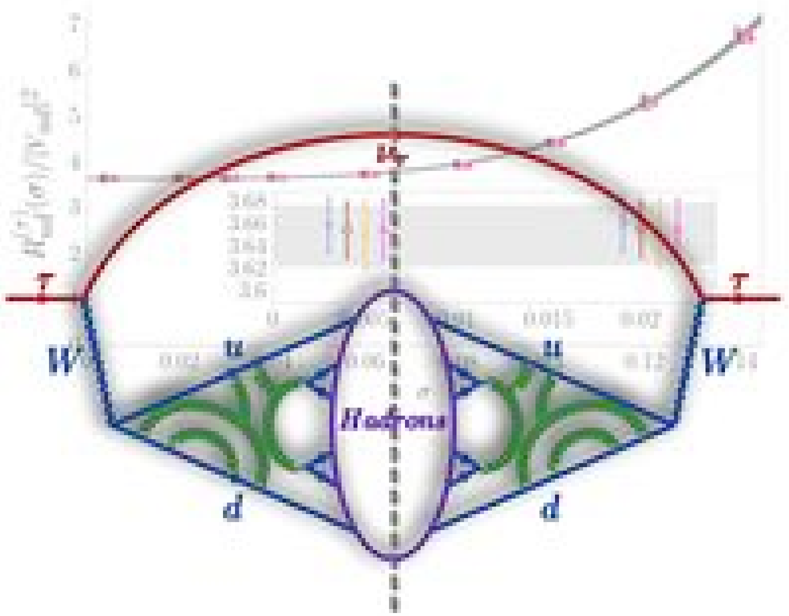
The Cabibbo Angle from Inclusive τ Decays

Francesco Sanfilippo – INFN Roma Tre

XXXVIII Convegno Nazionale di Fisica Teorica, 22/05/2025

Based on

[A. evangelista et al.
Phys Rev. D 108, 074513 (2023)]



EDITORS' SUGGESTION

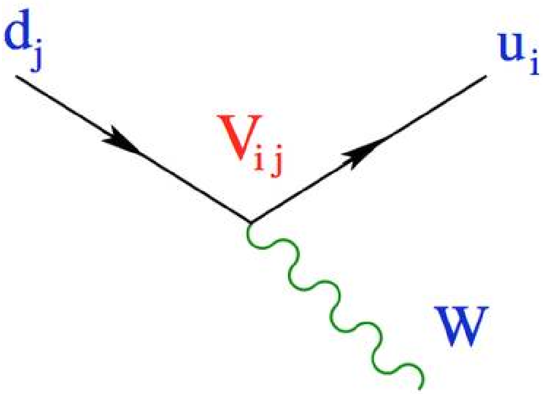
Inclusive hadronic decay rate of the τ lepton from lattice QCD

The authors express the inclusive hadronic decay rate of the tau lepton as an integral over the spectral density of the two-point correlator of the weak $V - A$ hadronic current which they compute fully nonperturbatively in lattice QCD. In a lattice QCD computation with all systematic errors except for isospin breaking effects under control, they then obtain the CKM matrix element V_{ud} with subpercent errors showing that their nonperturbative method can become a viable alternative to superallowed nuclear beta decays for obtaining V_{ud}.

Flavor in the SM: the CKM matrix

In the SM, the charged change of flavors occur through the exchange of a W boson

$$\mathcal{L}_w = -\frac{g_w}{\sqrt{2}}\,(\bar{u}\ ,\ \bar{c}\ ,\ \bar{t}\,)\,\gamma^\mu P_L\, W_\mu^+\, V_{\rm CKM}\begin{pmatrix} d \\ s \\ b \end{pmatrix} + \text{ h.c. }, \qquad V_{\rm CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$



Unitary matrix connecting up-like and down-like quarks

Hierarchy of amplitude between different generation

$$\underbrace{\begin{pmatrix} |V_{ud}| & & \\ & |V_{cs}| & \\ & & |V_{tb}| \end{pmatrix}}_{\mathcal{O}(1)} \gg \underbrace{\begin{pmatrix} & |V_{us}| & \\ |V_{cd}| & & \\ & & \end{pmatrix}}_{\mathcal{O}(10^{-1})} \gg \underbrace{\begin{pmatrix} & & |V_{ub}| \\ & & |V_{cb}| \\ |V_{td}| & |V_{ts}| & \end{pmatrix}}_{\mathcal{O}(10^{-2}\text{–}10^{-3})}$$

We focus on the first row of the CKM matrix

First row unitarity

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 \ (?)$$

$|V_{ud}|$ from superallowed nuclear beta decays: 0.03%

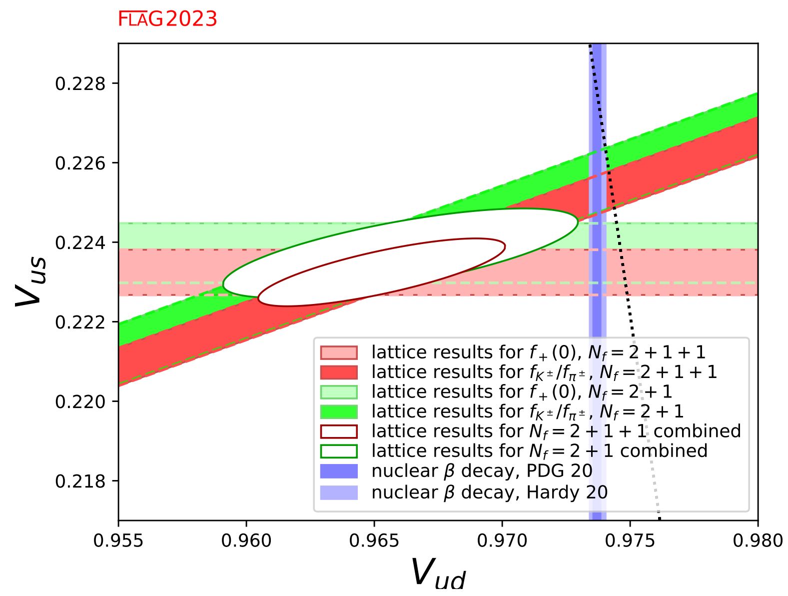
$|V_{ud}|$ from semileptonic $\pi \to \pi^0 e \nu_e$ beta decay: 0.3%

$|V_{ub}|$ is very small and we can drop it

$|V_{us}|$ from semiletponic decay $K \to \pi \ell \nu$: 0.2%

$\dfrac{|V_{us}|}{|V_{ud}|}$ from the ratio of semiletponic decay $K/\pi \to \ell \nu(\gamma)$: 0.3%

Tension between the value of $|V_{us}|$ from leptonic and semileptonic decays vs. $|V_{us}^{uni}|$



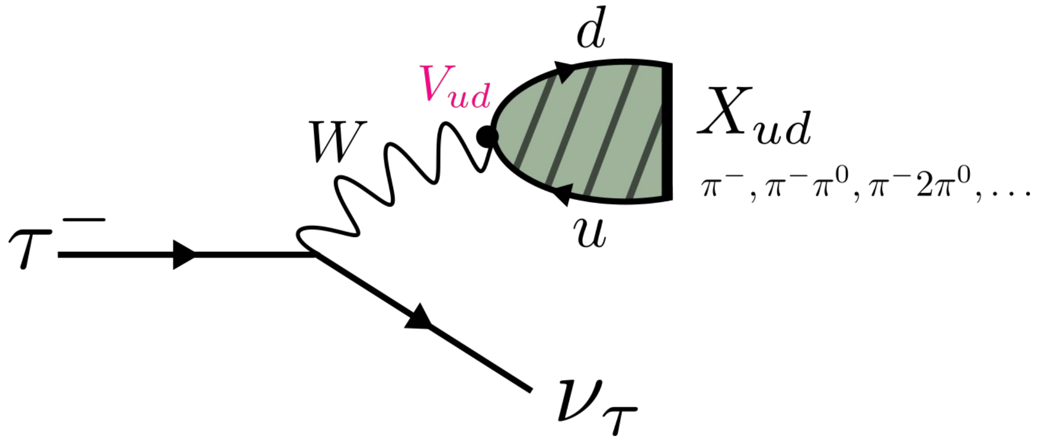
Usefuleness of τ decay

Three leptons are known experimentally, their measured masses is:

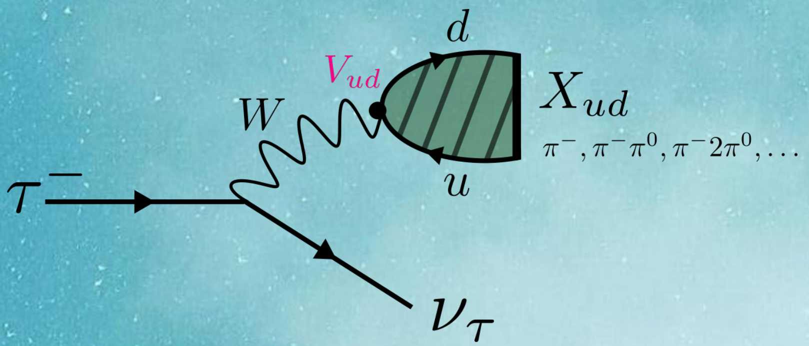
$$m_e = 0.5110 \text{ KeV}, \quad m_\mu = 105.66 \text{ MeV}, \quad m_\tau = 1.776 \text{ GeV}$$

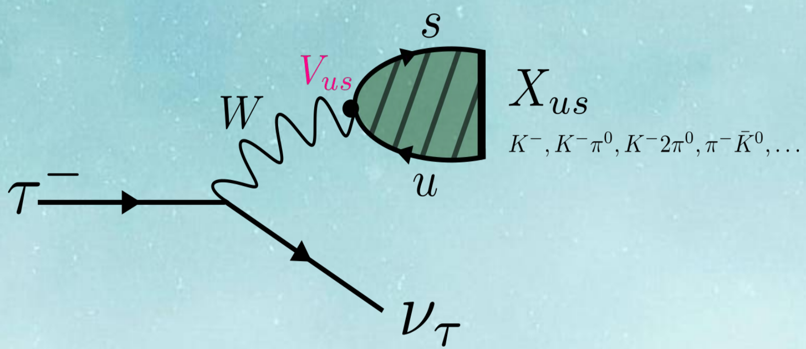
The lightest charged hadron is the **pion**, with mass: $m_\pi^\pm = 139.57 \text{ MeV}$

$\longrightarrow$ τ is unique among leptons, as **it can decay into hadrons**



Two hadronic decay channels



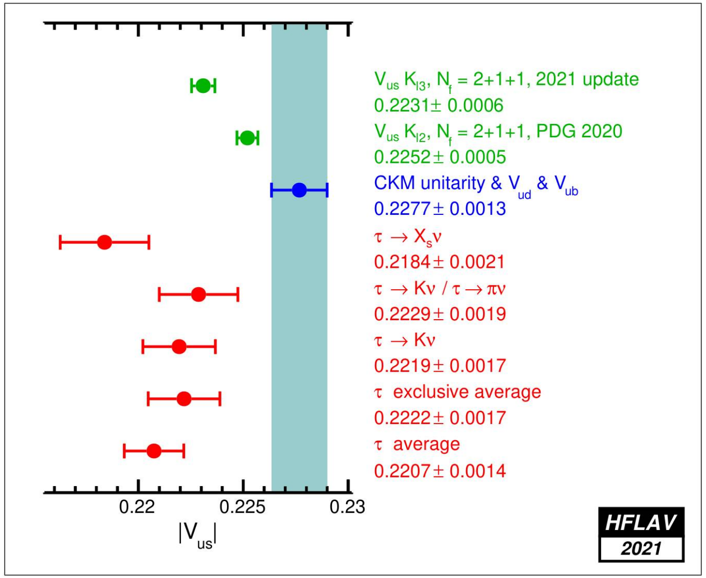


Studying the ud/us channels one one can determine $V_{ud/us}$

Comparison with $K_{\ell 2},\ K_{\ell 3}$ determinations shows tension

τ determinations also in tension with unitarity constraints

Tension is more pronounced for inclusive process $\tau \to X_{ux}$



Inclusive process, $\mathcal{B}_{X_{ud/us}} \equiv \Gamma[\tau \to X_{ud/us}\nu_\tau]/\Gamma_{\text{tot}}$

Sum of exclusive processes for a given channel

$\tau \to X_{us}$ channel:

Branching fraction	HFLAV 2021 fit (%)
$K^-\nu_\tau$	0.6957 ± 0.0096
$K^-\pi^0\nu_\tau$	0.4322 ± 0.0148
$K^-2\pi^0\nu_\tau$ (ex.K^0)	0.0634 ± 0.0219
$K^-3\pi^0\nu_\tau$ (ex.K^0, η)	0.0465 ± 0.0213
$\pi^-\overline{K}^0\nu_\tau$	0.8375 ± 0.0139
$\pi^-\overline{K}^0\pi^0\nu_\tau$	0.3810 ± 0.0129
$\pi^-\overline{K}^0 2\pi^0\nu_\tau$ (ex.K^0)	0.0234 ± 0.0231
$\overline{K}^0 h^-h^-h^+\nu_\tau$	0.0222 ± 0.0202
$K^-\eta\nu_\tau$	0.0155 ± 0.0008
$K^-\pi^0\eta\nu_\tau$	0.0048 ± 0.0012
$\pi^-\overline{K}^0\eta\nu_\tau$	0.0094 ± 0.0015
$K^-\omega\nu_\tau$	0.0410 ± 0.0092
$K^-\phi(K^+K^-)\nu_\tau$	0.0022 ± 0.0008
$K^-\phi(K^0_S K^0_L)\nu_\tau$	0.0015 ± 0.0006
$K^-\pi^-\pi^+\nu_\tau$ (ex.K^0, ω)	0.2924 ± 0.0068
$K^-\pi^-\pi^+\pi^0\nu_\tau$ (ex.K^0, ω, η)	0.0387 ± 0.0142
$K^-2\pi^-2\pi^+\nu_\tau$ (ex.K^0)	0.0001 ± 0.0001
$K^-2\pi^-2\pi^+\pi^0\nu_\tau$ (ex.K^0)	0.0001 ± 0.0001
$X_s^-\nu_\tau$	2.9076 ± 0.0478

Accuracy: 1.7%

$\tau \to X_{ud}$ channel:

In the past, obtained by <u>difference</u>

$$\mathcal{B}_{X_{ud}} = 1 - \mathcal{B}_{X_{us}} - \mathcal{B}_e - \mathcal{B}_\mu$$

with $\mathcal{B}_{e/\mu}$ branching to electrons/muons

Nowadays <u>directly measured</u>

$$\mathcal{B}_{X_{ud}} = 0.6183 \pm 0.0010$$

according to HFLAV average

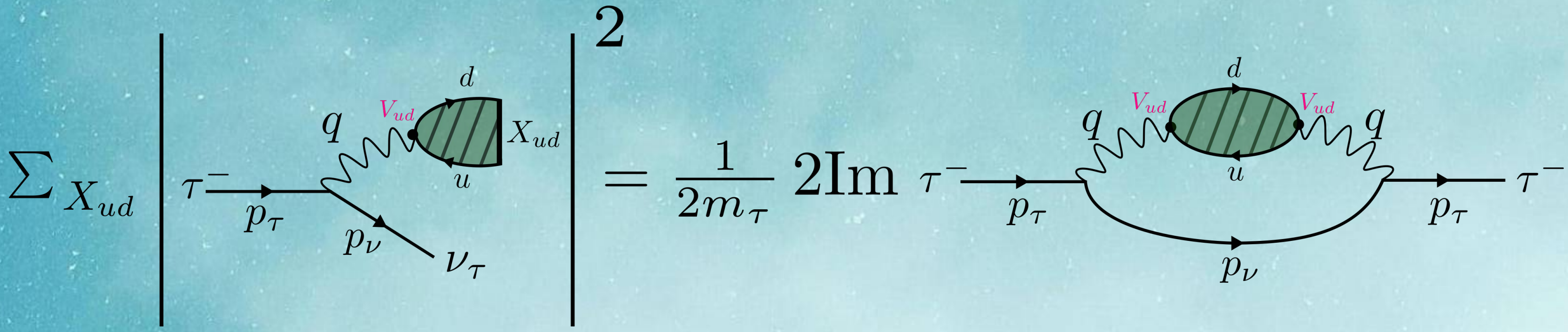
Weak Hamiltonian Effective theory

$$H_{ud}^{\text{eff}}(x) = \frac{G_F}{\sqrt{2}} V_{ud} \sqrt{S_{\text{EW}}} \; \underbrace{\bar{\nu}_\tau(x)\gamma^\alpha(1-\gamma^5)\tau(x)}_{J^\alpha_{\nu_\tau\tau}(x)} \times \underbrace{\bar{d}(x)\gamma^\alpha(1-\gamma^5)u(x)}_{(J^\alpha_{ud})^\dagger(x)} + \text{h.c.}$$

Factorization of the amplitude in leptonic and **nonperturbative** hadronic parts

$$\mathcal{A}(\tau \to X_{ud}\nu_\tau) = \frac{G_F}{\sqrt{2}} V_{ud} \sqrt{S_{\text{EW}}} \; \langle \nu_\tau | \, J^\alpha_{\nu_\tau\tau}(0) \, | \tau \rangle \; \langle X_{ud} | \, J^\alpha_{ud}(0)^\dagger \; | 0 \rangle$$

From matrix element to decay width



All that has been done so far is an application of the **optical theorem**

$$\Gamma[\tau \to X_{ud}\nu_\tau] = \frac{1}{2m_\tau} 2\mathrm{Im}[\Gamma_{\tau\tau}]$$

where $\Gamma_{\tau\tau} = \langle\tau|T|\tau\rangle$ is the forward amplitude $\mathrm{S} = 1 + iT$ related to ud states

The hadronic tensor

Squaring $\mathcal{A}(\tau \to X_{ud}\nu_\tau)$ and summing over all X_{ud} states:

$$|\mathcal{A}|^2 = \sum_{X_{ud}} |\mathcal{A}(\tau \to X_{ud}\,\nu_\tau)|^2 = \left(\frac{G_F}{\sqrt{2}}\right)^2 |\,V_{ud}\,|^2\, S_{\mathrm{EW}}\, L^{\alpha\beta}(p_\tau, p_\nu)\, \rho_{ud}^{\alpha\beta}(q)$$

with $p_\tau,\ p_\nu$ being the τ, ν_τ four-momentum, and $q = p_\tau - p_\nu$.

Leptonic tensor $L^{\alpha\beta}(p_\tau, p_\nu)$ obtained averaging/summing over τ/ν_τ polarizations:

$$L^{\alpha\beta}\,(p_\tau, p_\nu) = 4\left(p_\tau^\alpha p_\nu^\beta + p_\tau^\beta p_\nu^\alpha - g^{\alpha\beta} p_\tau \cdot p_\nu - i\,\varepsilon^{\alpha\beta\rho\sigma} p_{\tau\rho} p_{\nu\sigma}\right)$$

Missing ingredient: the <u>hadronic tensor</u>

$$\rho_{ud}^{\alpha\beta}(q) \equiv \sum_{X_{ud}} \langle 0|\; J_{ud}^{\alpha}(0)\; |X_{ud}(q)\rangle\; \langle X_{ud}(q)|\; J_{ud}^{\beta}(0)^\dagger\; |0\rangle$$

Determining the Hadronic Tensor

In Minkowsky spacetime, from the spectral representation (Fourier transform):

$$C^{\alpha\beta}(t,q) \overset{t\geq 0}{=} \int_0^\infty \frac{\mathrm{d}E}{2\pi} e^{-iEt} \rho_{ud}^{\alpha\beta}(E,q)$$

...of the current-current correlation function in time-momentum representation:

$$C^{\alpha\beta}\left(t,q\right) = \int d^3x\, e^{-iq\cdot x} \left\langle 0\right| T\left(J_{ud}^{\alpha}(-it,x)\, J_{ud}^{\beta}(0)^{\dagger}\right)\left|0\right\rangle$$

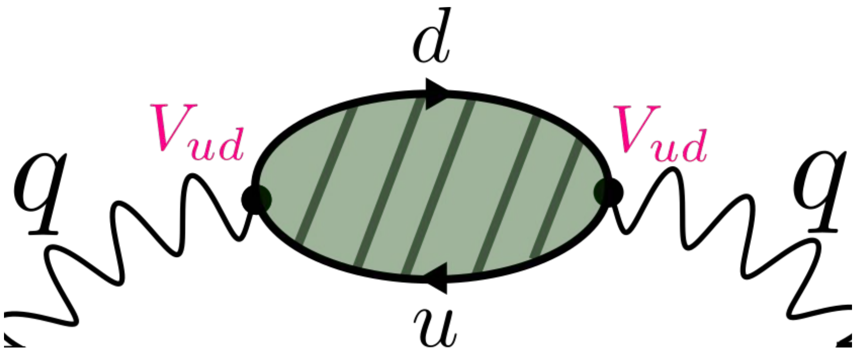
Determining the Hadronic Tensor

In Minkowsky spacetime, from the spectral representation (Fourier transform):

$$C^{\alpha\beta}(t,q) \stackrel{t\geq 0}{=} \int_0^\infty \frac{\mathrm{d}E}{2\pi} e^{-iEt} \rho^{\alpha\beta}_{ud}(E,q)$$

...of the current-current correlation function in time-momentum representation:

$$C^{\alpha\beta}(t,q) = \int d^3x\, e^{-iq\cdot x} \langle 0|\, T\left(J^{\alpha}_{ud}(-it,x)\, J^{\beta}_{ud}(0)^{\dagger}\right) |0\rangle$$



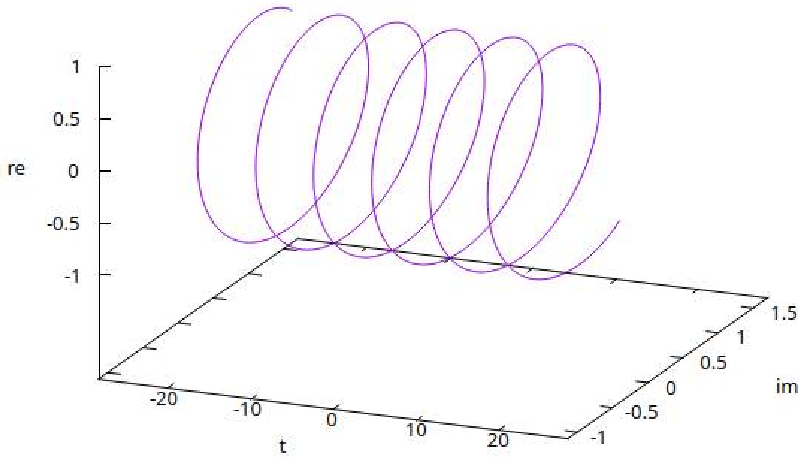
- Nonperturbative quantity
- Severe disease with a bad prognosis
- Call a latticist!

Hadronic amplitudes in Minkowskian

Correlation function, e.g. J_A, J_B currents on state $|P\rangle$

$$C(t) \equiv \langle 0|T\{J_A(t)J_B(0)\}|P\rangle \overset{t\geq 0}{=} \sum_{n=0}^{\infty} C_n\, e^{-iE_n t}$$

(Fourier transform)

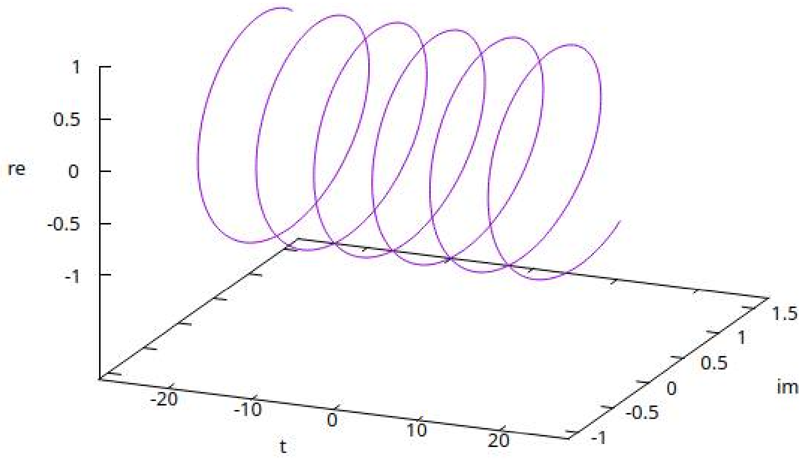


Hadronic amplitudes in Minkowskian

Correlation function, e.g. J_A, J_B currents on state $|P\rangle$

$$C(t) \equiv \langle 0|T\{J_A(t)J_B(0)\}|P\rangle \overset{t\geq 0}{=} \sum_{n=0}^{\infty} C_n\, e^{-iE_n t}$$

(Fourier transform)



Extracting Hadronic Amplitude $H(E)$

in **Minkowskian** continuum spacetime:

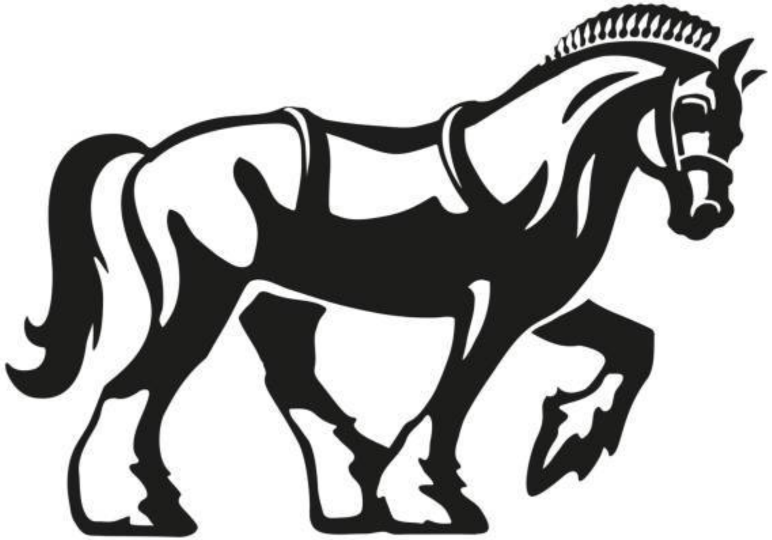
$$\rho(E) = i \lim_{T\to\infty} \int_0^T dt\, e^{iEt}\, C(t)$$

Hadronic amplitudes in Minkowskian

Correlation function, e.g. J_A, J_B currents on state $|P\rangle$

$$C(t) \equiv \langle 0|T\{J_A(t)J_B(0)\}|P\rangle \overset{t\geq 0}{=} \sum_{n=0}^{\infty} C_n\, e^{-iE_n t}$$

(Fourier transform)



Extracting Hadronic Amplitude $H(E)$

in **Minkowskian** continuum spacetime:

$$\rho(E) = i \lim_{T\to\infty} \int_0^T dt\, e^{iEt}\, C(t)$$

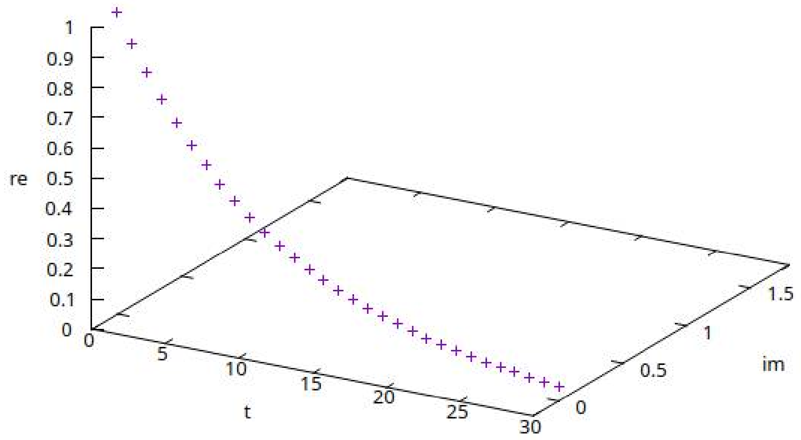
Inverse Fourier transform: a reliable horse

Hadronic amplitudes in Euclidean lattice

Analytic continuation to discrete Euclidean spacetime: $\tau = it$

$$C_e(\tau) \equiv C(-i\tau) = \sum_{n=0}^{\infty} C_n\, e^{-E_n \tau}$$

Laplace transform

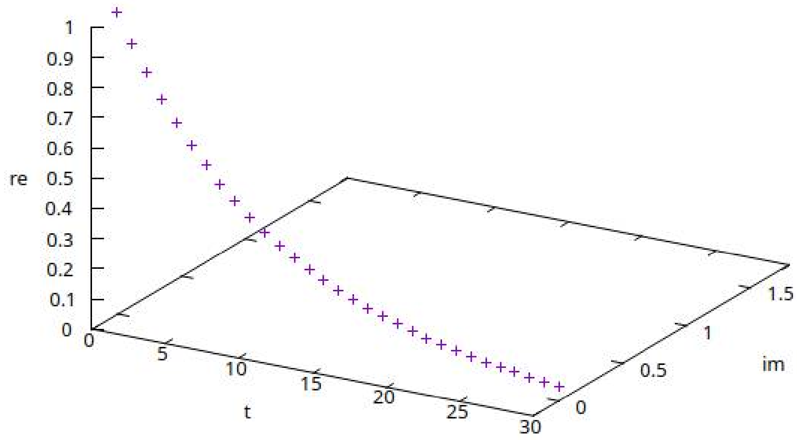


Hadronic amplitudes in Euclidean lattice

Analytic continuation to discrete Euclidean spacetime: $\tau = it$

$$C_e(\tau) \equiv C(-i\tau) = \sum_{n=0}^{\infty} C_n \, e^{-E_n \tau}$$

Laplace transform



Extracting Hadronic Amplitude

in discrete **Euclidean** spacetime of length T:

$$\rho^T(E) = \int_0^T d\tau \, e^{E\tau} \, C_e(\tau)$$

Hadronic amplitudes in Euclidean lattice

Analytic continuation to discrete Euclidean spacetime: $\tau = it$

$$C_e(\tau) \equiv C(-i\tau) = \sum_{n=0}^{\infty} C_n\, e^{-E_n \tau}$$

Laplace transform

Extracting Hadronic Amplitude

in discrete **Euclidean** spacetime of length T:

$$\rho^T(E) = \int_0^T d\tau\, e^{E\tau}\, C_e(\tau)$$

Inverse Laplace transform!??

A wild and unreliable horse!

Analytic continuation issues

In Euclidean spacetime, it holds true that

$$C^{\alpha\beta}(t,q) \overset{t\geq 0}{=} \int_0^\infty \frac{\mathrm{d}E}{2\pi} e^{-Et} \rho_{ud}^{\alpha\beta}(E,q)$$

i.e the current-current correlator is the Laplace transform of the needed Hadronic tensor

But

One cannot naively invert this relation if the energy E lies above the minimal threshold E_0

This requires an inverse Laplace transform, **ill-posed problem on finite noisy dataset**

CFR Nazario's talk

Spectral representation

$$\rho^T(E) = \int_0^T dt\, e^{Et}\, C_E(t) = \int_0^T dt\, e^{Et} \sum_{n=0}^{\infty} C_n e^{-E_n t} = \sum_{n=0}^{\infty} C_n \frac{1 - e^{-(E_n - E)T}}{E_n - E}$$

Let us break down two energy regimes

$$\boxed{E < E_0}$$

$$\rho^T(E) \underset{T\to\infty}{\rightarrow} \sum_{n=0}^{\infty} \frac{C_n}{E_n - E}$$

GOOD!

$$\boxed{E > E_0}$$

$$\rho^T(E) \underset{T\to\infty}{\rightarrow} \sum_{n=0}^{\infty} \frac{C_n}{E_n - E} + \sum_{n}^{E_n < E} C_n \frac{e^{(E-E_n)T}}{E - E_n}$$

BAD!!!

- How to subtract the divergent part?
- Needs all C_n to recover the imaginary part!

Bypassing the determination of $\rho(E)$

Setting the external spatial momentum to zero we can simplify the needed correlators

$$\rho^{00}_{ud}(q) = q^2 \rho_L(q^2) \ , \qquad \frac{1}{3}\sum_{i=1}^{3} \rho^{ii}_{ud}(q) = q^2 \rho_T(q^2)$$

and rewrite the decay rate in terms of a **convolution integral** over energy

$$R^{(\tau)}_{ud} \propto \int_0^\infty \mathrm{d}E \left[K_\mathrm{T}\left(\frac{E}{m_\tau}\right) E^2 \rho_\mathrm{T}(E^2) + K_\mathrm{L}\left(\frac{E}{m_\tau}\right) E^2 \rho_L(E^2) \right]$$

having incorporated back the leptonic term into the kernels:

$$K_\mathrm{L}(x) \equiv \frac{1}{x}\left(1 - x^2\right)^2 \theta(1 - x) \ , \qquad K_\mathrm{T}(x) \equiv \left(1 + 2x^2\right) K_L(x)$$

The explicit determination of the spectral density is avoided: only its **convolution** with K is needed

Inverse Laplace transform for $K(E)$

We don't need to determine the spectral densities, just its convolution with $K(E)$

$$R_{ud}^{(\tau)} = 12\pi S_{\mathrm{EW}} \frac{|V_{ud}|^2}{m_\tau^3} \int_0^\infty \mathrm{d}E \left[K_{\mathrm{T}}\left(\frac{E}{m_\tau}\right) E^2 \rho_{\mathrm{T}}(E^2) + K_{\mathrm{L}}\left(\frac{E}{m_\tau}\right) E^2 \rho_L(E^2) \right]$$

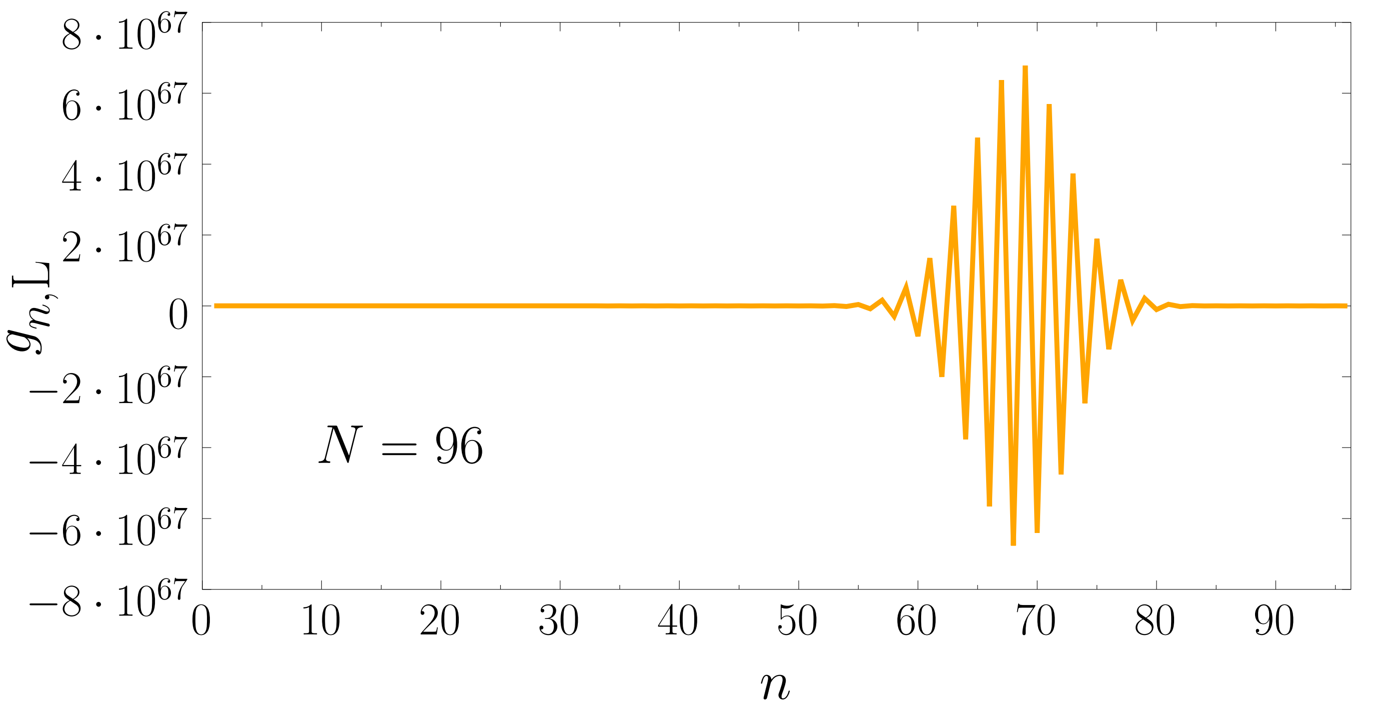
Both kernels can be expressed in terms of an approximated inverse Laplace transform

$$R_{ud}^{(\tau)} \propto \sum_{I\in\{L,T\}} \int_0^\infty dE K_I \left(\frac{E}{m_\tau}\right) E^2 \rho_I(E^2) = \sum_{I\in\{L,T\}} \int_0^\infty dE \sum_{n=1}^{N} g_{n,I} e^{-nE} E^2 \rho_I(E^2)$$

with coefficients $g_{n,I}$ determined imposing minimal L^2 distance

...will this be numerically stable…?

...not quite...



- This is due to the θ function, needed to cut out energy larger than m_τ
- Using this coefficients introduces uncontrolled error in the rate
- We need to smooth out the sharp θ function

Smoothing out the kernels

HLT framework to solve the inverse Laplace transform in the context of Lattice simulations
[M.Hansen, A.Lupo, N.Tantalo, PRD 96 (2017)]

Smoothing the kernel defining the rate has been tested in the inclusive semilptonic decays
[P.Gambino et al., PRL 125 (2020)]

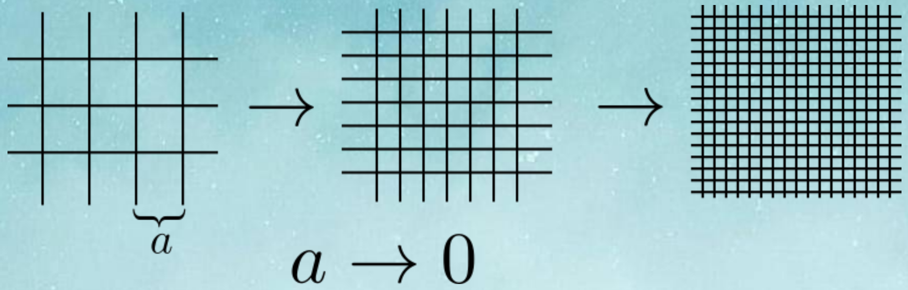
We introduce a smeared kernel in terms of a smoothed θ, and compute the rate

$$\theta(x) \to \Theta_\sigma(x) \equiv \frac{1}{1+e^{-x/\sigma}} \,, \qquad\qquad \lim_{\sigma \mapsto 0} \Theta_\sigma(x) = \theta(x)$$

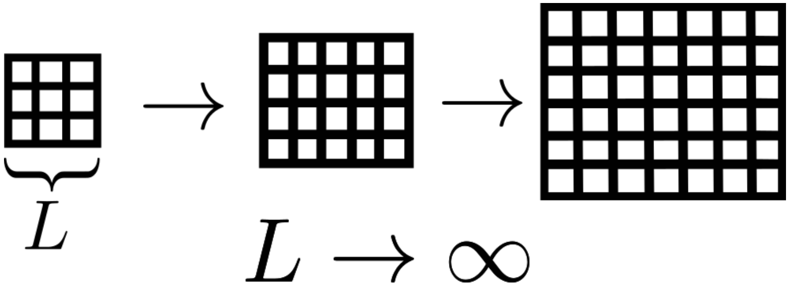
$$R_{ud}^{(\tau,\mathrm{I})}(\sigma) = 12\pi S_{\mathrm{EW}} \frac{|V_{ud}|^2}{m_\tau^3} \int_0^\infty \mathrm{d}E\, K_{\mathrm{I}}^{\sigma}\left(\frac{E}{m_\tau}\right) E^2 \rho_{\mathrm{I}}(E^2)$$

Infinite volume/continuum extrapolations

It is convenient to carry out the extrapolations to the continuum at fixed σ



The infinite volume limit involves no power-law correction: $R_{ud}^{(\tau,\mathrm{I})}(\sigma,\infty) - R_{ud}^{(\tau,\mathrm{I})}(\sigma,L) \sim \mathcal{O}(L^{-\infty})$



The extrapolation to zero smoothing is done at the end, exploiting the asymptotic relation

$$R_{ud}^{(\tau,\mathrm{I})}(\sigma) = R_{ud}^{(\tau,\mathrm{I})} + C\sigma^4 + \mathcal{O}(\sigma^6)$$

Numerical setup

Four physical-point Nf=2+1+1 Wilson-clover-twisted mass ensembles

Lattice spacing: $a \in [0.057\ \mathrm{fm} - 0.080\ \mathrm{fm}]$ Iwasaki action

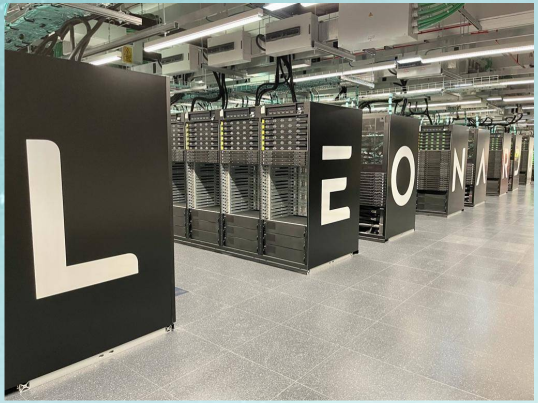
$L \sim 5.1\ \mathrm{fm}$ and $L \sim 7.6\ \mathrm{fm}$ to control Finite Size Effects

Automatic improved $\mathcal{O}(a)$ improvement of observables

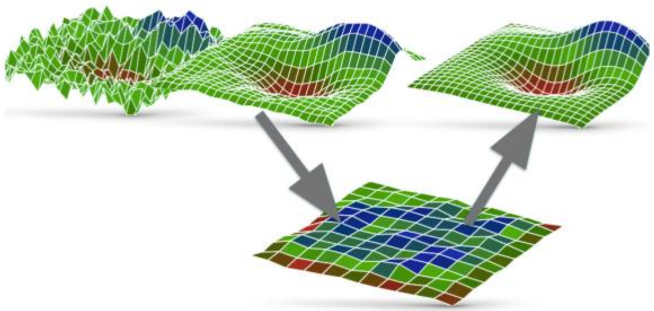
ID	V/a^4	a fm	L fm	m_π GeV
B64	$64^3 \cdot 128$	0.07957(13)	5.09	0.1352(2)
B96	$96^3 \cdot 192$	0.07957(13)	7.64	0.1352(2)
C80	$80^3 \cdot 160$	0.06821(13)	5.46	0.1349(3)
D96	$96^3 \cdot 192$	0.05692(12)	5.46	0.1351(3)

Numerical aspects

- Multi GPU simulations on Leonardo
- ISCRA & EuroHPC competitive grants
- INFN contingent of computing time



Multigrid algorithms to accelerate solution of the Dirac Equation



Optimally implemented in QUDA library by community + hardware vendor effort

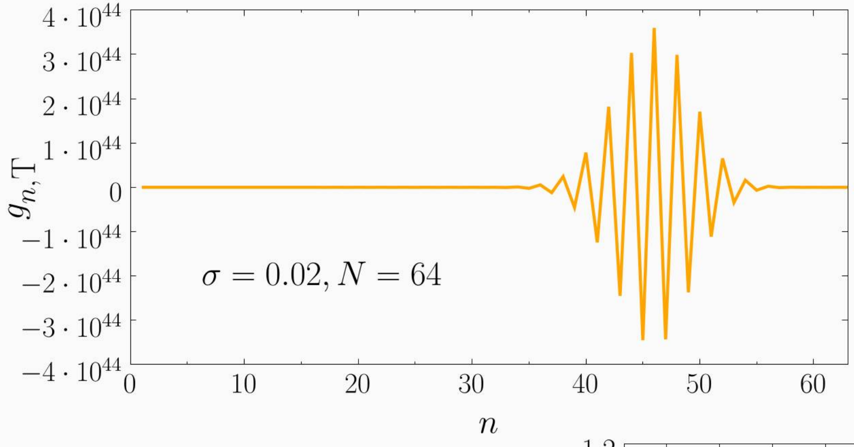
NVIDIA CUDA

...in a fully GPU-ported suite of ~100k lines of C++17 code

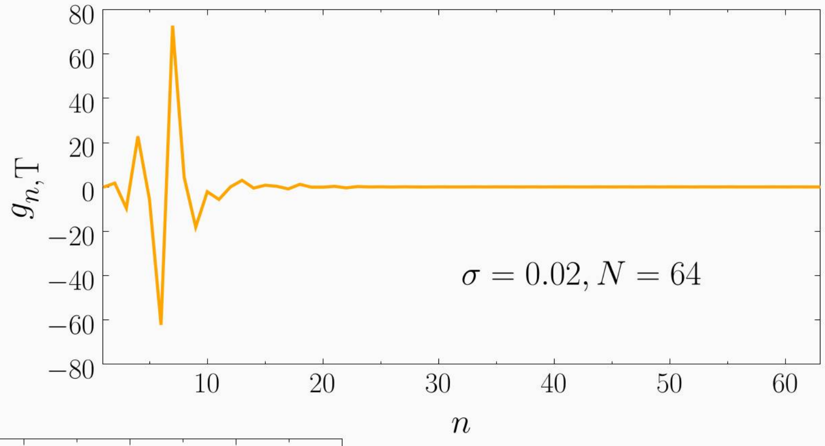
Scaling up to hundreds of GPUs

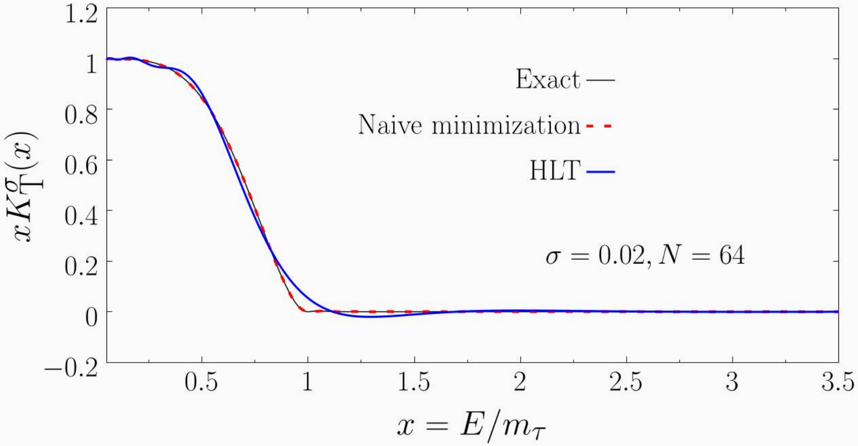
HLT method stabilization

Naive minimization



HLT method ($\alpha = 4, r_{\max} = 4$, B64)

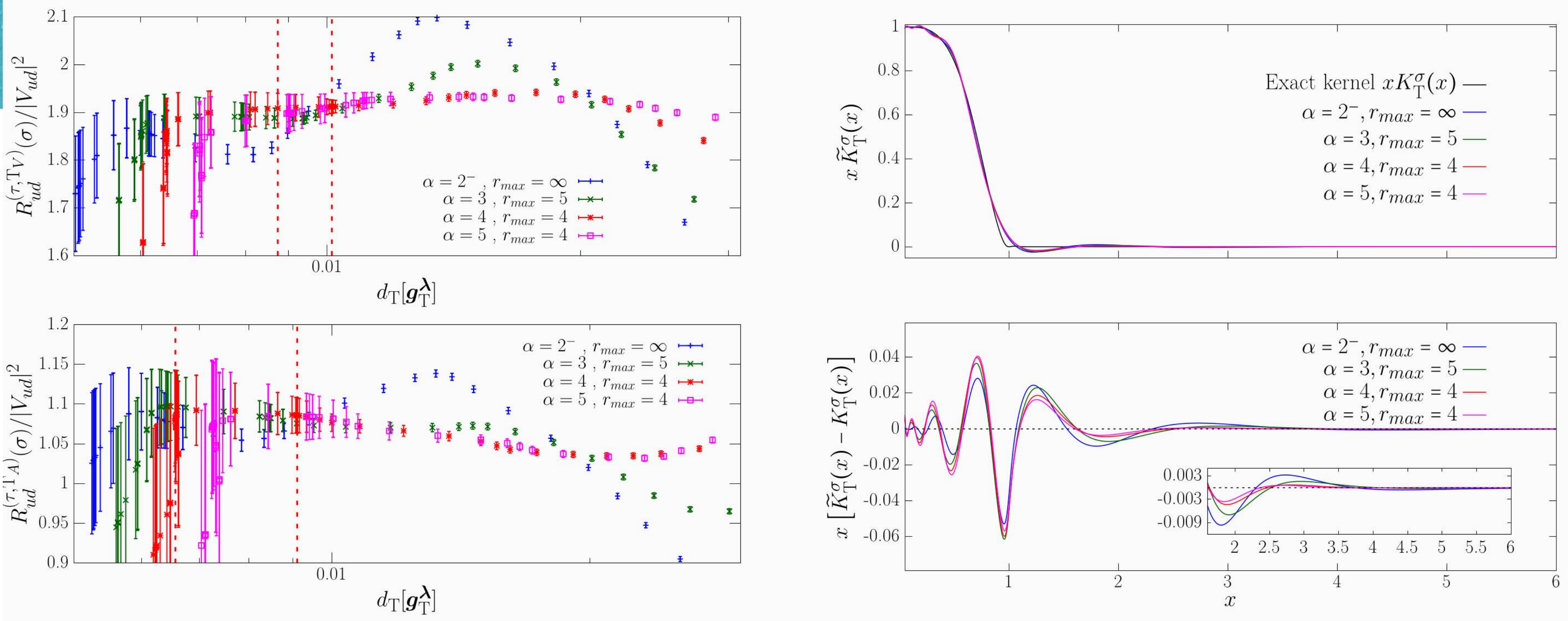




Stability analysis

Tuning of λ and tweaking the parameters $\{\alpha, E_{\min}, r_{\max}\}$ probes the stability of the reconstruction

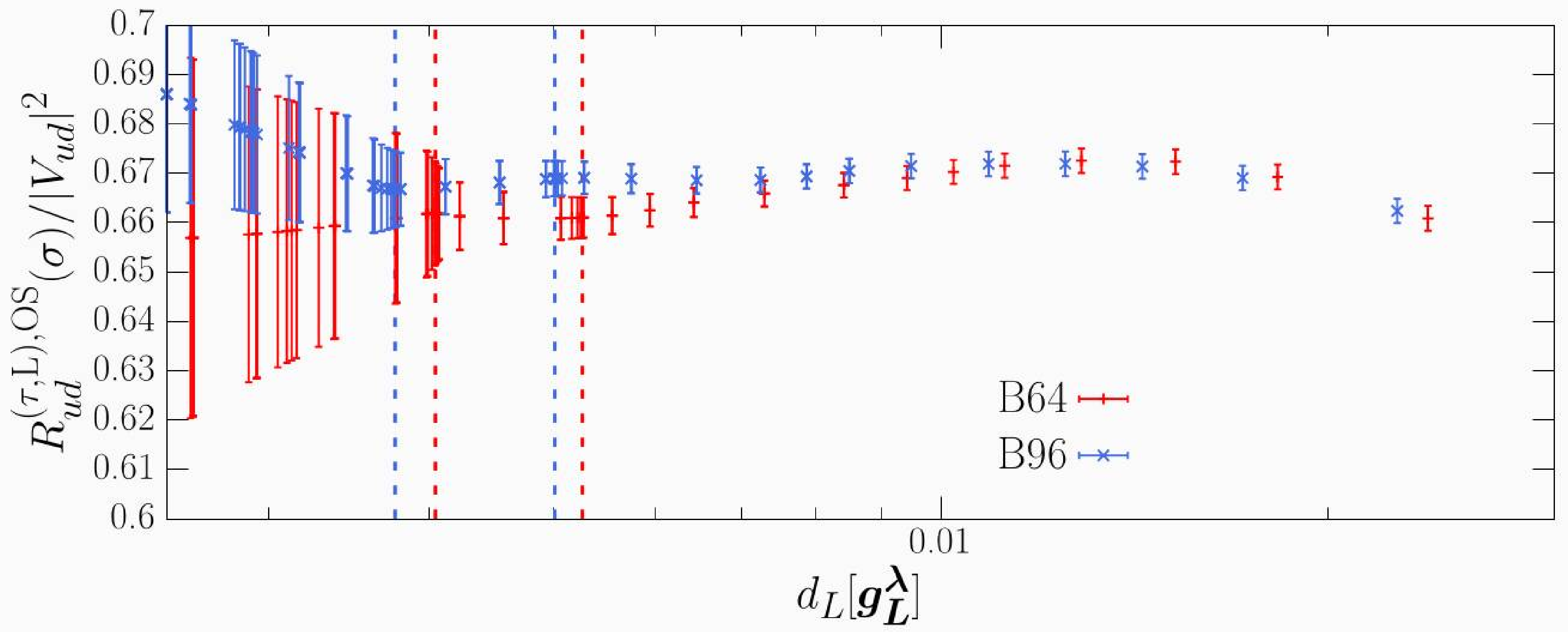
$$d_{\mathrm{I}}[g_{\mathrm{I}}^{\lambda}] \equiv \sqrt{A_{\mathrm{I}}^{0}[g_{\mathrm{I}}^{\lambda}]/A_{\mathrm{I}}^{0}[0]}$$



Comparison of different volumes

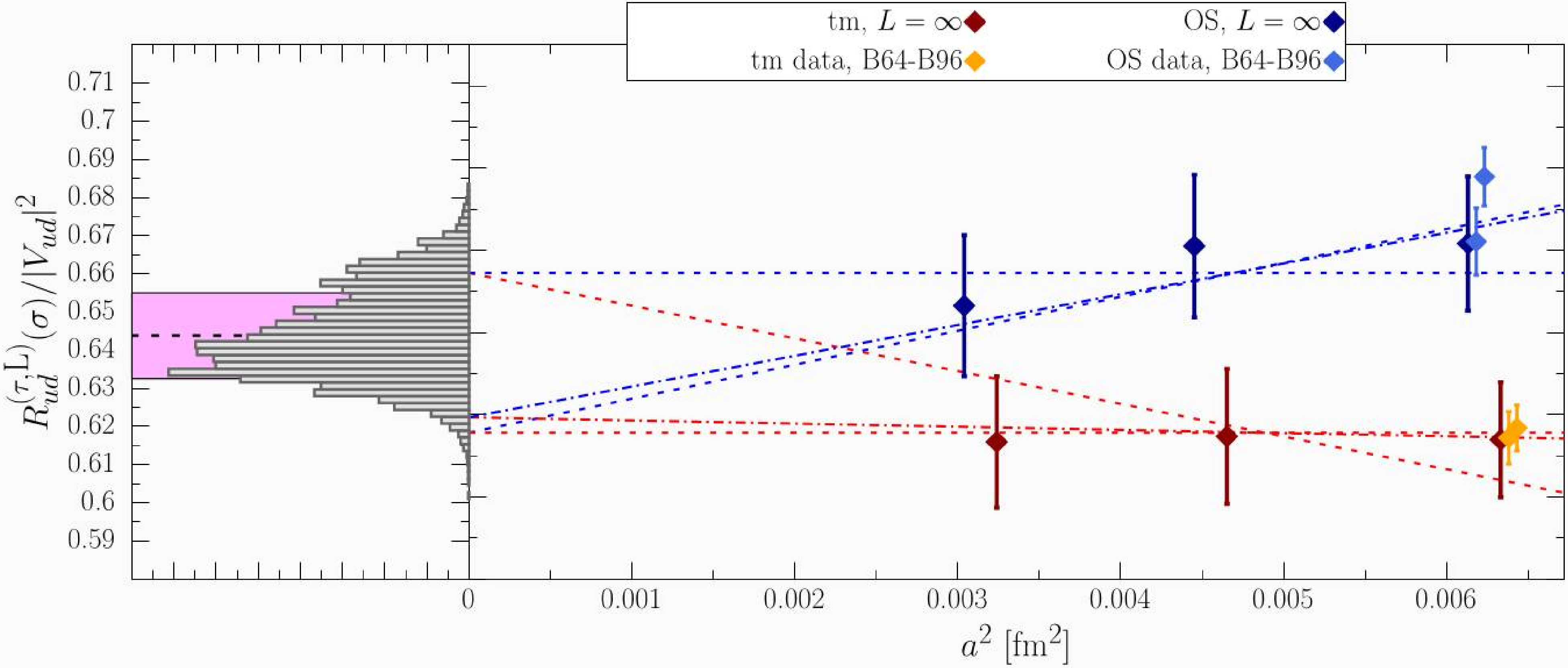
Systematic of FSE estimated by comparing two volumes

Observed difference added as an error

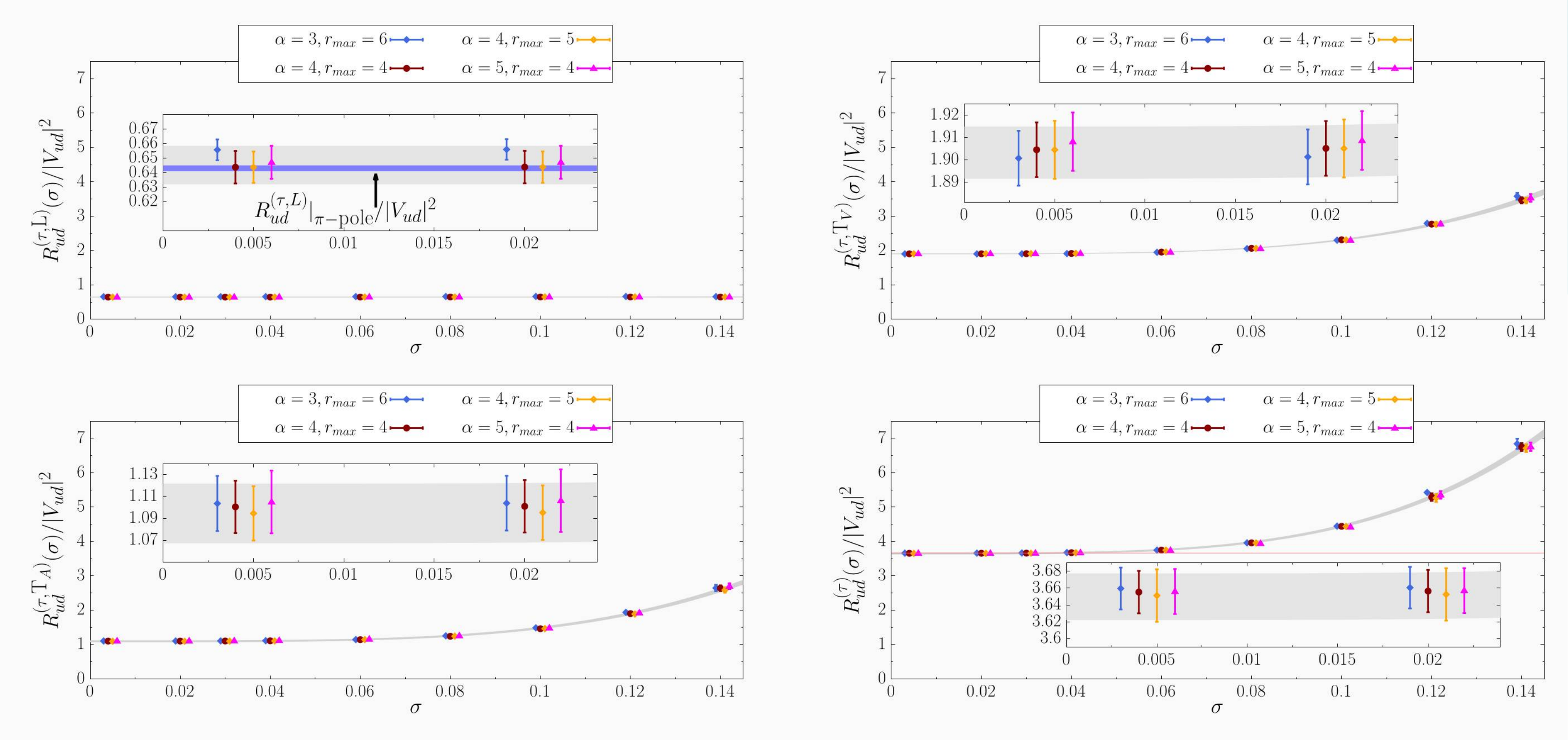


Continuum limit extrapolation

- Combination of two regularizations (OS and TM)
- Several fit ansatz variations

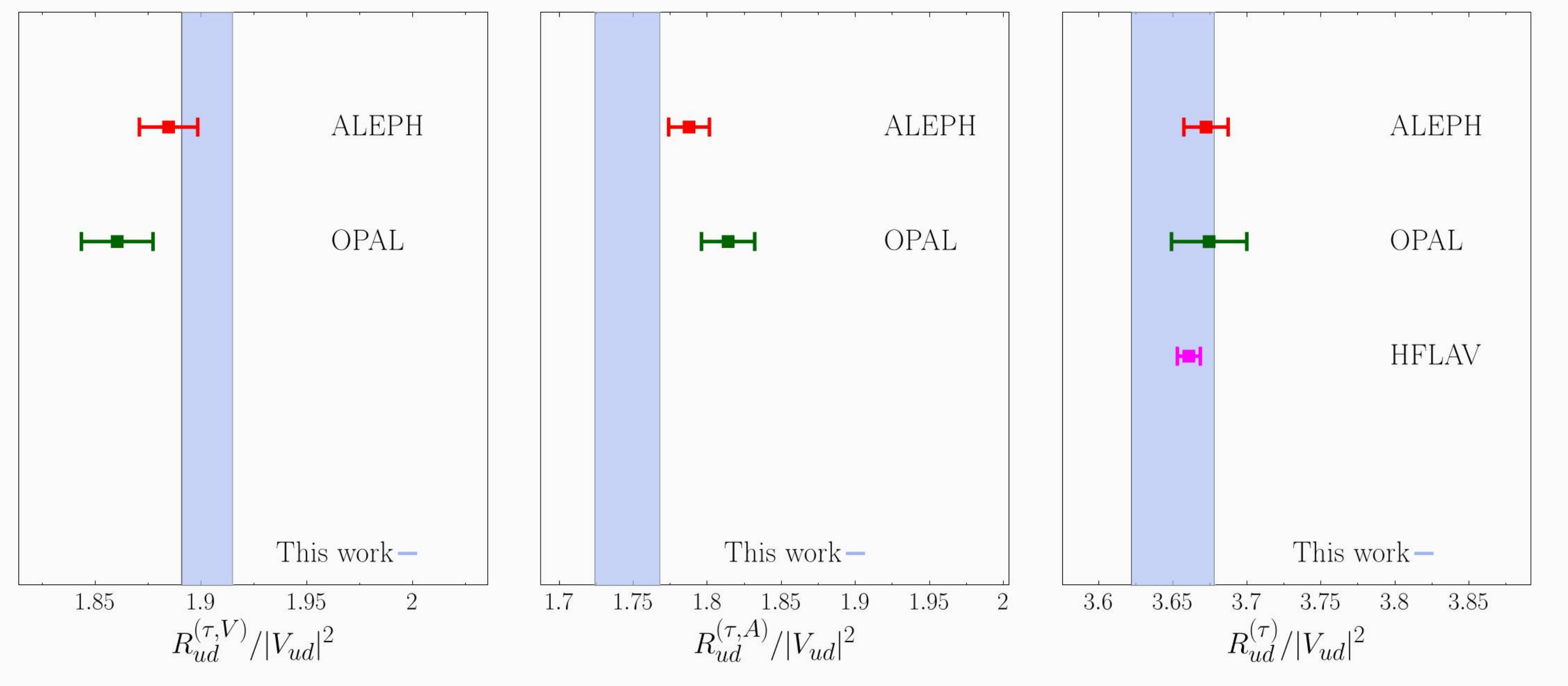


Extrapolating to vanishing smearing



Extrapolation to zero smearing well described by σ^4 asymptotic prediction

Comparison with the experiments



Using the average value $R_{ud}^{(\tau)}(\mathrm{HFLAV}) = 3.471(7)$ we obtain $|V_{ud}| = 0.9752(39)$

Good agreement with $|V_{ud}| = 0.97373(31)$ from superallowed β-decay.

The *us* channel

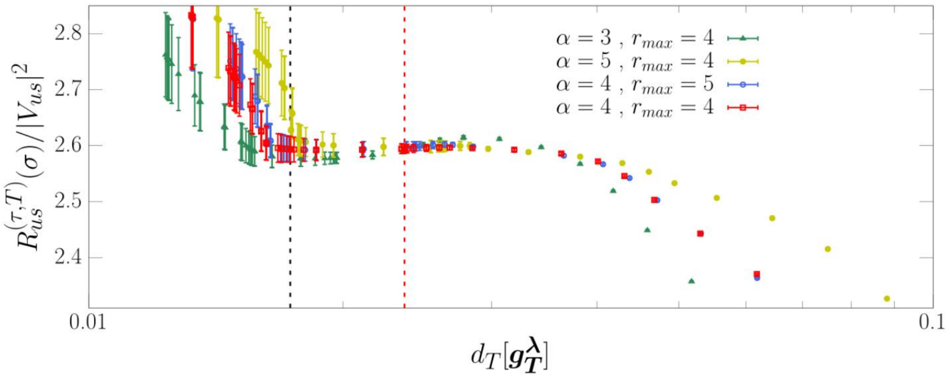
PHYSICAL REVIEW LETTERS **132,** 261901 (2024)

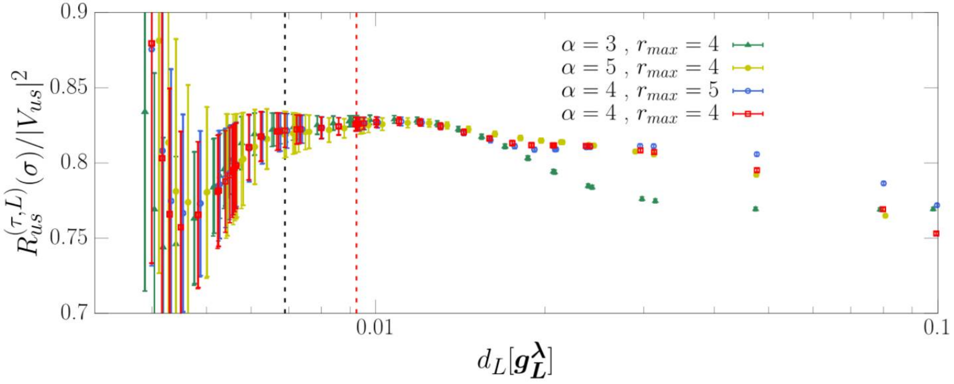
Inclusive Hadronic Decay Rate of the τ Lepton from Lattice QCD: The $\bar{u}s$ Flavor Channel and the Cabibbo Angle

Constantia Alexandrou,[1,2] Simone Bacchio,[2] Alessandro De Santis,[3] Antonio Evangelista,[3] Jacob Finkenrath,[4] Roberto Frezzotti,[3] Giuseppe Gagliardi,[5] Marco Garofalo,[6] Bartosz Kostrzewa,[7] Vittorio Lubicz,[8] Simone Romiti,[6] Francesco Sanfilippo,[5] Silvano Simula,[5] Nazario Tantalo,[3] Carsten Urbach,[6] and Urs Wenger[9]

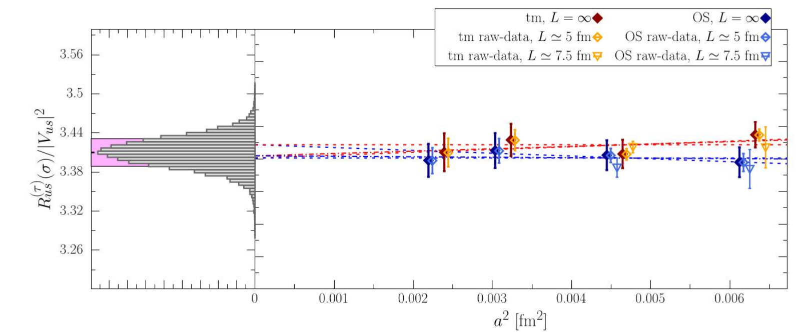
(Extended Twisted Mass Collaboration)

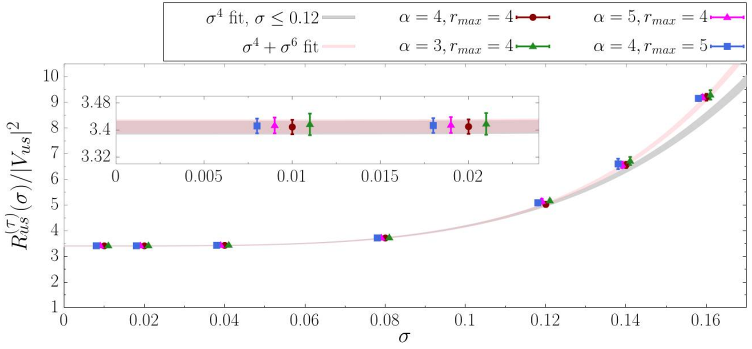
$$R_{us}(\sigma) = 12\pi S_{EW} |V_{ud}|^2 \int_0^\infty \frac{dE E^2}{m_\tau^3} \left\{ K_T^\sigma\left(\frac{E}{m_\tau}\right) \rho_T(E^2) + K_L^\sigma\left(\frac{E}{m_\tau}\right) \rho_L(E^2) \right\}$$

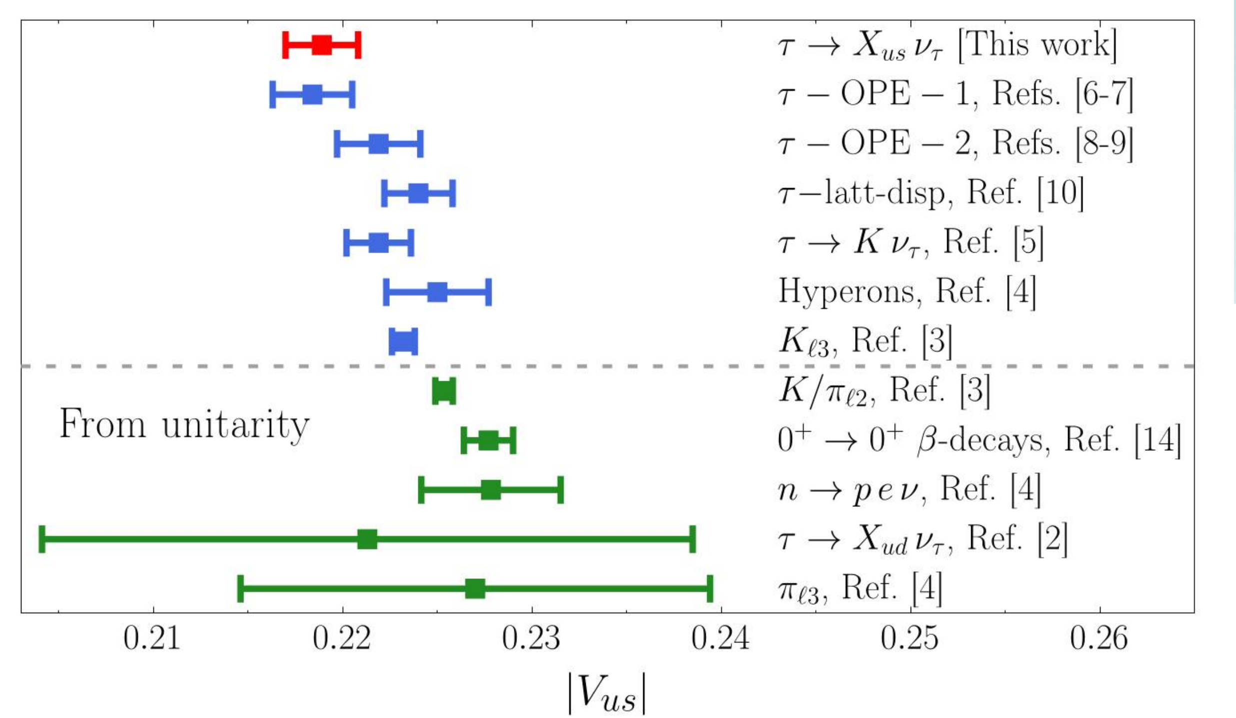




The *us* channel



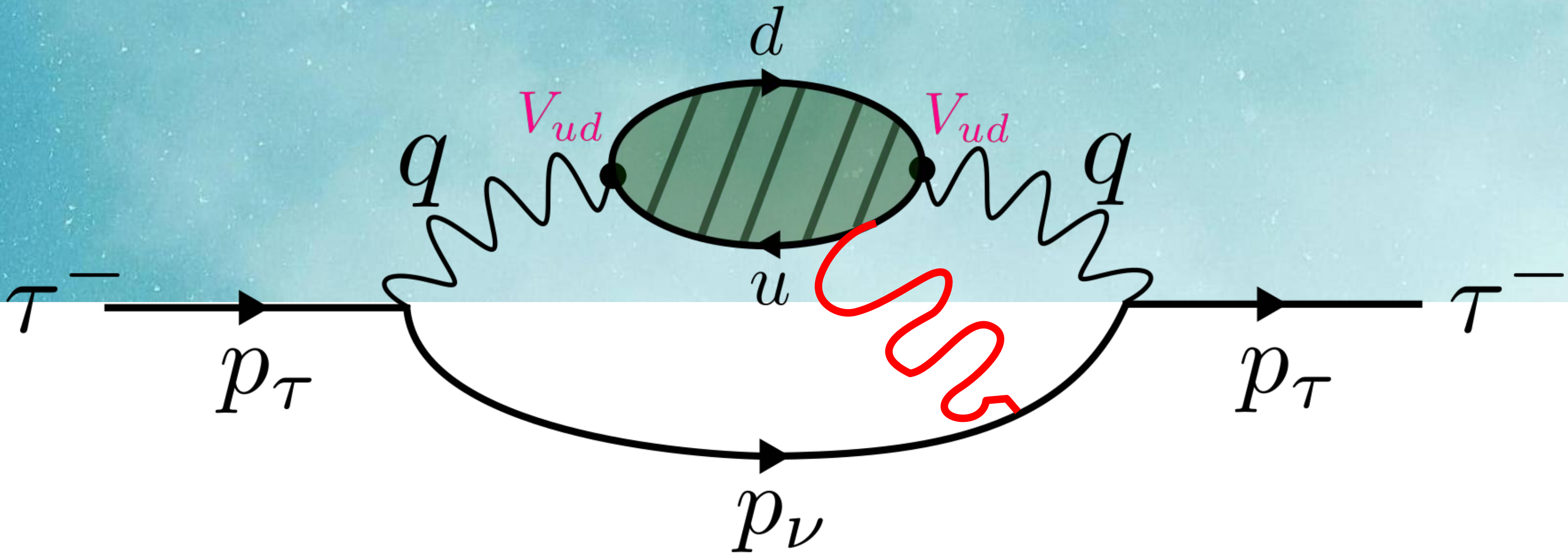




Alternative determination, not (yet) competitive with $K_{\ell 3}$ but independent

Some tension is visible indeed...

Perspective: Isospin Breaking



Needs to include the photon exchange between hadrons and tau

A few percent effect, but very important to perform effective prediction

Work in progress...

Thank you!