The moments of the spectral form factor in SYK

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New Frontiers in Theorethical Physics

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Quantum Chaos Beyond Universality

Quantum Chaos is ubiquitous across physics: many-body systems, nuclear physics, black holes, etc.

In contrast to classical chaos, quantum chaos **lacks a single definition**.
Currently, the term refers to different phenomena that co-occur (thermalization, scrambling, ergodicity)

BGS conjecture

Characterization of Chaotic Quantum Spectra and Universality of Level Fluctuation Laws

O. Bohigas, M. J. Giannoni, and C. Schmit
Division de Physique Théorique, Institut de Physique Nucléaire, F-91406 Orsay Cedex, France
(Received 2 August 1983)

prove the following conjecture: Spectra of time-reversal–invariant systems whose classical analogs are K systems show the same fluctuation properties as predicted by GOE (alternative

can be generalized to random matrix universality as an operational criterion:
The spectral statistics of a quantum-chaotic system match those of a random matrix ensemble.

We challenge this notion by computing the **moments of the spectral form factor** in the **SYK model**, revealing deviations from random matrix theory.

The SYK model

SYK (Sachadev-Ye-Kitaev) model is a all-to-all connected systems of N Majorana fermions:

$$\mathcal{H} = \mathrm{i}^{q/2} \sum_{1\leq i_1<\cdots<i_q\leq N} J_{i_1\ldots i_q}\psi_{i_1}\ldots\psi_{i_q}\,, \qquad \psi_i = \psi_i^\dagger\,, \qquad \{\psi_i,\psi_j\} = \delta_{ij} \tag{1}$$

The couplings $J_{i_1\ldots i_q}$ are i.i.d. Gaussian random variables with

$$\langle J^2_{i_1\ldots i_q}\rangle = \frac{J^2(q-1)!}{N^{q-1}}. \tag{2}$$

It gained a lot of attention recently since it is:

- Originally introduced as toy model for non-Fermi liquid. [Sachdev-Ye '92]
- Prototypical example of a many-body chaotic system for $q > 2$.
- Analytically tractable [Maldacena-Stanford '16].
- Holographically dual to near-extremal black holes. [Kitaev's talk '15, Sachdev '15]

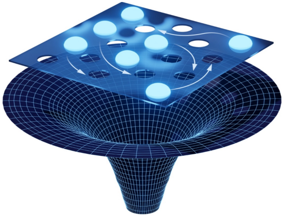

Figure by Lucy Reading-Ikkanda, Simons Foundation

Spectral form factor (SFF)

Since quantum chaos encompasses multiple phenomena, we need concrete probes to diagnose it. The SFF, introduced in random matrix theory, is a diagnostic tool for spectral correlations:

$$K(T) = \frac{|Z(iT)|^2}{|Z(0)|^2} = \frac{1}{L^2}\sum_{m,n=1}^{L} e^{i(E_m - E_n)T} \qquad Z(iT) = \mathrm{Tr}\left[e^{i\mathcal{H}t}\right] \tag{3}$$

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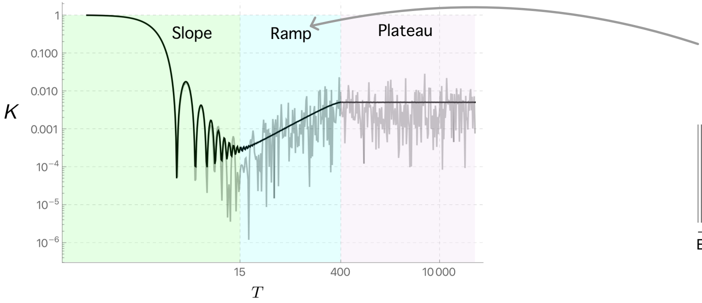
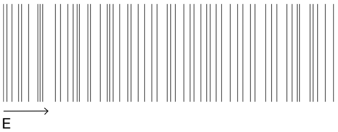
While $K(T)$ is a very erratic function, its disorder-averaged value $\langle K(T) \rangle$ exhibits a universal profile.





Review of [Saad-Shenker-Stanford '18]: SFF for the SYK model

The SFF for the SYK model can be represented as a path integral

$$Z(\mathrm{i}T)Z(-\mathrm{i}T) = \int D\psi \exp\left\{\mathrm{i}\int_0^T \mathrm{d}\,t\left[\frac{\mathrm{i}}{2}\psi_i^a\partial_t\psi_i^a - J_{i_1\ldots i_q}\left(\mathrm{i}^{\frac{q}{2}}\psi_{i_1}^L\ldots\psi_{i_q}^L - (-\mathrm{i})^{\frac{q}{2}}\psi_{i_1}^R\ldots\psi_{i_q}^R\right)\right]\right\}, \quad (4)$$

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If we average over the coupling and introduce collective fields G, Σ we can integrate out the fermions

$$\langle |Z(\mathrm{i}T)|^2 \rangle = \int DGD\Sigma\, e^{-I[G,\Sigma]}$$

$$\frac{1}{N} I[G,\Sigma] = -\frac{1}{2}\log\det(\partial_t - \Sigma) + \frac{1}{2}\int_0^T \int_0^T \mathrm{d}\,t\,\mathrm{d}\,t' \left[\Sigma_{ab}(t,t')G_{ab}(t,t') - \frac{J^2}{q} s_{ab} G_{ab}(t,t')^q\right] \quad (5)$$

$$s_{LL} = s_{RR} = -1, \qquad s_{LR} = s_{RL} = \mathrm{i}^q.$$

We introduced the two-point function $G_{ab}(t,t') = \frac{1}{N}\sum_i \psi_i^a(t)\psi_i^b(t')$ and its self-energy $\Sigma_{ab}(t,t')$.

Review of [Saad-Shenker-Stanford '18]: SFF for the SYK model

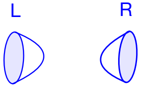
The large-N limit makes the saddle point approximation reliable:

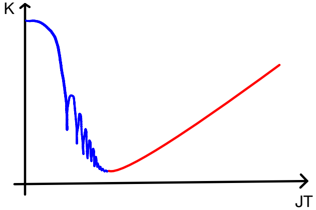
$$\begin{aligned} \Sigma_{ab}(t_1 - t_2) &= J^2 s_{ab} G_{ab}(t_1 - t_2)^{q-1}\,, \\ G_{ab}(\omega) &= (-i\omega - \Sigma(\omega))^{-1}_{ab}\,, \end{aligned} \tag{6}$$

There are two solutions to this equation, obtained numerically or for $\omega \to 0$:

$$G_*^{(2)} = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$$

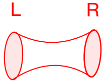
"Disconnected"





$$G_*^{(2)} = \begin{bmatrix} * & * \\ * & * \end{bmatrix}$$

"Wormhole"



Moments of the SFF in SYK: path integral

The calculation for the kth moment of the SFF is formally very similar

$$\langle Z(\mathrm{i}T)^k Z(-\mathrm{i}T)^k \rangle = \int DGD\Sigma\, e^{-I[G,\Sigma]}$$

$$\frac{1}{N} I[G,\Sigma] = -\frac{1}{2}\log\det(\partial_t - \Sigma) + \frac{1}{2}\iint_0^T \mathrm{d}\,t\,\mathrm{d}\,t' \left[\Sigma_{ab}(t,t')G_{ab}(t,t') - \frac{J^2}{q} s_{ab} G_{ab}(t,t')^q\right] \tag{7}$$

$$s_{L_\# L_\#} = s_{R_\# R_\#} = -1, \qquad s_{L_\# R_\#} = s_{R_\# L_\#} = \mathrm{i}^q.$$

Main features:

- There are k L and R fields $\Rightarrow$ G, Σ are $2k \times 2k$ matrices.
- There is a S_k^2 symmetry for the permutations of the L and R copies
- The saddle point equations are formally identical as the $k = 1$ case:

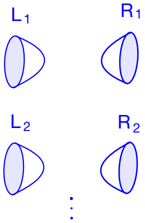
$$\begin{aligned} \Sigma_{ab}(t_1 - t_2) &= J^2 s_{ab} G_{ab}(t_1 - t_2)^{q-1}, \\ G_{ab}(\omega) &= (-i\omega - \Sigma(\omega))_{ab}^{-1}, \end{aligned} \tag{8}$$

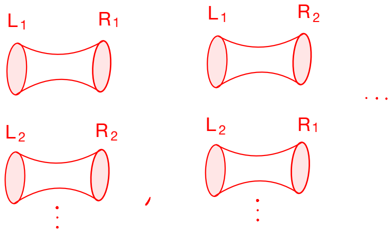
Moments of the SFF in SYK: saddle point

A simple solution is given by:

$$G(t) = \left[\begin{array}{c|c|c} G_*^{(2)}(t) & 0 & 0 \\ \hline 0 & G_*^{(2)}(t) & 0 \\ \hline 0 & 0 & \ddots \end{array}\right]. \tag{9}$$

In principle the $G_*^{(2)}$ can be either "Disconnected" or "Wormhole".
Far away from the point where they change dominance we have two dominant solutions:



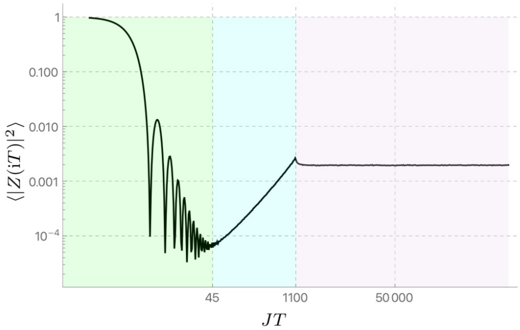


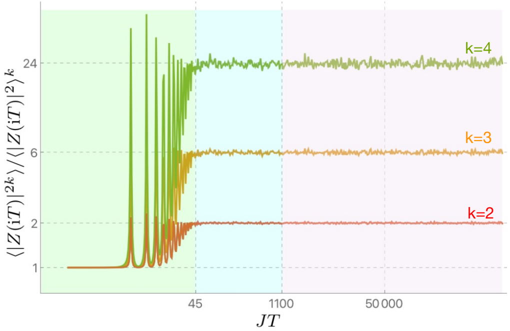
$\langle |Z(\mathrm{i}T)|^{2k}\rangle = |\langle Z(\mathrm{i}T)\rangle|^{2k}$

$\langle |Z(\mathrm{i}T)|^{2k}\rangle = k!\langle |Z(\mathrm{i}T)|^{2}\rangle^{k}$

Moments of the SFF in SYK: saddle points

No other saddle dominates, as we can see from a numerical analysis:





This result follows random matrix universality. But we expect that at high k there will be corrections.

Moments of the SFF in SYK: perturbative corrections

To calculate the corrections a perturbative analysis is needed.
Expanding around the wormhole saddle point we find:

$$\frac{\langle |Z(\mathrm{i}T)|^{2k}\rangle}{\langle |Z(\mathrm{i}T)|^2\rangle^k} = k!\left[1 + \frac{k(k-1)}{4}\,\frac{q!}{N^q}\,T^2|\overline{\Delta}_E|^2 + o(N^{2-q})\right] \tag{10}$$

1) $\overline{\Delta}_E$ can be seen as a difference of energies, it is extensive and it gets its contribution from the **edge of the spectrum**.

2) $\frac{N^q}{q!} \sim \binom{N}{q} = \#$ i.i.d. couplings in the Hamiltonian. We expect this to be true also for other disordered systems.

3) Deviation from random matrix theory for $k \sim N^{q/2-1}$.

Conclusions

Let's conclude with some remarks:

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- In disordered quantum many-body systems, the correction is inversely proportional to the **number of random variables** in the Hamiltonian (shown numerically in sparse SYK).
  How does this generalize to non-disordered systems?
- The spectral edge is the main source of universality violation [Altland et al. '24] .
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THANK YOU!

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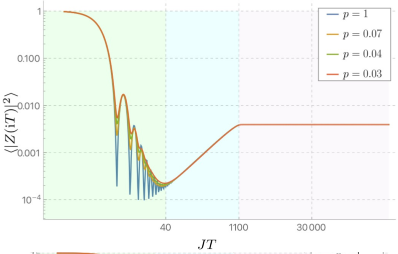
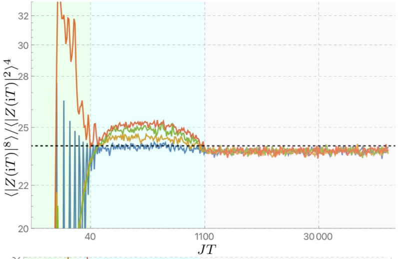
Moments of the SFF in SYK: perturbative corrections

We performed a numerical analysis using exact diagonalization for $N = 18$ fermions in sparse SYK:

Usual SFF





Filtered SFF

$Y(T) = \mathrm{Tr}[f(H)e^{-\mathrm{i}TH}]$

$f(H) = e^{-H^2/2\Delta^2}$