CALICO: parametric annihilators for loop integrals & special functions

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Introduction

Loop Integrals

- LEGO® blocks of perturbative QFT beyond tree level
- Key ingredient of phenomenological predictions
- Rich and interesting mathematical structures

Thousands of loop integrals appear when studying perturbative predictions!

- **Crucial:** Finding relations between them
- Loop integrals admit various integral representations with different tradeoffs and mathematical properties

This work

- Study and elaborate on the method of **parametric annihilators** for finding integral identities,
    - Focus on parametric representation of loop integrals
    - Extend applications to different representations
    - Illustrate a similar technique for finding differential equations
- Provide an **implementation** of annihilators and differential operators based on modern linear solvers relying on cutting edge **finite-fields** techniques
- Implement such techniques in a public Mathematica package: **`CALICO`**

Computing **A**nnihilators from **L**inear **I**dentities
Constraining (differential) **O**perators

Some useful definitions

A list of variables: $\mathbf{z} = (z_1, \ldots, z_n)$

A list of exponents: $\alpha = (\alpha_1, \ldots, \alpha_n)$

$\longrightarrow$

$$\mathbf{z}^\alpha = \prod_j z_j^{\alpha_j}$$

Monomials

$$|\alpha| = \sum_j \alpha_j$$

total degree

We are interested in families of integrals of the form

$$I_\alpha = \int \mathrm{d}^n\mathbf{z}\, \varphi_\alpha(\mathbf{z})\, u(\mathbf{z})$$

Rational factors raised to integer powers

Twist

Multivalued

$$u(\mathbf{z}) = \prod_j B_j(\mathbf{z})^{\gamma_j}$$

$$u(\mathbf{z}) = \exp F(\mathbf{z}) \prod_j B_j(\mathbf{z})^{\gamma_j}$$

These include many parametrizations of loop integrals

What do we want to do?

- Finding and solving **linear relations** satisfied by integrals having the form of I_α
- Express integrals within a family as a linear combination of a set of independent **master integrals (MIs)**

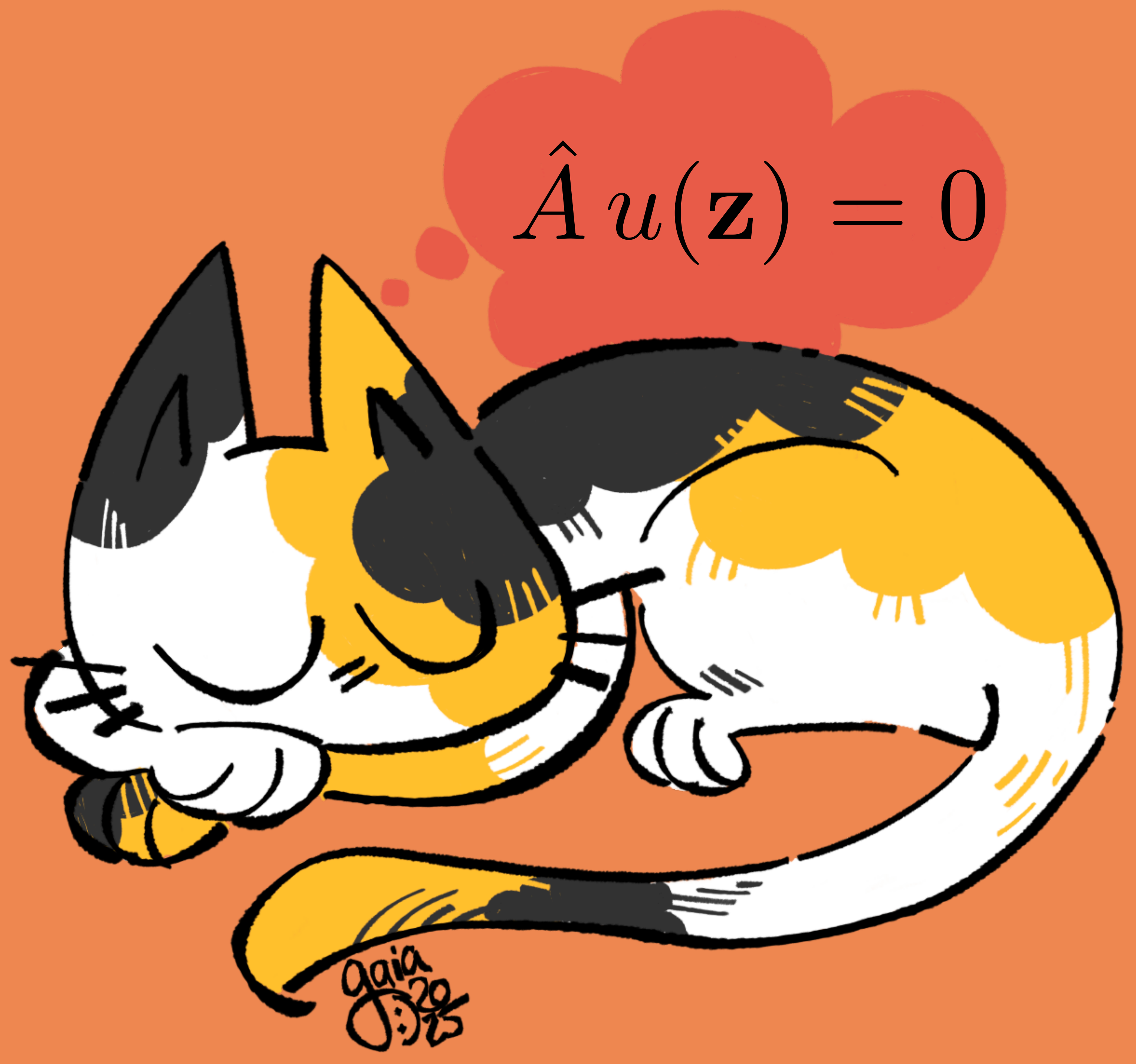
$$I_\alpha = \sum_{\beta \in \mathrm{MIs}} c_{\alpha\beta} I_\beta$$

Crucial ingredient are **Integration-By-Parts identities (IBP):**

$$\int \mathrm{d}^n \mathbf{z}\, \partial_j \Big(\varphi_\alpha(\mathbf{z})\, u(\mathbf{z}) \Big) = 0$$

Regulated integrals vanish at the integration boundary

Parametric annihilators
&
Differential operators



Integral identities via parametric annihilators

Parametric annihilator of order o of $u(\mathbf{z})$

[Baikov(1996), Lee(2014),
Bitoun, Bogner, Klausen, Panzer (2017)]

$$\hat{A} = c_0(\mathbf{z}) + \sum_j c_j(\mathbf{z})\,\partial_j + \sum_{j_1 \le j_2} c_{j_1 j_2}(\mathbf{z})\,\partial_{j_1}\partial_{j_2}$$

$$+ \cdots + \sum_{j_1 \le \cdots \le j_o} c_{j_1 \cdots j_o}(\mathbf{z})\,\partial_{j_1}\cdots\partial_{j_o}$$

$c_{j_1 j_2 \ldots}$ polynomials in $\mathbf{z}$

Such that $$\hat{A}\,u(\mathbf{z}) = 0$$

For any annihilator $\hat{A}$, we have infinitely many integral identities

$$\int \mathrm{d}^n\mathbf{z}\,\varphi_\alpha(\mathbf{z})\,\hat{A}\,u(\mathbf{z}) = 0,\ \ \forall\alpha$$

symbolic α

Using IBPs on derivatives, we get a **template identity** for symbolic α

$$\int u\,(\varphi_\alpha c_0) - \sum_j \int u\,(\partial_j c_j \varphi_\alpha) + \cdots + (-1)^o \sum_{j_1\le\cdots\le j_o} \int u\,\partial_{j_1}\cdots\partial_{j_o}(c_{j_1\cdots j_o}\varphi_\alpha) = 0$$

All the integrals belong to the family I_α

Laporta algorithm

[Chetyrkin, Tkachov (1981), Laporta (2000)]

$$\int u\,(\varphi_\alpha c_0) - \sum_j \int u\,(\partial_j c_j \varphi_\alpha) + \cdots + (-1)^o \sum_{j_1 \leq \cdots \leq j_o} \int u\,\partial_{j_1} \cdots \partial_{j_o} (c_{j_1 \cdots j_o} \varphi_\alpha) = 0$$

Seeding the template eq.s : replacing symbolic α with integer numbers

- Applying each template identity to a large number of seed integrals: obtain a **linear system** of equations
- Choice of an **ordering**: express **complex** integrals as a function of **simple** ones
- solving it: reduction to **master integrals**

$$I_\alpha = \sum_{\beta \in \mathrm{MIs}} c_{\alpha\beta} I_\beta$$

Differential equations

[Barucchi, Ponzano ’73; Kotikov ’91; Bern, Dixon, Kosower ‘94; Gehrmann, Remiddi 2000]

- Integrals in the form of I_α also depend on additional free parameters x (e.g. kinematic invariants)
- Studying of analytic structure & their numerical or analytical evaluation
- Reducing the derivative of MIs with respect to x to MIs, write a system of differential equations satisfied by the MIs themselves

$$\partial_x I_\alpha = \sum_{\beta \in \mathrm{MIs}} M_{\alpha\beta} I_\beta , \qquad \text{for } \alpha \in \mathrm{MIs}.$$

x **free parameter of the integrals**

Differential equations via differential operators

- Derive an operator $\hat{O}_x$ that realizes differentiation with respect to x

$$\hat{O}_x = c_0^{(x)}(\mathbf{z}) + \sum_j c_j^{(x)}(\mathbf{z})\,\partial_j + \sum_{j_1 \le j_2} c_{j_1 j_2}^{(x)}(\mathbf{z})\,\partial_{j_1}\partial_{j_2}$$

$$+ \cdots + \sum_{j_1 \le \cdots \le j_o} c_{j_1 \cdots j_o}^{(x)}(\mathbf{z})\,\partial_{j_1} \cdots \partial_{j_o}$$

$c_{j_1 j_2 \cdots}^{(x)}$ polynomials

Such that $\hat{O}_x\, u(\mathbf{z}) = \partial_x\, u(\mathbf{z})$

How to compute parametric annihilators

Computing annihilators via linear constraints

- Computing parametric annihilators up to a certain order and polynomial degree
- implemented in the `CALICO` package

$$\hat{A}\, u(\mathbf{z}) = 0 \;\Rightarrow \Big\{ \text{Linear system for } c_{\alpha;j_1 j_2 \dots}$$

`CALICO uses the efficient linear solver of FiniteFlow, based on finite-field methods`

[Peraro (2019)]

$$\hat{A} = c_0(\mathbf{z}) + \sum c_j(\mathbf{z})\partial_j + \dots$$

$$c_{j_1 j_2 \dots}(\mathbf{z}) = \sum_{\alpha} c_{\alpha;j_1 j_2 \dots}\, \mathbf{z}^{\alpha}$$

Applications

Hypergeometric ${}_2F_1$

$$I_\alpha = \int_0^1 \mathrm{d}z\, \varphi_\alpha(z)\, u(z)$$

$$\varphi_\alpha(z) = z^\alpha, \qquad u(z) = z^{b_2-1}\,(1-z)^{b_3-b_2-1}\,(1-x\,z)^{-b_1}$$

Related to the hypergeometric ${}_2F_1$

$$I_\alpha = \frac{\Gamma(b_2+\alpha)\Gamma(b_3-b_2)}{\Gamma(b_3+\alpha)}\, {}_2F_1(b_1,\, b_2+\alpha,\, b_3+\alpha;\, x)$$

1. 1 first-order annihilator → reduction to 2 MIs $\{I_0, I_1\}$
2. First-order differential operator

$$\partial_x \begin{pmatrix} I_1 \\ I_0 \end{pmatrix} = \begin{pmatrix} \frac{b_1\,x-b_3}{(1-x)\,x} & \frac{b_2}{(1-x)\,x} \\ \frac{b_1-b_3}{1-x} & \frac{b_2}{1-x} \end{pmatrix} \cdot \begin{pmatrix} I_1 \\ I_0 \end{pmatrix}$$

Generalised to Hypergeometric ${}_{n+1}F_n$

Loop integrals

Momentum-space representation

$$J_{\alpha} = J_{\alpha_1 \cdots \alpha_n} = \int \prod_{i=1}^{\ell} \frac{\mathrm{d}^d k_i}{i\pi^{d/2}} \frac{1}{D_1^{\alpha_1} \ldots D_n^{\alpha_n}}$$

$$D_{F,j} = l_j \cdot v_j - m_j^2$$

$$D_{F,j} = l_j^2 - m_j^2$$

l_j linear combination of k_j,
v_j linear combination of p_j

D_js are generalised denominators

- **Proper denominators:** D_j such that $\alpha_j > 0$
- Irreducible scalar products (ISPs): D_j such that $\alpha_j \leq 0$

IBPs in momentum space

[Tkachov (1981), Chetyrkin, Tkachov (1981)]

$$\int \prod_{i=1}^{\ell} \frac{\mathrm{d}^d k_i}{i\pi^{d/2}} \frac{\partial}{\partial k_j^{\mu}} \frac{v^{\mu}}{D_1^{\alpha_1} \ldots D_n^{\alpha_n}} = 0, \quad \text{with } v^{\mu} = k_i^{\mu}, p_i^{\mu}$$

Parametric representations of loop integrals

Baikov

$$I_\alpha = \int \mathrm{d}^n\mathbf{z}\, \frac{1}{\mathbf{z}^\alpha}\, B(\mathbf{z})^\gamma$$

Also Loop-by-Loop Baikov
& Duals of loop integrals

Lee-Pomeransky

$$I_\alpha = \int \mathrm{d}^n\mathbf{z} \left(\prod_{j=1}^{n} \frac{1}{\Gamma(\alpha_j)} \right) \mathbf{z}^{\alpha-1} G(\mathbf{z})^{-d/2}$$

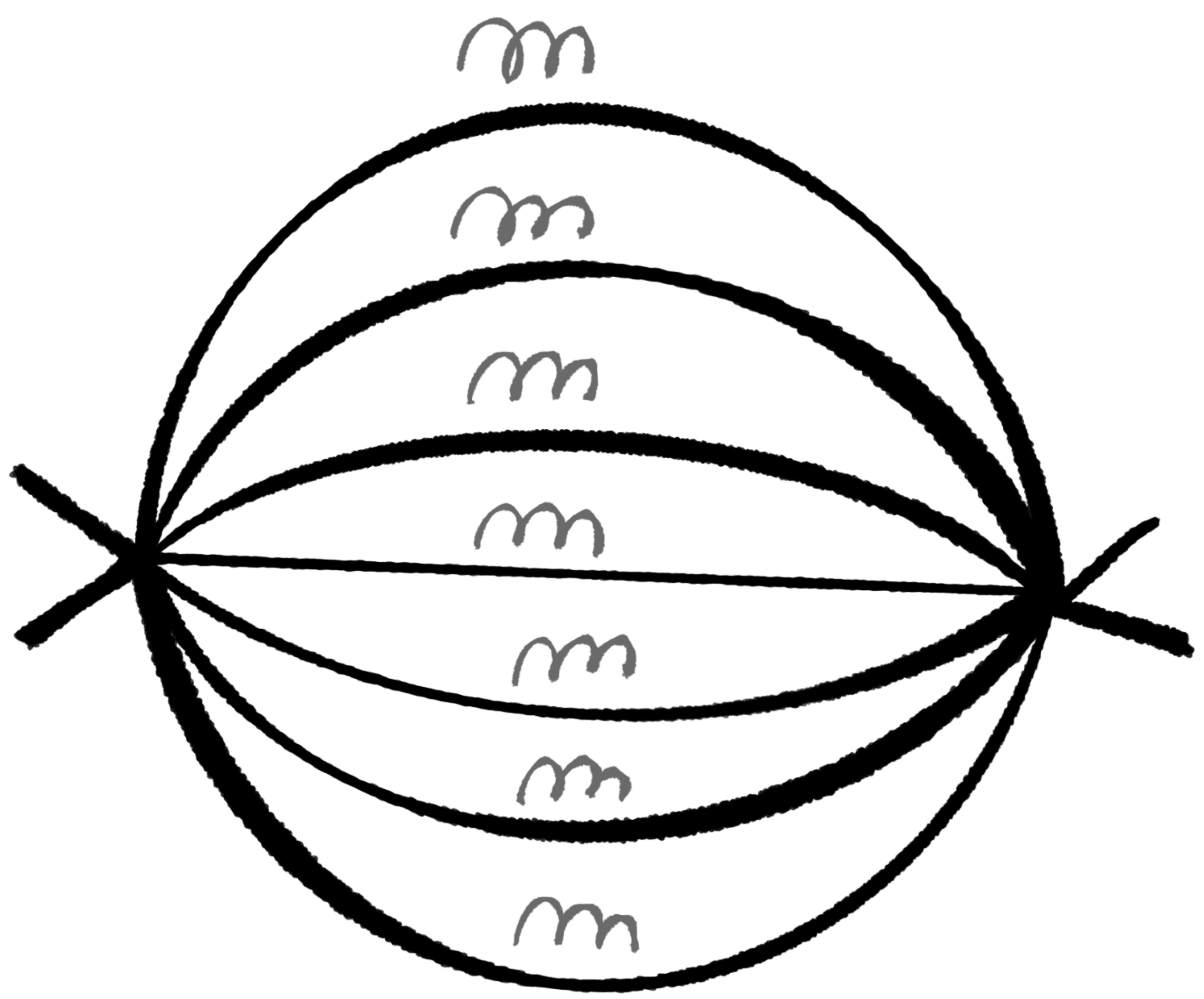
$$G(\mathbf{z}) = \mathcal{U}(\mathbf{z}) + \mathcal{F}(\mathbf{z})$$

Schwinger

$$I_\alpha = \int \mathrm{d}^n\mathbf{z} \left(\prod_{j=1}^{n} \frac{1}{\Gamma(\alpha_j)} \right) \mathbf{z}^{\alpha-1} \exp\left[-\mathcal{F}(\mathbf{z})/\mathcal{U}(\mathbf{z}) \right] \mathcal{U}(\mathbf{z})^{-d/2}$$

L loop bananas

ℓ-loop & one internal mass m, defined by the set of $\ell + 1$ proper denominators:



$$D_j = k_j^2 - m^2 \quad \text{for } j = 1,\ldots,\ell$$

$$D_{\ell+1} = (k_1 + \cdots + k_\ell - p)^2 - m^2$$

Momentum space has $(\ell + 2)(\ell - 1)/2$ ISPs → for $\ell = 6$, it has 20 ISPs

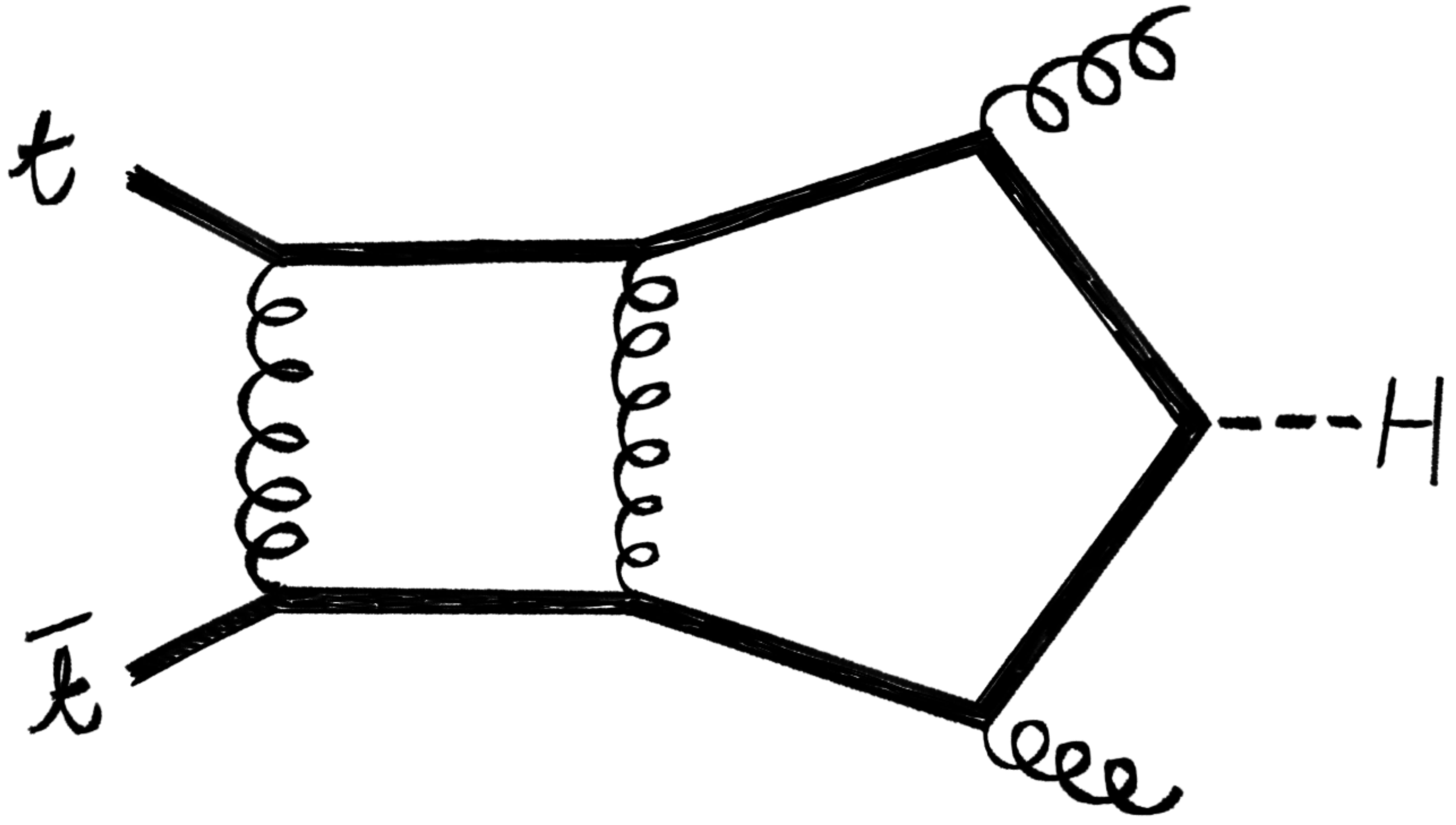
Use Lee-Pomeranski or Schwinger representation to

- Reduce to MIs
- Derive DEQs

Without the need of additional ISPs!

Done in a couple of minutes on a laptop up to 6 loops (mostly spent computing annihilators and template eqs)

Family for $t\bar{t}H$ production



Cutting-edge example

- Many different scales
- Many external legs

Using the integral representations:

- Schwinger
- Lee-Pomeranski

Tested: numerical reduction on a laptop with up to 3 extra powers of denominators

Finding relations between integrals with constant numerator and higher powers of proper denominators

Useful for finding integrals with good properties

- Quasi-finite [von Manteuffel, Panzer, Schabinger (2014)]
- Pure functional form [Henn (2013)]

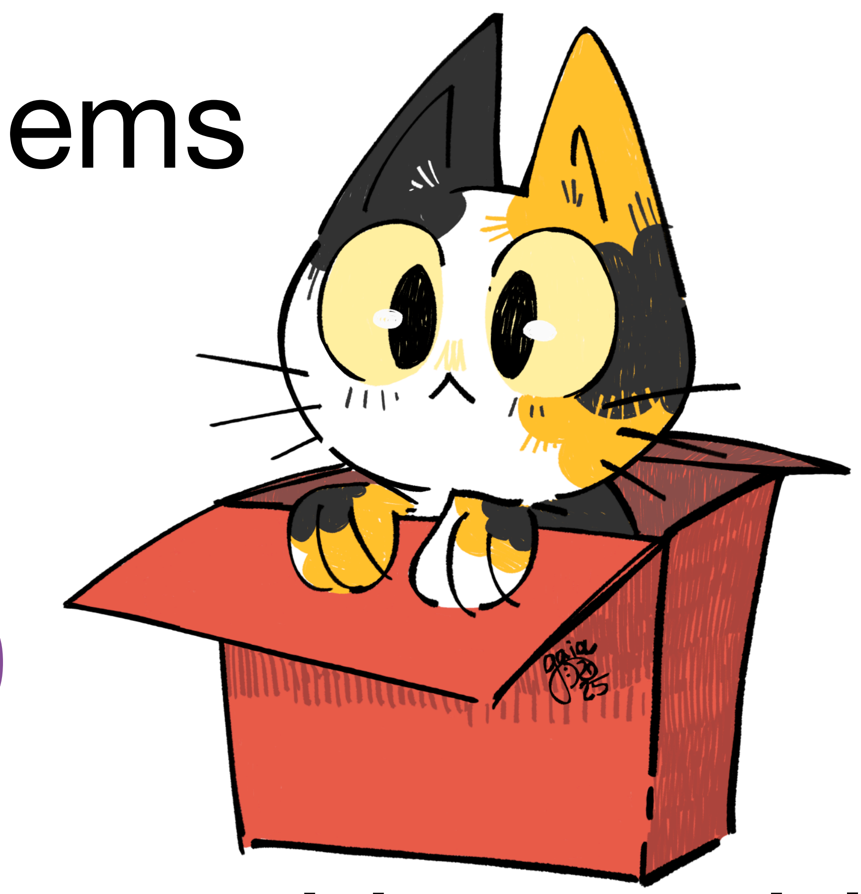
Conclusions & Outlook

- Parametric annihilators are a useful tool for finding linear relations between integrals

- Allow to use integral parametrizations tailored to specific problems

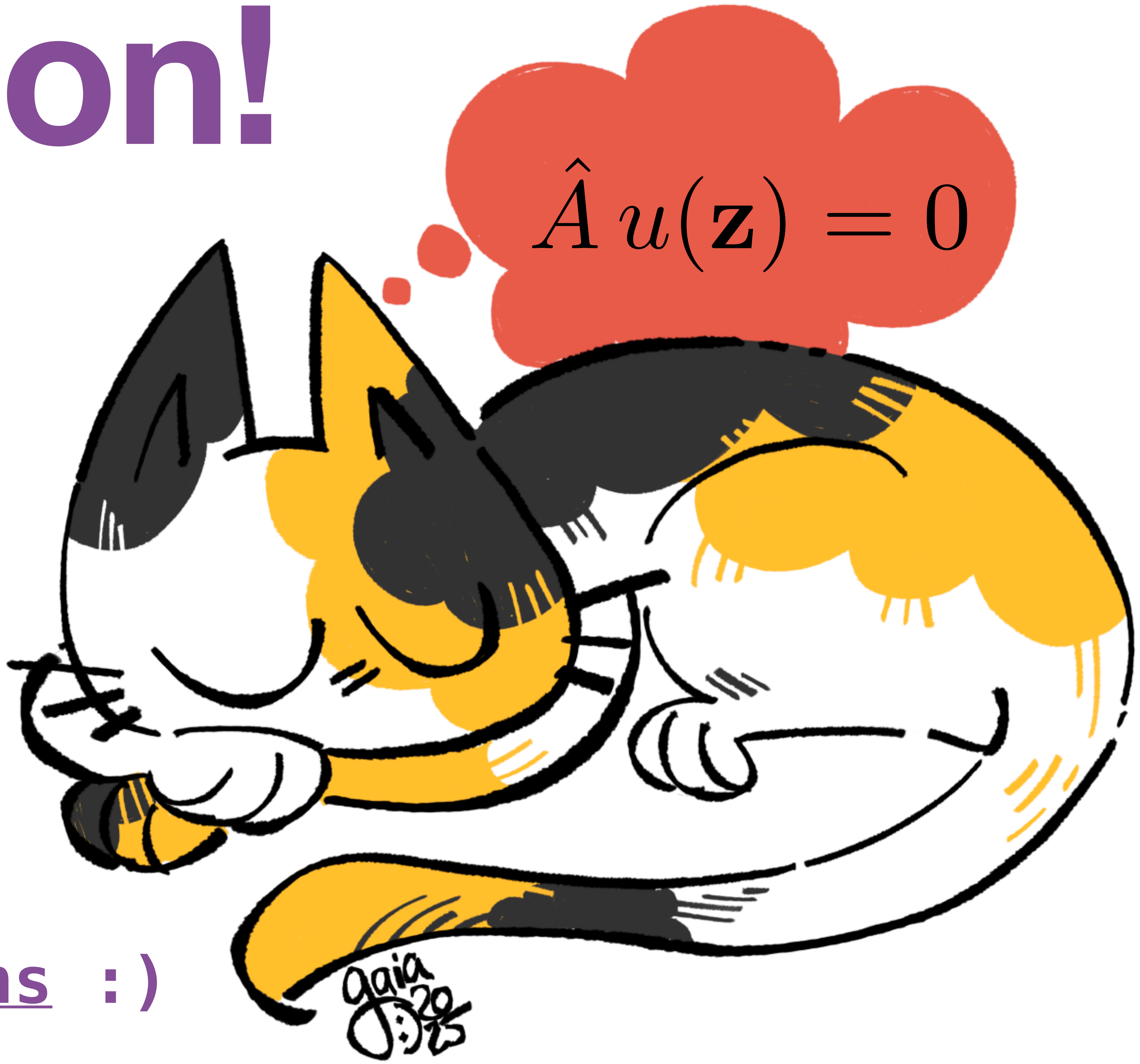
- Still WIP: study of more cutting-edge integral families

- Implementation will be released in the public package **CALICO**



✦ **Bonus:** can also solve syzygy equations and polynomial decomposition problems

Thank you for your attention!



… more drawings @qftoons :)