RG RUNNING AND MIXING FOR $\Delta F = 2$ FOUR-FERMION OPERATORS IN χSF SCHEMES

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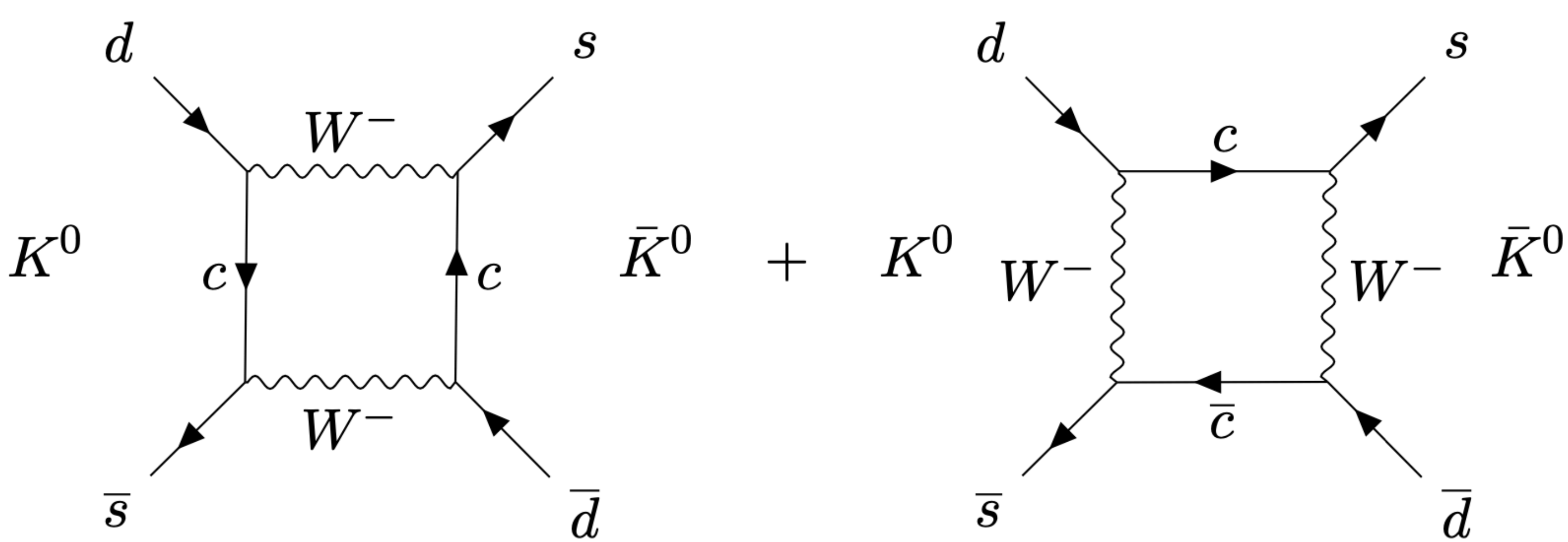
MOTIVATIONS

- **Future goal: accurate evaluation of the CP-violating angle δ of the CKM matrix**
- **$K^0 - \bar{K}^0$ oscillations in the SM are sensitive to loop effects, and so to BSM contributions**



Indirect investigation of CP violation: ε parameter

$$\varepsilon^{\text{theor}}(\delta) = \frac{A(K_L \to (\pi\pi)_{I=0})}{A(K_S \to (\pi\pi)_{I=0})} \propto \tilde{U}(\mu)\langle \bar{K}^0 \,|\, \mathbf{Q}(\mu) \,|\, K^0 \rangle F(\delta)$$

To be evaluated non-perturbatively

Comparing $\varepsilon^{\text{theor}}$ with its experimental estimate we obtain

in the SM:

1. **new estimate of the phase δ**
2. **non-perturbative uncertainties**

beyond the SM:

1. **δ kept to the current estimate**
2. **bounds to BSM contributions**

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$K^0 - \bar{K}^0$ OSCILLATIONS

- **Effective Hamiltonian for K oscillations:**

SM: $H_{\rm eff}^{\Delta S=2} = \tilde{U}_1 \mathbf{Q}_1$

Only one relevant operator

BSM: $H_{\rm eff}^{\Delta S=2} = \sum_{i=1}^{5} \tilde{U}_i \mathbf{Q}_i + \sum_{i=1}^{3} \tilde{U}'_i \tilde{\mathbf{Q}}_i$

An operator basis $\{Q_i\}$

- **Transition amplitudes are calculated with** $\langle \bar{K}^0 | H_{\rm eff}^{\Delta S=2} | K^0 \rangle$

- **The renormalisation introduces an energy-scale in the matrix elements and in the Wilson coefficients:**

$$\langle \bar{K}^0 | H_{\rm eff}^{\Delta S=2} | K^0 \rangle^{\rm parity}_{\rm cons.} = \tilde{U}_i(\mu) \langle \bar{K}^0 | \mathbf{Q}_i(\mu) | K^0 \rangle$$

FOUR-FERMION OPERATORS RENORMALISATION

- **Dirac operators:**

$$\Gamma_V = \gamma_\mu\,, \quad \Gamma_A = \gamma_\mu\gamma_5\,, \quad \Gamma_s = \mathbf{1}\,,$$
$$\Gamma_P = \gamma_5\,, \quad \Gamma_T = \sigma_{\mu\nu}\,, \quad \Gamma_{\tilde{T}} = \frac{1}{2}\epsilon_{\mu\nu\rho\tau}\sigma_{\rho\tau}\,,$$

- **Four-Fermion Operators:**

$$\mathcal{O}^{\pm}_{[\Gamma_1\Gamma_2 \pm \Gamma_2\Gamma_1]} := \mathcal{O}^{\pm}_{[\Gamma_1\Gamma_2]} \pm \mathcal{O}^{\pm}_{[\Gamma_2\Gamma_1]}\,,$$
$$\mathcal{O}^{\pm}_{[\Gamma_1\Gamma_2]} := \frac{1}{2}\Big[\big(\bar{\psi}_1\Gamma_1\psi_2\big)\big(\bar{\psi}_3\Gamma_2\psi_4\big) \pm \big(\bar{\psi}_1\Gamma_1\psi_4\big)\big(\bar{\psi}_3\Gamma_2\psi_2\big)\Big]$$

- **Parity-odd operators:**

$$\mathcal{Q}_1^{\pm} = \mathcal{O}^{\pm}_{[VA+AV]} \quad \mathcal{Q}_3^{\pm} = \mathcal{O}^{\pm}_{[PS-SP]} \quad \mathcal{Q}_5^{\pm} = -2\mathcal{O}^{\pm}_{[T\tilde{T}]}$$
$$\mathcal{Q}_2^{\pm} = \mathcal{O}^{\pm}_{[VA-AV]} \quad \mathcal{Q}_4^{\pm} = \mathcal{O}^{\pm}_{[PS+SP]}$$

- **Renormalisation pattern of parity-odd operators:**

$$\begin{pmatrix} \bar{\mathcal{Q}}_1^{\pm} \\ \bar{\mathcal{Q}}_2^{\pm} \\ \bar{\mathcal{Q}}_3^{\pm} \\ \bar{\mathcal{Q}}_4^{\pm} \\ \bar{\mathcal{Q}}_5^{\pm} \end{pmatrix} = \begin{pmatrix} \mathcal{Z}_{11} & 0 & 0 & 0 & 0 \\ 0 & \mathcal{Z}_{22} & \mathcal{Z}_{23} & 0 & 0 \\ 0 & \mathcal{Z}_{32} & \mathcal{Z}_{33} & 0 & 0 \\ 0 & 0 & 0 & \mathcal{Z}_{44} & \mathcal{Z}_{45} \\ 0 & 0 & 0 & \mathcal{Z}_{54} & \mathcal{Z}_{55} \end{pmatrix}^{\pm} \begin{pmatrix} \mathcal{Q}_1^{\pm} \\ \mathcal{Q}_2^{\pm} \\ \mathcal{Q}_3^{\pm} \\ \mathcal{Q}_4^{\pm} \\ \mathcal{Q}_5^{\pm} \end{pmatrix}$$

EVOLUTION MATRICES

- **Evolution matrices between two scales:** $\bar{Q}_i(\mu_2) = U_{ij}(\mu_2, \mu_1)\bar{Q}_j(\mu_1)$

- **Evolution matrices down to a scale $\hat{U}(\mu)$:** $\mathbf{U}(\mu_2, \mu_1) =: \left[\hat{\mathbf{U}}(\mu_2)\right]^{-1} \hat{\mathbf{U}}(\mu_1)$

- **Problem: for $N_\mathrm{f} = 3$ (and 30) two eigenvalues of γ_0/β_0 satisfy the resonance condition $\lambda_i - \lambda_j = 2$, making it impossible to adopt the usual definition**

$$\tilde{\mathbf{U}}(\mu) = \left[\frac{\bar{g}^2(\mu)}{4\pi}\right]^{-\frac{\gamma_0}{2b_0}} \mathbf{W}(\mu)$$

$\mathbf{W}(\mu)$ DEFINITION ISSUES

- **W(μ) solves the equation**

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu}\mathbf{W} = [\gamma, \mathbf{W}] - \beta\left(\frac{\gamma}{\beta} - \frac{\gamma_0}{g\beta_0}\right)\mathbf{W}$$

must the perturbative expansion

$$\mathbf{W} = \mathbf{1} + g^2\mathbf{J}_1 + g^4\mathbf{J}_2 + \ldots$$

therefore the NLO $\mathbf{J}_1$ solves

$$2\mathbf{J}_1 - \left[\frac{\gamma_0}{\beta_0}, \mathbf{J}_1\right] = \frac{\beta_1}{\beta_0}\frac{\gamma_0}{\beta_0} - \frac{\gamma_1}{\beta_0}$$

- **Non-invertible system in the 2|3 submatrix if $N_\mathrm{f} = 3$**
- **Method still feasible for nonresonant submatrices**

WILSON COEFFICIENTS FROM THE POINCARÉ-DULAC THEOREM

- $\mathbf{A}(g) \equiv \frac{\gamma(g)}{\beta(g)}$ can be set [2013.16220v3] in its canonical form

$$\mathbf{A}^{\mathrm{can}}(g) = \frac{1}{g}\left(\Lambda + g^2\mathbf{N}\right)$$

Λ → diagonal; $\mathbf{N}$ → upper-diagonal

through a change of operator basis $\mathbf{Q}' = \mathbf{S}(g)\mathbf{Q}$, with

$$\mathbf{S}(g) = \left(\mathbf{1} + g^2\mathbf{H}_2 + g^4\mathbf{H}_4 + \dots\right)\mathbf{S}_{\mathrm{D}}$$

- The evolution operator can be evaluated and then rotated back to the original operator basis:

$$\hat{\mathrm{U}}(u) = \mathbf{S}_{\mathrm{D}}^{-1} u^{-\mathbf{N}/2} u^{-\Lambda/2}\left(\mathbf{1} + u\mathbf{H}_2 + u^2\mathbf{H}_4\right)\mathbf{S}_{\mathrm{D}} \qquad u \equiv g^2$$

WILSON COEFFICIENTS CORRESPONDENCE

Wilson coefficient (new approach)

$$\hat{\mathrm{U}}(u) = \mathbf{S}_\mathrm{D}^{-1} u^{-\mathrm{N}/2} u^{-\Lambda/2} \left(\mathbf{1} + u\mathbf{H}_2 + u^2\mathbf{H}_4\right) \mathbf{S}_\mathrm{D}$$

With no resonance, $\mathrm{N} = 0$

$$\tilde{\mathrm{U}}(u) = u^{-\gamma_0/2\beta_0} \left(\mathbf{1} + u\mathbf{J}_1 + u^2\mathbf{J}_2\right)$$

Wilson coefficient (traditional approach)

In absence of resonances, the Wilson coefficients correspond exactly, with the dictionary

$$\mathbf{J}_k = \mathbf{S}_D^{-1}\mathbf{H}_{2k}\mathbf{S}_D$$

Perturbation theory has been fixed...

STEP-SCALING FUNCTIONS

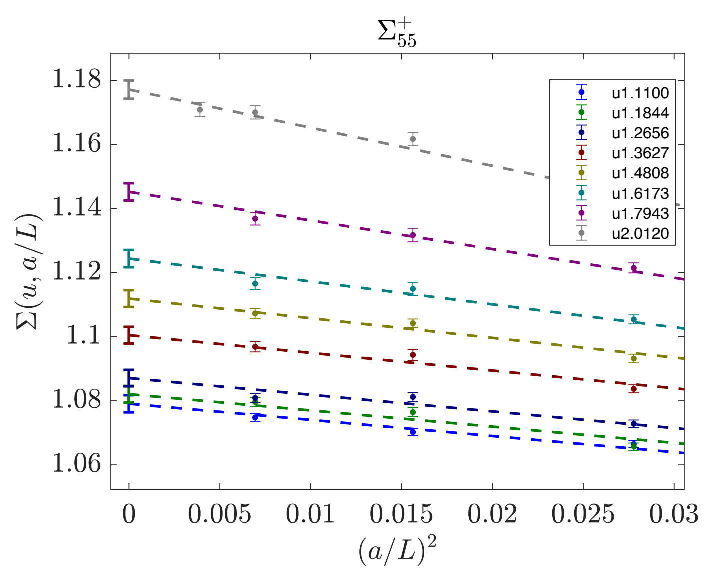
- **Non-perturbative evolution from the step-scaling functions (SSF):**

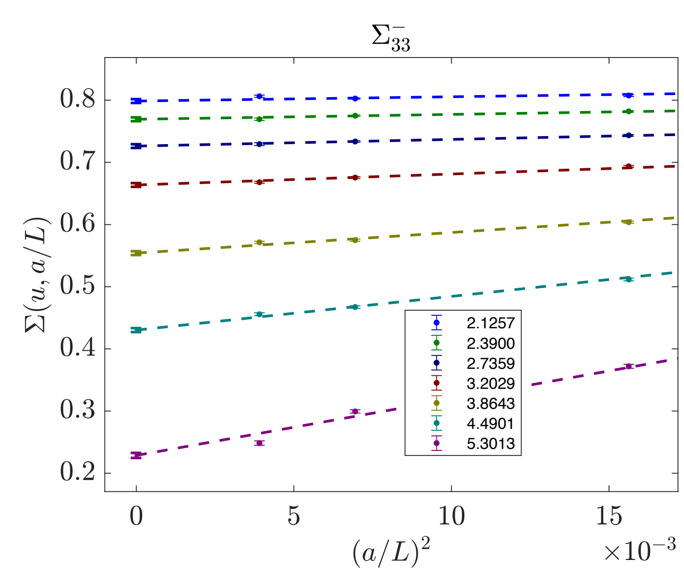
$$\boldsymbol{\sigma}(u) := \mathbf{U}(\mu/2,\mu)\Big|_{\bar{g}^2(\mu)=u} \longrightarrow \mathbf{U}(u_{\text{had}} \equiv u_1, u_{\text{pt}} \equiv u_N) = \boldsymbol{\sigma}(u_1)\ldots\boldsymbol{\sigma}(u_N)$$

- **Discrete step-scaling functions:** $\boldsymbol{\Sigma}\left(g_0^2, \frac{a}{L}\right) := \boldsymbol{\mathcal{Z}}\left(g_0^2, \frac{a}{2L}\right)\left[\boldsymbol{\mathcal{Z}}\left(g_0^2, \frac{a}{L}\right)\right]^{-1}$

Obtained on the lattice, lattice spacing dependence!

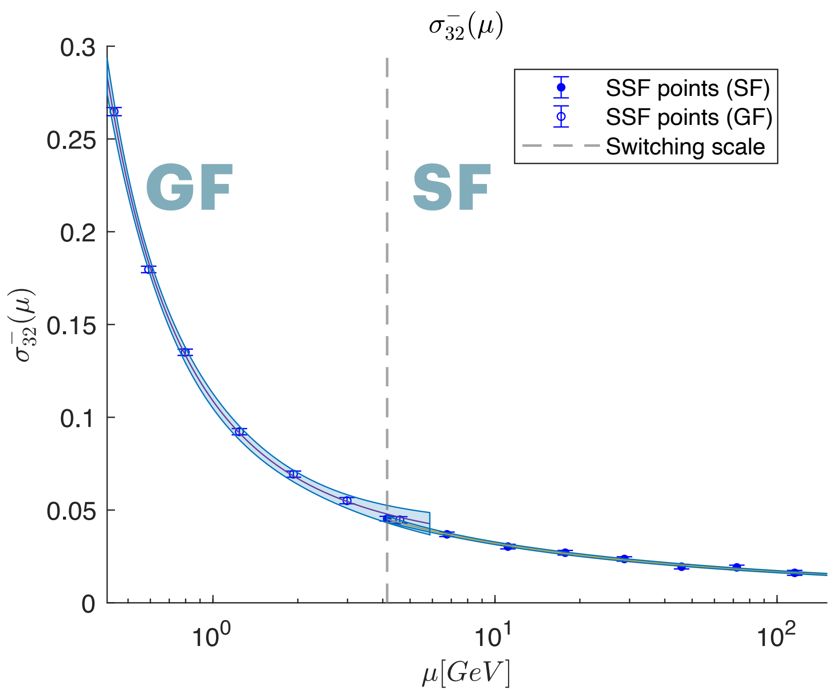
- **Continuum extrapolations fitting data obtained for different lattice dimensions**

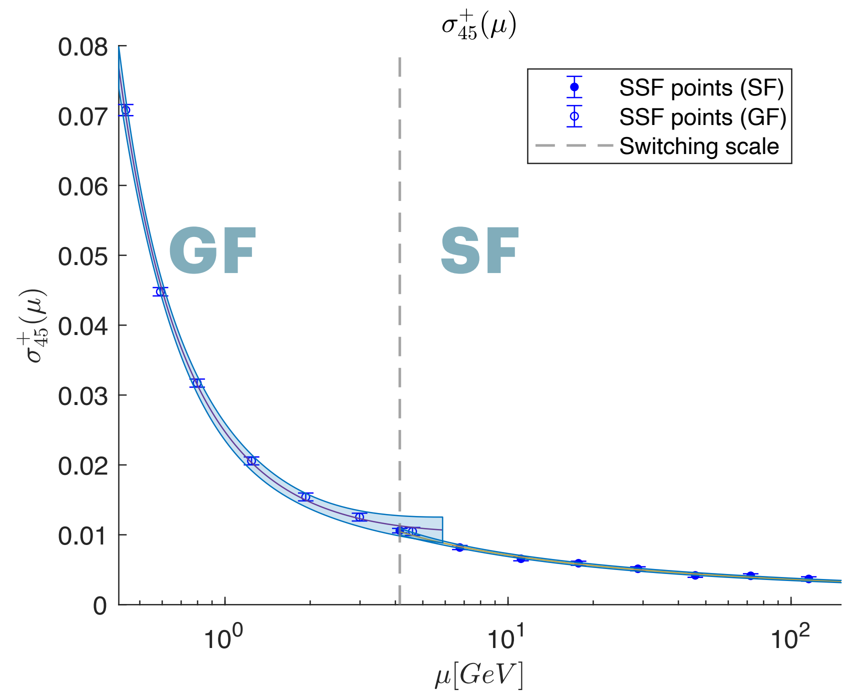




EVOLUTION MATRICES FROM SSF

- **Continuum Step-scaling functions $\sigma(\mu)$ as functions of the scale:**





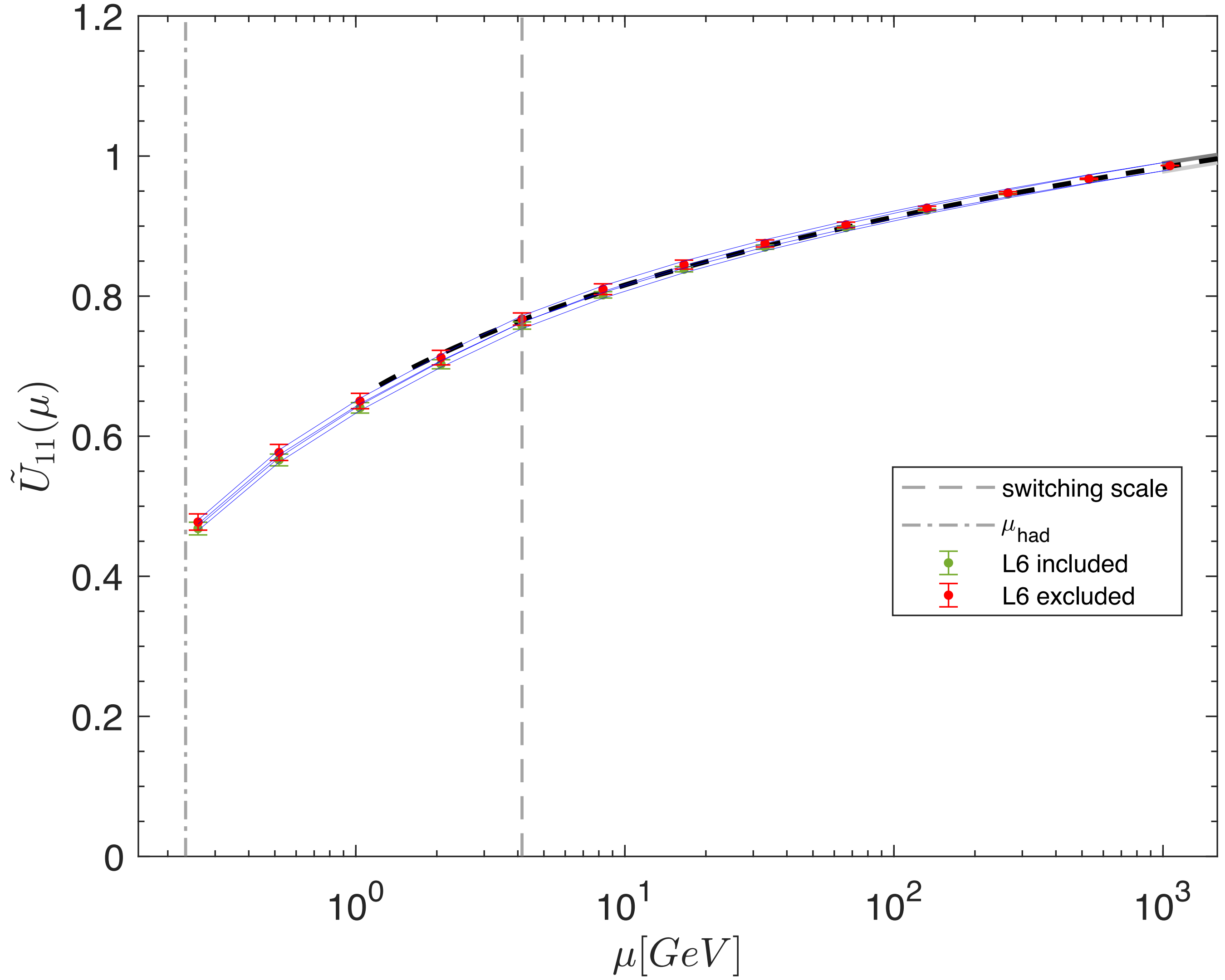
- **N couplings evaluated as** $$u_n(\mu) \equiv u(\mu_n) \equiv u(2^n\mu_0)$$

- **Evolution operator between u_1 and u_N:** $$\mathrm{U}(u_1, u_N) = \boldsymbol{\sigma}(u_1)\ldots\boldsymbol{\sigma}(u_N)$$

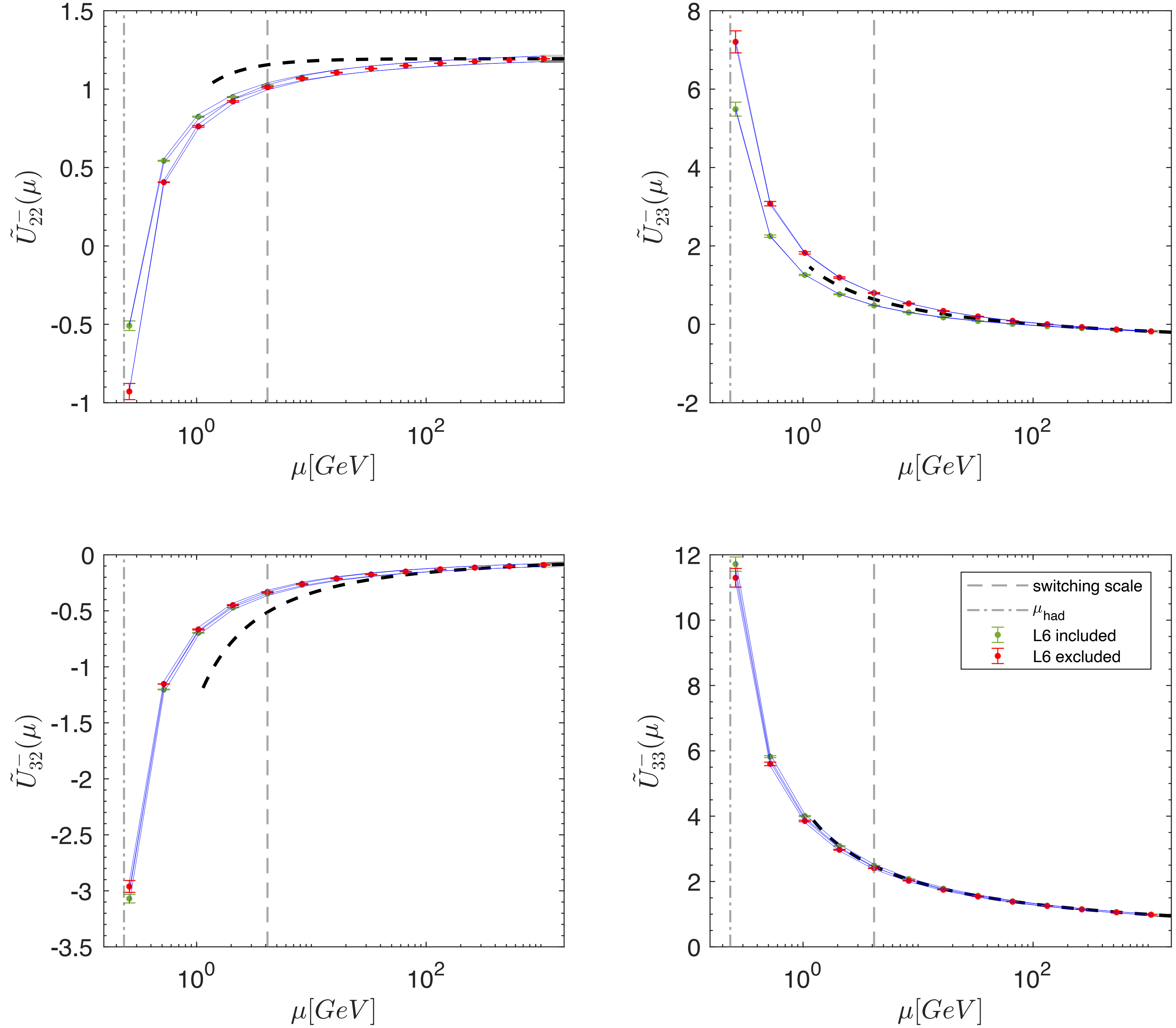
NP RUNNING: SM 1|3 INDICES

Final non-perturbative running:

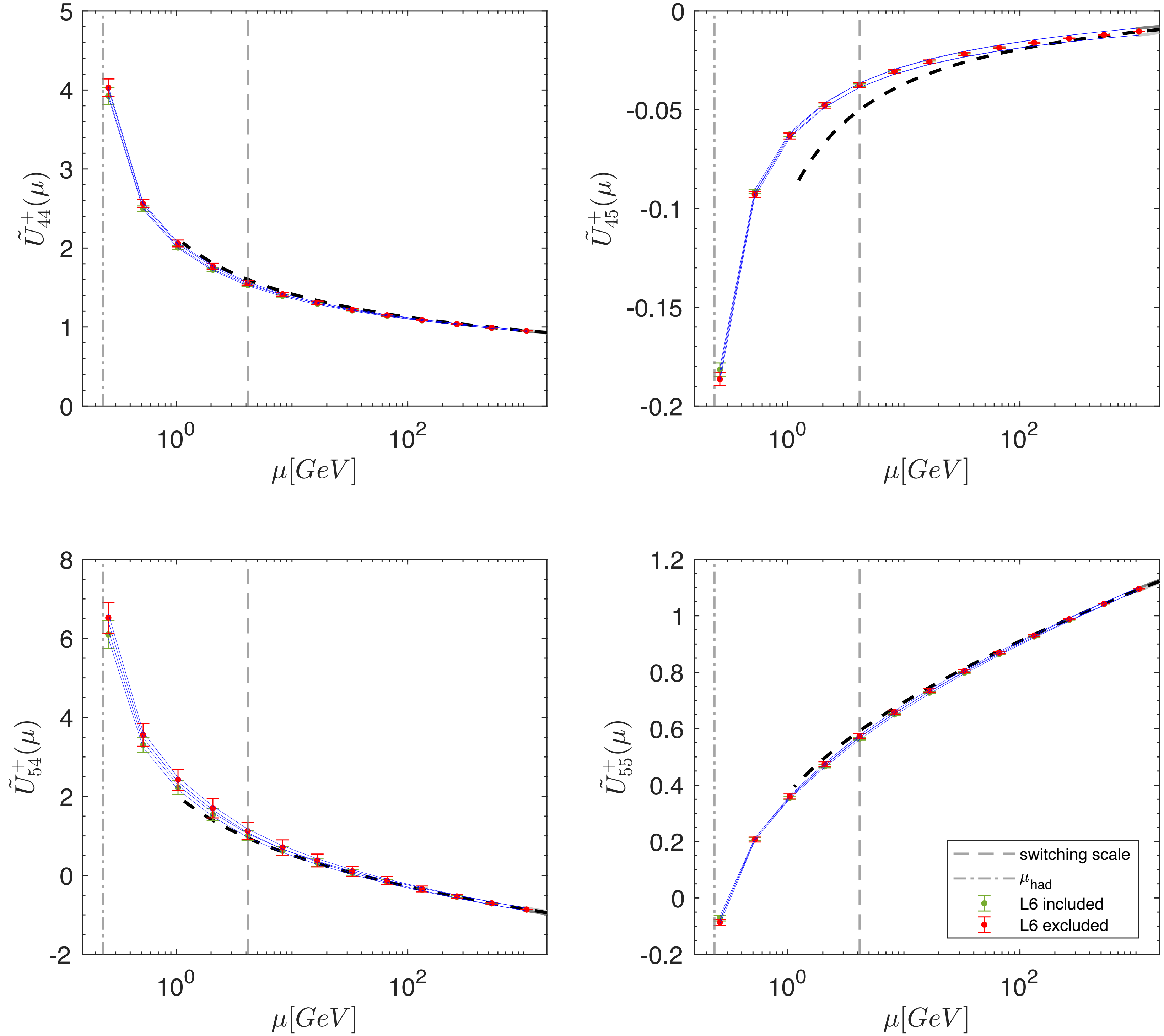
$$\hat{\mathbf{U}}(u) = \mathbf{S}_{\mathrm{D}}^{-1} \exp\left(-\frac{\mathbf{\Lambda}}{2}\ln u_{\mathrm{pt}}\right) \exp\left(-\frac{\mathbf{N}_2}{2}\ln u_{pt}\right) \mathbf{s}_n(g)\, \mathbf{S}_{\mathrm{D}}[\mathbf{U}(u,u_{\mathrm{pt}})]^{-1}$$



NP RUNNING: BSM 2|3 INDICES



NP RUNNING: BSM 4|5 INDICES



FUTURE DEVELOPMENTS

- **Preliminary analysis computing non-perturbatively the running down to a scale $\mathcal{O}(200\,\mathrm{MeV})$ incorporating the NLO in the perturbative part of the study and solving the problem that appears for $N_f = 3$.**

In order to evaluate the value of $\varepsilon^{\mathrm{theor}}$ we are planning to perform the following computations:

- **bare tm-QCD matrix elements estimated on Wilson gauge configurations (CLS);**
- **a non-perturbative evaluation of the renormalisation constants in the χSF at the lattice spacings of the CLS ensembles.**

Thank you

BACKUP

[FLAG2021]

$$B_i \propto \langle \mathbf{Q}_i(\mu = 3\text{GeV}) \rangle$$

Collaboration	Ref.	N_f	publication status	continuum extrapolation	chiral extrapolation	finite volume	renormalization	running	B_2	B_3	B_4	B_5
ETM 15	[55]	2+1+1	A	★	○	○	★	a	0.46(1)(3)	0.79(2)(5)	0.78(2)(4)	0.49(3)(3)
RBC/UKQCD 16	[60]	2+1	A	○	○	○	★	b	0.488(7)(17)	0.743(14)(65)	0.920(12)(16)	0.707(8)(44)
SWME 15A	[58]	2+1	A	★	○	★	○†	–	0.525(1)(23)	0.773(6)(35)	0.981(3)(62)	0.751(7)(68)
SWME 14C	[508]	2+1	C	★	○	★	○†	–	0.525(1)(23)	0.774(6)(64)	0.981(3)(61)	0.748(9)(79)
SWME 13A‡	[495]	2+1	A	★	○	★	○†	–	0.549(3)(28)	0.790(30)	1.033(6)(46)	0.855(6)(43)
RBC/ UKQCD 12E	[502]	2+1	A	■	○	★	★	b	0.43(1)(5)	0.75(2)(9)	0.69(1)(7)	0.47(1)(6)
ETM 12D	[59]	2	A	★	○	○	★	c	0.47(2)(1)	0.78(4)(2)	0.76(2)(2)	0.58(2)(2)

- **Inconsistencies between different estimates**
- **Some results refer to perturbative renormalisation**

THE XSF

- **In the continuum we map the SF into the χSF with a chiral rotation:**

$$\psi' = R\left(\frac{\pi}{2}\right)\psi , \quad \bar{\psi}' = \bar{\psi}R\left(\frac{\pi}{2}\right) , \quad R(\alpha) = e^{\frac{i}{2}\alpha\gamma_5\tau^3}$$

- **Correspondence between correlation functions in the SF and χSF:**

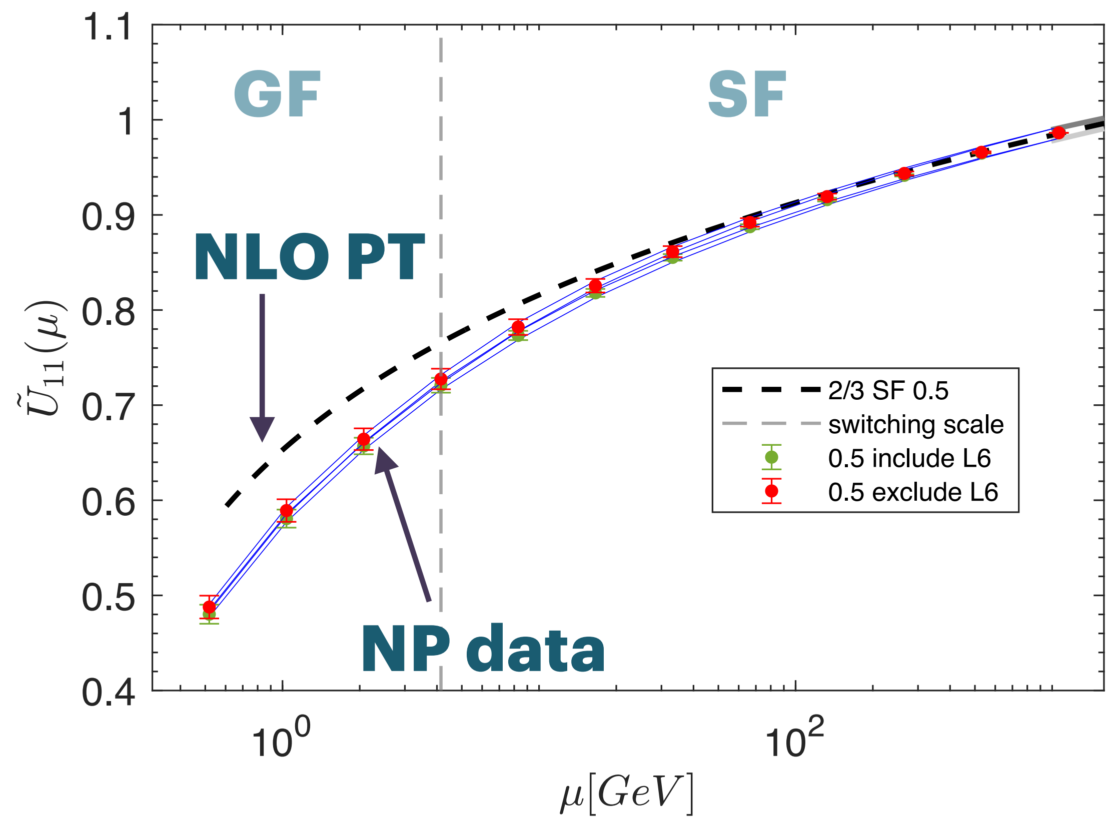
$$\langle O[\psi,\bar{\psi}]\rangle_{\mathrm{SF}}^{\mathrm{cont}} = \left\langle O\left[R\left(\pi/2\right)\psi, \bar{\psi}R\left(\pi/2\right)\right]\right\rangle_{\chi\mathrm{SF}}^{\mathrm{cont}}$$

- **The boundary rotation removes $\mathcal{O}(a)$ effects in the observables!**

$$\left\langle O_{\mathrm{even}}\right\rangle_{\mathrm{c}} = \left\langle O_{\mathrm{even}}\right\rangle_{\mathrm{c}}^{\mathrm{cont}} + \mathcal{O}(a^2)$$

NP FEATURES

- **Running evaluation with 3 quark flavours in the sea down to $\sim 4\mathrm{GeV}$ with SF coupling;**
- **Running evaluation down to $\sim 200 MeV$ with Gradient Flow (GF) coupling;**
- **New theoretical formulation of the operator running and mixing in the perturbative regime for $N_f = 3$.**



A difference often observed between PT and non-PT results at 3GeV (the scale at which matrix elements in FLAG are renormalised), could be relevant in the estimate of quantities like $\varepsilon(\delta)$

ERROR ESTIMATES

- **The non-perturbative running is finally given by**

$$\hat{\mathbf{U}}(u) = \mathbf{S}_{\mathrm{D}}^{-1} \exp\left(-\frac{\boldsymbol{\Lambda}}{2} \ln u_{\mathrm{pt}}\right) \exp\left(-\frac{\mathbf{N}_2}{2} \ln u_{pt}\right) \mathbf{s}_n(g)\, \mathbf{S}_{\mathrm{D}} [\mathbf{U}(u, u_{\mathrm{pt}})]^{-1}$$

- **Statistical errors: propagation from the fits**
- **Systematic errors (guess) :**
  - **Lack of knowledge on higher orders of the anomalous dimension**
  - **Differences arising if L=6 is included or not**