Embedding interacting dark energy models in trace-free Einstein gravity

Marco de Cesare

Scuola Superiore Meridionale (SSM), Napoli, Italy

INFN, Sezione di Napoli, Italy

talk based on:

MdC, E. Wilson-Ewing, Phys.Rev. D 106 (2022), 023527 [arxiv:2112.12701]

Y. Zhai, MdC, C. van de Bruck, E. Di Valentino, E. Wilson-Ewing [arxiv:2503.15659]

New Frontiers in Theoretical Physics—XXXVIII Convegno Nazionale di Fisica Teorica

May 20th 2025

Energy-momentum conservation in trace-free gravity?

In general relativity, energy-momentum conservation is a consequence of the field equations:

$$G_{ab} = \kappa T_{ab} \implies \nabla^a T_{ab} = 0$$

In trace-free Einstein gravity, the Einstein field equations are replaced by

$$R_{ab} - \frac{1}{4} R\, g_{ab} = \kappa \left(T_{ab} - \frac{1}{4} T\, g_{ab} \right)$$

Using the Bianchi identities, we no longer get a conservation law:

$$\kappa \nabla^c T_{ac} = \frac{1}{4} \nabla_a (R + \kappa T) \equiv J_a$$

energy-momentum transfer

$$\boxed{\text{In trace-free Einstein gravity, energy-momentum conservation for matter fields becomes an independent assumption, and } \nabla^a T_{ab} \neq 0 \text{ in general.}}$$

[Ellis, van Elst, Murugan, Uzan (2011); Josset, Perez, Sudarsky (2017)]

Effective dark energy from energy-momentum non-conservation

The field equations can be recast in the form of effective Einstein equations

$$R_{ab} - \frac{1}{2} R\, g_{ab} + \left(\Lambda_\infty + \int_\ell J \right) g_{ab} = \kappa T_{ab}$$

To ensure that the geometry only depends on the point (and not on the path ℓ !) the energy-momentum transfer must be integrable

$$dJ = 0 \underset{\text{(locally)}}{\Longrightarrow} J_a = -\nabla_a Q$$

energy-momentum transfer potential

Then, the field equations read as $G_{ab} = \kappa\,(T_{ab} + \tilde{T}_{ab})$ with $\kappa\,\tilde{T}_{ab} = Q\, g_{ab}$

(dark energy with $w = -1$)

By construction, the total energy-momentum tensor is conserved $\nabla^a (T_{ab} + \tilde{T}_{ab}) = 0$

IDE models with $w = -1$ and integrable transfer are embedded in trace-free gravity

[Perez, Sudarsky, Josset, Wilson-Ewing (2020)]

Large scale instabilities?

A large class of interacting dark energy models suffers from well-known non-adiabatic large scale instabilities for super-horizon modes

[Valiviita, Majerotto, Maartens (2008)]

Goal 1: show that we can build instability-free IDE models that can be embedded in trace-free Einstein gravity.

Goal 2: Test one such model with observational data and search for evidence of new physics

Transfer models

In this context, a model consists of a specific proposal for the transfer potential Q .
This does not follow from the field equations and therefore must be prescribed separately.

There are proposals for the possible microscopic origin of energy-momentum non-conservation due to spacetime discreteness at the Planck scale.

[Perez, Sudarsky (2017)]

Energy non-conservation in quantum gravity

Due to spacetime discreteness, diffeomorphism invariance may not hold on all scales.

$\Longrightarrow$ we may have $\nabla^a T_{ab} \neq 0$ at a fundamental level.

'Energy diffusion' resulting from the interaction of matter with a 'granular' spacetime structure at the Planck scale

It is argued that massive fields with spin can probe such a structure, and feel a friction force

[Perez, Sudarsky (2017)]

$$u^c \nabla_c u^a = \alpha \frac{m}{M_{Pl}^2} \mathrm{sign}(s \cdot \xi) R\, s^a \qquad (\, \xi \equiv \partial_t \,)$$

Assuming a gas of such particles at thermal equilibrium, one obtains for the current:

$$J_a = \kappa \nabla^c T_{ca} \approx 2\pi\alpha\hbar \frac{T}{M_{Pl}^2} R^2 (dt)_a$$

NB: both in Sorkin's causal sets approach and in Jacobson's spacetime thermodynamics one actually recovers the trace-free Einstein equations, which are consistent with the above.

Energy non-conservation in quantum gravity

Such diffusion effects may explain the observed value of the cosmological constant

[Perez, Sudarsky (2017); Perez, Sudarsky, Bjorken (2018)]

Furthermore, friction due to spacetime granularity may also spin down BHs.

[Perez, Sudarsky (2019)]

Diffusion effects can potentially resolve the Hubble tension.

The background dynamics has been analyzed in two models, where diffusion kicks in after recombination:

1) sudden transfer from $\rho_{\rm m}$ to Λ ; 2) anomalous decay of $\rho_{\rm m}$

[Perez, Sudarsky, Wilson-Ewing (2020)]

Comparison of some diffusion models with CMB data

(deviations from standard cosmology enter at the background level)

[Landau, Benetti, Perez, Sudarsky (2022)]

A full Bayesian analysis including the effects of diffusion on perturbations is still missing

Cosmological perturbations in unimodular gravity

We focus on the scalar sector, because that's where the instability found in [Valiviita, Majerotto, Maartens '08] shows up, and also where modifications to GR play a role.

(scalar) metric perturbations:

$$ds^2 = a(\eta)^2 \left\{ -(1+2\phi)d\eta^2 + 2B_{,i}dx^i d\eta + \left[(1-2\psi)\delta_{ij} + 2E_{,ij}\right] dx^i dx^j \right\}$$

matter perturbations:

$$\delta T^{a}_{A\ b} = (\delta\rho_A + \delta p_A)\bar{u}^a \bar{u}_b + (\bar{\rho}_A + \bar{p}_A)\left(\delta u^a_A \bar{u}_b + \bar{u}^a \delta u^A_b\right) + \delta p_A\, \delta^a_{\ b} + \pi^{a}_{A\ b}$$

dark energy perturbations (recall $\rho_x = -\kappa^{-1} Q$) : $\quad \delta \tilde{T}^a_{\ b} = -\,\delta\rho_x\, \delta^a_{\ b}$

Background evolution

$$\mathscr{H}^2 = \frac{\kappa}{3} a^2 \left(\bar{\rho} + \bar{\rho}_x\right) \qquad \bar{\rho}_A' + 3\mathscr{H}(\bar{\rho}_A + \bar{p}_A) = \kappa^{-1}\bar{Q}_A' \qquad \sum_A Q^A = Q$$

$$\bar{\rho}' + 3\mathscr{H}(\bar{\rho} + \bar{p}) = \kappa^{-1} Q' = -\,\bar{\rho}_x'$$

NB: no dynamical equation for $\bar{Q}$.

Cosmological perturbations in unimodular gravity

perturbed field equations (longitudinal gauge $E = B = 0$):

$$-\triangle\psi + 3\mathscr{H}(\psi' + \mathscr{H}\phi) = -\frac{\kappa}{2}a^2(\delta\rho + \delta\rho_x)\ , \quad \mathscr{H}\phi + \psi' = -\frac{\kappa}{2}a^2(\bar{\rho} + \bar{p})v\ , \quad \psi - \phi = \kappa\, a^2\pi\ ,$$

$$\psi'' + \mathscr{H}(\phi' + 2\psi') + \left(2\frac{a''}{a} - \mathscr{H}^2\right)\phi = \frac{\kappa}{2}a^2\left(\delta p - \delta\rho_x + \frac{2}{3}\triangle\pi\right)$$

continuity equations: $\theta_A \equiv \triangle(v_A + B)\ ,\ \ \delta_A \equiv \dfrac{\delta\rho_A}{\bar{\rho}_A}$

$$\delta_A' + \left(3\mathscr{H}(c_{sA}^2 - w_A) + \kappa^{-1}\frac{\bar{Q}_A'}{\bar{\rho}_A}\right)\delta_A + (w_A + 1)\theta_A - 3(w_A + 1)\psi' + 3\mathscr{H}(c_{sA}^2 - c_{aA}^2)\left[3\mathscr{H}(1 + w_A) - \kappa^{-1}\frac{\bar{Q}_A'}{\bar{\rho}_A}\right]\frac{\theta_A}{k^2} = \kappa^{-1}\frac{\delta Q_A'}{\bar{\rho}_A}\ ,$$

$$\theta_A' + \mathscr{H}(1 - 3c_{sA}^2)\theta_A - k^2\phi - \frac{k^2}{1 + w_A}c_{sA}^2\delta_A + \frac{2k^4}{3(1 + w_A)\bar{\rho}_A}\pi_A = \kappa^{-1}\left[\frac{k^2}{\bar{\rho}_A(1 + w_A)}\delta Q_A - \left(\frac{1 + c_{sA}^2}{1 + w_A}\right)\frac{\bar{Q}_A'}{\bar{\rho}_A}\theta_A\right]$$

Note that there is no additional equation for $\delta_x = \delta Q/\bar{Q}$!

θ_x is not defined for a fluid with $w = -1$.

We can identify it with the velocity perturbation for the total fluid $\theta_x \equiv \theta$.

A simple transfer model

In order to solve the equations, we also need to model the energy-momentum transfer.

We choose a model where the violation of energy-momentum conservation is only due to dark matter and the transfer potential is

$$Q = -\Lambda_\infty + \epsilon\,\kappa\,\rho_c$$

cosmological constant at future infinity (integration constant)

energy density of CDM

With this, dark energy evolves adiabatically w.r.t. CDM $\zeta_x = \zeta_c \implies S_{xc} = 0$

The CDM (non)conservation equations then read as:

background: $(1-\epsilon)\bar{\rho}_c' + 3\mathscr{H}\bar{\rho}_c = 0$

perturbations: $(1-\epsilon)\delta_c' + \theta_c - 3\psi' = 0\ , \quad \theta_c' + \left(\frac{1-4\epsilon}{1-\epsilon}\right)\mathscr{H}\theta_c - k^2\phi - \epsilon\,k^2\delta_c = 0$

Due to diffusion, CDM effectively behaves as a fluid with $c_{s,\mathrm{eff}}^2 = w_{\mathrm{eff}} = \dfrac{\epsilon}{1-\epsilon}$

Effective sound speed of DM

CDM effectively behaves as a fluid with $c^2_{s,\mathrm{eff}} = w_{\mathrm{eff}} = \dfrac{\epsilon}{1-\epsilon}$

To avoid gradient instabilities, we shall require $\epsilon \geq 0$.

This condition also tells us that **energy flows from dark matter to dark energy**
(the effective cosmological 'constant' must be increasing)

The characterisation of DM with parameters $c^2_{s,\mathrm{eff}}$, w_{eff} resembles
the *generalized dark matter* phenomenological model (in the inviscid case)
[Hu (1998)] [Kopp, Skordis, Thomas (2016)]

Radiation dominated era

We solve the gravity+matter equations during RDE at tight coupling for super-horizon modes, taking into account baryons, photons, neutrinos in addition to DE and CDM

Constant mode: $\psi = \left(1 + \frac{2}{5}\Omega_\nu\right)\phi + \frac{4}{15}S_{\nu\gamma}\Omega_\nu(1-\Omega_\nu)$

Decaying mode: $\psi \approx (k\eta)^n$ $\quad n \approx -3 + \frac{8}{5}\Omega_\nu$ $\quad \phi = (1 - \frac{8}{5}\Omega_\nu)\,\psi$

Matter perturbations θ_A , δ_A , σ_ν are also well behaved in both cases

DE density perturbations are also bounded and decreasing in magnitude:

$$\delta_x = -3\frac{\epsilon}{1-\epsilon}\left(\frac{\bar{\rho}_c}{\bar{\rho}_x}\right)\psi$$

No instability!

Matter dominated era

Assume only interacting CDM and DE as matter fields.

There is no anisotropic stress, so $\phi = \psi$.

$$\mathscr{H} = \left(\frac{1-\epsilon}{1+2\epsilon}\right)\frac{2}{\eta} \qquad\qquad c^2_{s,\mathrm{eff}} = \frac{\epsilon}{1-\epsilon}$$

$$\psi'' + 3\mathscr{H}(1+c^2_{s,\mathrm{eff}})\psi' + \left(2\mathscr{H}' + \mathscr{H}^2(1+3c^2_{s,\mathrm{eff}})\right)\psi + c^2_{s,\mathrm{eff}}k^2\psi = 0$$

Super-horizon modes ($c_{s,\mathrm{eff}}\, k\eta \ll 1$)

$$\psi = \psi_0 + C\,\eta^{-\left(\frac{5+3c^2_{s,\mathrm{eff}}}{1+3c^2_{s,\mathrm{eff}}}\right)} \qquad\qquad \delta_c \sim a^{1+3c^2_{s,\mathrm{eff}}}$$

same as in Hu's GDM model

Also in this case there are no instabilities.
The ΛCDM limit $\epsilon \to 0$ is continuous on these scales.

Matter dominated era

Sub-horizon modes ($c_{s,\mathrm{eff}} k\eta \gg 1$)

on these scales, the potential and the density contrast oscillate, the latter with increasing amplitude

$$\psi \sim \cos(c_{s,\mathrm{eff}} k\eta + \varphi_k) \qquad \delta_c \sim a^{\frac{1}{2}(1+3c^2_{s,\mathrm{eff}})} \sin(c_{s,\mathrm{eff}} k\eta + \varphi_k)$$

different power compared to super-horizon modes

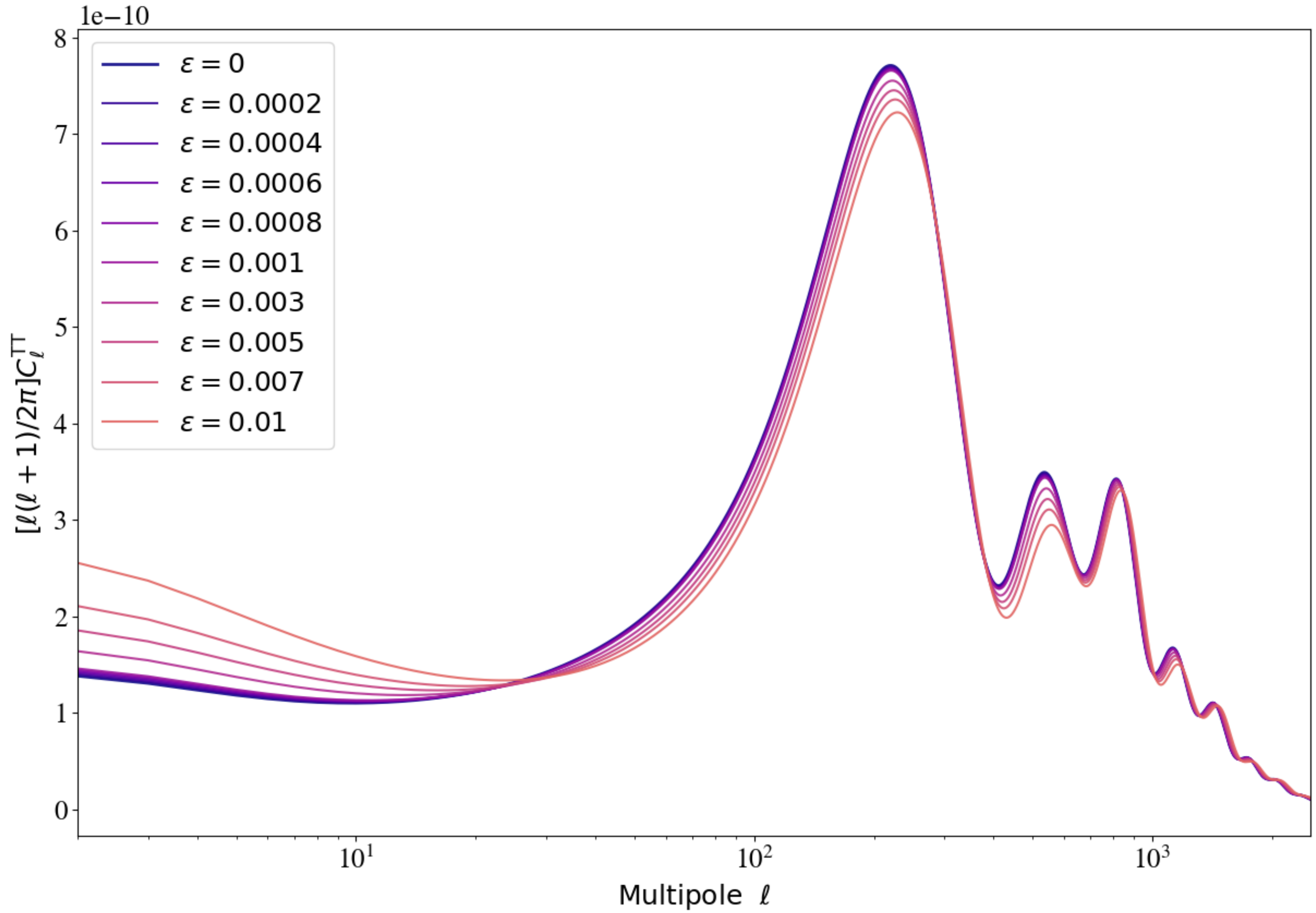
in ΛCDM, the sound speed of CDM is exactly zero and $\delta_c \sim a$ on all scales

Therefore, $\epsilon \to 0$ is a singular limit. For any finite non-zero values of ϵ (no matter how small), the behaviour of sub-horizon modes is qualitatively different from the $\epsilon = 0$ case.

This offers an opportunity to test the model and constrain the coupling ϵ.

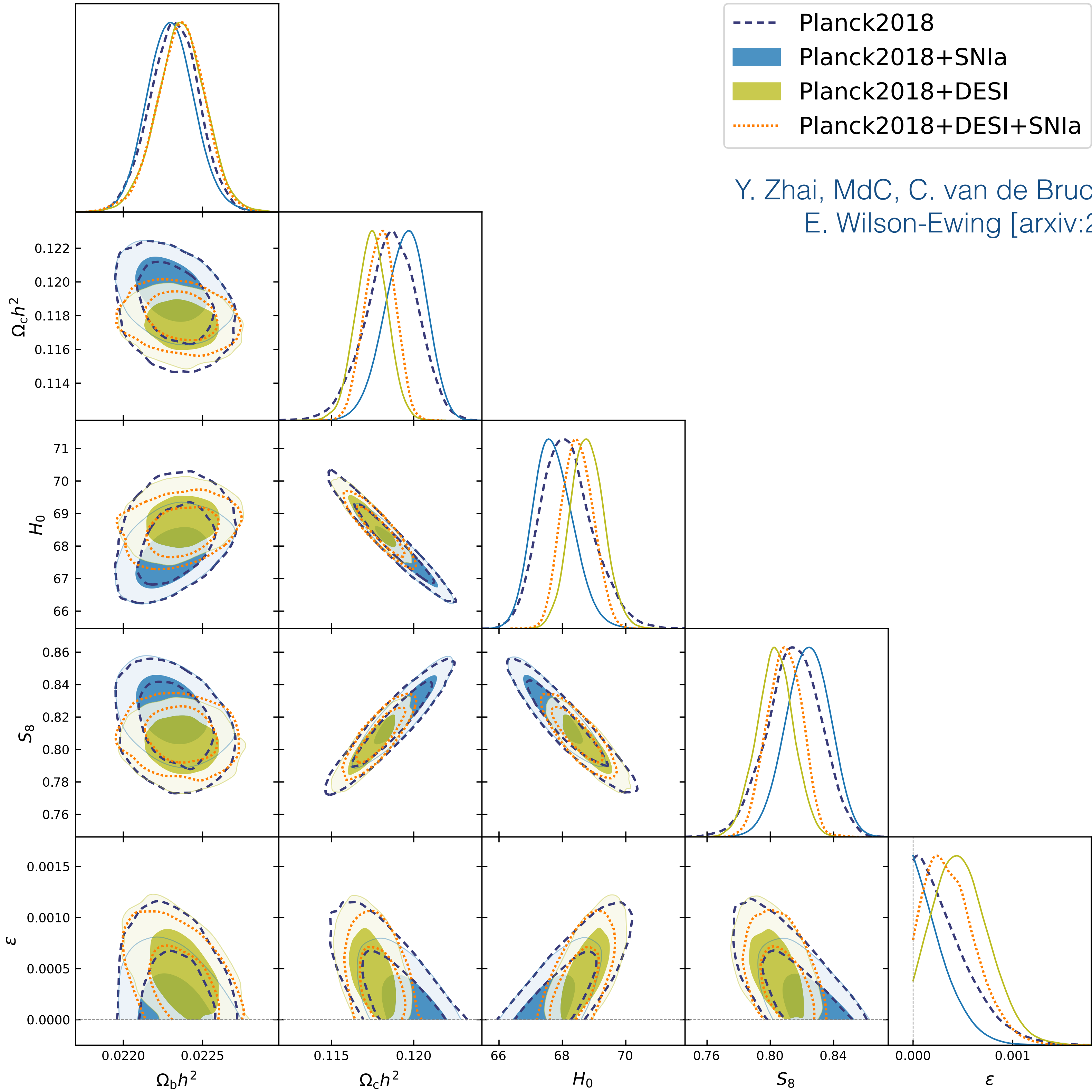
also similar to [Pan, Yang, Di Valentino, Mota, Silk (2022)] (non-interacting case for DM with $w \neq 0$)

Observational tests of $Q = -\Lambda_\infty + \epsilon\,\kappa\,\rho_c$



Y. Zhai, MdC, C. van de Bruck, E. Di Valentino, E. Wilson-Ewing [arxiv:2503.15659]

Observational tests of $Q = -\Lambda_\infty + \epsilon \kappa \rho_c$



Y. Zhai, MdC, C. van de Bruck, E. Di Valentino, E. Wilson-Ewing [arxiv:2503.15659]

Observational tests of $Q = -\Lambda_\infty + \epsilon \kappa \rho_c$

A uniform prior is imposed on ϵ in the interval [0.,0.1].

Negative values are not allowed so as to avoid a gradient instability.

Parameter	Planck2018	Planck2018+DESI	Planck2018+SNIa	Planck2018+DESI+SNIa
$\Omega_b h^2$	0.02232 ± 0.00015	0.02237 ± 0.00015	0.02230 ± 0.00015	0.02234 ± 0.00015
$\Omega_c h^2$	$0.1186^{+0.0017}_{-0.0014}$	0.11744 ± 0.00095	$0.1195^{+0.0013}_{-0.0012}$	0.11800 ± 0.00088
$\tau_{\rm reio}$	0.0542 ± 0.0076	$0.0567^{+0.0069}_{-0.0077}$	0.0529 ± 0.0071	0.0560 ± 0.0071
n_s	0.9660 ± 0.0044	0.9684 ± 0.0036	0.9646 ± 0.0040	0.9674 ± 0.0036
$\log(10^{10} A_s)$	3.043 ± 0.015	$3.047^{+0.014}_{-0.016}$	3.042 ± 0.014	3.046 ± 0.014
H_0	$68.16^{+0.67}_{-0.93}$	68.78 ± 0.52	$67.72^{+0.56}_{-0.69}$	68.47 ± 0.47
Ω_m	$0.305^{+0.012}_{-0.009}$	0.2970 ± 0.0063	$0.3107^{+0.0088}_{-0.0076}$	0.3008 ± 0.0058
σ_8	0.8082 ± 0.0060	$0.8071^{+0.0058}_{-0.0064}$	0.8095 ± 0.0058	0.8079 ± 0.0058
S_8	$0.815^{+0.018}_{-0.016}$	0.803 ± 0.011	0.824 ± 0.014	0.809 ± 0.011
ϵ	$< 0.000432 (< 0.000907)$	$0.00049^{+0.00022}_{-0.00033} (< 0.000970)$	$< 0.000293 (< 0.000635)$	$0.00041^{+0.00015}_{-0.00035} (< 0.000880)$

TABLE II. Constraints at 68% (95%) CL on parameters from various combinations of datasets.

Non-zero values of ϵ are mildly favoured by the combined datasets.

The situation does not change if negative values are allowed in the prior.

Y. Zhai, MdC, C. van de Bruck, E. Di Valentino, E. Wilson-Ewing [arxiv:2503.15659]

Further generalizations

We could choose a more general model for the transfer potential. Some examples are:

$$Q_{II} = -\Lambda_\infty + \alpha(\nabla^a u_a^{(c)})^2 \qquad\qquad Q_{III} = -\Lambda_\infty + L^{-1}\, \nabla^a u_a^{(c)}$$

$$Q_{IV} = -\Lambda_{\mathrm{f}} + \alpha\, u^c \nabla_c (\nabla^a u_a) \qquad \ldots\ldots$$

One could also include a functional dependence on further geometric invariants or additional interactions with other matter species.

In any case, we would still have $w = -1$, but such modifications may introduce further features to the effective fluid description of DM.

Models motivated from fundamental physics
(e.g., energy diffusion due to spacetime discreteness) are of primary interest.

Conclusions

- IDE models with $w = -1$ and integrable transfer are embedded in unimodular gravity (trace-free Einstein equations)

- For this class of models, we derived the general equations for scalar cosmological perturbations. These are a special case of [Valiviita, Majerotto, Maartens '08]. However, the models at hand are free from large-scale non-adiabatic instabilities.

- We assume a simple model $Q = -\Lambda_\infty + \epsilon\,\kappa\,\rho_c$ and examine the evolution of perturbations
  - ✦ CDM effectively behaves as a fluid with $c^2_{s,\mathrm{eff}} = w_{\mathrm{eff}} = \epsilon/(1-\epsilon)$
  - ✦ $\epsilon \geq 0$ ensures that there is no gradient instability. Energy flows from CDM to DE.
  - ✦ We solved the equations analytically for super-horizon modes during RDE, and for both super-horizon and sub-horizon modes during MDE. No large scale instabilities.
  - ✦ The model predicts small deviations from ΛCDM. Data favour small but non-zero values of $\epsilon \sim \mathcal{O}(10^{-4})$.

Open questions

- Several generalisations are possible, providing a general framework for phenomenology and to study diffusion effects from fundamental physical processes.
- What about the H_0 and S_8 tensions? [Perez, Sudarsky, Wilson-Ewing (2020)]
- Can we explain DE evolution as observed by DESI in DR2 within this class of IDE models?

Backup slides

Aside on the unimodularity condition

The trace-free Einstein equations can be obtained from the action

$$S = \int d^4x \sqrt{-g}\, \left[\frac{1}{2\kappa} R + \lambda \underbrace{\left(\omega - \sqrt{-g} \right)}_{\text{'unimodularity constraint'}} \right]$$

In this approach one finds, in addition to the trace-free equations, a constraint that breaks the diffeomorphism invariance of GR down to volume-preserving diffeomorphisms.

The linearized unimodularity constraint for cosmological perturbations reads $\nabla^2 E + \phi - 3\psi = 0.$

Following this approach, the class of allowed gauges is clearly restricted.

[Gao, Brandenberger, Cai, Chen (2014)]

In our approach, we **do not** assume an action principle.

Instead, we take only the trace-free Einstein equations as a starting point.

$\Longrightarrow$ full diffeomorphism invariance is retained.

Aside: equivalence with generalized dark matter

more in general, i.e. without assuming a specific transfer model

total stress-energy of the dark sector:

$$T_{ab}^{\text{dark}} = T_{ab}^{\text{DM}} + T_{ab}^{\text{DE}}$$

$$\boxed{\begin{aligned} T_{ab}^{\text{DE}} &= -\rho_x\, g_{ab} \\ T_{ab}^{\text{DM}} &= (\rho_d + p_d) u_a u_b + p_d\, g_{ab} + \pi_{ab} \end{aligned}}$$

$$T_{ab}^{\text{DM}} + T_{ab}^{\text{DE}} = \check{T}_{ab} - \kappa^{-1} \Lambda_\infty\, g_{ab}$$

(late-time) cosmological constant

generalized dark matter

$$\boxed{\check{T}_{ab} = (\check{\rho} + \check{p}) u_a u_b + \check{p}\, g_{ab} + \pi_{ab}}$$

$$\check{\rho} = \rho_d + \tilde{\rho}_x\,, \quad \check{p} = p_d - \tilde{\rho}_x$$

This rewriting is possible for all ‘interacting vacuum’ models

Aside: equivalence with generalized dark matter

more in general, i.e. without assuming a specific transfer model

total stress-energy of the dark sector:

$$T_{ab}^{\text{dark}} = T_{ab}^{\text{DM}} + T_{ab}^{\text{DE}}$$

$$\boxed{\begin{aligned} T_{ab}^{\text{DE}} &= -\rho_x\, g_{ab} \\ T_{ab}^{\text{DM}} &= (\rho_d + p_d)u_a u_b + p_d\, g_{ab} + \pi_{ab} \end{aligned}}$$

$$T_{ab}^{\text{DM}} + T_{ab}^{\text{DE}} = \check{T}_{ab} - \kappa^{-1}\Lambda_\infty\, g_{ab}$$

NB: there may still be entropy perturbations if ρ_x depends on extra fields other than DM

The absence of instabilities depends of course on the transfer model.

However, it remains true in general that $\theta_x \equiv \theta$ and therefore the velocity instability found by Majerotto *et al.* cannot arise in this class of models.