). But an entangled state cannot be generated using LOCC $\Rightarrow$ **Entanglement is a (quantum) resource**.

Quantum Correlations

Quantum systems exhibit properties that are beyond our understanding of reality. They show correlations that have no classical counterpart.

Entanglement is the most known of these correlations. But the terminology *quantum correlations* refers to a broader concept:

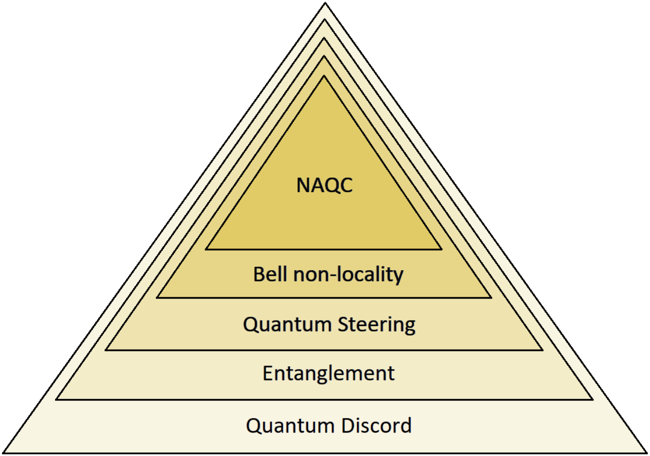
Quantum correlations related to entanglement:

- Bell non-locality
- Entanglement
- Quantum steering

Quantum correlations beyond entanglement:

- Quantum discord

Quantum Correlations[§]



Hierarchy of quantum correlations (figure adapted from G.Adesso et al., J. Phys. A (2016))

[§] G. Adesso, T.R. Bromley and M. Cianciaruso, J. Phys. A (2016)

Quantum correlations in neutrino oscillations

- Recently, quantum correlations in neutrino oscillations have been thoroughly investigated. A partial list of publications include:

A.K. Alok et al., *Quantum correlations in terms of neutrino oscillation probabilities*, Nuc. Phys. B (2016)

F. Ming et al. *Quantification of quantumness in neutrino oscillations*, Eur. Phys. J. C (2020)

M.B., S.De Siena and C.Matrella, *Wave packet approach to quantum correlations in neutrino oscillations*, Eur. Phys. J. C (2021)

V. Bittencourt, M.B., S.De Siena and C.Matrella, *Complete complementarity relations for quantum correlations in neutrino oscillations*, Eur. Phys. J. C (2022)

Y.W.Li et al. *Genuine tripartite entanglement in three-flavor neutrino oscillations* Eur. Phys. J. C (2022)

V. Bittencourt, M.B., S.De Siena and C.Matrella, *Quantifying quantumness in three-flavor neutrino oscillations*, Eur. Phys. J. C (2024)

Non-local Advantage of Quantum Coherence†

- A state is said to be coherent provided that there are non-zero non-diagonal elements in its matrix representation.

Coherence can be quantified by means of the l_1-norm of coherence:*

$$C_{l_1}(\rho) = \sum_{i \neq j} |\rho_{i,j}|$$

If the qubit is prepared in either spin up or down state along z, it is incoherent in z-basis ($C_{l_1}^z = 0$) and fully coherent in x- and y-basis ($C_{l_1}^{x(y)} = 1$).

Upper bound beyond which the effects of non-locality emerge:

$$\sum_{i=x,y,z} C_i^{l_1}(\rho) \leq C_{max}.$$

* T.Baumgratz, M.Cramer and M.B.Plenio, Phys. Rev. Lett. (2014).
† D. Mondal, T. Pramanik, A.K. Pati, Phys. Rev. A (2017).

Non-local Advantage of Quantum Coherence

Consider a bipartite system made of two spatially separated subsystems. Alice performs a measurement Π_i^b on σ_i eigenbasis with outcome $b = \{0, 1\}$ and probability $p_{\Pi_i^b} = \mathrm{Tr}[(\Pi_i^b \otimes \mathbf{1})\rho_{AB}]$.

Measured state for the two-qubit state is$\rho_{AB|\Pi_i^b} = (\Pi_i^b \otimes \mathbf{1})\rho_{AB}(\Pi_i^b \otimes \mathbf{1})/p_{\Pi_i^b}$ and the conditional state for qubit B is $\rho_{B|\Pi_i^b} = \mathrm{Tr}_A(\rho_{AB|\Pi_i^b})$.

Then Alice tells Bob her measurement choice and Bob has to measure the coherence of qubit B at random in the eigenbases of the other two Pauli matrices σ_j and σ_k.

If the above condition for locality is violated then we cannot have a single-system description of the coherence of subsytem B.

The criterion for achieving a NAQC of qubit B can be written as:

$$N_{l_1}(\rho_{AB}) = \frac{1}{2}\sum_{i,j,b} p(\rho_{B|\Pi^b_{j\neq i}}) C^{\sigma_i}_{l_1}(\rho_{B|\Pi^b_{j\neq i}}) > \sqrt{6}.$$

Quantification of quantumness in neutrino oscillations

- Quantumness in neutrino oscillations has been quantified through various correlation measures‡: Non-local Advantage of Quantum Coherence (NAQC), quantum steering and Bell non-locality.

- The criterion for NAQC is:

$$N^{l_1}(\rho_{AB}) = \frac{1}{2}\sum_{i,j,b} p(\rho^b_{\Pi_{j\neq i}}) C^{\sigma_i}_{l_1}(\rho_{B|\Pi_{j\neq i}}) > \sqrt{6}.$$

- Bell non-locality (violation of CHSH inequality):

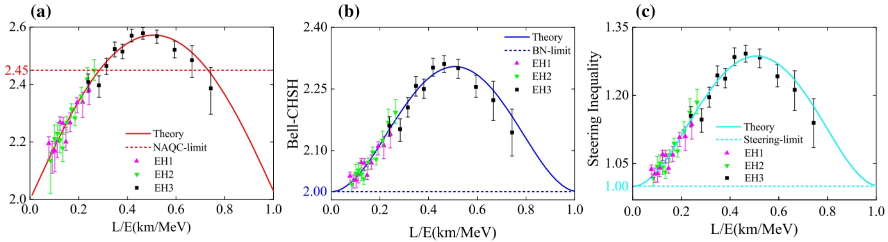
$$B(\rho_{AB}) = |\langle B_{CHSH}\rangle| \leq 2.$$

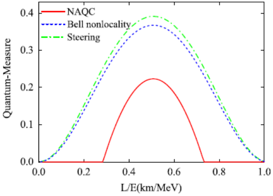
- Quantum steering:

$$F_n(\rho_{AB}, \varsigma) = \frac{1}{\sqrt{n}}\left|\sum_{i=1}^{n} Tr(\rho_{AB} A_i \otimes B_i)\right| \leq 1.$$

‡F. Ming, X-K. Song, D. Wang, Eur. Phys. J. C (2020)

Quantumness in neutrino oscillations (Daya Bay) §



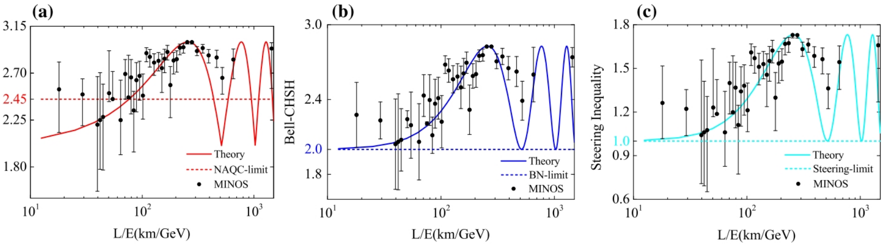


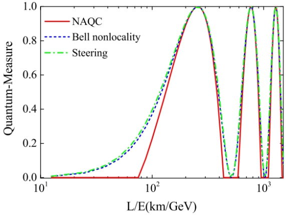
Daya Bay: $\sin^2 2\theta_{13} = 0.084$ and $\Delta m^2_{ee} = 2.42 \times 10^{-3} eV^2$

NAQC is a stronger nonclassical correlation than Bell non-locality and quantum steering.

§F. Ming, X-K. Song, D. Wang, Eur. Phys. J., (2020)

Quantumness in neutrino oscillations (MINOS) ¶





MINOS: $\sin^2 2\theta_{23} = 0.95$ and $\Delta m^2_{32} = 2.32 \times 10^{-3} eV^2$.

NAQC is a stronger nonclassical correlation than Bell non-locality and quantum steering.

¶F. Ming, X-K. Song, D. Wang, Eur. Phys. J., (2020)

NAQC & Bell nonlocality in the wave packet approach**

• We have extended the studies on quantumness of neutrino oscillations through NAQC using the wave packet approach.‖

Neutrino with definite flavor:

$$|\nu_\alpha(x,t)\rangle = \sum_j U^*_{\alpha j}\psi_j(x,t)|\nu_j\rangle$$

where:

$$\psi_j(x,t) = \frac{1}{\sqrt{2\pi}}\int dp\ \psi_j(p)e^{ipx-iE_j(p)t}$$

with:

$$\psi_j(p) = \left(2\pi{\sigma_p^P}^2\right)^{-\frac{1}{4}}\exp-\frac{(p-p_j)^2}{4{\sigma_p^P}^2}$$

‖C. Giunti, C.W. Kim, *Fundamentals of Neutrino Physics and Astrophysics*, Oxford University Press (2007)

**M.B., S. De Siena and C. Matrella, Eur. Phys. J. C (2021)

Wave packet description of neutrino oscillations

Assume the condition $\sigma_p^P \ll E_j^2(p_j)/m_j$. Then we have:

$$E_j(p) \simeq E_j + v_j(p - p_j)$$

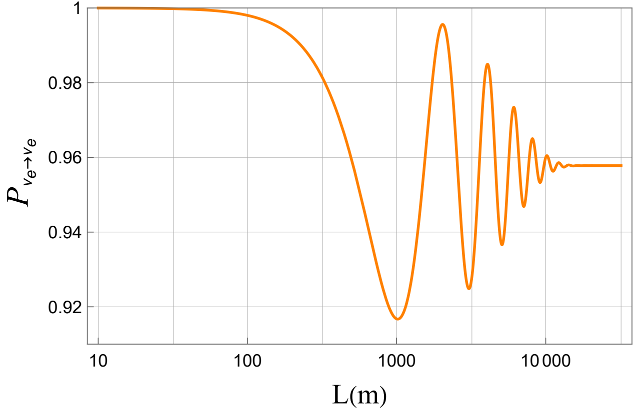
Integrating on p, one gets the wave packet in coordinate space:

$$\psi_j(x,t) = \left(2\pi{\sigma_x^P}^2\right)^{-\frac{1}{4}} \exp\left[-iE_j t + ip_j x - \frac{(x - v_j t)^2}{4{\sigma_x^P}^2}\right]$$

Write density matrix operator $\rho_\alpha(x,t) = |\nu_\alpha(x,t)\rangle\langle\nu_\alpha(x,t)|$. After time integration, one gets the oscillation formula in space

$$P_{\alpha\beta}(L) = \sum_{j,k} U_{\alpha j}^* U_{\alpha k} U_{\beta j}^* U_{\beta k} \exp\left[-2\pi i \frac{L}{L_{jk}^{osc}} - \left(\frac{L}{L_{jk}^{coh}}\right)^2 - 2\pi^2(1-\xi)^2\left(\frac{\sigma_x}{L_{jk}^{osc}}\right)^2\right]$$

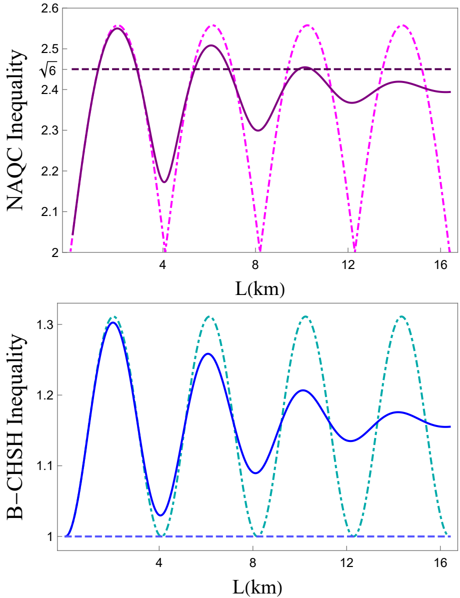
Wave packet description of neutrino oscillations††



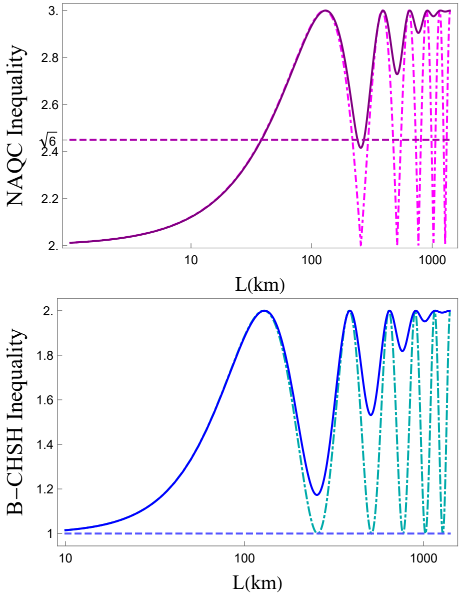
• Survival probability in the wave packet approach. $E = 2\,MeV$, $\xi = 0$, $\sin^2 2\theta_{13} = 0.084$ and $\Delta m^2_{ee} = 2.42 \times 10^{-3} eV^2$ and $\sigma_x = 3.3 \times 10^{-6} m$.

††C. Giunti, C.W. Kim, *Fundamentals of Neutrino Physics and Astrophysics*, Oxford University Press (2007)

NAQC in the wave packet approach (Daya Bay)



NAQC in the wave packet approach (MINOS)



Results

- Our treatment based on wave packets leads to a improved agreement with experimental data in the case of MINOS.*
- NAQC has a different long-distance behaviour for the two experiments, due to the different values of the mixing angle.
- Existence of a "critical" angle for which NAQC exceeds the bound.

* M.B., S. De Siena and C. Matrella, Eur. Phys. J. C (2021)

Complete Complementarity Relations

To better understand the above results, we resort to the recently introduced concept of CCR.

- N.Bohr The quantum postulate and the recent development of atomic theory, Nature (1928)
- W.K.Wootters and W.H.Zurek,*Complementarity in the double-slit experiment: quantum nonseparability and a quantitative statement of Bohr's principle*, Phys. Rev. D (1979)
- M.Jakob and J.A.Bergou, *Quantitative complementarity relations in bipartite systems: entanglement as a physical reality*, Opt. Comm. (2010)
- M.L.W.Basso and J.Maziero, *Complete complementarity relations for multipartite pure states*, J. Phys. A (2020)

Complementarity Principle†

- Complementarity: a quantum system may possess properties which are equally real but mutually exclusive.

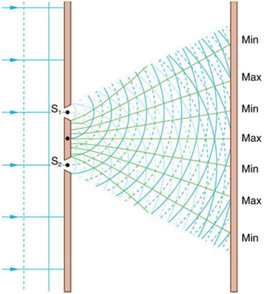
It is often associated with wave-particle duality, the complementarity aspect between propagation and detection.

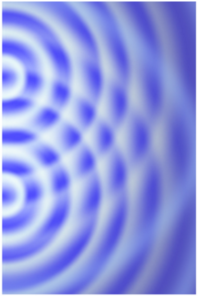
In the double-slit interferometer, the wave aspect is characterized by the *interference fringes visibility*, while the particle nature is given by the *which-way information* of the path along the interferometer.

† N. Bohr, Nature (1928)

Double-slit

Usual view on complementarity: The complete knowledge of the path destroys the interference pattern visibility and vice-versa.





Quantitative wave-particle duality

• Wootters and Zurek *: first quantitative version of the wave-particle duality. A path-detecting device can give incomplete which-way information and a sharply interference pattern can still be retained.

Their work was then extended and formulated in terms of a complementarity relation†

$$P^2 + V^2 \leq 1$$

where P is the predictability and V is the visibility.

• A "**quanton**"‡ may behave partially as a wave or as a particle at the same time.

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*W.K.Wootters and W.H.Zurek, Phys. Rev. D (1979)

†D.M.Greenberger and A.Yasin, Phys.Lett. A (1988); B.-G. Englert, PRL (1996).

‡J.-M.Lévy-Leblond, Physica (1988)

Triality relation

- For bipartite systems a complete complementarity relation (CCR) can be obtained by including the correlations between A and B subsystems[§]:

$$V_k^2 + P_k^2 + C^2 = 1$$

V_k and P_k, $k = 1, 2$, generate *local* single-partite realities which can be related to wave-particle duality.

C is the entanglement measure **concurrence** which generate an exclusive bipartite *nonlocal* reality.

[§] M.Jakob and J.A.Bergou, Opt. Comm. (2010)

Concurrence[¶]

The concurrence for a generic qubit system described by the density matrix ρ is given by

$$C(\rho) \equiv \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$$

where the λ_i are the square root of the eigenvalues λ_i^2 of the operator $\rho\tilde{\rho}$ in decreasing order, with

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$$

[¶] S. A. Hill, W. K. Wootters, Phys. Rev. Lett. (1997)

CCR for bipartite systems

Consider the most general bipartite state of two qubits:

$$|\Theta\rangle = a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle$$

One obtains:

$$C = |\langle\Theta|\tilde{\Theta}\rangle| = 2|ad - bc|$$

$$V_k = 2|\langle\Theta|\sigma_k^\dagger|\Theta\rangle| \rightarrow \begin{cases} V_1 = 2|ac^* + bd^*| \\ V_2 = 2|ab^* + cd^*| \end{cases}$$

$$P_k = |\langle\Theta|\sigma_{z,k}|\Theta\rangle| \rightarrow \begin{cases} P_1 = |(|c|^2 + |d|^2) - (|a|^2 + |b|^2)| \\ P_2 = |(|b|^2 + |d|^2) - (|a|^2 + |c|^2)| \end{cases}$$

where: $|\tilde{\Theta}\rangle = (\sigma_y \otimes \sigma_y)\,|\Theta^*\rangle$, $\quad \sigma_k^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\quad \sigma_{z,k} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The complementarity relation is satisfied, since the left hand side is just the square norm of the general *pure* bipartite state $|\Theta\rangle$:
$(|a|^2 + |b|^2 + |c|^2 + |d|^2)^2 = 1$.

Examples

- Bell states (maximally entangled states)

$$\Phi^{\pm} = \frac{1}{\sqrt{2}} \left(|00\rangle \pm |11\rangle \right), \quad \Psi^{\pm} = \frac{1}{\sqrt{2}} \left(|01\rangle \pm |10\rangle \right)$$

We have $C = 1, \; V_1 = V_2 = P_1 = P_2 = 0.$

- Separable state

$$|\Theta_1\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle + |01\rangle \right) = \frac{1}{\sqrt{2}} |0\rangle \left(|0\rangle + |1\rangle \right)$$

In this case $C = 0, \; V_1 = P_2 = 0, \; V_2 = P_1 = 1.$

- Unbalanced state

$$|\Theta_2\rangle = \frac{1}{2} |00\rangle + \frac{\sqrt{3}}{2} |11\rangle.$$

In this case $C = \frac{\sqrt{3}}{2}, \; V_1 = V_2 = 0, \; P_1 = P_2 = \frac{1}{2}.$

Examples

- A separable state with all four terms

$$|\Theta_3\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) \;=\; \frac{1}{2}(|0\rangle + |1\rangle)(|0\rangle + |1\rangle).$$

We have $C = 0, \;\; V_1 = \; V_2 = 1, \;\; P_2 = P_2 = 0.$

- Unbalanced state with all four terms

$$|\Theta_4\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2\sqrt{2}}|10\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|11\rangle.$$

In this case we have $C = \frac{\sqrt{3}-1}{2\sqrt{2}}, \; V_1 = \frac{\sqrt{3}+1}{2\sqrt{2}}, \;\; V_2 = \frac{\sqrt{3}+2}{4}, \;\; P_1 = 0, \;\; P_2 = \frac{1}{4}.$

Complete Complementarity Relation for pure states

Alternative form of CCR for multipartite states*.

Consider a bipartite pure state in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$:

$$\rho_{A,B} = \sum_{i,k=0}^{d_A-1} \sum_{j,l=0}^{d_B-1} \rho_{ij,kl} \left| i, j \right\rangle \left\langle k, l \right| .$$

If the state of subsystem A is mixed:

$$P_{hs}(\rho_A) + C_{hs}(\rho_A) < \frac{d_A - 1}{d_A} .$$

where $P_{hs}(\rho_A)$ and $C_{hs}(\rho_A)$ are the predictability and the Hilbert-Schmidt quantum coherence (generalization of the visibility†).

*M.L.W.Basso and J.Maziero, J. Phys. A (2020)

†T. Qureshi, Quanta (2019).

CCR for pure states

- The missing information about subsystem A is being shared via correlations with the subsystem B:

$$P_{hs}(\rho_A) + C_{hs}(\rho_A) + C_{hs}^{nl}(\rho_{A|B}) = \frac{d_A - 1}{d_A}$$

– Predictability

$$P_{hs}(\rho_A) \equiv \sum_{i=0}^{d_A - 1} (\rho_{ii}^A)^2 - \frac{1}{d_A}\,,$$

– Quantum coherence (visibility)

$$C_{hs}(\rho_A) \equiv \sum_{i \neq k}^{d_A - 1} |\rho_{ik}^A|^2$$

– Non-local quantum coherence (entanglement)

$$C_{hs}^{nl}(\rho_{A|B}) = \sum_{i \neq k, j \neq l} |\rho_{ij,kl}|^2 - 2 \sum_{i \neq k, j < l} \Re(\rho_{ij,kj}\rho_{il,kl}^*)$$

$C_{hs}^{nl}(\rho_{A|B})$ is equivalent to the linear entropy of subsystem A.

CCR for pure states - entropic formulation

• Another form of CCR can be obtained by defining the predictability and the coherence measures in terms of the von Neumann entropy:

$$C_{\mathrm{re}}(\rho_A) + P_{vn}(\rho_A) + S_{vn}(\rho_A) = \log_2 d_A$$

where

$$C_{\mathrm{re}}(\rho_A) = S_{vn}(\rho_{A\mathrm{diag}}) - S_{vn}(\rho_A)$$

$$P_{vn}(\rho_A) \equiv \log_2 d_A - S_{vn}(\rho_{A\mathrm{diag}})$$

For pure states $S_{vn}(\rho_A) = -\mathrm{Tr}\,(\rho_A \log_2 \rho_A)$ is a measure of entanglement between A and B.

CCR for mixed states*

- For mixed states, $S_{vn}(\rho_A)$ does not quantify entanglement, but it is just a measure of mixedness of A. CCR have to be modified:

$$P_{vn}(\rho_A) + C_{re}(\rho_A) + I_{A:B}(\rho_{AB}) + S_{A|B}(\rho_{AB}) = \log_2 d_A,$$

where:

– $P_{vn}(\rho_A) \equiv \ln d_A - S_{vn}(\rho_{A diag})$ is the **predictability**;

– $C_{re}(\rho_A) = S_{vn}(\rho_{A diag}) - S_{vn}(\rho_A)$ is the **relative entropy of coherence**;

– $I_{A:B}(\rho_{AB}) = S_{vn}(\rho_A) + S_{vn}(\rho_B) - S_{vn}(\rho_{AB})$ is the **mutual information** of A and B;

– $S_{A|B}(\rho_{AB}) = S_{vn}(\rho_{AB}) - S_{vn}(\rho_B)$ is the **conditional entropy**:
It tells how much it is convenient knowing about subsystem A with respect to the whole system.

*M.L.W.Basso and J.Maziero, EPL (2021)

CCR for oscillating neutrinos[†]

• We now consider the CCR for neutrino oscillations, both for pure and mixed states.

Let us consider a two-flavor neutrino state:

$$|\nu_\alpha(t)\rangle = a_{\alpha\alpha}(t)\,|\nu_\alpha\rangle + a_{\alpha\beta}(t)\,|\nu_\beta\rangle$$

We can use the following correspondence:

$$|\nu_\alpha\rangle = |1\rangle_\alpha \otimes |0\rangle_\beta = |10\rangle$$
$$|\nu_\beta\rangle = |0\rangle_\alpha \otimes |1\rangle_\beta = |01\rangle$$

For an initial electronic neutrino, we have:

$$|\nu_e(t)\rangle = a_{ee}\,|10\rangle + a_{e\mu}\,|01\rangle$$

[†] V.Bittencourt, M.B., S.De Siena and C.Matrella, Eur. Phys. J. C. (2022)

CCR for oscillating neutrinos

The corresponding density matrix is:

$$\rho_{e\mu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |a_{e\mu}|^2 & a_{ee}a_{e\mu}^* & 0 \\ 0 & a_{e\mu}a_{ee}^* & |a_{ee}|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The state of subsystems e and μ are:

$$\rho_e = \begin{pmatrix} |a_{ee}|^2 & 0 \\ 0 & |a_{e\mu}|^2 \end{pmatrix}; \qquad \rho_\mu = \begin{pmatrix} |a_{e\mu}|^2 & 0 \\ 0 & |a_{ee}|^2 \end{pmatrix}$$

CCR for oscillating neutrinos

We verify that the CCRs for pure states are verified in the case of neutrino. We find:[‡]

$$
\begin{aligned}
P_{hs}(\rho_e) &= P_{ee}^2 + P_{e\mu}^2 - \frac{1}{2} \\
C_{hs}(\rho_e) &= 0 \\
C_{hs}^{nl}(\rho_{e\mu}) &= 2P_{ee}P_{e\mu}
\end{aligned}
$$

where $|a_{ee}|^2 = P_{ee}$, $|a_{e\mu}|^2 = P_{e\mu}$ and $P_{ee} + P_{e\mu} = 1$.

Thus:

$$
P_{hs}(\rho_e) + C_{hs}(\rho_e) + C_{hs}^{nl}(\rho_{e\mu}) = \frac{1}{2}
$$

as expected.

[‡] V.Bittencourt, M.B., S.De Siena and C.Matrella, Eur. Phys. J. C. (2022)

CCR for oscillating neutrinos

Analogously:

$$
\begin{aligned}
P_{vn}(\rho_e) &= 1 + |a_{ee}|^2 \log_2 |a_{ee}|^2 + |a_{e\mu}|^2 \log_2 |a_{e\mu}|^2 \\
C_{re}(\rho_e) &= 0 \\
S_{vn}(\rho_e) &= -|a_{ee}|^2 \log_2 |a_{ee}|^2 - |a_{e\mu}|^2 \log_2 |a_{e\mu}|^2
\end{aligned}
$$

and the CCR is verified:

$$P_{vn}(\rho_e) + C_{re}(\rho_e) + S_{vn}(\rho_e) = 1$$

CCR for neutrino mixed state

In a wave-packet description of neutrino oscillations, one starts with a pure state $\rho_\alpha(x,t)$ which become mixed after time integration:

$$\rho_\alpha(x) = \sum_{k,j} U_{\alpha k} U^*_{\alpha j} f_{jk}(x) \left|\nu_j\right\rangle \left\langle\nu_k\right| ,$$

where:

$$f_{jk}(x) = \exp\left[-i\frac{\Delta m^2_{jk} x}{2E} - \left(\frac{\Delta m^2_{jk} x}{4\sqrt{2}E^2\sigma_x}\right)^2\right]$$

By considering:

$$\left|\nu_i\right\rangle = \sum_\alpha U_{\alpha i} \left|\nu_\alpha\right\rangle , \qquad \left|\nu_\alpha\right\rangle = \left|\delta_{\alpha e}\right\rangle_e \left|\delta_{\alpha\mu}\right\rangle_\mu \left|\delta_{\alpha\tau}\right\rangle_\tau$$

we can write:

$$\rho_\alpha(x) = \sum_{\beta\gamma} F^\alpha_{\beta\gamma}(x) \left|\delta_{\beta e}\delta_{\beta\mu}\delta_{\beta\tau}\right\rangle \left\langle\delta_{\gamma e}\delta_{\gamma\mu}\delta_{\gamma\tau}\right|$$

where:

$$F^\alpha_{\beta\gamma}(x) = \sum_{kj} U^*_{\alpha j} U_{\alpha k} f_{jk}(x) U_{\beta j} U^*_{\gamma k}$$

CCR for neutrino mixed state

- We consider the CCR in the case of a two-flavor neutrino oscillation, for an initial electron neutrino

$$P_{vn}(\rho_e) + C_{re}(\rho_e) + I_{A:B}(\rho_{e\mu}) + S_{e|\mu}(\rho_{e\mu}) = \log_2 d_e,$$

where:

$$\begin{aligned} P_{vn}(\rho_e) &= \log_{d_e} - S_{vn}(\rho_{e_{diag}}) \\ C_{re}(\rho_e) &= S_{vn}(\rho_{e_{diag}}) - S_{vn}(\rho_e) \\ I_{A:B}(\rho_{e\mu}) &= S_{vn}(\rho_e) + S_{vn}(\rho_\mu) - S_{vn}(\rho_{e\mu}) \\ S_{e|\mu}(\rho_{e\mu}) &= S_{vn}(\rho_{e\mu}) - S_{vn}(\rho_\mu) \end{aligned}$$

For a generic matrix ρ, the von Neumann entropy is defined as $S_{vn}(\rho) = -\sum_i \lambda_i \log_2 \lambda_i$, where λ_i are the eigenvalues of ρ.

CCR for neutrino mixed state

The starting density matrix is:

$$\rho_{e\mu}(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & F^e_{ee} & F^e_{e\mu} & 0 \\ 0 & F^e_{\mu e} & F^e_{\mu\mu} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the reduced density matrices are:

$$\rho_e(x) = \begin{pmatrix} F^e_{ee} & 0 \\ 0 & F^e_{\mu\mu} \end{pmatrix} \qquad \rho_\mu(x) = \begin{pmatrix} F^e_{\mu\mu} & 0 \\ 0 & F^e_{ee} \end{pmatrix}$$

By evaluating the eigenvalues of these matrices, we obtain:

$$P_{vn}(\rho_e) = 1 + F^e_{ee}\log_2 F^e_{ee} + F^e_{\mu\mu}\log_2 F^e_{\mu\mu}$$
$$C_{re}(\rho_e) = 0$$
$$I_{e:\mu}(\rho_{e\mu}) + S_{e|\mu}(\rho_{e\mu}) = -F^e_{ee}\log_2 F^e_{ee} - F^e_{\mu\mu}\log_2 F^e_{\mu\mu}$$

By adding all the terms we find that the CCR for mixed states is satisfied for a neutrino state.

CCR for neutrino mixed state

The sum of the non-local terms of the CCR is equal to the Quantum Discord, defined as:

$$QD(\rho_{AB}) = I(\rho_{AB}) - CC(\rho_{AB}),$$

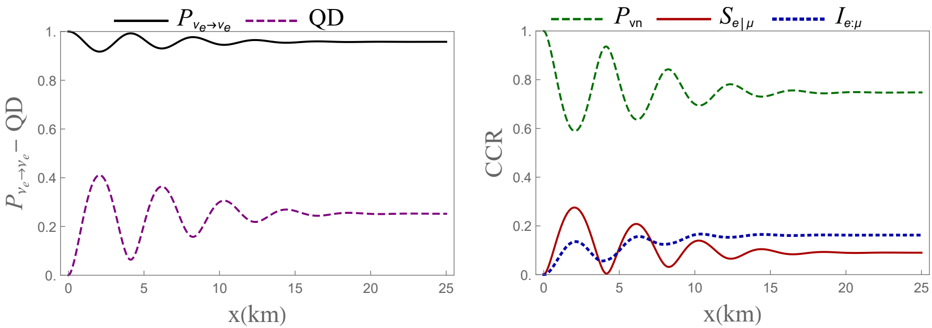
where $I(\rho_{AB})$ is the total correlations between the subsystems A and B; and $CC(\rho_{AB})$ quantifies the classical correlations. We have

$$QD(\rho_{AB}) = S_{\text{vn}}(\rho_A) - S_{\text{vn}}(\rho_{AB}) + \min_{\{\Pi_i^b\}} S_{\text{vn},\{\Pi_i^b\}}(\rho_{A|B})$$

that, for the neutrino density matrix under consideration, gives

$$QD(\rho_{e\mu}) = -F_{ee}^e \log_2 F_{ee}^e - F_{\mu\mu}^e \log_2 F_{\mu\mu}^e$$

CCR for neutrino oscillations* - DAYA BAY

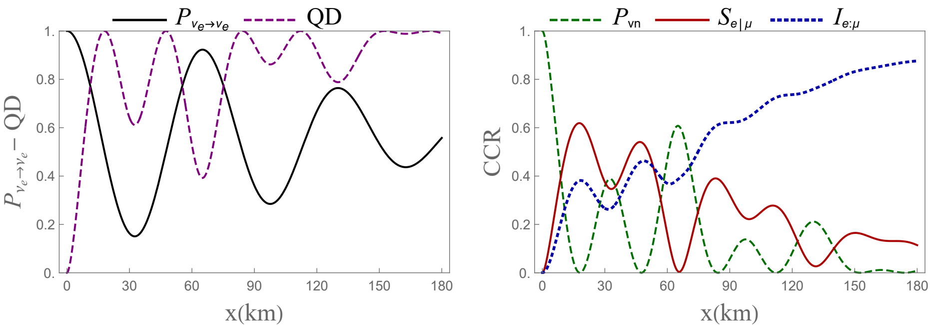


(a) *DAYA BAY* ($L \in [364m, 1912m]$)

$$\Delta m^2_{ee} = 2.42 \times 10^{-3} eV^2, \ \sin^2 2\theta_{13} = 0.084, \ E = 4MeV$$

*V.Bittencourt, M.B., S.De Siena and C.Matrella, Eur. Phys. J. C. (2022)

CCR for neutrino oscillations† - KamLAND

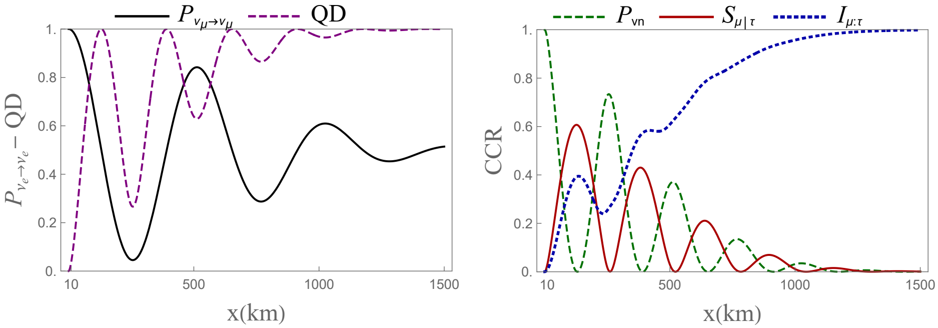


(b) *KamLAND (L = 180 Km)*

$$\Delta m^2_{12} = 7.49 \times 10^{-5} eV^2, \; \tan^2 2\theta_{12} = 0.47, \; E = 2MeV$$

†V.Bittencourt, M.B., S.De Siena and C.Matrella, Eur. Phys. J. C. (2022)

CCR for neutrino oscillations[‡] - MINOS



(c) *MINOS (L = 735 km)*

$$\Delta m^2_{32} = 2.32 \times 10^{-3} eV^2, \; \sin^2 2\theta_{23} = 0.95, \; E = 0.5 GeV$$

[‡]V.Bittencourt, M.B., S.De Siena and C.Matrella, Eur. Phys. J. C. (2022)

Results

- We have studied CCR for the oscillating neutrino systems, both in the pure and in the mixed case.

- Complete characterization of quantum correlations in neutrino oscillations.

- Interesting long-distance behaviour of the correlations, depending on the mixing angle.

CCR for neutrino oscillations - 3 flavors*

Tripartite pure state:

$$\rho_{ABC} = \sum_{i,l=0}^{d_A-1} \sum_{j,m=0}^{d_B-1} \sum_{k,n=0}^{d_C-1} \rho_{ijk,lmn} \left|i,j,k\right\rangle_{ABC} \left\langle l,m,n\right| .$$

State of subsystem A:

$$\rho_A = \sum_{i,l=0}^{d_A-1} \left(\sum_{j=0}^{d_B-1} \sum_{k=0}^{d_C-1} \rho_{ijk,ljk} \right) \left|i\right\rangle_A \left\langle l\right| \equiv \sum_{i,l=0}^{d_A-1} \rho^A_{il} \left|i\right\rangle_A \left\langle l\right| ,$$

CCR

$$P_{\text{hs}}(\rho_A) + C_{\text{hs}}(\rho_A) + C^{nl}_{\text{hs}}(\rho_{A|BC}) = \frac{d_A - 1}{d_A}$$

The non local coherence is given by:

$$C^{nl}_{\text{hs}}(\rho_{A|BC}) = \sum_{i\neq l} \left(\sum_{\substack{j\neq m \\ k\neq n}} + \sum_{\substack{j=m \\ k\neq n}} + \sum_{\substack{j\neq m \\ k=n}} \right) \left|\rho_{ijk,lmn}\right|^2 - 2 \sum_{i\neq l} \left(\sum_{\substack{j=m \\ k<n}} + \sum_{\substack{j<m \\ k=n}} + \sum_{\substack{j<m \\ k\neq n}} \right) \Re(\rho_{ijk,ljk} \rho^*_{imn,lmn}).$$

*V.Bittencourt, M.B., S.De Siena and C.Matrella, Eur. Phys. J. C (2024)

CCR for 3 flavor neutrino oscillations

- Entropic form of CCR still valid for the single-partite subsystems A, B and C. For the subsystem AB, we have:

$$C_{\mathrm{re}}(\rho_{AB}) + P_{\mathrm{vn}}(\rho_{AB}) + S_{\mathrm{vn}}(\rho_{AB}) = \log_2(d_A d_B),$$

and similar ones for AC and BC.

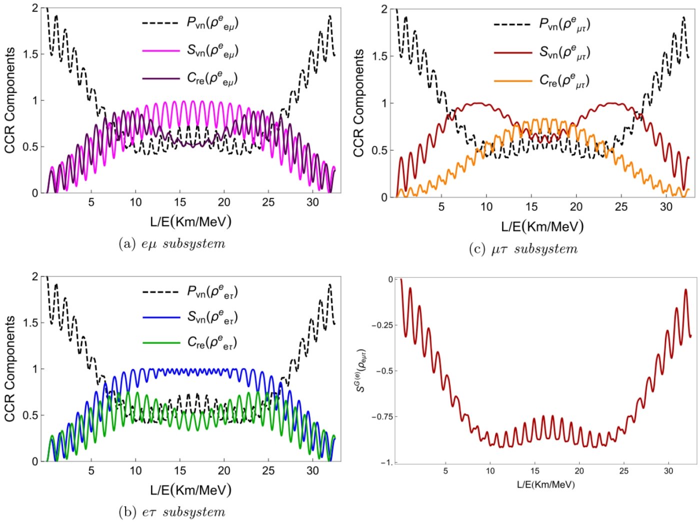
- For tripartite mixed states, CCR for subsystem AB takes the form:

$$P_{vn}(\rho_{AB}) + C_{re}(\rho_{AB}) + S_{AB|C}(\rho_{ABC}) + I_{AB:C}(\rho_{ABC}) = \log_2(d_A d_B),$$

- The state for the subsystem C, on the other hand, satisfy the CCR:

$$P_{vn}(\rho_C) + C_{re}(\rho_C) + S_{C|AB}(\rho_{ABC}) + I_{C:AB}(\rho_{ABC}) = \log_2(d_C).$$

CCR 3 flavors – plane waves*



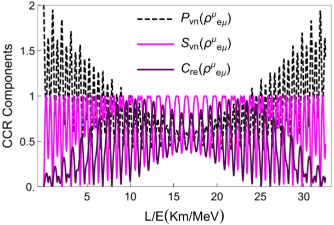
(a) *eμ subsystem*

(c) *μτ subsystem*

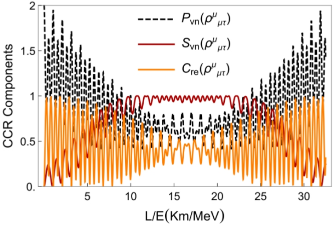
(b) *eτ subsystem*

• CCR terms and tripartite entanglement for an initial electron neutrino state.

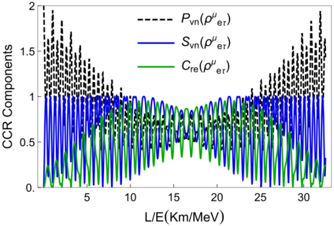
CCR 3 flavors – plane waves*



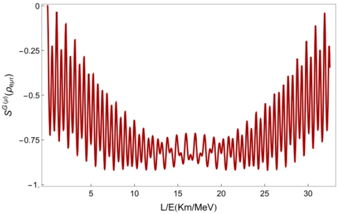
(a) *eμ subsystem*



(c) *μτ subsystem*



(b) *eτ subsystem*



- CCR terms and tripartite entanglement for an initial muon neutrino state.

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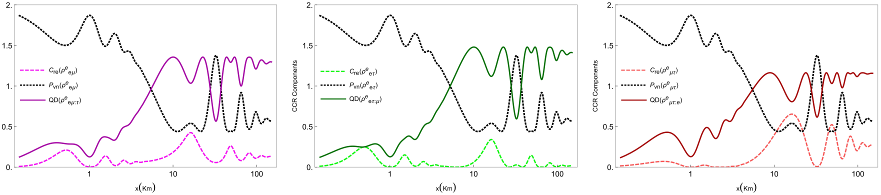
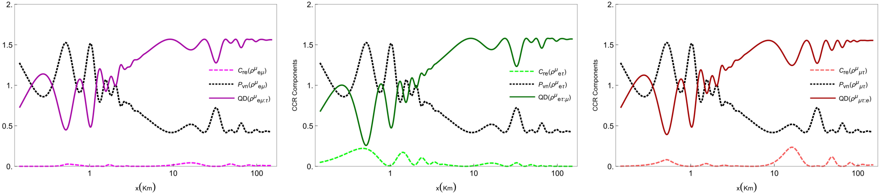
*V.Bittencourt, M.B., S.De Siena and C.Matrella, Eur. Phys. J. C (2024)

CCR 3 flavors – wave packets*



(a) (b) (c)

Figure 1: CCR terms for two flavor subsystems $e\mu$ (a), $e\tau$ (b) and $\mu\tau$ (c) as function of x - electron neutrino.



(a) (b) (c)

Figure 2: CCR terms for two flavor subsystems $e\mu$ (a), $e\tau$ (b) and $\mu\tau$ (c) as function of x - muon neutrino.

*V.Bittencourt, M.B., S.De Siena and C.Matrella, Eur. Phys. J. C (2024)

CCR 3 flavors with CP violation*

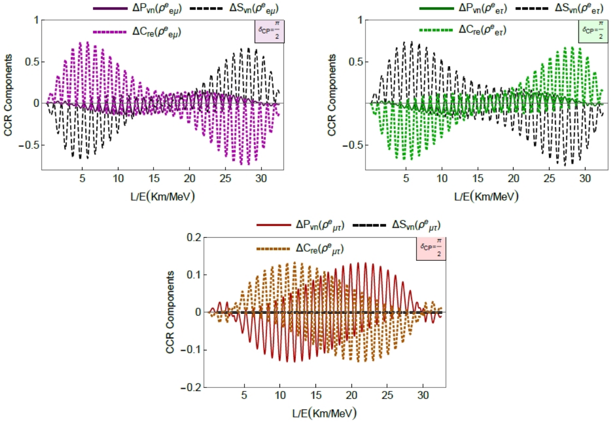


Fig. 6. Difference between neutrino and anti-neutrino CCR terms for bipartite subsystems $e\mu$, $e\tau$ and $\mu\tau$ as function of L/E in the case of an initial electronic neutrino.

- Recent review of neutrino correlations for three flavors:

W. Guangjie, S. Xueke, Y. Liu and W. Dong, Acta Phys. Sin. (2025)

*M.B., S. De Siena and C. Matrella, Int. J. Quant. Inf. (2024).

Violations of macrorealism in neutrino oscillations

- The notion of Macroscopic realism (Macrorealism) tries to encode our intuition of macroscopic world*
- Violations of macrorealism tested by Leggett-Garg inequalities (LGIs): temporal analogue of Bell inequalities
- Violations of LGIs in neutrino oscillations have been proved by using the MINOS data†
- Bell vs LGIs:
  - Bell inequalities: necessary and sufficient for local realism‡,
  - LGIs: only necessary but not sufficient for macrorealism.

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* A. J. Leggett and A. Garg, Phys. Rev. Lett. (1985)
† J. Formaggio, D. Kaiser, M. Murskyj, and T. Weiss, Phys. Rev. Lett. (2016)
‡ A. Fine, Phys. Rev. Lett. (1982).

Violations of macrorealism in neutrino oscillations

- A necessary and sufficient condition can be formulated in terms of two set of *equalities*[*]: non-signaling in time (NSIT) and arrow of time (AoT).
- We computed NSIT/AoT in the case of two-flavor neutrino oscillations in the wave-packet formalism[†] and in the case of meson oscillations[‡]
- NSIT/AoT reveal violations of macrorealism hidden by LGIs.

[*] L. Clemente and J. Kofler, Phys. Rev. A (2015).
[†] M.B., F.Illuminati, L.Petruzziello, K.Simonov and L.Smaldone. Eur. Phys. J. C (2023)
[‡] M.B., F.Illuminati, L.Petruzziello, K.Simonov and L.Smaldone. Phys. Rev. A (2024).

Chiral oscillations

Chiral oscillations

- Taking into account (bi)spinorial nature of neutrinos and chiral nature of weak interaction, one naturally gets chiral oscillations *
- Interplay with flavor oscillations in the non-relativistic region†
- For CνB, chiral oscillations reduce detection by a factor of 2.‡

* A. Bernardini and S. De Leo, Phys. Rev. D (2005)

†V.A.Bittencourt, A.Bernardini and M.B.,Eur.Phys.J.C(2021);EPL Persp.(2022);
M. W. Li, Z. L. Huang and X. G. He, Phys. Lett. B (2024);
K. Kimura and A. Takamura, Annals Phys. (2025).
T. Morozumi, and T. Tahara, Prog. Theor. Exp. Phys. (2025)
V. Bittencourt, M. B. and G. Zanfardino, Phys. Lett. B (2025)

‡S.-F. Ge and P.Pasquini, Phys. Lett. B (2020)

Chiral oscillations

Chiral representation of the Dirac matrices

$$\alpha_i = \begin{bmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{bmatrix}, \qquad \beta = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix},$$

and $\gamma_5 = (I_2, -I_2)$. Any bispinor $|\xi\rangle$ can be written in this representation as

$$|\xi\rangle = \begin{bmatrix} |\xi_R\rangle \\ |\xi_L\rangle \end{bmatrix},$$

The Dirac equation $H_D\,|\xi\rangle = i|\dot{\xi}\rangle$ can then be written as

$$i\partial_t\,|\xi_R\rangle - \mathbf{p}\cdot\sigma\,|\xi_R\rangle = m\,|\xi_L\rangle\,,$$
$$i\partial_t\,|\xi_L\rangle + \mathbf{p}\cdot\sigma\,|\xi_L\rangle = m\,|\xi_R\rangle\,,$$

• Evolution under the free Dirac Hamiltonian $\hat{H}_D$ induces left-right chiral oscillations.

Take initial state $|\psi(0)\rangle = [0,\, 0,\, 0,\, 1]^T$ which has negative helicity and negative chirality: $\hat{\gamma}_5 \, |\psi(0)\rangle = - \, |\psi(0)\rangle$.

The time evolved state $|\psi_m(t)\rangle = e^{-i\hat{H}_D t} \, |\psi(0)\rangle$ is given by

$$
\begin{aligned}
|\psi_m(t)\rangle \quad = \quad \sqrt{\frac{E_{p,m}+m}{4E_{p,m}}} \Bigg[\left(1 + \frac{p}{E_{p,m}+m}\right) e^{-iE_{p,m}t} \, |u_-(p,m)\rangle \\
- \left(1 - \frac{p}{E_{p,m}+m}\right) e^{iE_{p,m}t} \, |v_-(-p,m)\rangle \Bigg],
\end{aligned}
$$

with (for one-dimensional propagation along the $\mathbf{e}_z$ direction)

$$
|u_\pm(p,m)\rangle = \sqrt{\frac{E_{p,m}+m}{4E_{p,m}}} \begin{bmatrix} \left(1 \pm \frac{p}{E_{p,m}+m}\right) |\pm\rangle \\ \left(1 \mp \frac{p}{E_{p,m}+m}\right) |\pm\rangle \end{bmatrix},
$$

$$
|v_\pm(p,m)\rangle = \sqrt{\frac{E_{p,m}+m}{4E_{p,m}}} \begin{bmatrix} \left(1 \pm \frac{p}{E_{p,m}+m}\right) |\pm\rangle \\ -\left(1 \mp \frac{p}{E_{p,m}+m}\right) |\pm\rangle \end{bmatrix},
$$

with $|\pm\rangle$ eigenstates of σ_z.

- Survival probability of initial left-handed state

$$\mathcal{P}(t) = |\,\langle \psi_m(0)|\psi_m(t)\rangle\,|^2 = 1 - \frac{m^2}{E_{p,m}^2}\sin^2(E_{p,m}t)\,,$$

Average value of the chiral operator $\langle\hat{\gamma}_5\rangle(t)$

$$\langle\hat{\gamma}_5\rangle(t) = \langle\psi_m(t)|\hat{\gamma}_5\,|\psi_m(t)\rangle = -1 + \frac{2m^2}{E_{p,m}^2}\sin^2(E_{p,m}t)\,.$$

– Chiral oscillation period: $T_{ch} = \frac{2\pi}{E_{p,m}}$

– Chiral oscillation length: $L_{ch} = v\,\frac{2\pi}{E_{p,m}} = \frac{2\pi p}{E_{p,m}^2}$

Chiral and flavor oscillations

- State of a neutrino of flavor α at a given t:

$$|\nu_\alpha(t)\rangle = \sum_i U_{\alpha,i} \, |\psi_{m_i}(t)\rangle \otimes |\nu_i\rangle \, ,$$

where $|\psi_{m_i}(t)\rangle$ are bispinors.

- The state at $t = 0$ reads

$$|\nu_\alpha(0)\rangle = |\psi(0)\rangle \otimes \sum_i U_{\alpha,i} \, |\nu_i\rangle = |\psi(0)\rangle \otimes |\nu_\alpha\rangle \, ,$$

where $|\psi(0)\rangle$ is a left handed bispinor.

- Survival probability:

$$\mathcal{P}_{\alpha\to\alpha} = |\langle\nu_\alpha(0)|\nu_\alpha(t)\rangle|^2 = \left|\sum_i |U_{\alpha,i}|^2 \, \langle\psi(0)|\psi_{m_i}(t)\rangle\right|^2 .$$

Two flavor mixing:

$$|\nu_e(t)\rangle \;=\; \left[\cos^2\theta\,|\psi_{m_1}(t)\rangle + \sin^2\theta\,|\psi_{m_2}(t)\rangle\right] \otimes |\nu_e\rangle$$
$$+ \sin\theta\cos\theta\left[|\psi_{m_1}(t)\rangle - |\psi_{m_2}(t)\rangle\right] \otimes |\nu_\mu\rangle\,,$$

- The survival probability can be decomposed as

$$\mathcal{P}_{e\to e}(t) = \mathcal{P}^S_{e\to e}(t) + \mathcal{A}_e(t) + \mathcal{B}_e(t).$$

$\mathcal{P}^S_{e\to e}(t)$ is the standard flavor oscillation formula

$$\mathcal{P}^S_{e\to e}(t) = 1 - \sin^2 2\theta \sin^2\left(\frac{E_{p,m_2} - E_{p,m_1}}{2}t\right)$$

and

$$\mathcal{A}_e(t) \;=\; -\left[\frac{m_1}{E_{p,m_1}}\cos^2\theta\sin\left(E_{p,m_1}t\right) + \frac{m_2}{E_{p,m_2}}\sin^2\theta\sin\left(E_{p,m_2}t\right)\right]^2,$$

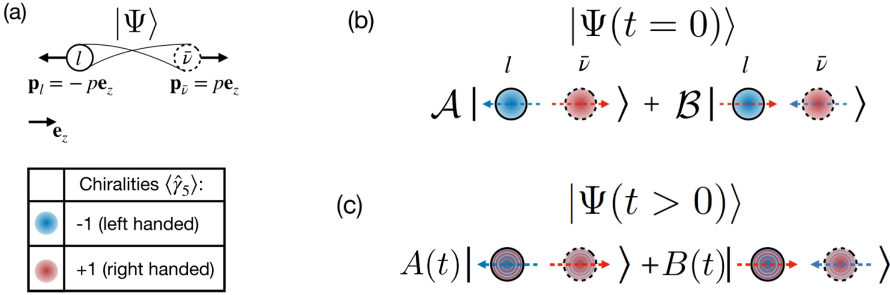
$$\mathcal{B}_e(t) \;=\; \frac{1}{2}\sin^2 2\theta\sin(E_{p,m_1}t)\sin(E_{p,m_2}t)\left(\frac{p^2 + m_1 m_2}{E_{p,m_1}E_{p,m_2}} - 1\right),$$

are correction terms due to the bispinorial structure.

- Agreement with the QFT formula.

$l - \bar{\nu}$ entanglement and chiral oscillations*

Chiral oscillations, we consider induced spin correlations in pion decay products ($\pi \to l + \bar{\nu}$)



*V.A.Bittencourt, A.Bernardini and M.B., Universe (2021)

Spin entanglement at $t = 0$

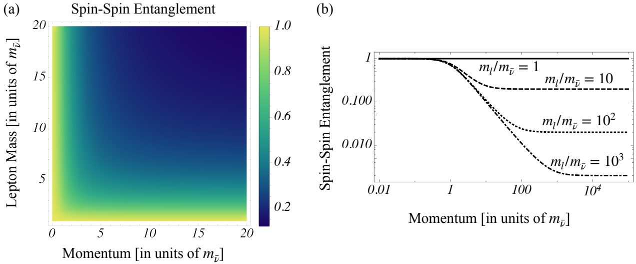
- The state of the lepton-antineutrino pair is then described in the composite Hilbert space $\mathcal{H}_{C_{\bar{\nu}}} \otimes \mathcal{H}_{S_{\bar{\nu}}} \otimes \mathcal{H}_{C_l} \otimes \mathcal{H}_{S_l}$.

- It is a 4-qubit entangled state.

- We can write $|\Psi(0)\rangle = |+_{C_{\bar{\nu}}}\rangle \otimes |-_{C_l}\rangle \otimes |\Psi_{S_{\bar{\nu}},S_l}\rangle$, with $|\pm_A\rangle$ denoting the positive (negative) chirality of $A = C_{\bar{\nu},l}$, and

$$|\Psi_{S_{\bar{\nu}},S_l}\rangle = \mathcal{A}(p, m_l, m_{\bar{\nu}})|\uparrow_{S_{\bar{\nu}}}\rangle \otimes |\downarrow_{S_l}\rangle - \mathcal{B}(p, m_l, m_{\bar{\nu}})|\downarrow_{S_{\bar{\nu}}}\rangle \otimes |\uparrow_{S_l}\rangle$$

is the joint spin state at $t = 0$.



Spin entanglement at $t \neq 0$

- The reduced matrix $\rho_{S_{\bar{\nu}},S_l}(t) = \mathrm{Tr}_{\mathrm{Chirality}}\left[|\Psi(t)\rangle\langle\Psi(t)|\right]$ describes a mixed state with entanglement dynamics directly affected by chiral oscillations.

- Entanglement between the spins at time t

$$\mathcal{N}_{S_{\bar{\nu}},S_l}(t) \equiv \mathcal{N}[\rho_{S_{\bar{\nu}},S_l}(t)] = ||\rho^T_{S_{\bar{\nu}},S_l}(t)|| - 1 = \mathcal{N}_{S_{\bar{\nu}},S_l}(0)\Gamma(t)$$

with

$$\Gamma(t) = \prod_{j=\bar{\nu},l}\left[1 - \frac{p^2}{m_j^2}\left(\langle\hat{\gamma}_5\rangle_j(t) - 1\right)^2\right]^{\frac{1}{2}}.$$

The average chiralities are given by $\langle\hat{\gamma}_5\rangle_A(t) = \mathrm{Tr}_A[\rho_A(t)]$ with $A = \bar{\nu}, l$:

$$\begin{aligned}\langle\hat{\gamma}_5\rangle_{\bar{\nu}}(t) &= 1 - \frac{m_{\bar{\nu}}^2}{E_{p,m_{\bar{\nu}}}^2}\left[1 - \cos\left(2E_{p,m_{\bar{\nu}}}t\right)\right],\\ \langle\hat{\gamma}_5\rangle_l(t) &= -1 + \frac{m_l^2}{E_{p,m_l}^2}\left[1 - \cos\left(2E_{p,m_l}t\right)\right].\end{aligned}$$

Spin entanglement at $t \neq 0$

- $\mathrm{Tr}[\rho^2_{S_{\bar{\nu}},S_l}(t)] < 1 \Rightarrow$ entanglement initially encoded only in the spins redistributes into spin-chirality entanglement.

- Entanglement encoded in the bipartition $(C_{\bar{\nu}}, S_{\bar{\nu}}); (C_l, S_l)$ is conserved:

$$
\begin{aligned}
\mathrm{Tr}[\rho^2_{\bar{\nu}}(t)] = \mathrm{Tr}[\rho^2_l(t)] &= \mathcal{A}^4(p, m_l, m_{\bar{\nu}}) + \mathcal{B}^4(p, m_l, m_{\bar{\nu}}) \\
&= 1 - \frac{\mathcal{N}^2_{S_{\bar{\nu}},S_l}(0)}{2} < 1.
\end{aligned}
$$

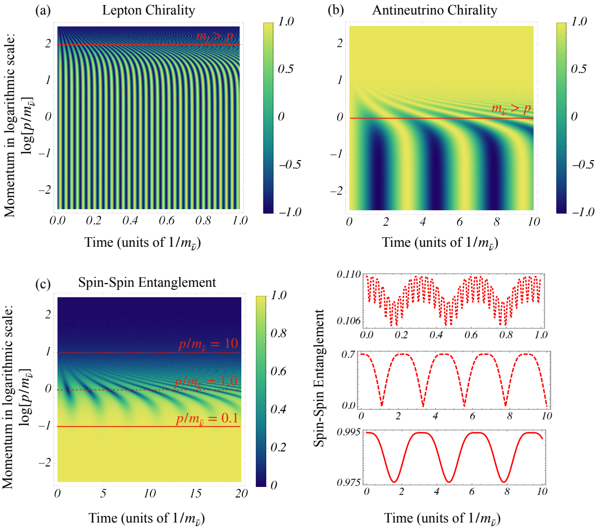


Figure 1: (a) Average lepton chirality, (b) average antineutrino chirality and (c) spin-spin entanglement as a function of the momentum and of time.

Chiral oscillations and Bell spin correlations

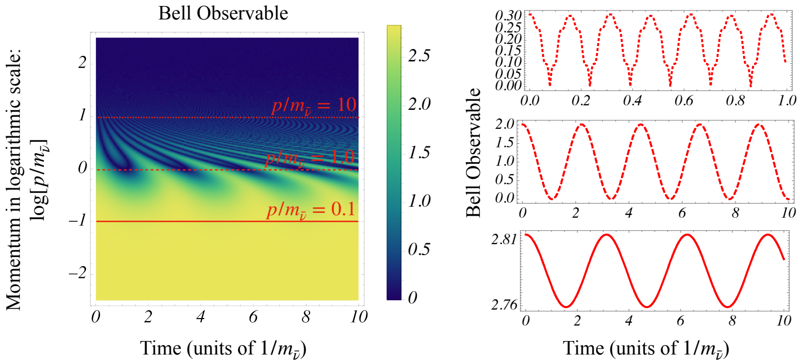
The quantity

$$B[\rho(t)] = |\langle \hat{S}_{\bar{\nu},1} \otimes \hat{S}_{l,1} \rangle + \langle \hat{S}_{\bar{\nu},1} \otimes \hat{S}_{l,2} \rangle + \langle \hat{S}_{\bar{\nu},2} \otimes \hat{S}_{l,1} \rangle - \langle \hat{S}_{\bar{\nu},2} \otimes \hat{S}_{l,2} \rangle|,$$

is the Bell observable first proposed to investigate non-local correlations*.

For pure states, $B[\rho] > 2$ indicates that the correlations shared between the spins are non-local and that the state is entangled.

*N.Brunner et al., Rev. Mod. Phys. (2014)



• Bell observable as a function of the momentum (in units of the antineutrino mass and in log scale) and of time for $m_l/m_{\bar{\nu}} = 10^2$.

Results

- We find that chiral oscillations do affect spin-spin correlations for the entangled lepton–antineutrino couple.

- Resonance of oscillation amplitude at neutrino mass: possibility of extracting fundamental information via quantum correlations.

- We have extended the study to the case in which flavor mixing is included: neutrino state is an *hyperentangled state* with three DoFs: chirality, spin and flavor.[†]

[†]V.A.Bittencourt, M.B. and G.Zanfardino, Physica Scripta (2024)

Quantum field theory of chiral oscillations[‡]

- Dirac Lagrangian density

$$\mathcal{L} = \overline{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\psi$$

Invariance under global phase transformations ⇒ conserved charge

$$Q = \int d^3\mathbf{x}\,\psi^\dagger(x)\psi(x)$$

Dirac field ψ can be split as $\psi = \psi_L + \psi_R$ where

$$\psi_L \equiv P_L\psi(x) = \frac{1-\gamma^5}{2}\psi(x), \qquad \psi_R \equiv P_R\psi(x) = \frac{1+\gamma^5}{2}\psi(x)$$

and hence Dirac Lagrangian can be written as

$$\mathcal{L} = \overline{\psi_L}\,i\gamma^\mu\,\partial_\mu\,\psi_L + \overline{\psi_R}\,i\gamma^\mu\partial_\mu\psi_R - m\left(\overline{\psi_L}\,\psi_R + \overline{\psi_R}\,\psi_L\right)$$

[‡]V. Bittencourt, M. B. and G. Zanfardino, Phys. Lett. B (2025)

• Chiral symmetry is explicitly broken by the Dirac mass term.

• Separate global phase transformations for ψ_L and ψ_R lead to the non-conserved chiral charges

$$Q_L(t) = \int d^3\mathbf{x}\, \psi_L^\dagger(x)\psi_L(x), \qquad Q_R(t) = \int d^3\mathbf{x}\, \psi_R^\dagger(x)\psi_R(x).$$

• The total (conserved) charge is equal to the sum of the (time dependent) chiral charges

$$Q \,=\, Q_L(t) \,+\, Q_R(t).$$

Diagonalization of chiral charges

- Introduce the following canonical (Bogoliubov) transformation:

$$
\begin{aligned}
\alpha_{\mathbf{k},L} &= \cos\theta_k\,\alpha^2_{\mathbf{k}} - e^{i\phi_k}\,\sin\theta_k\,\beta^{2\dagger}_{-\mathbf{k}} \\
\beta^\dagger_{-\mathbf{k},L} &= \cos\theta_k\,\beta^{1\dagger}_{-\mathbf{k}} - e^{-i\phi_k}\,\sin\theta_k\,\alpha^1_{\mathbf{k}} \\
\alpha_{\mathbf{k},R} &= \cos\theta_k\,\alpha^1_{\mathbf{k}} + e^{i\phi_k}\,\sin\theta_k\,\beta^{1\dagger}_{-\mathbf{k}} \\
\beta^\dagger_{-\mathbf{k},R} &= \cos\theta_k\,\beta^{2\dagger}_{-\mathbf{k}} + e^{-i\phi_k}\,\sin\theta_k\,\alpha^1_{\mathbf{k}}
\end{aligned}
$$

- Condition for diagonalization

$$
\begin{aligned}
\cos^2\theta_k &= \frac{1}{2}\left(1+\frac{|\mathbf{k}|}{\omega_k}\right), \quad \sin^2\theta_k = \frac{1}{2}\left(1-\frac{|\mathbf{k}|}{\omega_k}\right), \\
\cos 2\theta_k &= \frac{|\mathbf{k}|}{\omega_k}, \quad \sin 2\theta_k = -\frac{m}{\omega_k}, \quad \phi_k = 2\omega_k t.
\end{aligned}
$$

• Chiral charges are diagonal in the new operators

$$Q_L(t) \;=\; \int d^3\mathbf{k}\, \left(\alpha^{\dagger}_{\mathbf{k},L}(t)\alpha_{\mathbf{k},L}(t) - \beta^{\dagger}_{-\mathbf{k},L}(t)\beta_{-\mathbf{k},L}(t)\right),$$

$$Q_R(t) \;=\; \int d^3\mathbf{k}\, \left(\alpha^{\dagger}_{\mathbf{k},R}(t)\alpha_{\mathbf{k},R}(t) - \beta^{\dagger}_{-\mathbf{k},R}(t)\beta_{-\mathbf{k},R}(t)\right).$$

• The above defined chiral ladder operators are time-dependent and satisfy (equal time) canonical anticommutation relations (CAR):

$$\left\{\alpha^{r}_{\mathbf{k},L}(t), \alpha^{s\dagger}_{\mathbf{k},L}(t)\right\} = \delta^3(\mathbf{k}-\mathbf{p})\delta_{rs}\,, \qquad \left\{\beta^{r}_{\mathbf{k},L}(t), \beta^{s\dagger}_{\mathbf{k},L}(t)\right\} = \delta^3(\mathbf{k}-\mathbf{p})\delta_{rs}$$

Dirac field expansion:

$$
\begin{aligned}
\psi(x) \;=\; \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \Big[& u^1_{\mathbf{k}} \left(\cos\theta_k \, \alpha_{\mathbf{k},R} \,-\, e^{i\phi_k} \, \sin\theta_k \, \beta^\dagger_{-\mathbf{k},L} \right) e^{-i\omega_k t} \\
& + u^2_{\mathbf{k}} \left(\cos\theta_k \, \alpha_{\mathbf{k},L} \,+\, e^{i\phi_k} \, \sin\theta_k \, \beta^\dagger_{-\mathbf{k},R} \right) e^{-i\omega_k t} \\
& + v^1_{-\mathbf{k}} \left(\cos\theta_k \, \beta^\dagger_{-\mathbf{k},L} \,+\, e^{-i\phi_k} \, \sin\theta_k \, \alpha_{\mathbf{k},R} \right) e^{i\omega_k t} \\
& + v^2_{-\mathbf{k}} \left(\cos\theta_k \, \beta^\dagger_{-\mathbf{k},R} \,-\, e^{-i\phi_k} \, \sin\theta_k \, \alpha_{\mathbf{k},L} \right) e^{i\omega_k t} \Big]
\end{aligned}
$$

can be rearranged in the following form (using $\phi_k = 2\omega_k t$)

$$
\begin{aligned}
\psi(x) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Big[u_{\mathbf{k},L} \, \alpha_{\mathbf{k},L}(t) \, e^{-i\omega_k t} \,+\, v_{-\mathbf{k},L} \, \beta^\dagger_{-\mathbf{k},L}(t) \, e^{i\omega_k t} \Big] e^{i\mathbf{k}\cdot\mathbf{x}} \\
&+ \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Big[u_{\mathbf{k},R} \, \alpha_{\mathbf{k},R}(t) \, e^{-i\omega_k t} \,+\, v_{-\mathbf{k},R} \, \beta^\dagger_{-\mathbf{k},R}(t) \, e^{i\omega_k t} \Big] e^{i\mathbf{k}\cdot\mathbf{x}} \\
&= \psi_L(x) \,+\, \psi_R(x)
\end{aligned}
$$

with

$$u_{\mathbf{k},L} \equiv \cos\theta_k\, u^2_{\mathbf{k}} - \sin\theta_k\, v^2_{-\mathbf{k}}, \quad u_{\mathbf{k},R} \equiv \cos\theta_k\, u^1_{\mathbf{k}} + \sin\theta_k\, v^1_{-\mathbf{k}}$$
$$v_{-\mathbf{k},L} \equiv \cos\theta_k\, v^1_{-\mathbf{k}} - \sin\theta_k\, u^1_{\mathbf{k}}, \quad v_{-\mathbf{k},R} \equiv \cos\theta_k\, v^2_{-\mathbf{k}} + \sin\theta_k\, u^2_{\mathbf{k}}$$

$$u^\dagger_{\mathbf{k},L} u_{\mathbf{k},L} = u^\dagger_{\mathbf{k},R} u_{\mathbf{k},R} = 1\,, \quad v^\dagger_{-\mathbf{k},L} v_{-\mathbf{k},L} = v^\dagger_{-\mathbf{k},R} v_{-\mathbf{k},R} = 1$$
$$u^\dagger_{\mathbf{k},L} u_{\mathbf{k},R} = v^\dagger_{-\mathbf{k},L} v_{-\mathbf{k},R} = 0\,, \quad u^\dagger_{\mathbf{k},L} v_{-\mathbf{k},L} = u^\dagger_{\mathbf{k},R} v_{-\mathbf{k},L} = 0$$

and the completeness relation:

$$u_{\mathbf{k},R}\, u^\dagger_{\mathbf{k},R} + u_{\mathbf{k},L}\, u^\dagger_{\mathbf{k},L} + v_{-\mathbf{k},R}\, v^\dagger_{-\mathbf{k},R} + v_{-\mathbf{k},L}\, v^\dagger_{-\mathbf{k},L} = 1\!\!1$$

Consistency relations:

$$P_L\, u_{\mathbf{k},L} = u_{\mathbf{k},L}\,, \qquad P_L\, v_{-\mathbf{k},L} = v_{-\mathbf{k},L}$$
$$P_R\, u_{\mathbf{k},R} = u_{\mathbf{k},R}\,, \qquad P_R\, v_{-\mathbf{k},R} = v_{-\mathbf{k},R}$$
$$P_R\, u_{\mathbf{k},L} = P_R\, v_{-\mathbf{k},L} = P_L\, u_{\mathbf{k},R} = P_L\, v_{-\mathbf{k},R} = 0$$

The Bogoliubov transformation is written as

$$\alpha_{\mathbf{k},L} = G_t^{-1}\alpha^2_{\mathbf{k}}G_t \quad , \quad \beta_{\mathbf{k},L} = G_t^{-1}\beta^1_{\mathbf{k}}G_t$$
$$\alpha_{\mathbf{k},R} = G_t^{-1}\,\alpha^1_{\mathbf{k}}\,G_t \quad , \quad \beta_{\mathbf{k},R} = G_t^{-1}\,\beta^2_{\mathbf{k}}\,G_t$$

with generator

$$G_t(\theta,\phi) = \exp\left[\sum_r \int d^3\mathbf{k}\,\theta_k \epsilon^r \left(e^{-i\phi_k}\alpha^r_{\mathbf{k}}\beta^r_{-\mathbf{k}} - e^{i\phi_k}\beta^{r\dagger}_{-\mathbf{k}}\alpha^{r\dagger}_{\mathbf{k}}\right)\right]$$

• Explicit form for the *massive chiral vacuum*:

$$|\tilde{0}(t)\rangle_{LR} = \prod_{\mathbf{k},r}\left[\cos\theta_k + \epsilon^r e^{i\phi_k}\sin\theta_k \alpha^{r\dagger}_{\mathbf{k}}\beta^{r\dagger}_{-\mathbf{k}}\right]|0\rangle$$

• The massive chiral vacuum $|\tilde{0}(t)\rangle_{LR}$ and the Dirac vacuum $|0\rangle$ are orthogonal in the infinite volume limit:

$$\lim_{V\to\infty}\langle 0|\tilde{0}(t)\rangle_{LR} = 0,$$

generating *unitarily inequivalent representations* of the field algebra.

Chiral oscillation formula

Define the state $|\alpha_L\rangle \equiv \alpha^\dagger_{\mathbf{k},L}|\tilde{0}\rangle_{LR}$, with $|\tilde{0}\rangle_{LR} \equiv |\tilde{0}(0)\rangle_{LR}$.

Left chiral operator at time t

$$\alpha_{\mathbf{k},L}(t) = \cos\theta_k\, e^{-i\omega_k t}\, \alpha^2_{\mathbf{k}} \,-\, \sin\theta_k\, e^{i\omega_k t}\, \beta^{2\dagger}_{-\mathbf{k}}$$

- Chiral oscillation formula

$$\langle \alpha_{\mathbf{k},L}|Q_L(t)|\alpha_{\mathbf{k},L}\rangle = |\{\alpha_{\mathbf{k},L}(t), \alpha^\dagger_{\mathbf{k},L}(0)\}|^2$$

with

$$\{\alpha_{\mathbf{k},L}(t), \alpha^\dagger_{\mathbf{k},L}(0)\} = \cos^2\theta_k e^{-i\omega_k t} + \sin^2\theta_k e^{i\omega_k t}$$

We obtain

$$\langle \alpha_{\mathbf{k},L}|Q_L(t)|\alpha_{\mathbf{k},L}\rangle = 1 - \sin^2(2\theta_k)\sin^2(\omega_k t) = 1 - \frac{m^2}{\omega_k^2}\sin^2(\omega_k t)$$

Ontology of weak interactions: flavor/mass, chirality/energy ?

- unitary inequivalence between mass and flavor representations and between chiral and energy representations ⇒ nontrivial nature of weak interactions.

Chiral oscillations: perturbative approach

Fermion Hamiltonian density*

$$\mathcal{H}_0 \ = \ \sum_{\sigma=L,R} \overline{\psi}_\sigma i\partial\!\!\!/ \psi_\sigma \, , \quad \mathcal{H}_{int} \ = \ -m \left(\overline{\psi}_L \psi_R + \overline{\psi}_R \psi_L\right)$$

Massless fields with definite chirality

$$\psi_\sigma(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \left(u_{\mathbf{k},\sigma} \, \alpha_{\mathbf{k},\sigma} \, e^{-ikx} + v_{\mathbf{k},\sigma} \, \beta^\dagger_{\mathbf{k},\sigma} \, e^{ikx} \right)$$

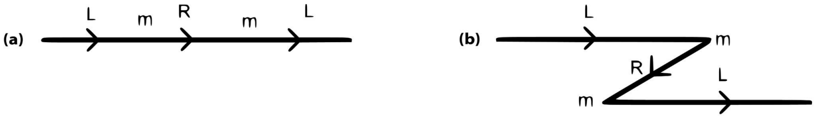
Chiral states

$$|\psi_{\mathbf{p},\sigma}\rangle \ \equiv \ \alpha^\dagger_{\mathbf{p},\sigma} |0\rangle$$

with $\alpha_{\mathbf{k},\sigma}|0\rangle = 0 = \beta_{\mathbf{k},\sigma}|0\rangle$

*M. B., F. Giacosa, L. Smaldone and G. Torrieri, Eur. Phys. J. C (2025)

Survival diagrams



• Second-order diagrams for the L survival probability. Time flows from left to right.

Chiral oscillations probability

At the quadratic order in m

$$\mathcal{A}_{L\to L}(\mathbf{p};t_i,t_f) \ = \ 1-\frac{1}{2}\mathcal{A}^{(2)}_{L\to L}(\mathbf{p};t_i,t_f)\,,$$

with

$$\mathcal{A}^{(2)}_{L\to L}(\mathbf{p};t_i,t_f)\approx m^2 \int_{t_i}^{t_f}\mathrm{d}t_1 \int_{t_i}^{t_f}\mathrm{d}t_2\, e^{2i|\mathbf{p}|(t_1-t_2)}$$

Chiral oscillation probability

$$\mathcal{P}_{L\to L}(\mathbf{p};\Delta t) \ = \ 1-\frac{m^2}{|\mathbf{p}|^2}\sin^2\left(|\mathbf{p}|\,\Delta t\right)\,.$$

It coincides with the one computed in QM and QFT (at $o(m^2)$ order).

QFT flavor/chiral oscillation formula†

- By considering expectation values of flavor/chiral charges, we obtain:

$$\langle Q_e^L(t)\rangle \equiv P_{e\to e}(t) = P_S^{e\to e}(t) + A_e(t) + B_e(t).$$

where $P_S^{e\to e}(t)$ is the standard (Pontecorvo) flavor oscillation formula:

$$P_S^{e\to e}(t) = 1 - \sin^2(2\theta)\sin^2\left(\frac{\omega_2-\omega_1}{2}t\right),$$

and

$$A_e(t) = -\left(\frac{m_1}{\omega_1}\cos^2(\theta)\sin(\omega_1 t) + \frac{m_2}{\omega_2}\sin^2(\theta)\sin(\omega_2 t)\right)^2,$$

$$B_e(t) = \frac{1}{2}\sin^2(2\theta)\sin(\omega_1 t)\sin(\omega_2 t)\left(\frac{k^2+m_1 m_2}{\omega_1\omega_2} - 1\right),$$

- Agreement with formula obtained by Dirac equation

† V.Bittencourt, M.B. and G.Zanfardino, arXiv:2507.09645 [hep-ph]

Criticism to our work

Recently, some papers appeared which deny the existence of chiral oscillations:

– A. Y. Smirnov, *Chiral interactions, chiral states and "chiral neutrino oscillations"*, [arXiv:2505.06116 [hep-ph]].

– E. Akhmedov, *On chirality and chiral neutrino oscillations*, [arXiv:2505.20982 [hep-ph]].

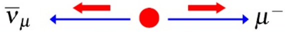
Work in progress in replying to these criticisms...

Standard argument for pion decay (Thomson)

★ Hence $u_\uparrow = P_R u_\uparrow + P_L u_\uparrow = \frac{1}{2}\left(1+\frac{|\vec{p}|}{E+m}\right) u_R + \frac{1}{2}\left(1-\frac{|\vec{p}|}{E+m}\right) u_L$

RH Helicity ($u_\uparrow$) — RH Chiral (u_R) — LH Chiral (u_L)

- In the limit $E \gg m$, as expected, the RH chiral and helicity states are identical
- Although only LH chiral particles participate in the weak interaction the contribution from RH **Helicity** states is not necessarily zero !



$m_\nu \approx 0$: RH Helicity ≡ RH Chiral

$m_\mu \neq 0$: RH Helicity has LH Chiral Component

★ Expect matrix element to be proportional to **LH chiral component of RH Helicity** electron/muon **spinor**

$$M_{fi} \propto \frac{1}{2}\left(1-\frac{|\vec{p}|}{E+m}\right) = \frac{m}{m_\pi + m}$$

(from the kinematics of pion decay at rest)

★ Hence because the electron mass is much smaller than the pion mass the decay $\pi^- \rightarrow e^- \overline{\nu}_e$ **is heavily suppressed.**

Criticism to our work: reply (preliminary)

The argument by Smirnov is based on the construction of neutrino state as superposition of helicity eigenstates.

However, if we consider the case of massless neutrino, the helicity of the associated lepton is fixed, and chiral projector applies, so chiral oscillations occur.

In this case, there is no possibility to define a state which is superposition of helicity states!