Determination of the Λ–parameter of SU(N) Yang–Mills theories in the Twisted Gradient Flow scheme on the lattice

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Università di Pisa

Introduction – The SU(N) Λ–parameter

β-function of the pure-gauge SU(N) Yang-Mills theory in a renormalization scheme *s* :

$$\beta_s(\lambda_s(\mu)) = \frac{d\lambda_s(\mu)}{d\log(\mu^2)} \underset{\lambda_s \to 0}{\sim} -\lambda_s^2 \left(b_0 + b_1 \lambda_s + b_2^{(s)} \lambda_s^2 + \dots \right) \qquad (\lambda_s(\mu) = N g_s^2(\mu))$$

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Renormalization-group invariant, scheme dependent energy scale:

$$\Lambda_s = \mu[b_0\lambda_s(\mu)]^{-\frac{b_1}{2b_0^2}} e^{-\frac{1}{2b_0\lambda_s(\mu)}} \exp\left\{-\int_0^{\lambda_s(\mu)} dx\left(\frac{1}{2\beta_s(x)} + \frac{1}{2b_0^2x^2} - \frac{b_1}{2b_0^2x}\right)\right\}$$

$$\Lambda_{\overline{\mathrm{MS}}} = \Lambda_s e^{-\frac{c_s}{2b_0}}\,; \qquad \lambda_{\overline{\mathrm{MS}}}(\mu) \underset{\lambda_s\to 0}{\sim} \lambda_s(\mu)(1 + c_s\lambda_s(\mu) + \dots)$$

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Energy scale of **nonperturbative** dynamics, can be determined from **lattice simulations**

Main interest for the **N = 3** theory (can be related to the scale of QCD) and **large-N limit**

Introduction – Determining the Λ–parameter

Twisted Gradient Flow (TGF) scheme: computationally convenient (smaller volumes), matching with $\overline{\text{MS}}$ is known ('t Hooft, 1980; González-Arroyo and Okawa, 1983)

Gradient-flow scale $\sqrt{t_0}$ as reference length: only dimensionless quantities can be computed, a conventional choice is $\Lambda_{\mathrm{TGF}}\sqrt{8t_0}$

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First part

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Second part

arXiv:2501.18449
this talk
(preliminary results)

Determination of $\mu_{\text{had}}\sqrt{8t_0}$ for N = 3, 5, 8 and computation of

$$\Lambda_{\text{TGF}}\sqrt{8t_0} = \Lambda_{\text{TGF}}/\mu_{\text{had}} \cdot \mu_{\text{had}}\sqrt{8t_0}$$

First part – Step scaling method

We want Λ in units of a low-energy scale $\mu_{\rm had} \sim \Lambda$ and use perturbation theory only at $\mu_{\rm pt} \gg \Lambda$

$$\frac{\Lambda_s}{\mu_{\rm had}} = \frac{\Lambda_s}{\mu_{\rm pt}} \frac{\mu_{\rm pt}}{\mu_{\rm had}} = \frac{\Lambda_s}{\mu_{\rm pt}} \exp\left\{ - \int_{\lambda_s(\mu_{\rm pt})}^{\lambda_s(\mu_{\rm had})} \frac{dx}{2\beta_s(x)} \right\}$$

First terms of the perturbative expansion evaluated with known coefficients of β-function

Requires integration of RG-flow in non-perturbative region, only possible with lattice methods

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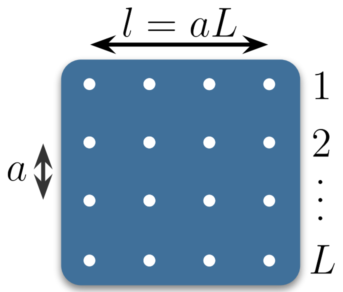
Problem: lattice simulations have IR cutoff $\sim 1/l$ and UV cut-off $\sim 1/a$, covering the range from $\mu_{\rm had}$ to $\mu_{\rm pt}$ in a single simulation requires too much computational power

Idea: find a recursive procedure to match simulations which cover different smaller sub-ranges



First part – Step scaling method

The TGF is a *finite-volume renormalization scheme* :

$$\mu \equiv (0.3l)^{-1} \quad (l = aL)$$

Consider the continuum the *step scaling function* :

$$\sigma(u) = \lambda(\mu/2)\Big|_{\lambda(\mu)=u}$$

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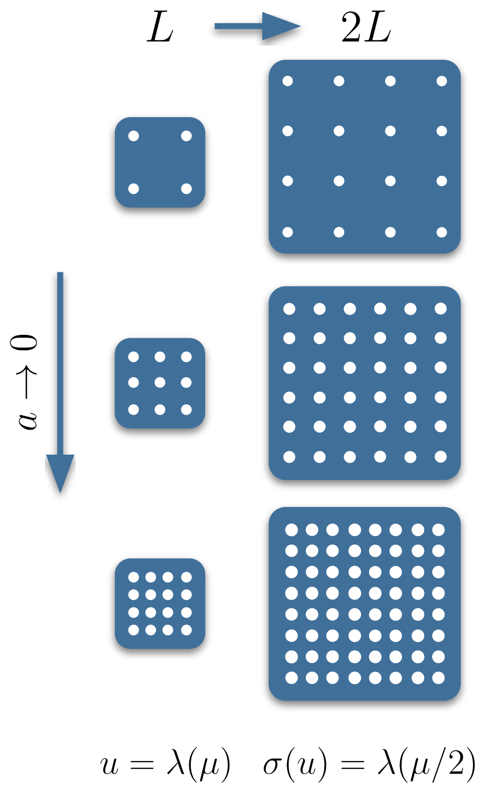
Step scaling procedure:

1. Choose target value $u = \lambda(\mu)$
2. Fine-tune bare coupling to measure u with several L
3. Repeat simulations with doubled L to measure $\sigma(u) = \lambda(\mu/2)$ after continuum extrapolation
4. Iterate from 1. with $u \longrightarrow \sigma(u)$, that is $\mu \longrightarrow \mu/2$

After n steps, reached scale $2^{-n}\mu \ll \mu$



First part – Results

Step scaling function measured for several values of u,
parametrized and best-fitted to be evaluated at arbitrary points

From a chosen μ_{had} and measured $\lambda(\mu_{\text{had}})$:

$$\lambda(\mu_{\text{pt}}) = \sigma^{-n}(\lambda(\mu_{\text{had}})), \; \mu_{\text{pt}} = 2^n \mu_{\text{had}}$$

$$\frac{\Lambda}{\mu_{\text{had}}} = \frac{\mu_{\text{pt}}}{\mu_{\text{had}}} \frac{\Lambda}{\mu_{\text{pt}}}\bigg|_{\lambda(\mu_{\text{pt}})} = 2^n \frac{\Lambda}{\mu_{\text{pt}}}\bigg|_{\lambda(\mu_{\text{pt}})}$$

$$\frac{\Lambda}{\mu_{\text{pt}}}\bigg|_{\lambda(\mu_{\text{pt}})} = [b_0 \lambda(\mu_{\text{pt}})]^{-\frac{b_1}{2b_0^2}} e^{-\frac{1}{2b_0\lambda(\mu_{\text{pt}})}} \left(1 + O\left(\lambda(\mu_{\text{pt}})\right)\right)$$

Matching with perturbation theory done for several $\lambda(\mu_{\text{pt}})$
and extrapolated for $\lambda(\mu_{\text{pt}}) \to 0$

N	$\Lambda_{\overline{MS}}/\mu_{\text{had}}$
3	$0.416(17)$
5	$0.478(17)$
8	$0.494(54)$

(Dasilva Golán et al., 2023)

Second part – Setting the scale t_0

Gradient flow : evolution of the gauge field in an auxiliary time with a diffusion-like equation, driven by the gradient of the action (Narayanan and Neuberger, 2006; Lüscher, 2010)

$$\frac{dA_\mu(t,x)}{dt} = -\frac{\delta S[A]}{\delta A_\mu} = D_\nu F_{\nu\mu}(t,x)$$

At $t > 0$ gauge observables are regularized

Fields smoothed in a radius $\sqrt{8t}$ (notice $[t] = [\mathrm{length}]^2$)

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Flowed energy density:

$$E(t) = \frac{1}{2}\mathrm{Tr}\left[F_{\mu\nu}(t,x)F_{\mu\nu}(t,x)\right]$$

Gradient-flow scale t_0 defined for SU(3) as

$$\langle t^2 E(t)\rangle\big|_{t=t_0} = 0.3 \implies \sqrt{8t_0} \simeq 0.5\ \mathrm{fm}$$

Definition can be extended to SU(N) as

$$\frac{N}{N^2-1}\langle t^2 E(t)\rangle\big|_{t=t_0} = 0.1125$$

Second part – Setting the scale t_0

Problem: $E(t)$ of a gauge configuration correlated with its topological charge $Q \in \mathbb{N}$

$$Q = \int d^4x \, \frac{g^2}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}[F_{\mu\nu}(x) F_{\rho\sigma}(x)]$$

Sampling Q becomes harder near continuum limit, standard algorithms stuck in $Q = 0$ (*topological freezing*)

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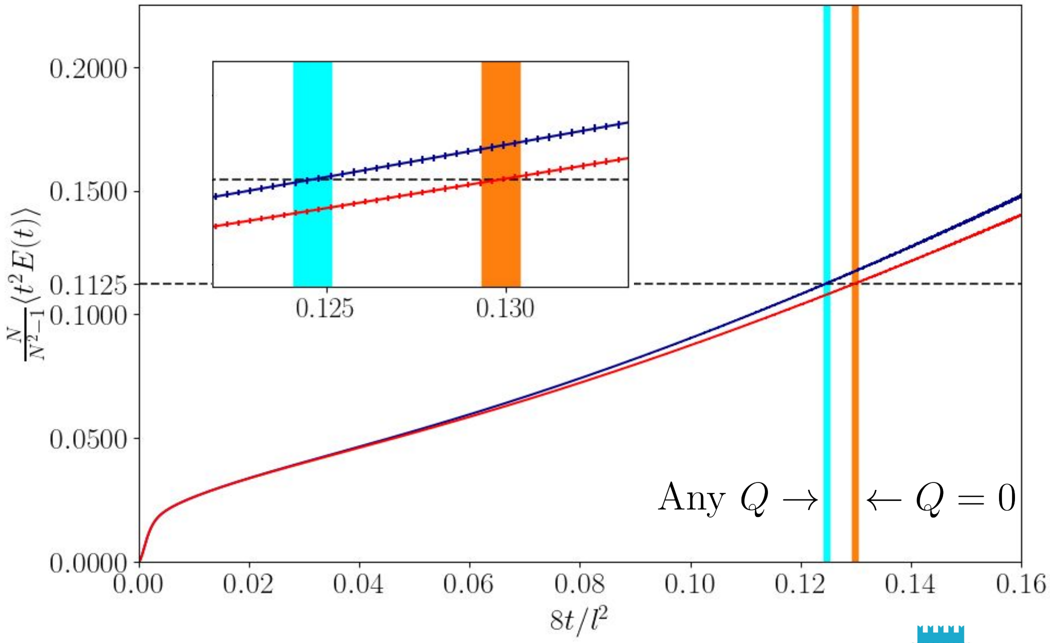
Idea: Use *Parallel Tempering on Boundary Conditions* (PTBC) algorithm, designed to mitigate freezing (Hasenbusch, 2017; Bonanno et al., 2021)

Evaluate systematics by comparing t_0 with all Q and projecting to $Q = 0$
Same infinite-volume limit expected (Brower et al., 2003)

$$Q = \int d^4x \, \frac{g^2}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}[F_{\mu\nu}(x) F_{\rho\sigma}(x)]$$

$N = 5\,, \;\; L = 30\,, \;\; l \simeq 1.4\,\mathrm{fm} \;\; (a \simeq 0.047\,\mathrm{fm})$

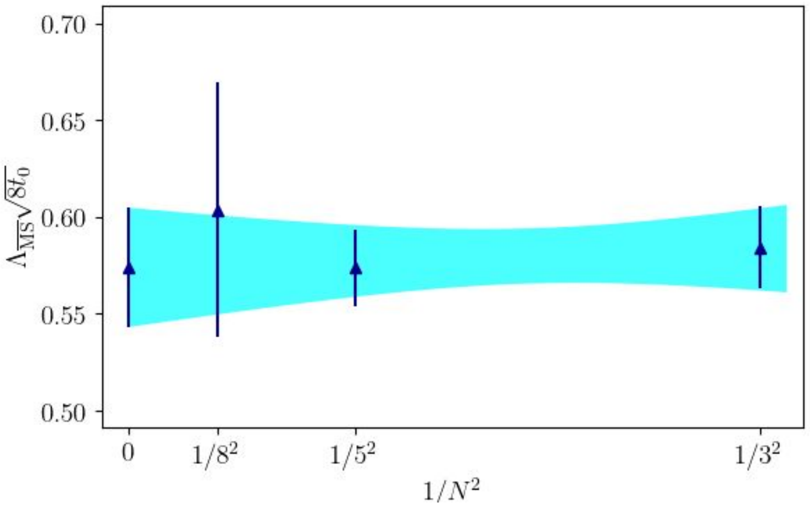


Second part – Preliminary results

With same bare couplings (lattice spacings) used for the step scaling, continuum extrapolation:

$$\mu_{\rm had}\sqrt{8t_0} = \lim_{a\to 0} a\mu_{\rm had} \times \sqrt{8t_0(a)}/a$$

Final result $\Lambda_{\overline{\rm MS}}\sqrt{8t_0} = \Lambda_{\overline{\rm MS}}/\mu_{\rm had} \cdot \mu_{\rm had}\sqrt{8t_0}$:



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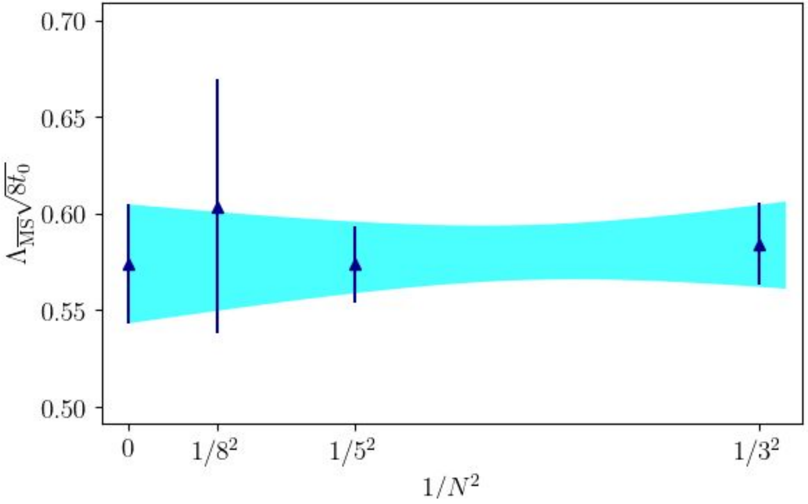
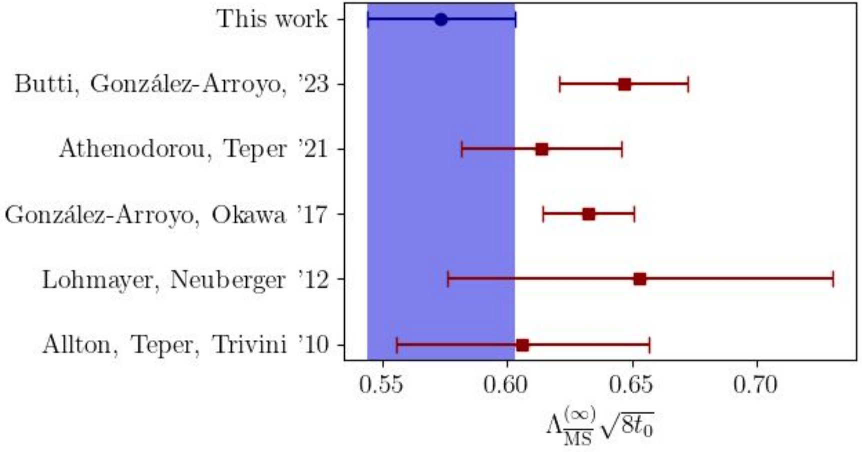


SU(3) value agrees with literature, little tension in large-N extrapolation:

$$\Lambda^{(3)}_{\overline{\rm MS}}\sqrt{8t_0} = 0.591(18)$$ (this work)

$$\Lambda^{(3)}_{\overline{\rm MS}}\sqrt{8t_0} = 0.610(12)$$ (FLAG review, 2024)



Conclusions

Take-home messages

Step scaling method and scale setting of Yang–Mills theory in the TGF scheme in order to determine the Λ–parameter in units of the gradient-flow scale

Parallel tempering on boundary conditions to mitigate and evaluate the systematics of topological freezing

Agreement of SU(3) value with literature, little tension in large-N extrapolation. Systematics in the matching with perturbation theory?

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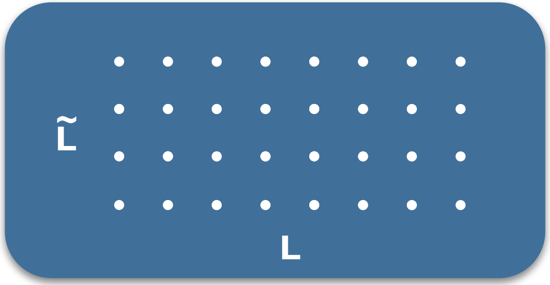
Future outlooks

Refine results, especially for SU(8), for more meaningful large-N extrapolation

Study the reduction of finite-volume effects in the scale setting achieved with TGF, try other gradient-flow scales

Backup – TGF scheme

SU(N) theory discretized on lattice of size $L^2 \times \tilde{L}^2$
with $\tilde{L} = L/N$ on directions 1,2



Twisted Boundary Conditions ('t Hooft, 1980; González-Arroyo and Okawa, 1983):
periodic up to a gauge transformation (the twist) for links $U_\mu(n)$
on plane (1,2). With appropriate choice of twist:

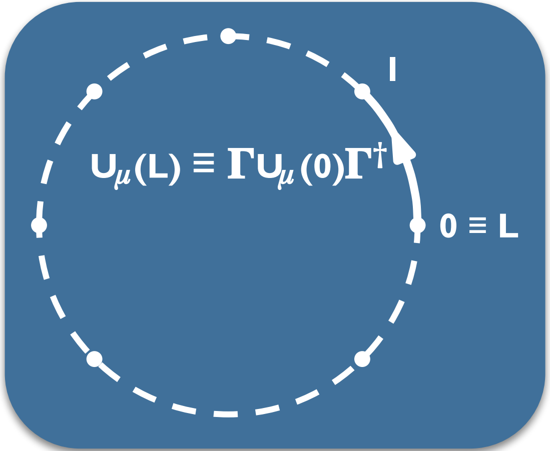
$$U_\mu(n + \tilde{L}\hat{\nu}) \equiv \Gamma_\nu U_\mu \Gamma_\nu^\dagger \quad (\mu, \nu = 1, 2)$$

Consistency relation:

$$\Gamma_1 \Gamma_2 = Z_{12} \Gamma_2 \Gamma_1 \, , \quad Z_{12} = e^{i \frac{2\pi}{N} k}$$

k, N coprime integers. To avoid instabilities in large-N limit,
best choice of k scales with N (Chamizo and González-Arroyo, 2017):

$$N = 3 \implies k = 1 \quad N = 5 \implies k = 2, \ldots$$



Backup – PTBC algorithm

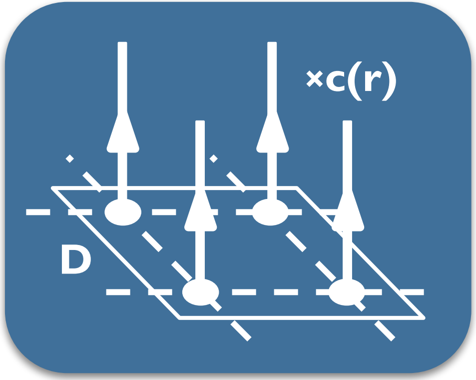
Proposed for 2d $CP^{(N-1)}$ (Hasenbusch, 2017), later for 4d SU(N) Yang–Mills (Bonanno et al., 2021) and full QCD (Bonanno et al., 2024)

N_r replicas of the lattice simulated in parallel

Replicas differ for boundary conditions on small 3d sub-lattice, the *defect* D:
links crossing D multiplied by $c(r) \quad (r = 0, 1, ..., N_r - 1)$
Periodic: $c(0) = 1$ Open: $c(N_r - 1) = 0$ Others: $0 < c(r) < 1$

Replicas are updated independently with standard methods for some steps, then swaps among configurations are proposed via Metropolis test: decorrelation of Q transferred from open to periodic replica

Observables computed only on periodic replica: easier to keep finite-size effects under control



To improve performance, the defect is translated randomly and updates are more frequent around it
Tuning of $c(r)$ and size of D to have 20% uniform acceptance of swaps

Backup – Decorrelation of topological charge

Comparison of MC histories of Q with PTBC and standard algorithm (heathbath + overrelaxation)
With PTBC, lattice sweeps counted on all replicas to account for extra computational effort

Standard algorithm frozen even with coarsest lattice spacing in this work

Integrated autocorrelation time τ_{Q^2} of Q^2 for quantitative comparison:

	PTBC	Standard
Q^2	$0.084(2)$	$0.08(2)$
τ_{Q^2}	$2.5(3)\cdot 10^2$	$> 10^5$

Gain of PTBC gets larger in the continuum limit

$N = 5\,,\ \ l \simeq 1.4\ \mathrm{fm}\,,\ \ L = 20 \quad (a \simeq 0.07\ \mathrm{fm})$