New Frontiers in Theoretical Physics - XXXVIII Convegno Nazionale di Fisica Teorica

Work fluctuations for Active Particles: singularities, dynamical phase transitions and big jumps

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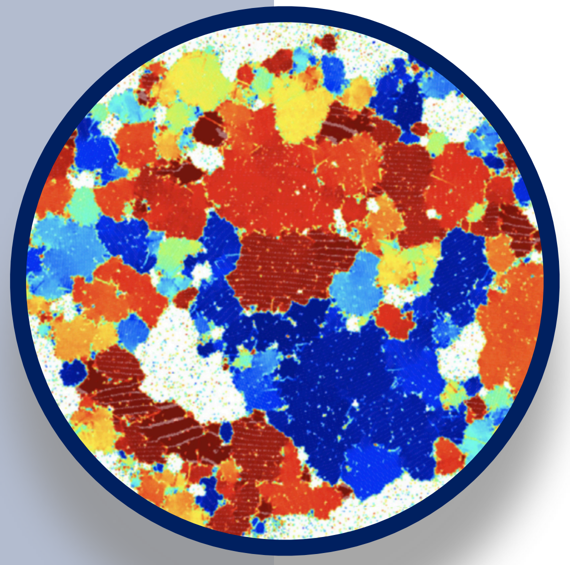
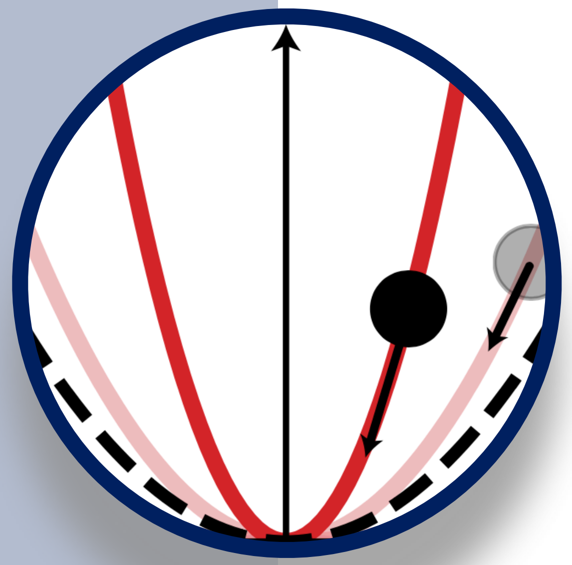
Cortona,
21st May 2025

Outline



Active Matter

Dynamical Phase Transitions



Harmonically confined AOUP

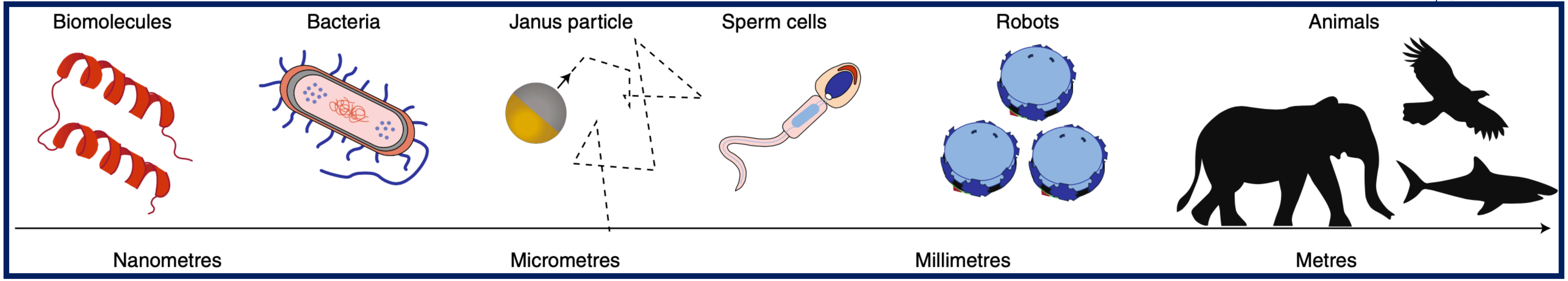
Take-home messages

Active Matter

- **Active Matter:** single components transform energy from internal reservoirs or from the surrounding environment to **self propel**

Ann Rev Cond Matt Phys 2010, Ramaswamy
Physica A 2018, Fodor et al.
Rev Mod Phys 2013, Marchetti et al.

Nature Mach Int 2020, Cichos et al.



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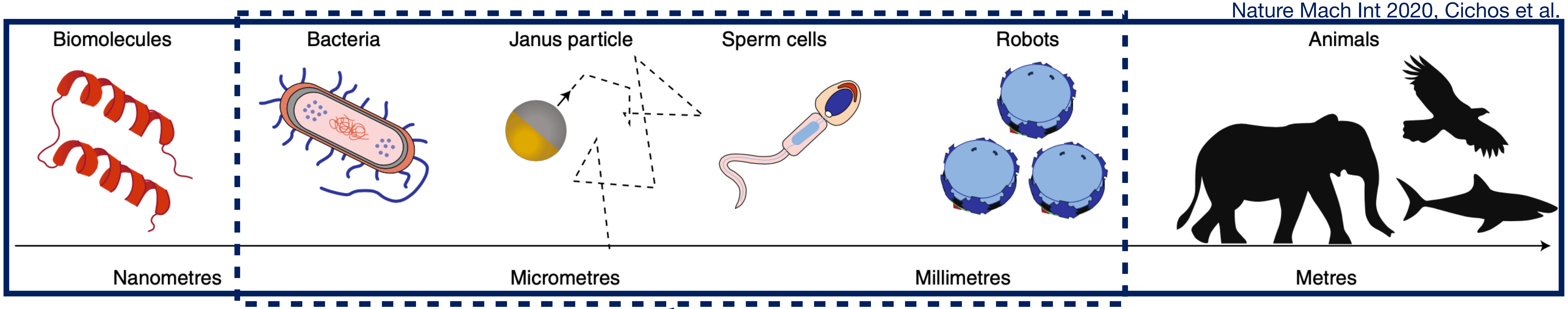
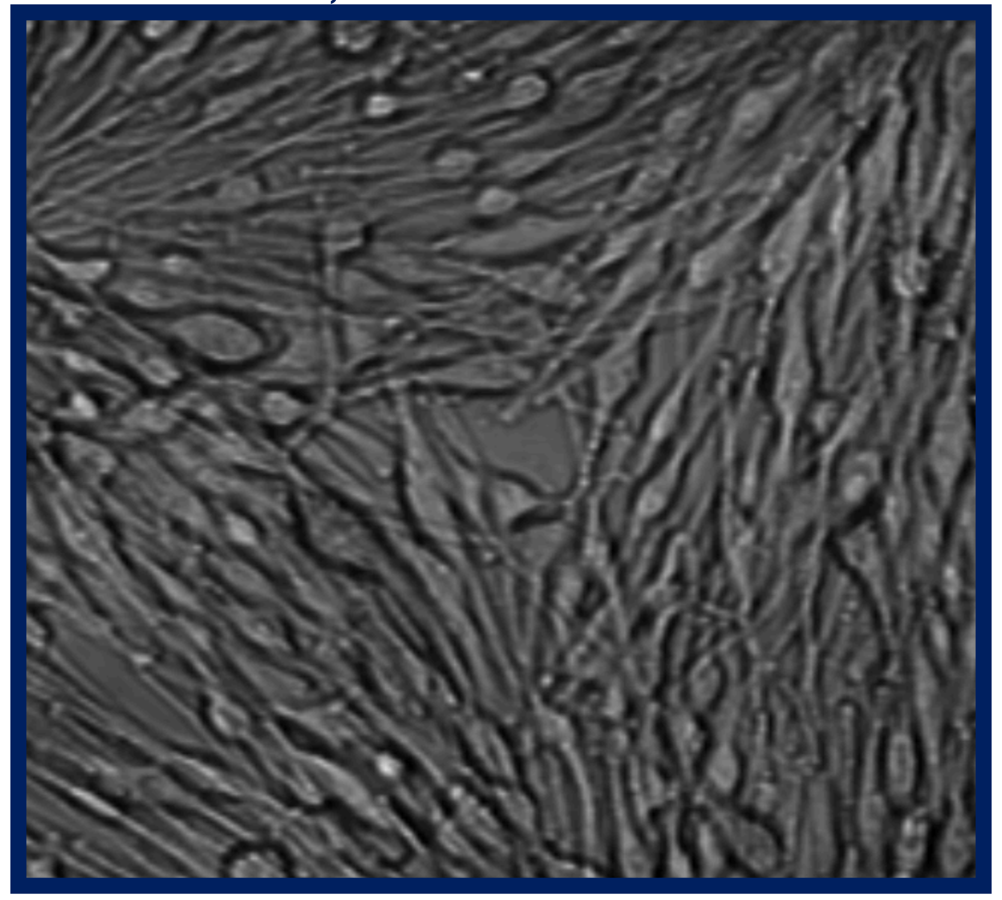


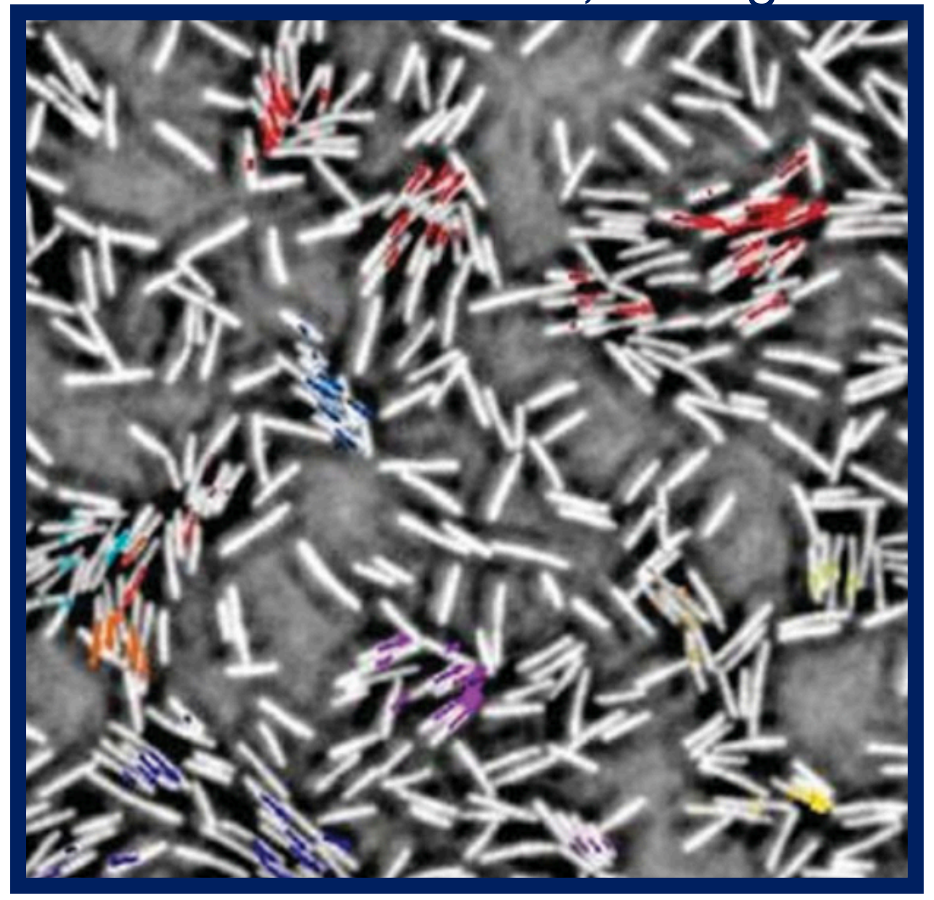
nature

EPJ E 2000, Kemkemer et al.



melanocytic cells

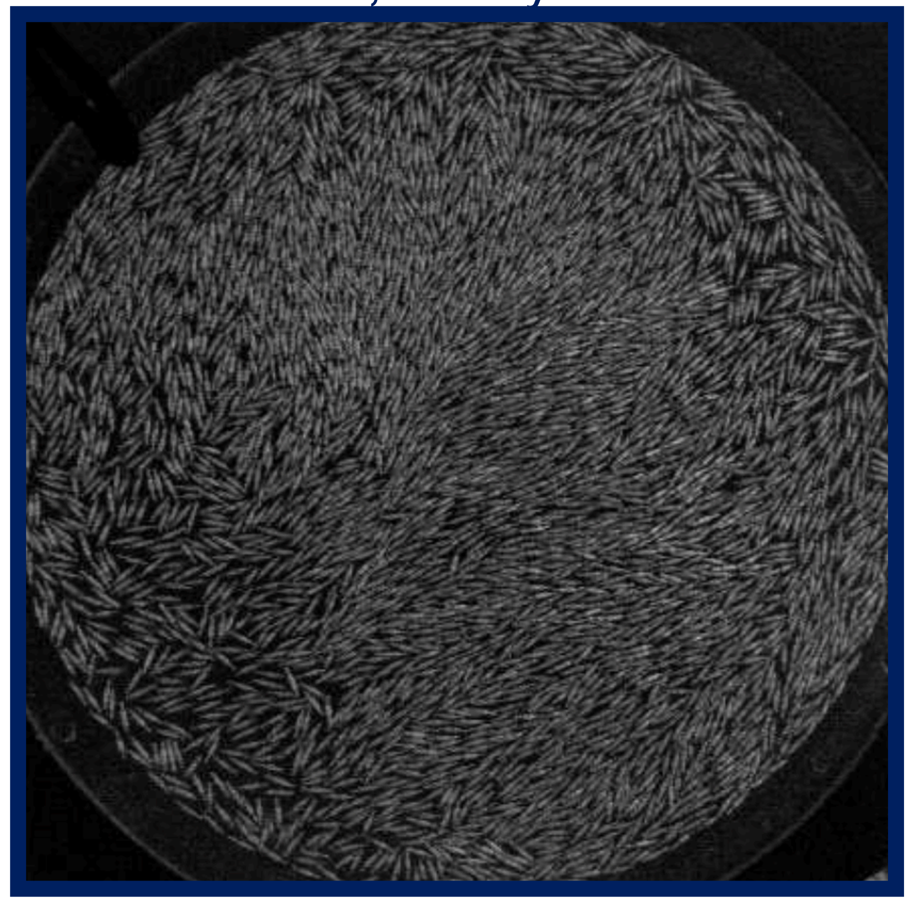
PNAS 2010, Zhang et al.



colonies of bacteria

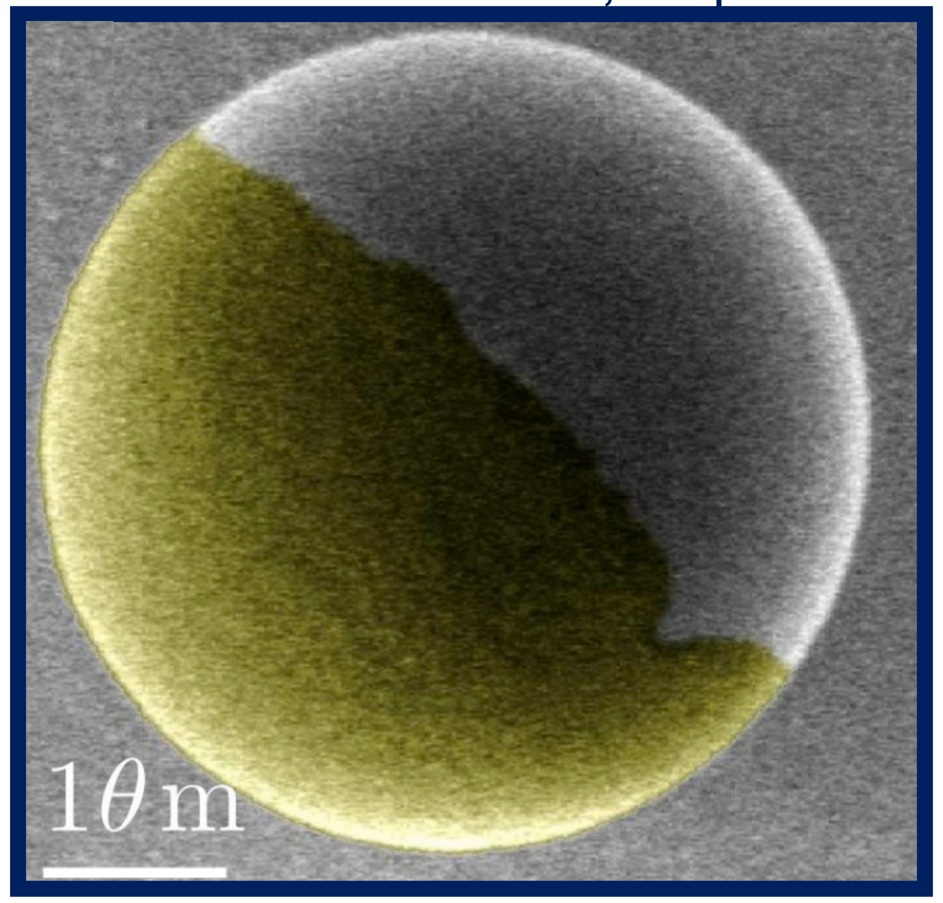
experiments

Science 2007, Narayan et al.



vibrated nanorods

Soft Matter 2011, Volpe et al.



Janus particle

Active Matter

- **Theoretical interest**
  - emergence of new features with no counterpart in passive systems
  - strong connection with biological systems
  - new paradigm of out-of-equilibrium systems

- **Experimental and technological interest**

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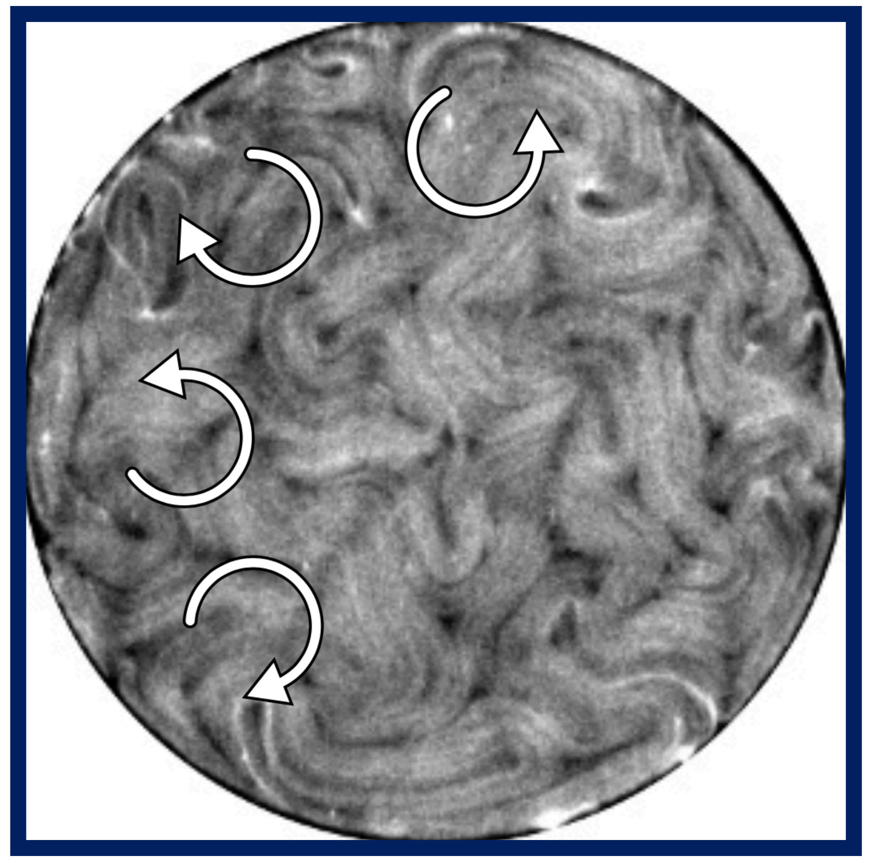
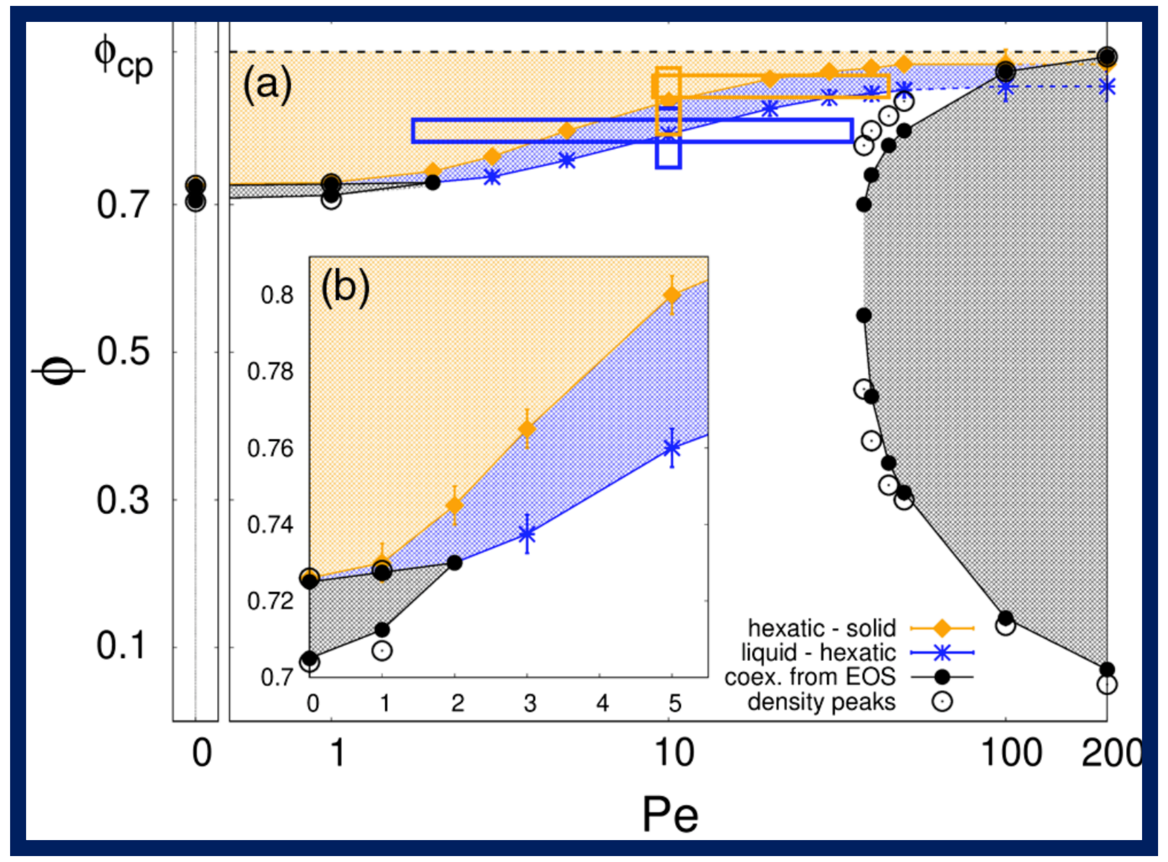
Nature Rev Phys 2022, Shankar et al.



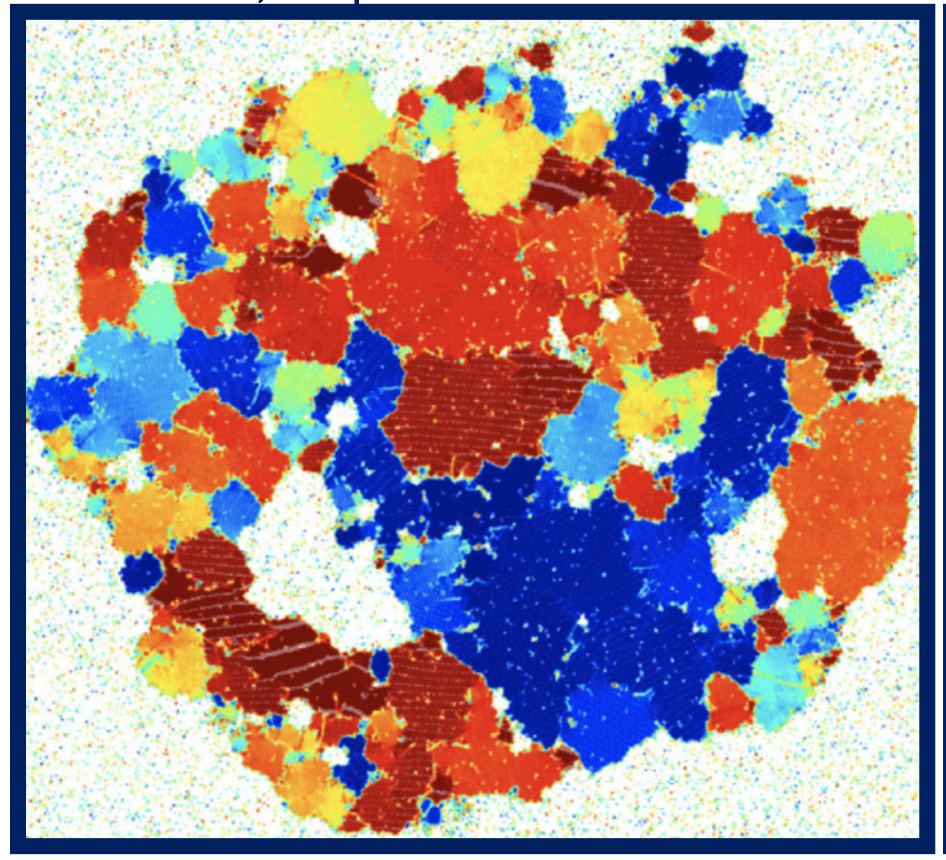
spontaneous flows

PRL 2018, Digregorio et al.

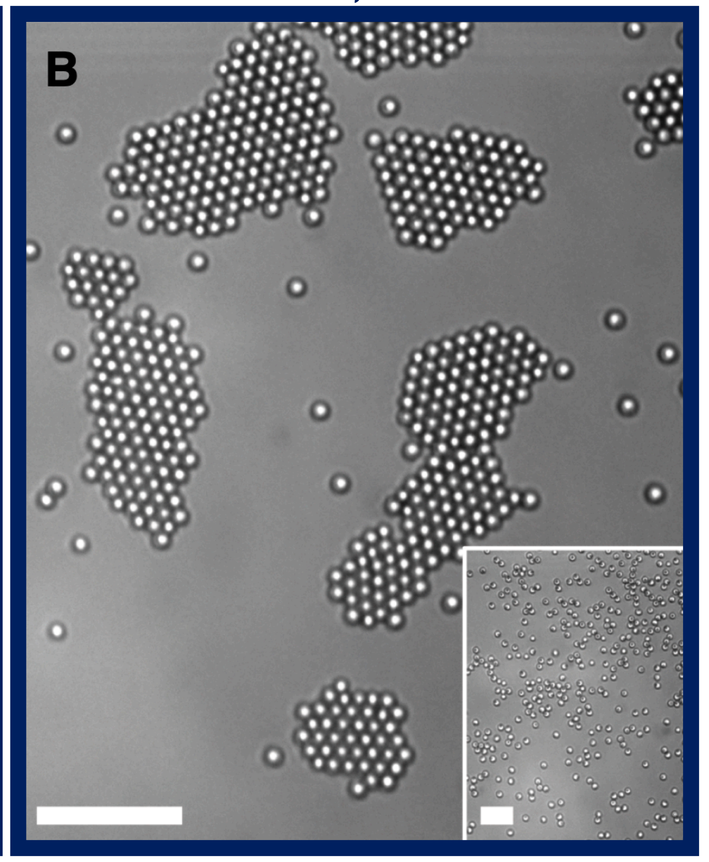


active, liquid, hexatic and solid phases

PRL 2020, Caporusso et al.



Science 2013, Palacci et al.



motility-induced phase separation

- **Experimental and technological interest**

Active Matter

- **Theoretical interest**
  - ▸ emergence of new features with no counterpart in passive systems
  - ▸ strong connection with biological systems
  - ▸ new paradigm of out-of-equilibrium systems
    - inherently out of equilibrium character
    - no imposed boundaries or external driving

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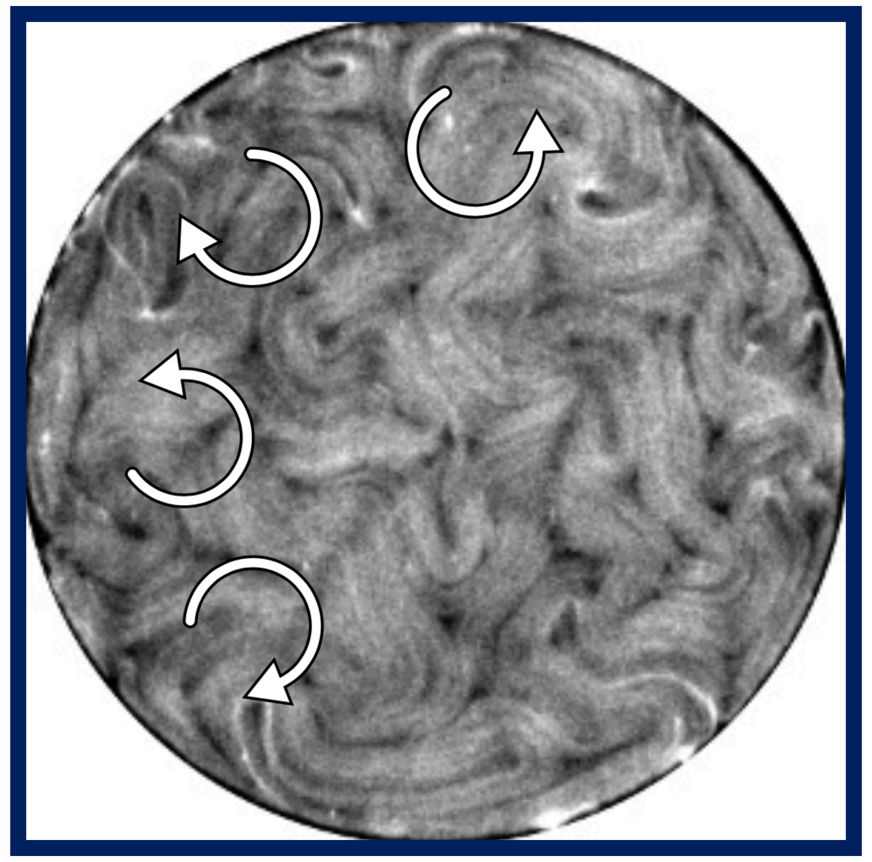

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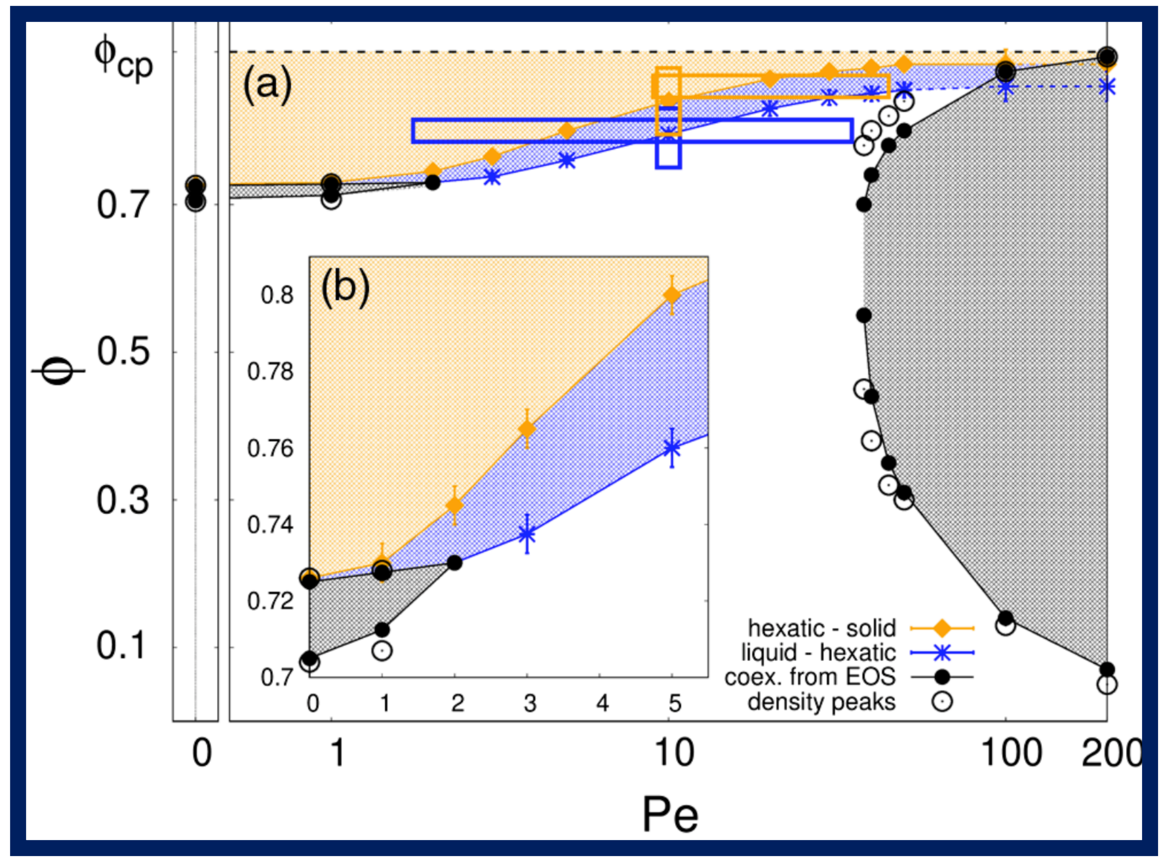

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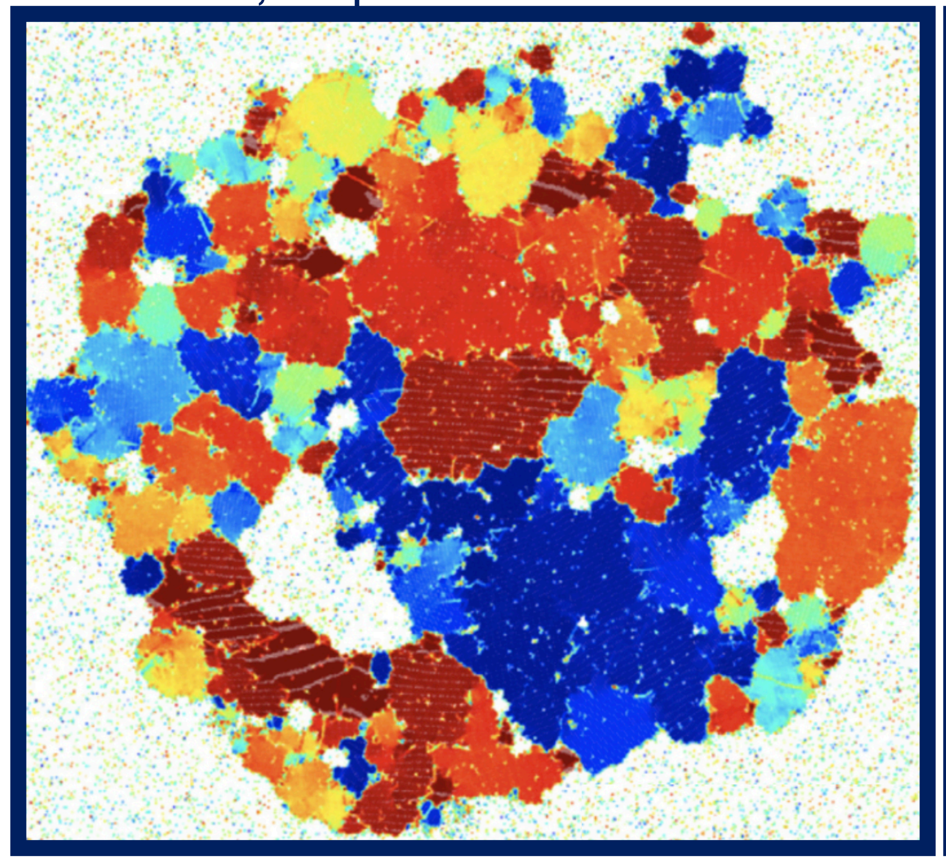

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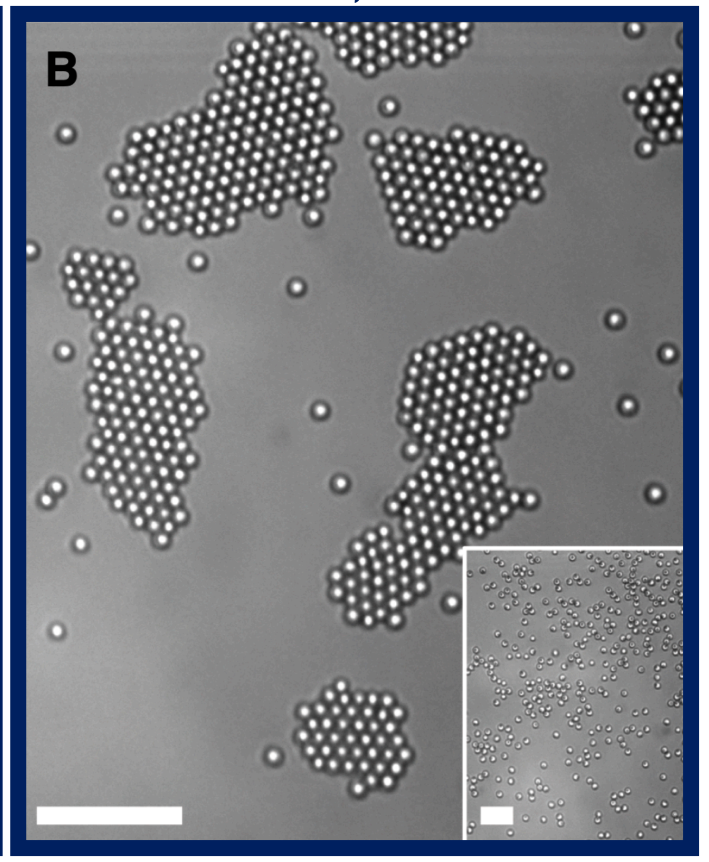

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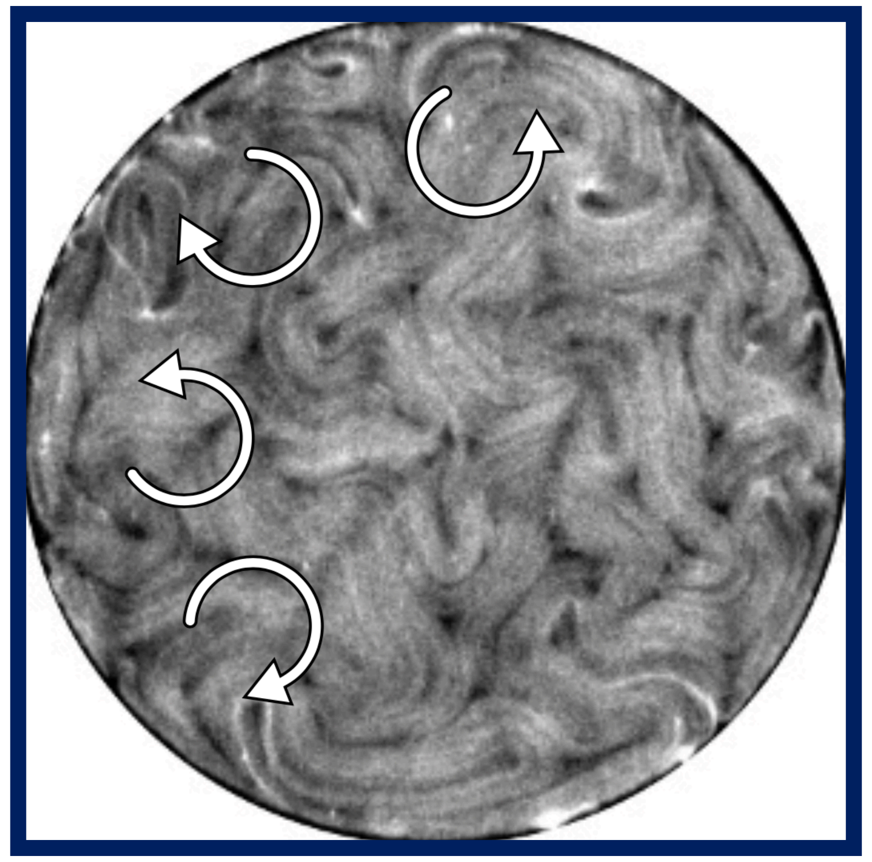

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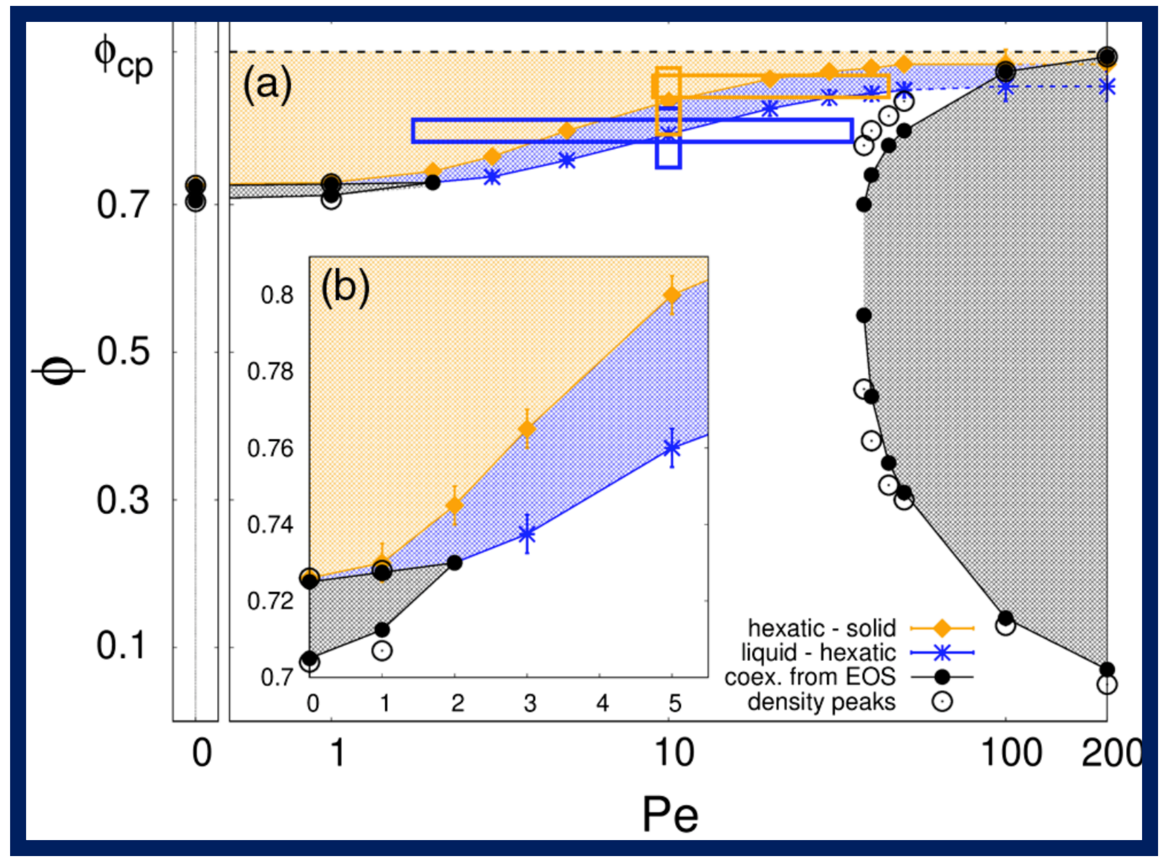

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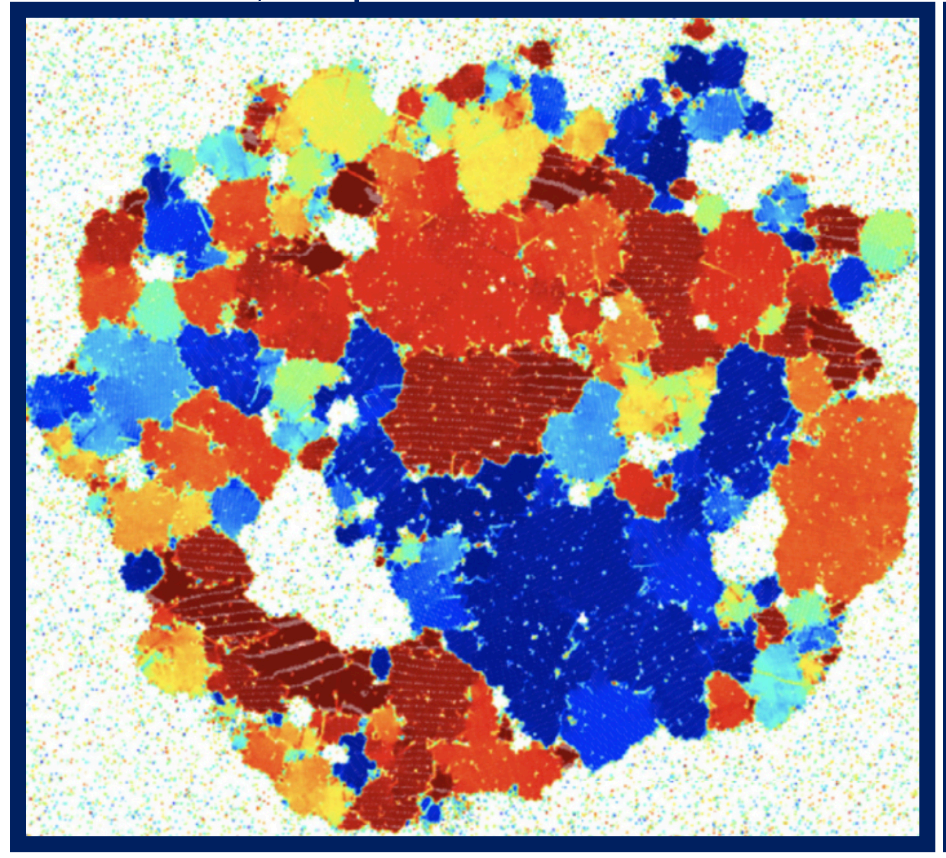

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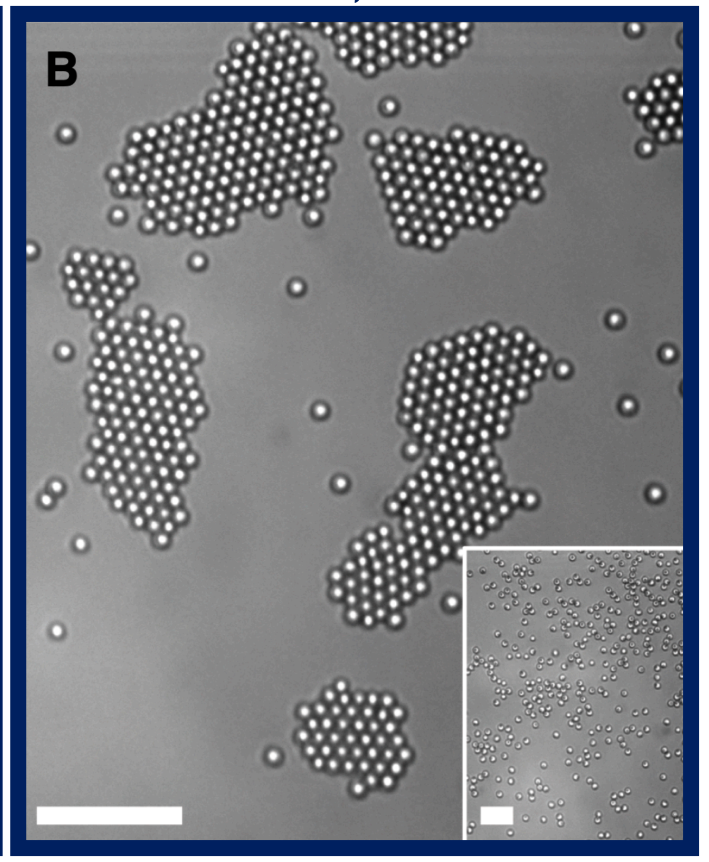

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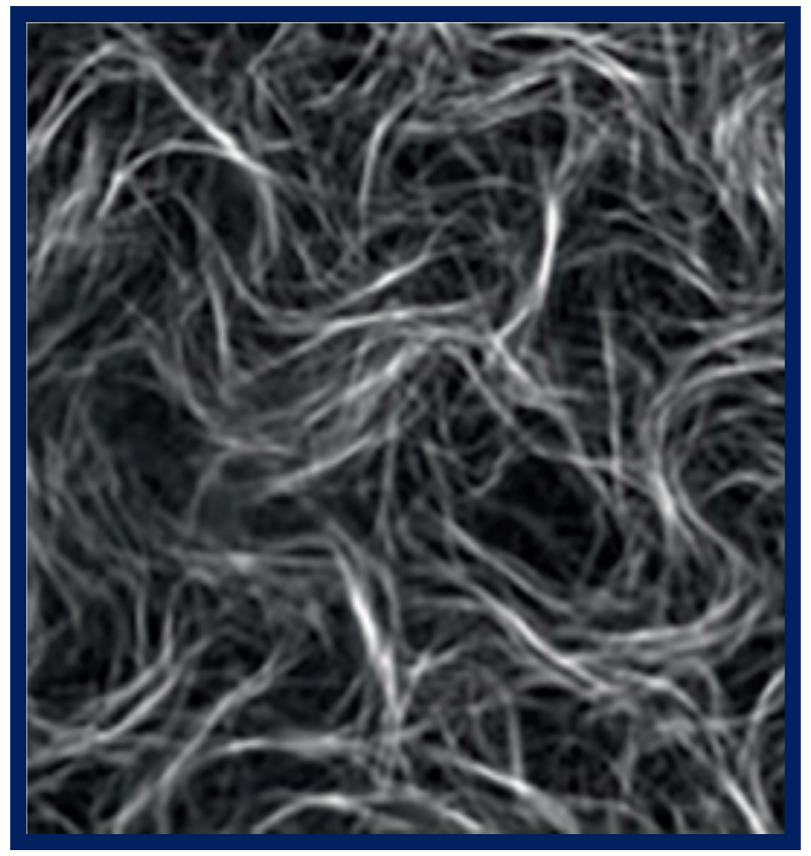
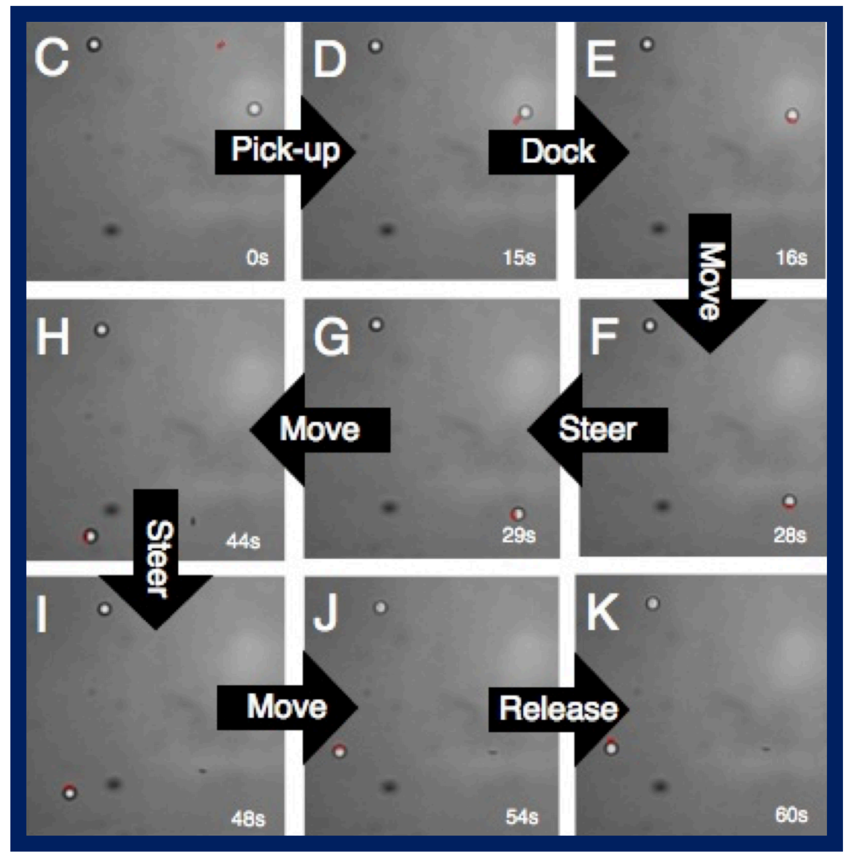
Nature Rev Mat 2017, Needleman et al.



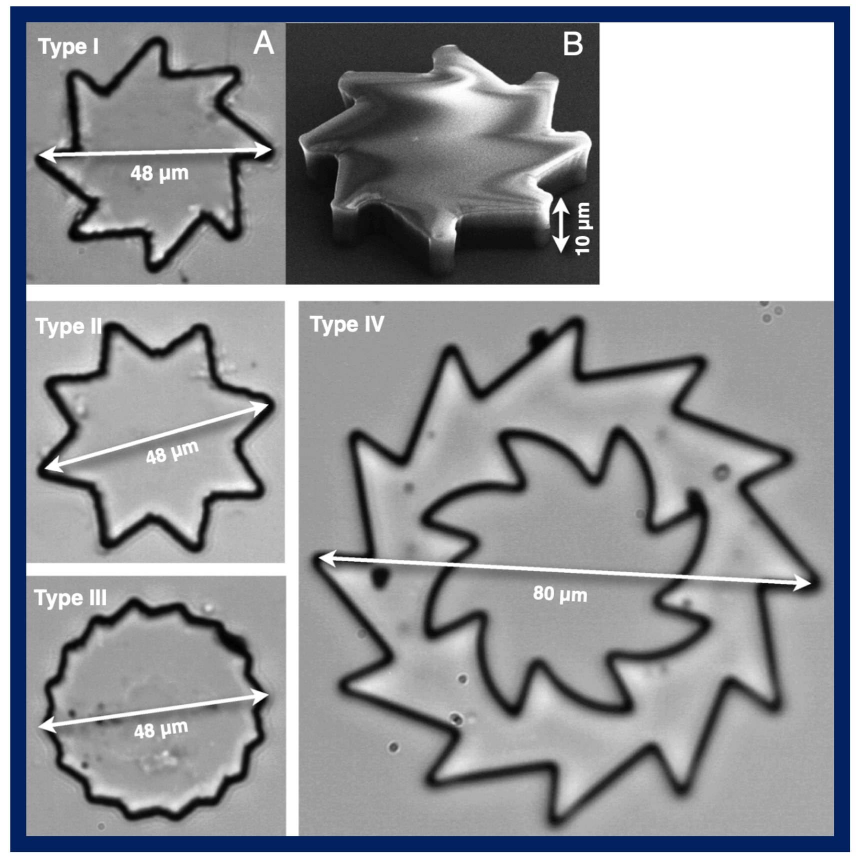
new bio-inspired materials

J American Chem Soc 2013, Palacci et al.



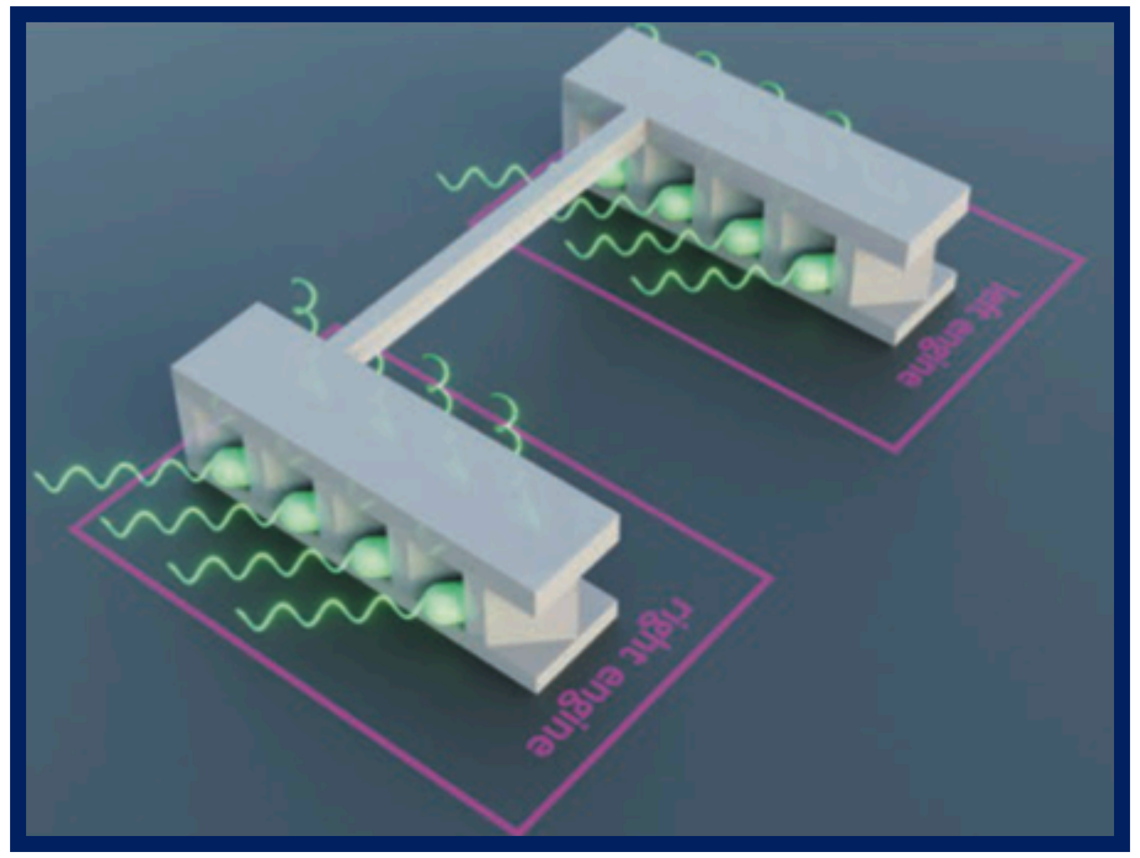
targeted delivery

Proc Nat Acad Sci Usa 2017, Di Leonardo et al.



active engines

Adv Funct Mat 2023, Pellicciotta et al.

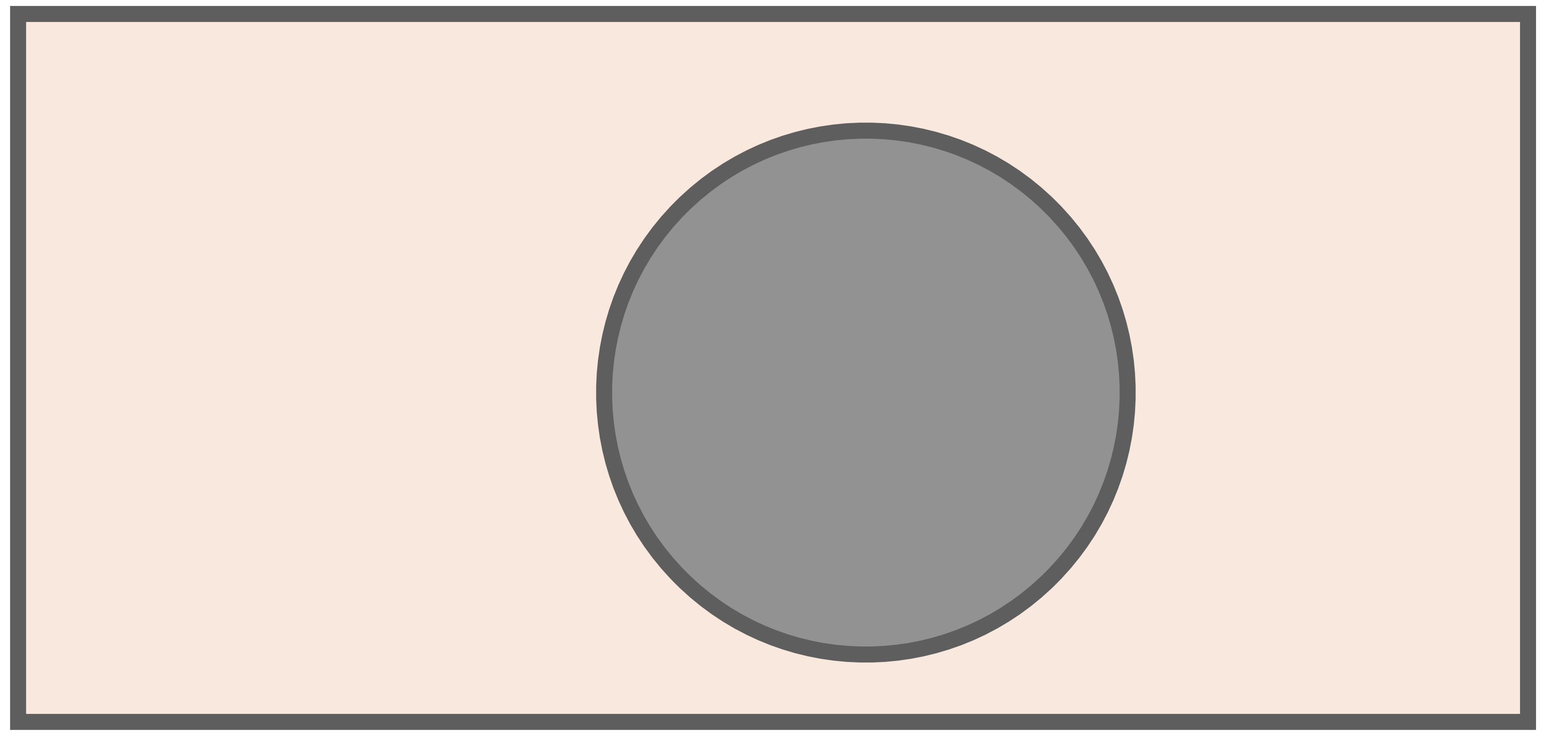


nanorobots

Brownian and Active particles models

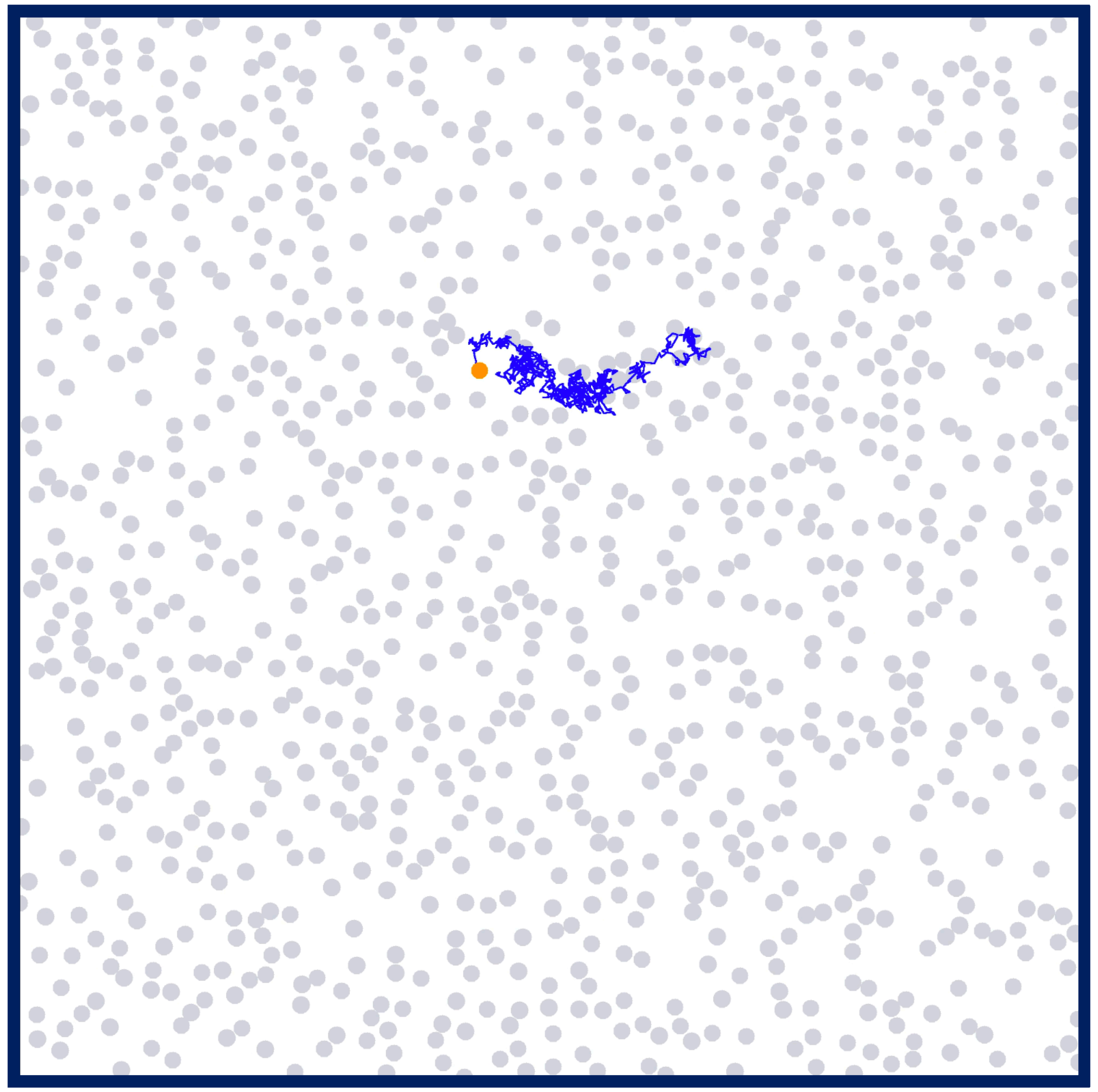
- **Langevin equations** Physics A 2017, Fodor et al

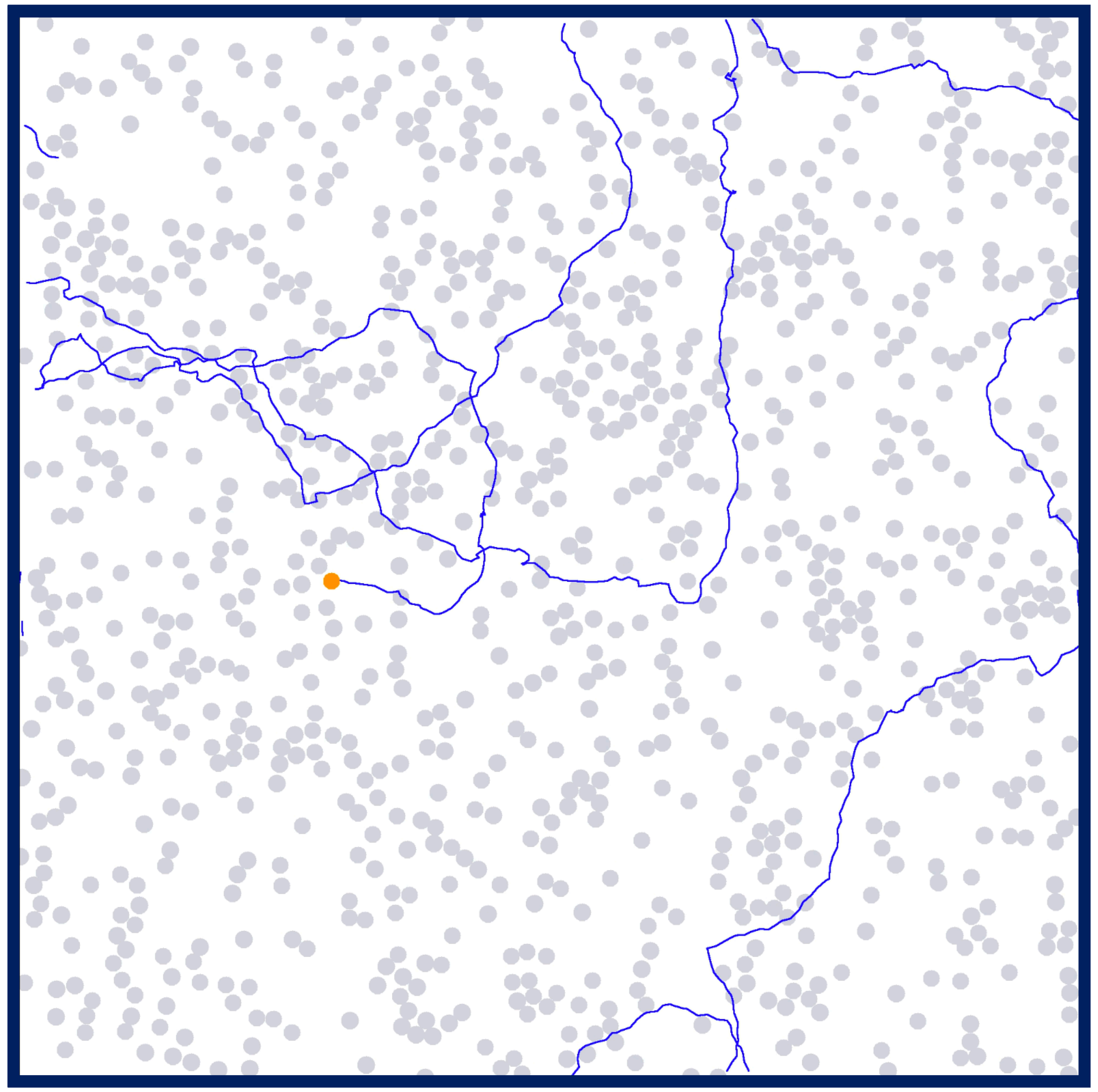
$$\ddot{x}_i(t) = -\frac{dU(x(t))}{dx_i(t)} + f_i(t) - \gamma \dot{x}_i(t) + \sqrt{2D}\ \xi_i(t)$$



- **Passive vs Active motion**

Passive Brownian Particle





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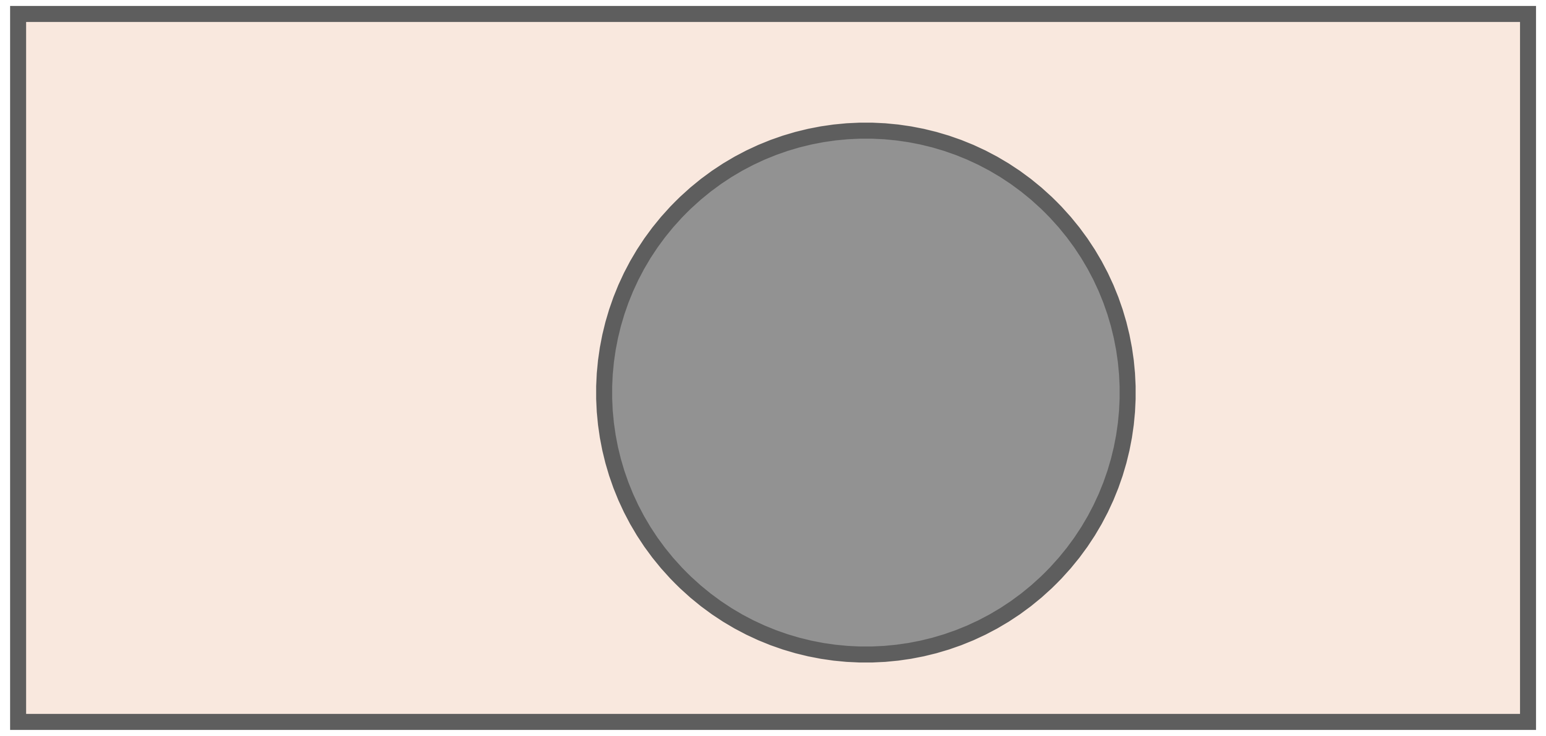

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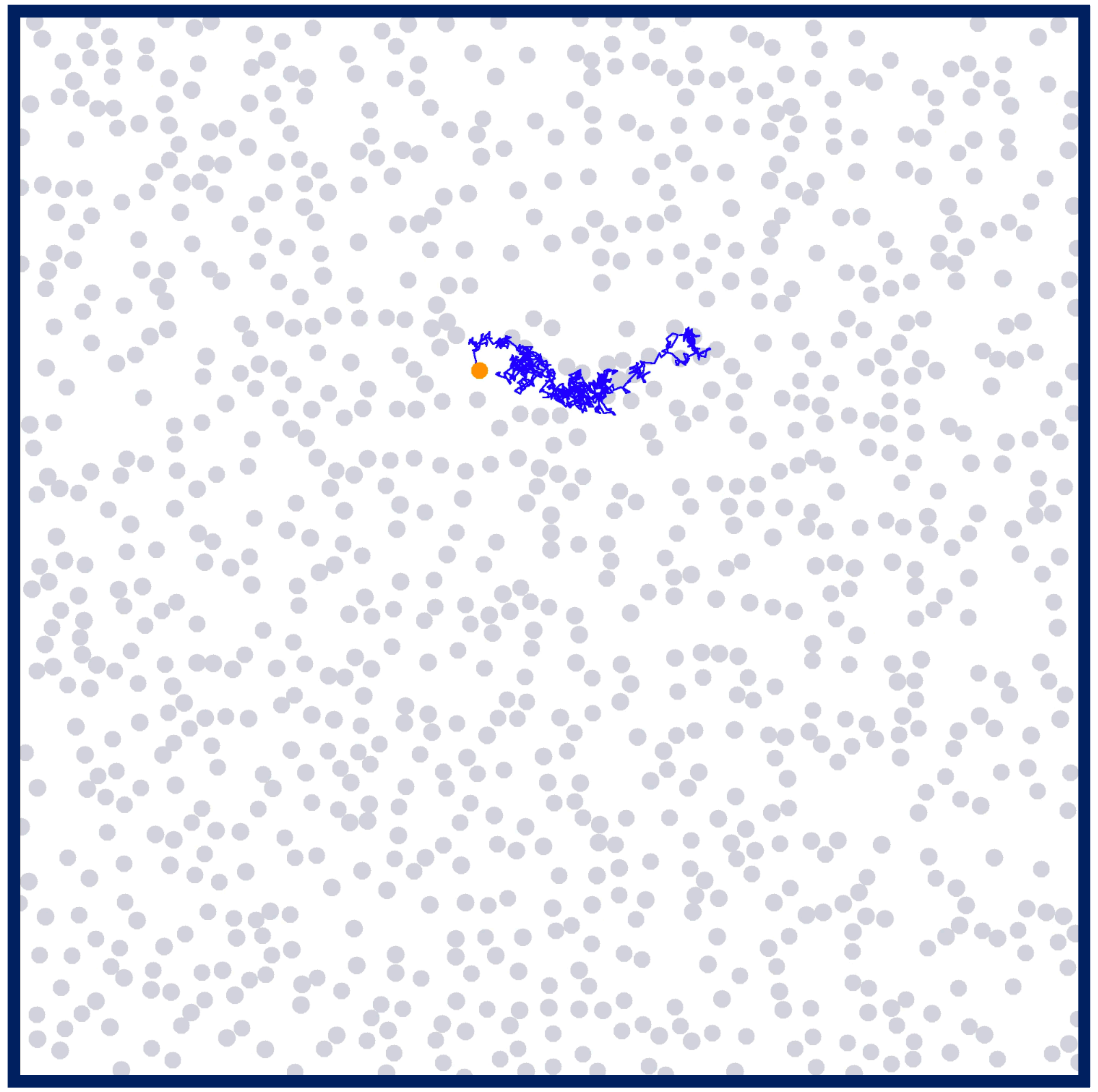

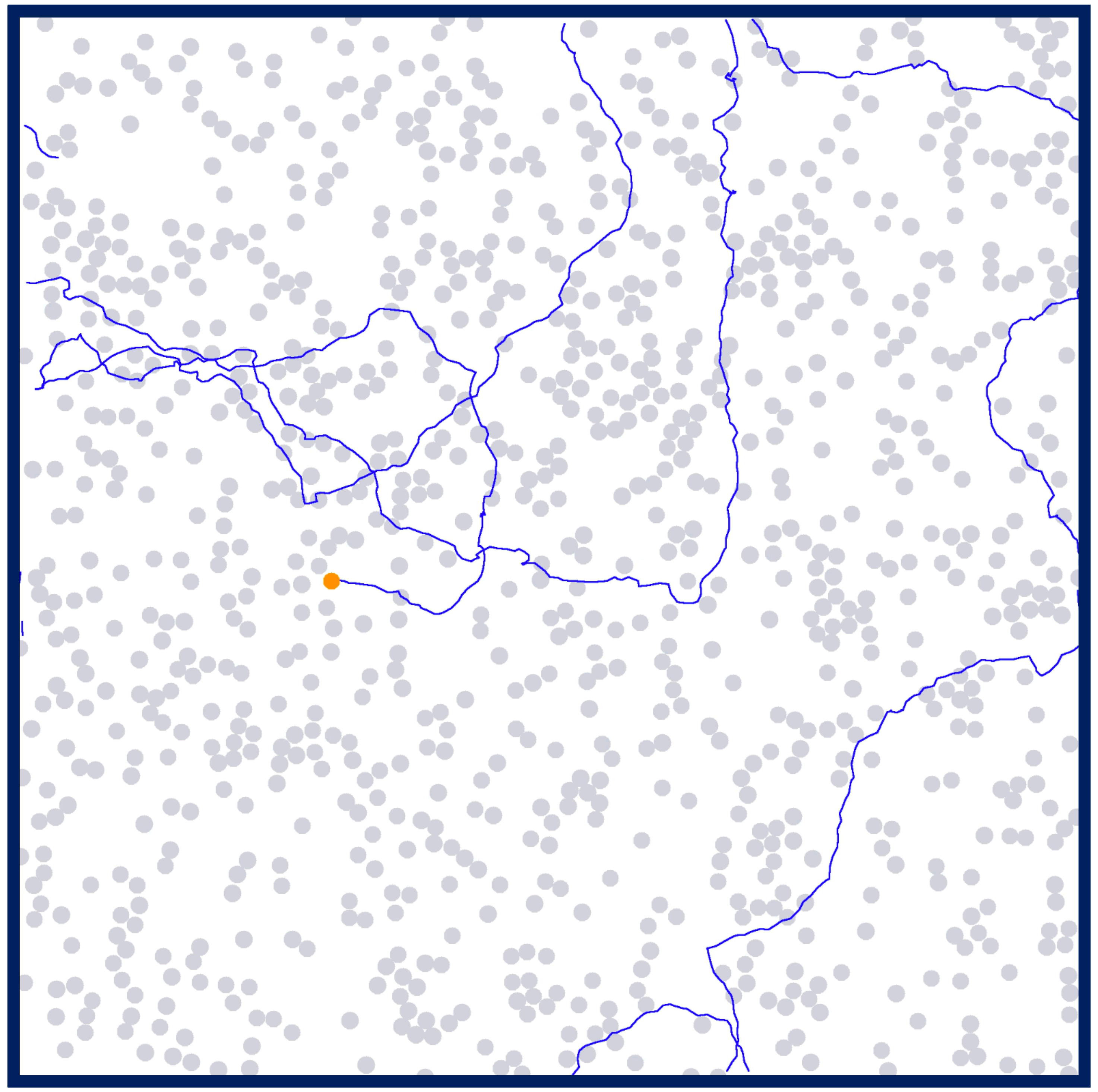

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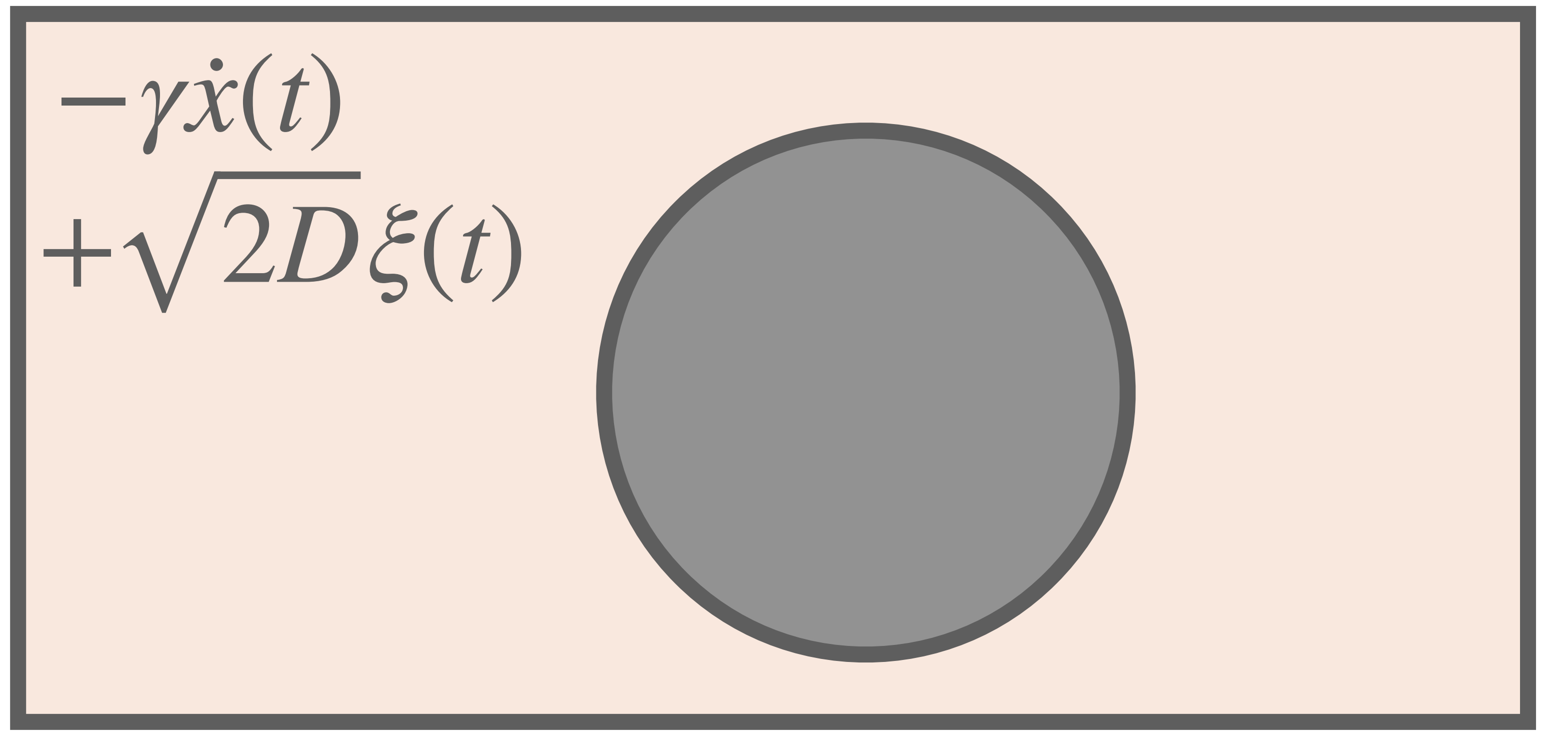

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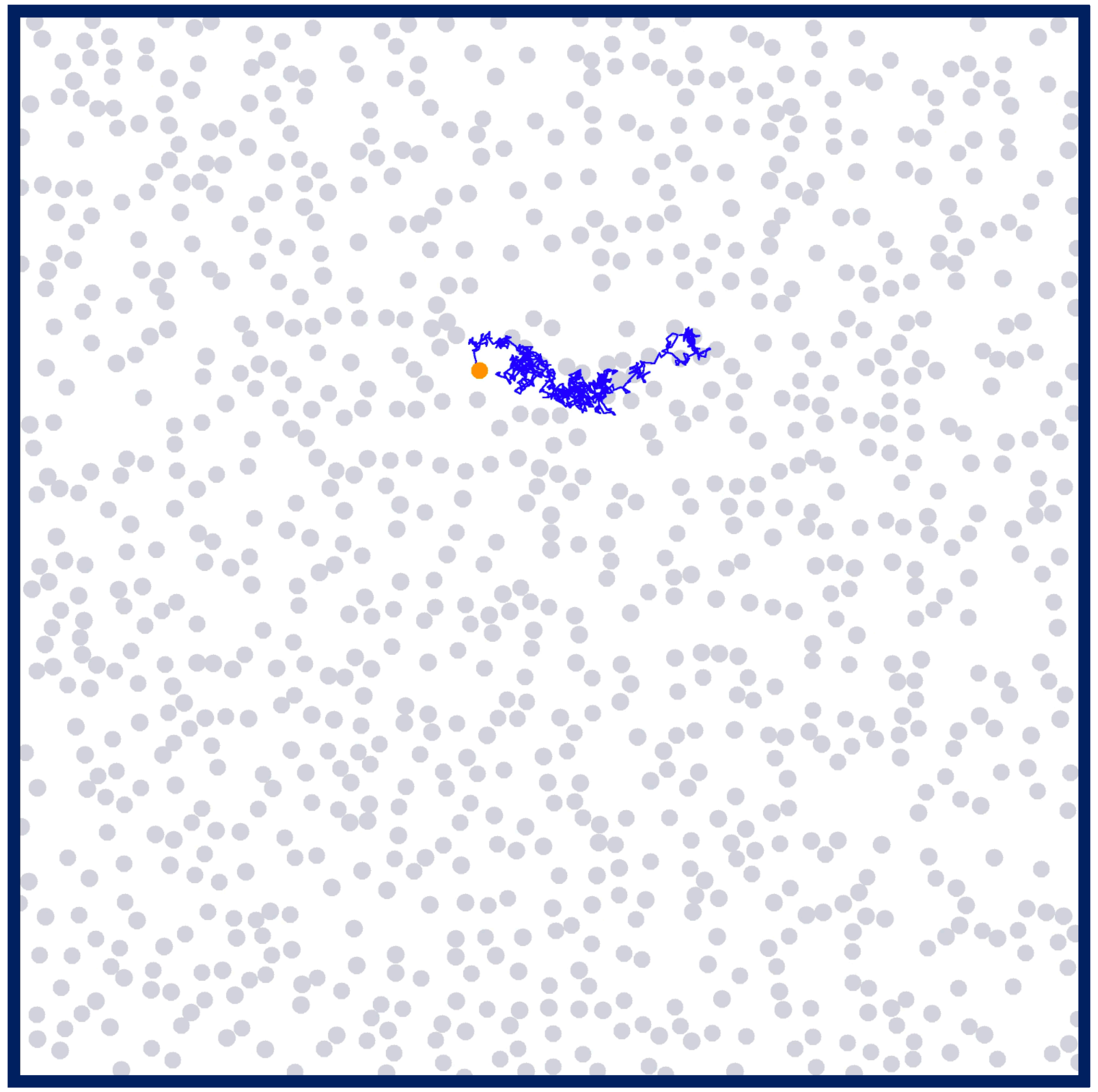

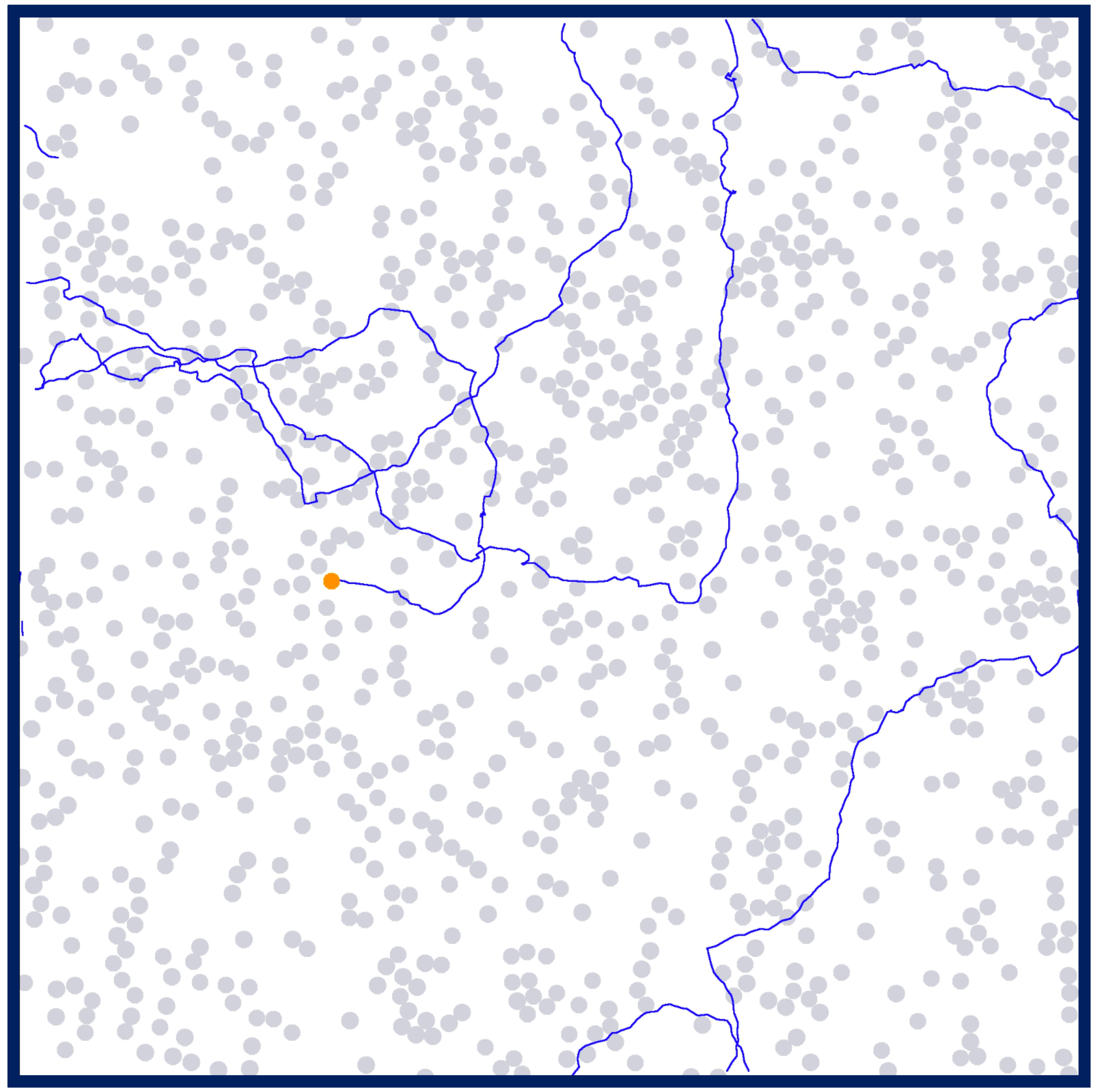

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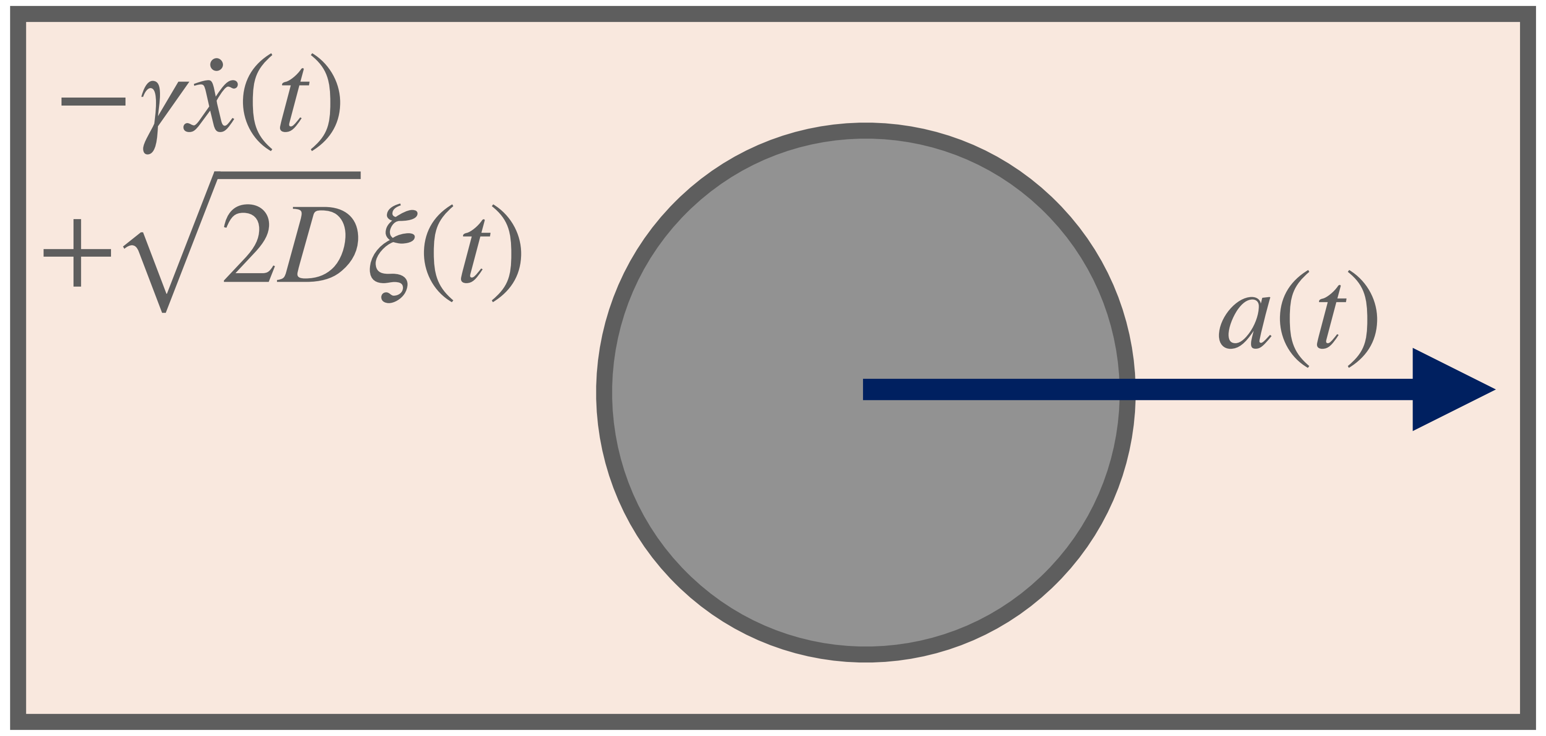

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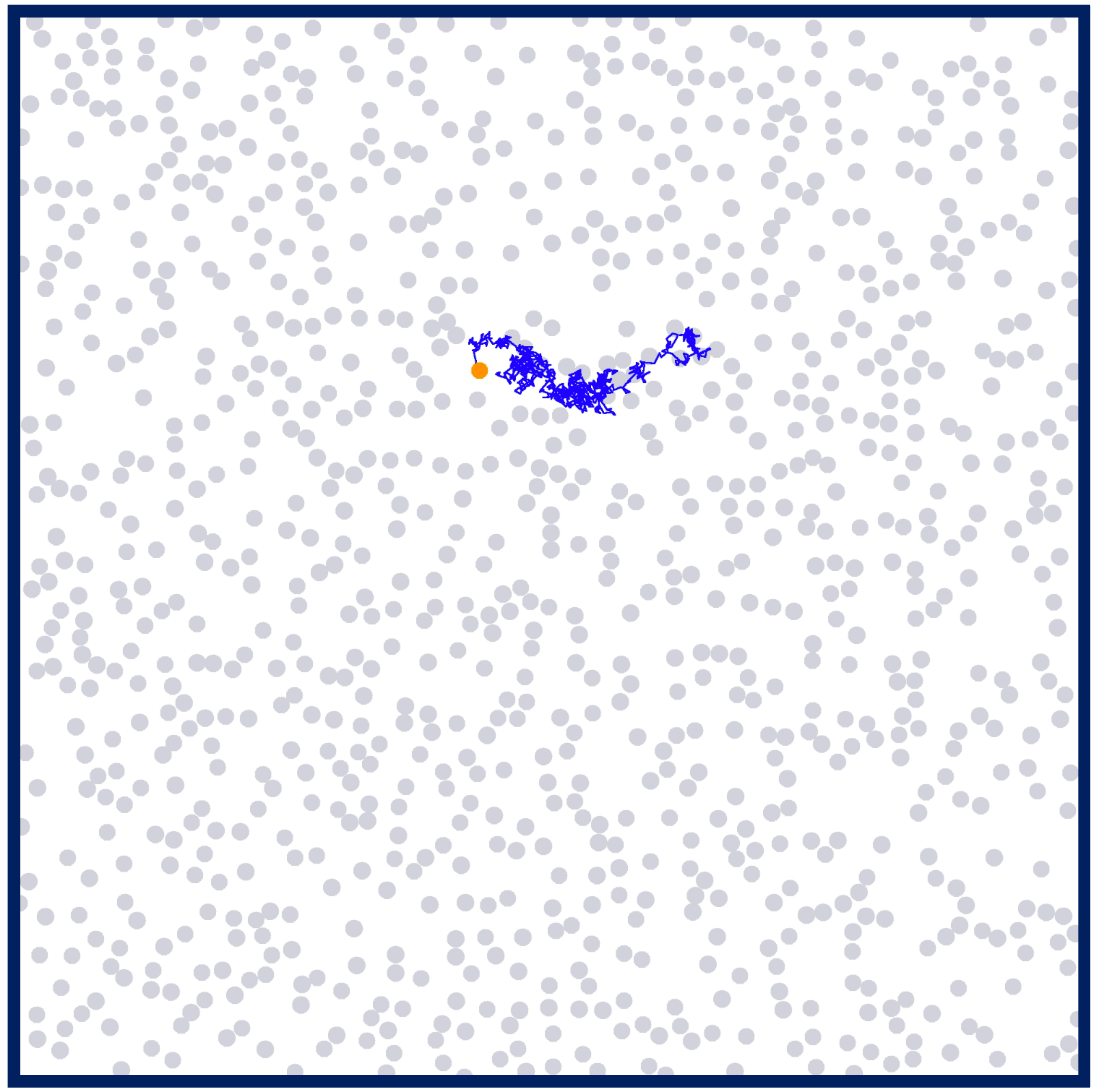

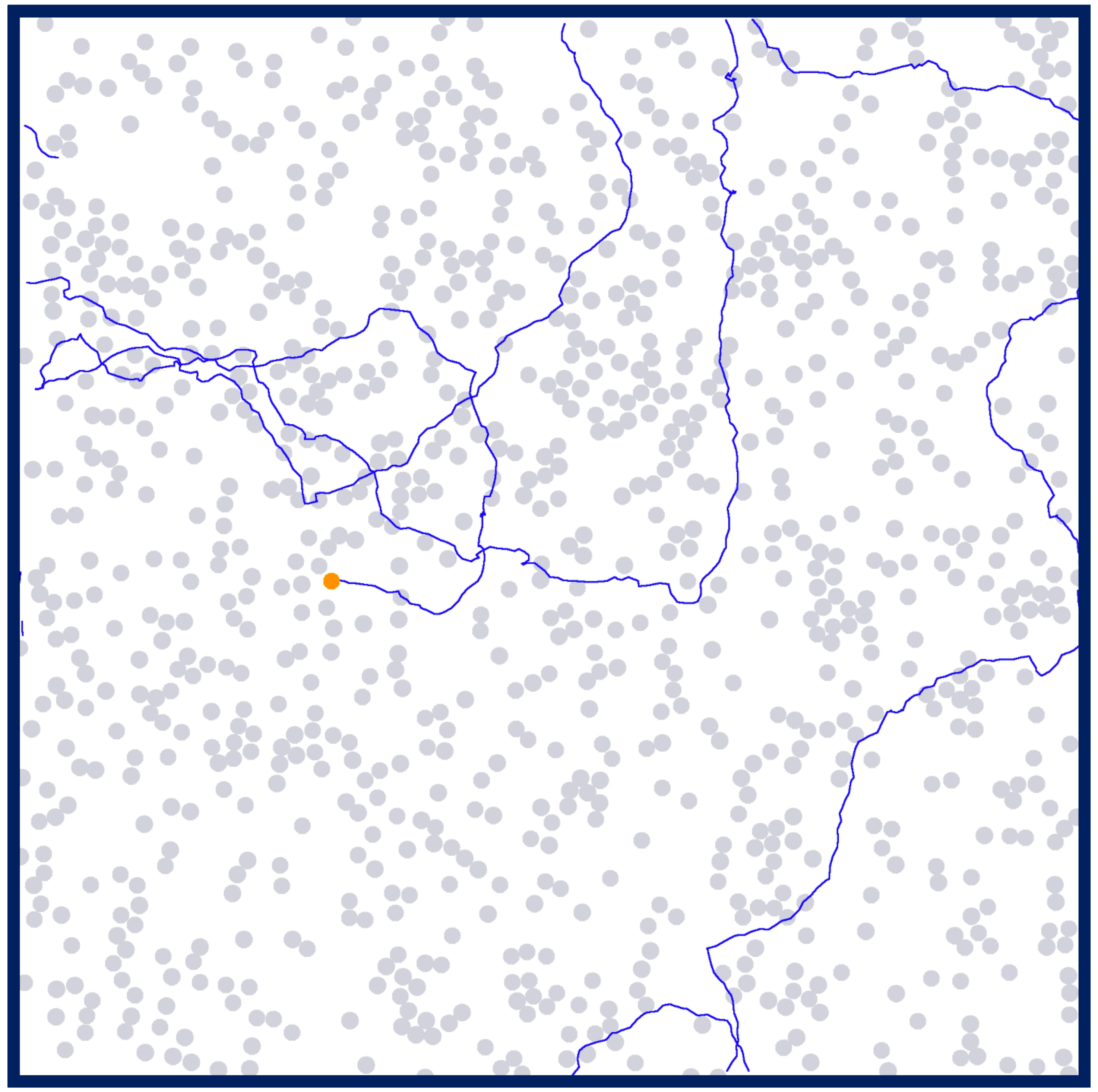

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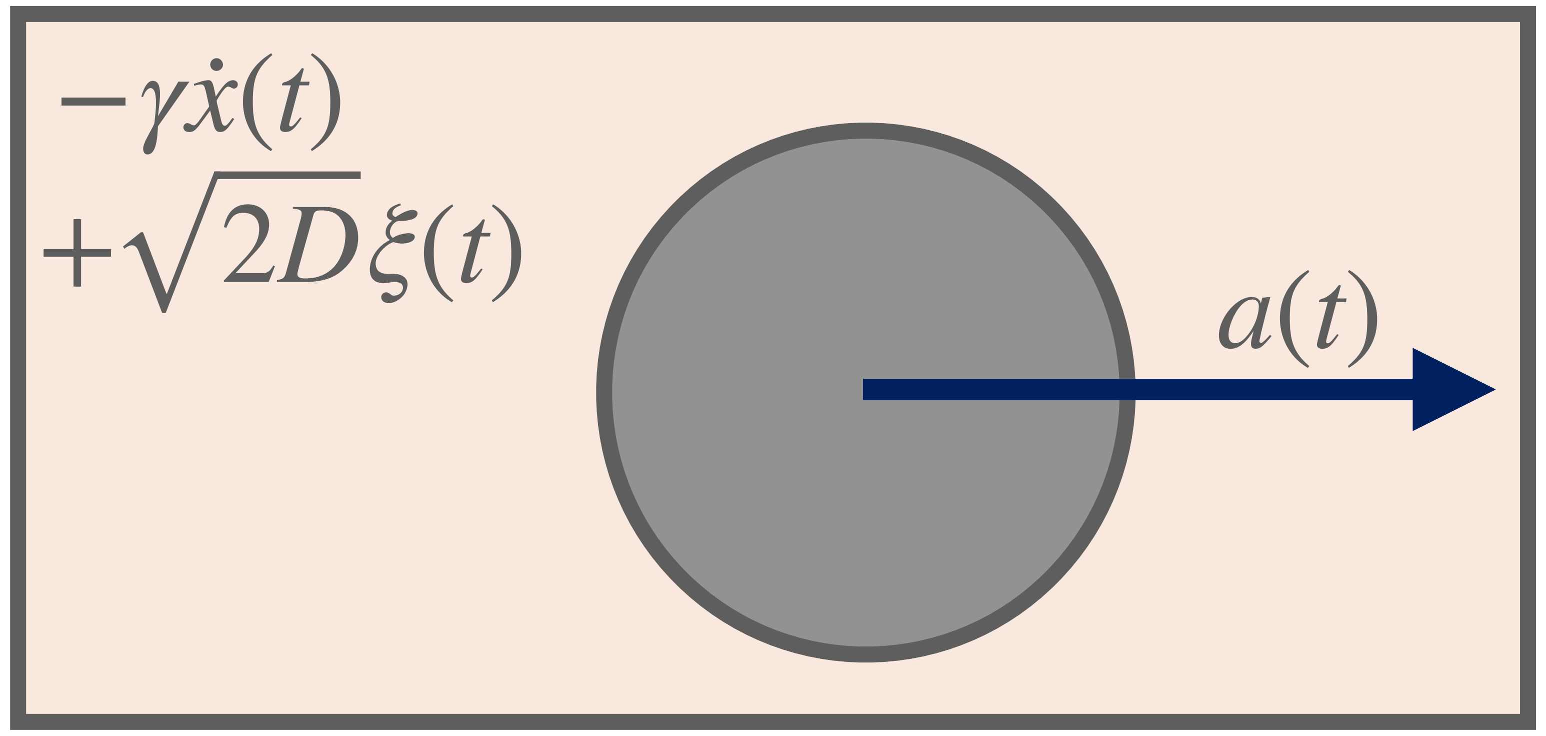

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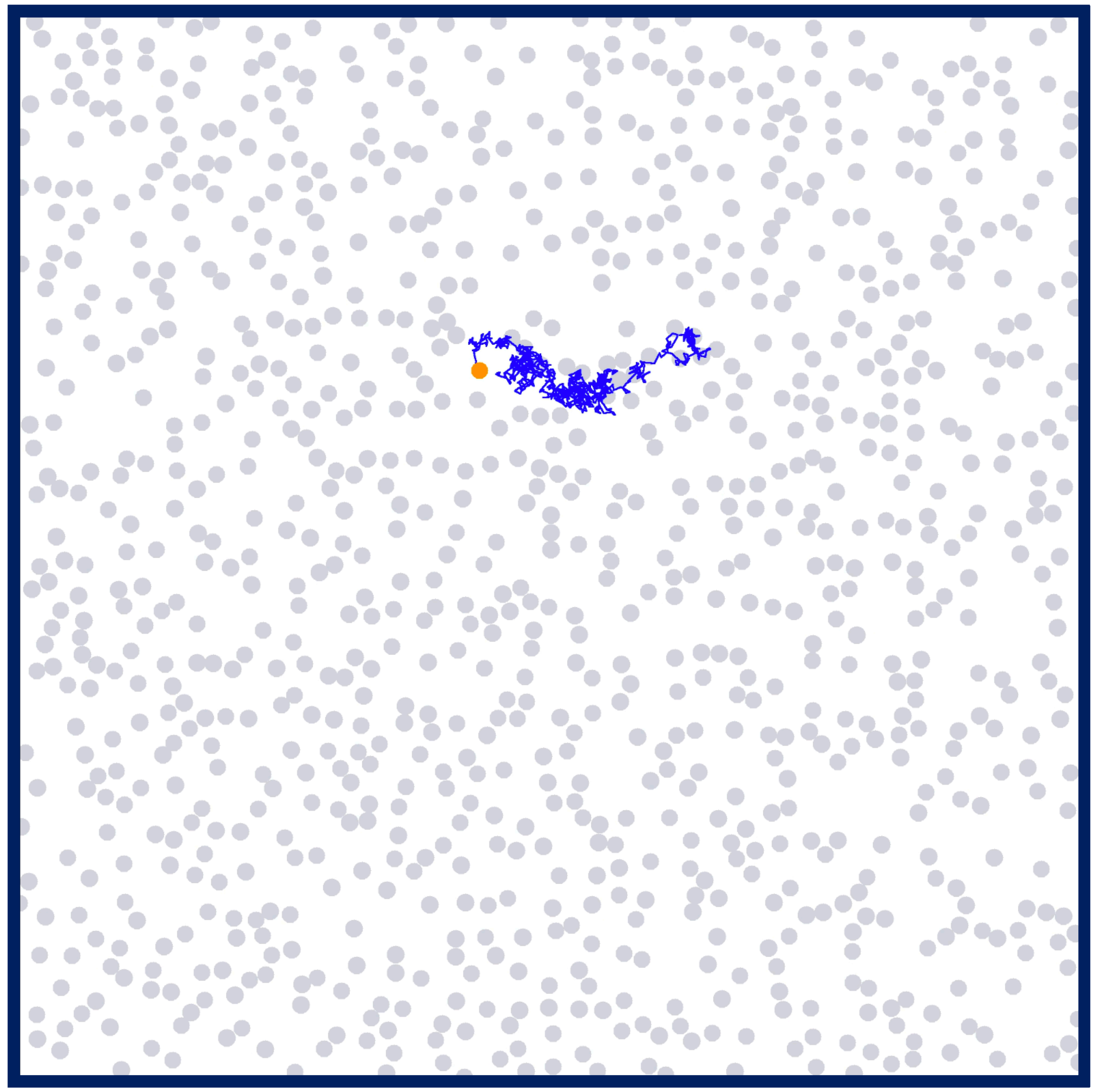
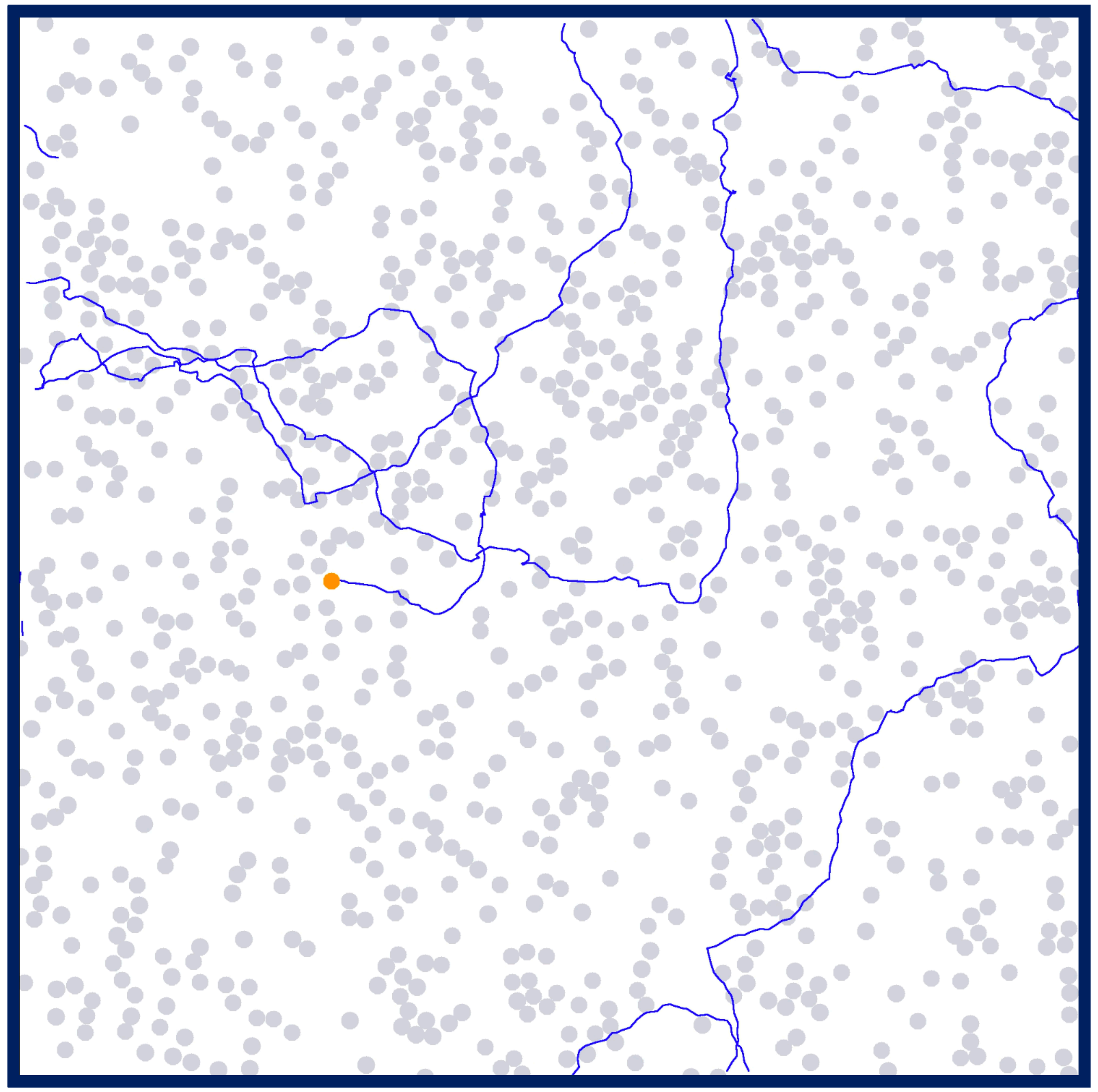




Active Brownian Particle

Large Deviations Theory

Dembo and Zeitouni 1988, Springer

Phys Rep 2009, Touchette

- **Dynamical observables**

$$\mathscr{W}_\tau = \frac{1}{\tau}\int_0^\tau G(x(s), \dot{x}(s), a(s)) \circ \underbrace{dx(s)}_{dx(s) \to \dot{x}(s)\, ds}$$

  - integrated observables measured along particle trajectories
  - G generic function of positions velocity and active force
  - $1/\tau$ essential to make $\mathscr{W}_\tau$ intensive in time

- **Large Deviations Theory**

$$p(\mathscr{W}_\tau = w) \boxed{\asymp} e^{-\tau I(w)}$$

  - asymptotic equivalence

$I(w)$ **Rate Function (RF)**

  - extension of thermodynamic potentials to out of equilibrium configurations

$$\phi(\lambda) = \lim_{\tau\to\infty} \frac{1}{\tau} \ln(\langle e^{\lambda \mathscr{W}_\tau}\rangle)$$

Scaled Cumulant Generating Function (SCGF)

  - function whose derivatives generate th distribution moments

$$I(w) = \sup_{\lambda \in O}\{\lambda w - \phi(\lambda)\}$$

  - RF and SCGF often related through Legendre-Fenchel transform

Dynamical Phase Transitions

- Rate Functions can be singular
  Entropy 2019, Corberi et al

- **Dynamical Phase Transitions**
  change in the physical mechanism producing fluctuations
- **Trajectory Separation**
  trajectories in different regions of the RF behave dynamically different

- Many examples in the in Brownian and active motion

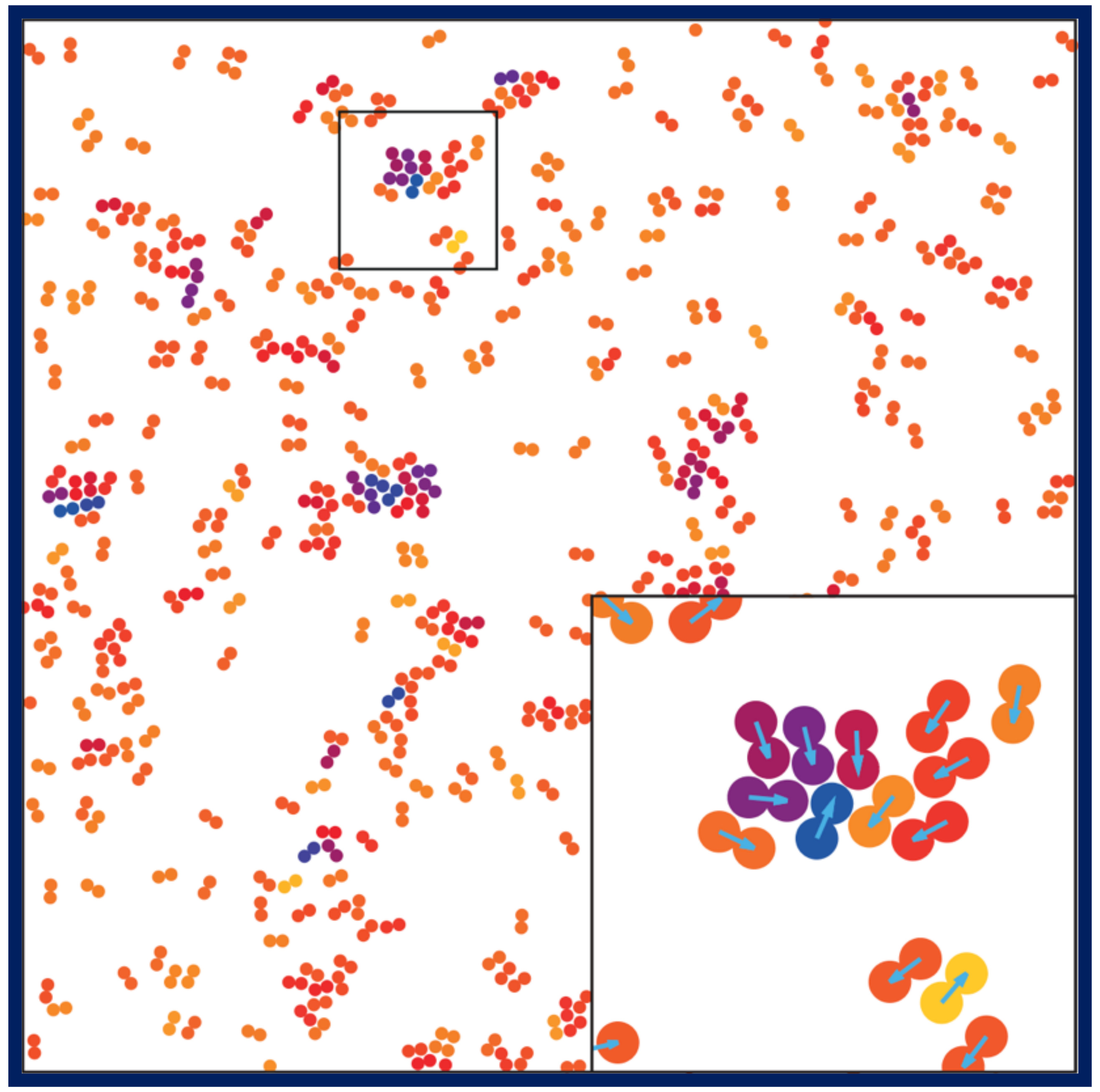
J Stat Phys 2022, Farago; PRL 2003, van Zon et al; J Stat Mech 2006, Visco; PRE 2018, Nyawo et al; J Stat Mech 2019, Gradenigo et al; PRE 2013, Pal et al

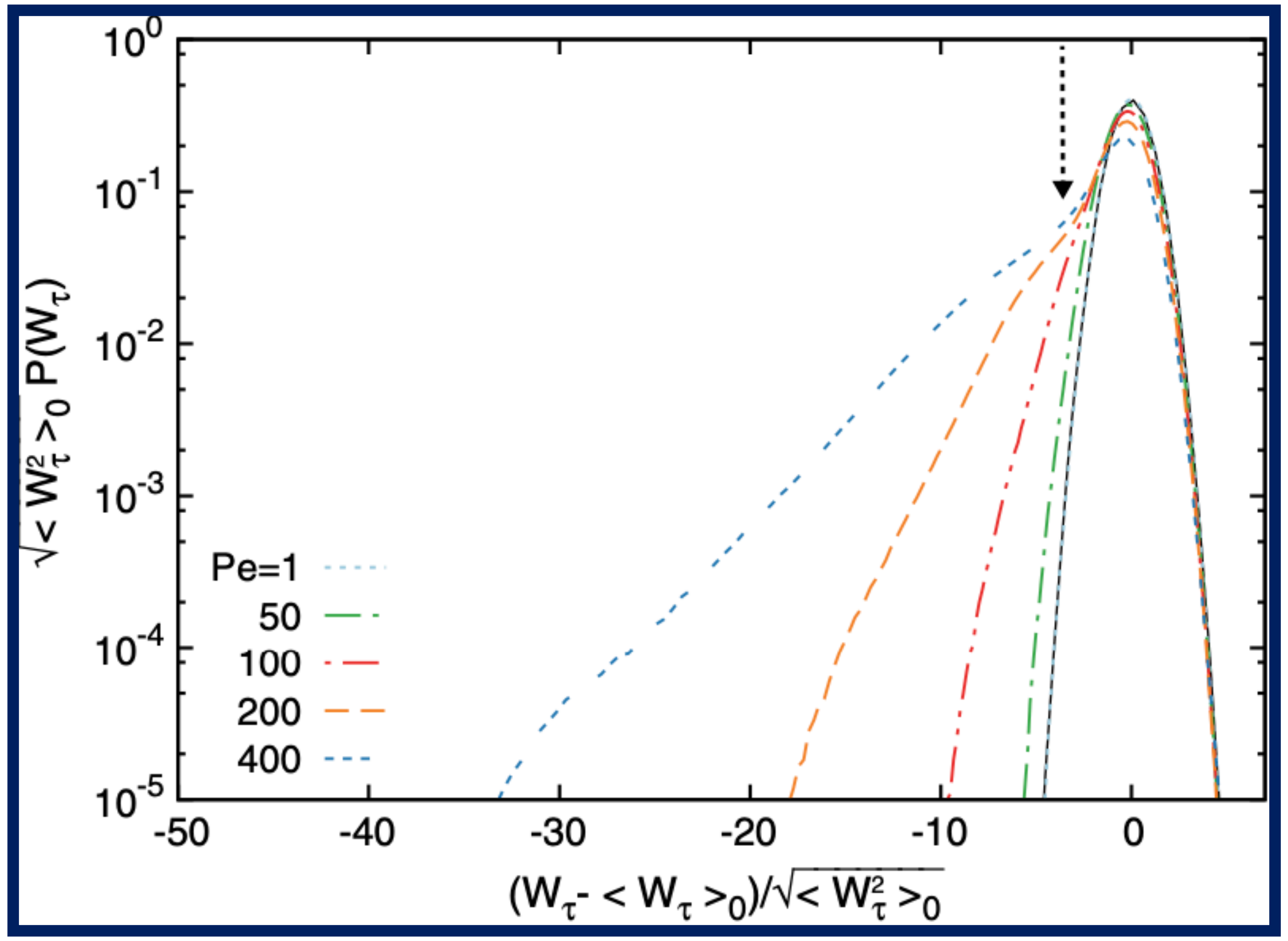
- Active Work in a system of interacting active particles

PRL 2017, Cagnetta et al

$$W_\tau = \frac{1}{\tau}\int_0^\tau a(t)v(t)\ dt$$

- In singular trajectories particles dragged against their active force





Analytical study of fluctuations of Active Work

- **Setting**
  single particle with external potential

  - confining potentials mimic the trapping of other particles at finite density — Soft Matter 2017, Nandi et al
  - experimental realisations — PRL 2002, Wang et al
  - analytical results feasible

- Active Ornstein-Uhlenbeck Particle **(AOUP)**
  free or with external harmonic potential

$$\begin{cases} \dot{x}(t) = F_a \gamma^{-1} a(t) - kx(t) + \sqrt{2T/\gamma}\ \xi(t) \\ \dot{a}(t) = -\gamma_R a(t) + \sqrt{2D_R}\ \eta(t) \end{cases}$$

- **Approach**
  analytical evaluation of the Rate Function through Large Deviations techniques

- **Interest**
  - **Theoretical** — PRX 2019, Pietzonka et al
    - energy cost to sustain self propulsion
  - **Practical** — EPL 2021, Fodor et al
    - thermodynamic efficiency of Active Engines

- **Scope**
  - investigation of distribution singularities and Dynamical Phase Transitions

Free AOUP

J Stat Mech 2021,
Semeraro, Suma, Petrelli, Cagnetta and Gonnella

- Free AOUP in d dimensions

$$\begin{cases} \dot{x}(t) = F_a \gamma^{-1} a(t) + \sqrt{2T/\gamma}\ \xi(t) \\ \dot{a}(t) = -\gamma_R a(t) + \sqrt{2D_R}\ \eta(t) \end{cases}$$

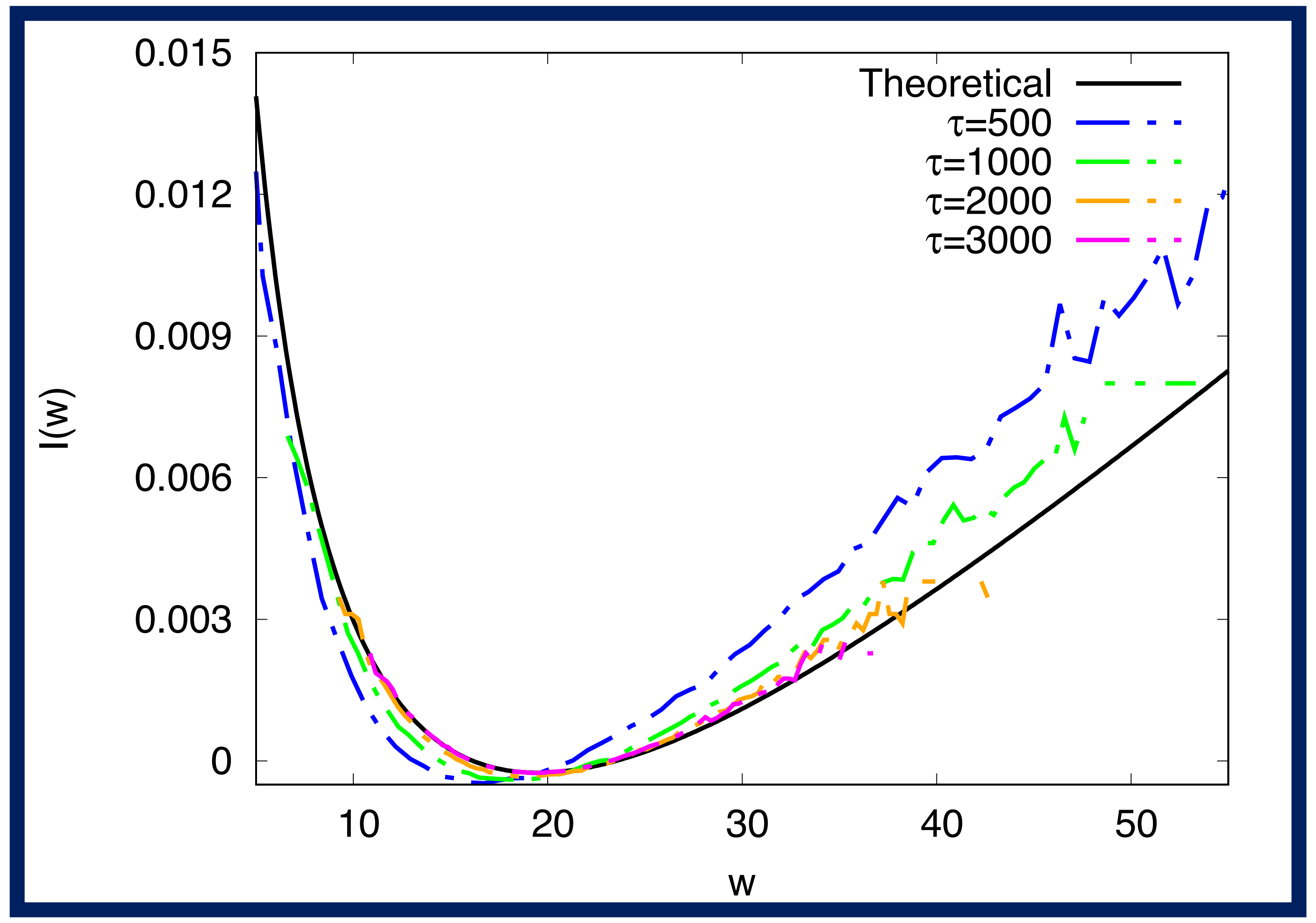
$$< a(t)a(t') > \simeq \left(e^{-\gamma_R(t-t')} - e^{-\gamma_R(t+t')}\right)$$

- Active work $\mathscr{W}_a = \frac{1}{\tau}\int_0^\tau a(t)\dot{r}(t)\ dt$

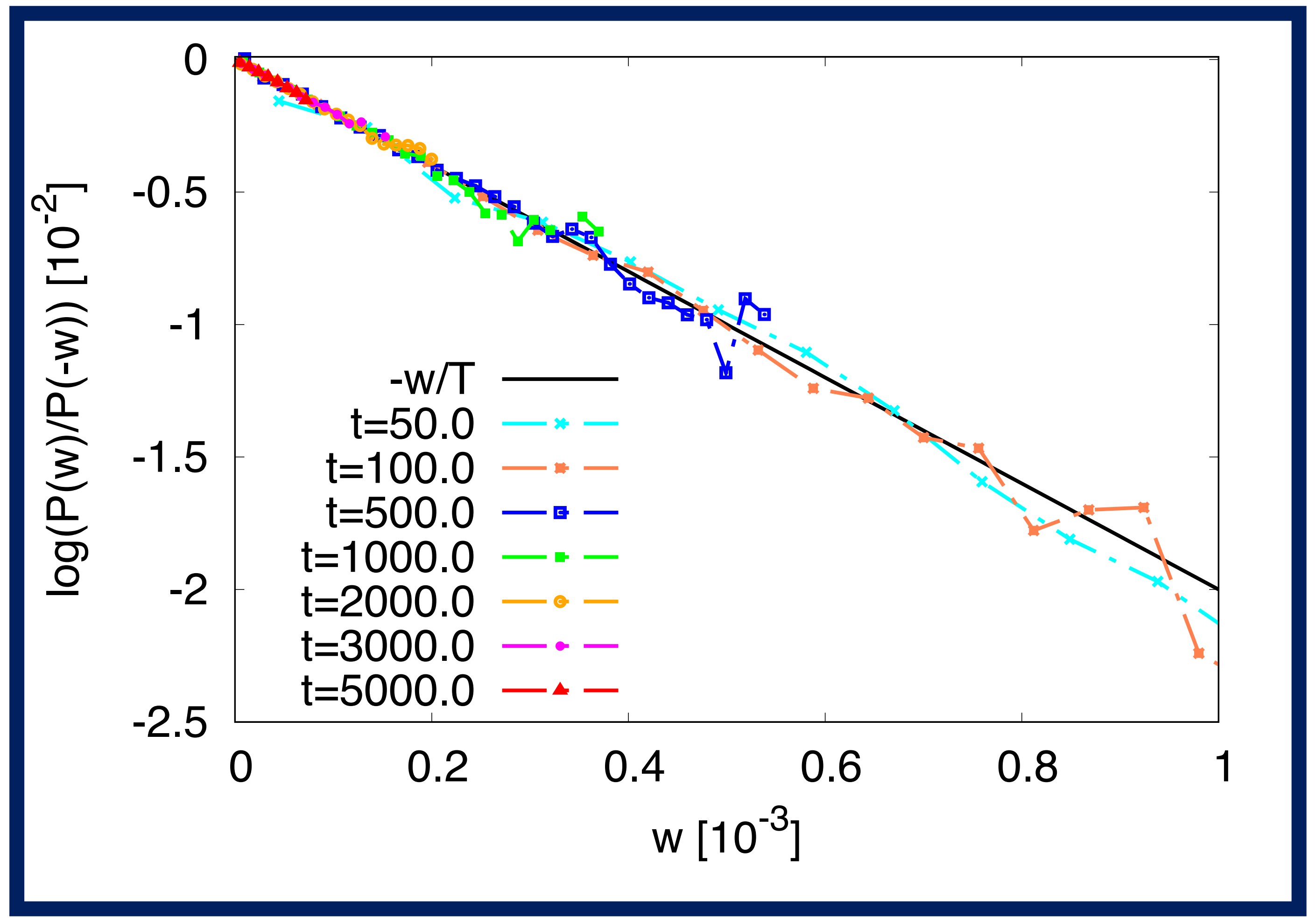
- Probability distribution $p(w) = \langle(\delta(\mathscr{W}_a - w)\rangle\rangle$ evaluated through **path integral** techniques

$$\mathscr{P}_\tau(x(\tau), a(\tau)) \propto \overbrace{p(r_0, a_0)}^{\text{initial conditions distribution}} \times$$

$$\underbrace{\times\, exp\left\{-\frac{1}{4D_T}\int_0^\tau [\dot{r}(s) - F_a\gamma^{-1}a(s)]^2\ ds\right\} exp\left\{-\frac{1}{4D_R}\int_0^\tau [\dot{a}(s) + \gamma_R a(s)]^2\ ds\right\}}_{\text{Onsager-Machlup weight for trajectories}}$$



- no singularities in $I(w)$

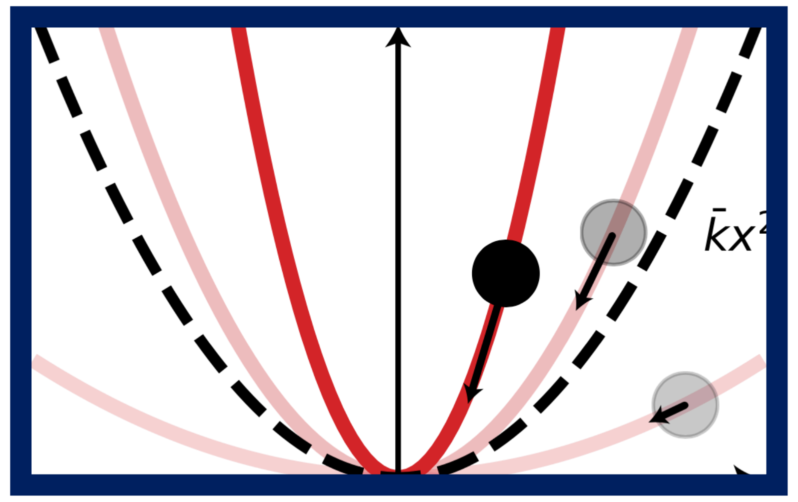


- Fluctuation Theorem $I(w) - I(-w) = -w/T$

Harmonically confined AOUP

- Harmonically-confined AOUP in 1 d

$$\begin{cases} \gamma \dot{x}(t) = a(t) - kx(t) + \sqrt{2\gamma T}\ \xi(t) \\ \dot{a}(t) = -\nu a(t) + F\sqrt{2\nu}\ \eta(t) \end{cases}$$



- Direct evaluation of $p(w)$through **path integral** techniques becomes difficult

  - Trajectory path probability

$$\mathscr{P}_\tau \propto \underbrace{\left\{ -\frac{1}{2}(x(0)\ a(0))\Sigma_0^{-1}\begin{pmatrix} x(0) \\ a(0) \end{pmatrix} \right\}}_{\text{initial conditions distribution}} \underbrace{exp\left\{ -\frac{1}{4}\int_0^\tau [\dot{x}(s) - a(s) + \kappa x(s)]^2\ ds \right\} exp\left\{ -\frac{1}{4Pe^2}\int_0^\tau [\dot{a}(s) + a(s)]^2\ ds \right\}}_{\text{Onsager-Machlup weight for trajectories}}$$

  - Laplace representation of the δ function

$$p(w) = \frac{1}{2\pi\imath}\int_{-\imath\infty}^{+\imath\infty} e^{-\tau\lambda w}\ \langle e^{\lambda \mathscr{W}_a} \rangle$$

  - Cumulant Generating Function

  - Saddle-point estimation of the RF

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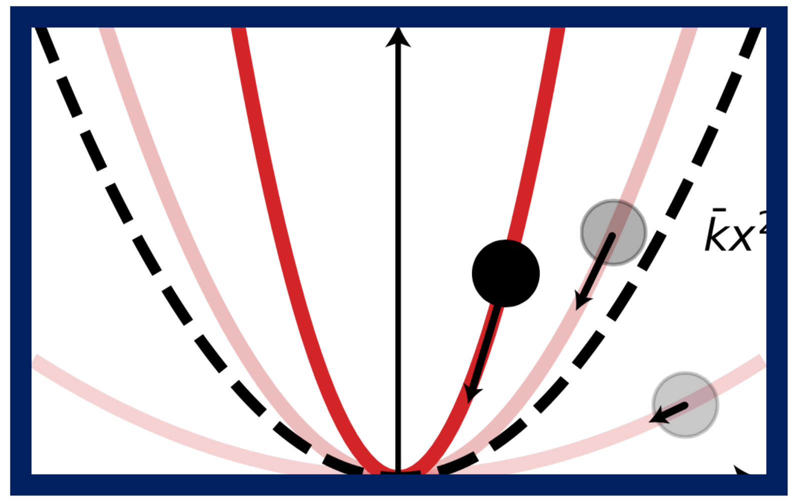

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$$\mathscr{P}_\tau \propto \underbrace{\left\{-\frac{1}{2}(x(0)\ a(0))\Sigma_0^{-1}\begin{pmatrix} x(0) \\ a(0)\end{pmatrix}\right\}}_{\text{initial conditions distribution}} \underbrace{exp\left\{-\frac{1}{4}\int_0^\tau [\dot{x}(s) - a(s) + \kappa x(s)]^2\ ds\right\} exp\left\{-\frac{1}{4Pe^2}\int_0^\tau [\dot{a}(s) + a(s)]^2\ ds\right\}}_{\text{Onsager-Machlup weight for trajectories}}$$

  - Laplace representation of the δ function

$$p(w) = \frac{1}{2\pi\imath}\int_{-\imath\infty}^{+\imath\infty} e^{-\tau\lambda w}\ \langle e^{\lambda \mathscr{W}_a}\rangle$$

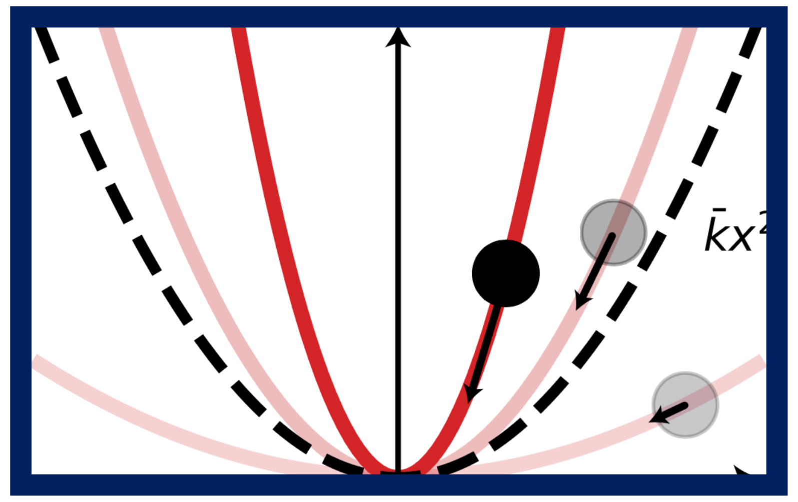
  - Cumulant Generating Function

  - Saddle-point estimation of the RF

Harmonically confined AOUP

- Harmonically-confined AOUP in 1 d

$$\begin{cases} \gamma \dot{x}(t) = a(t) - kx(t) + \sqrt{2\gamma T}\ \xi(t) \\ \dot{a}(t) = -\nu a(t) + F\sqrt{2\nu}\ \eta(t) \end{cases}$$



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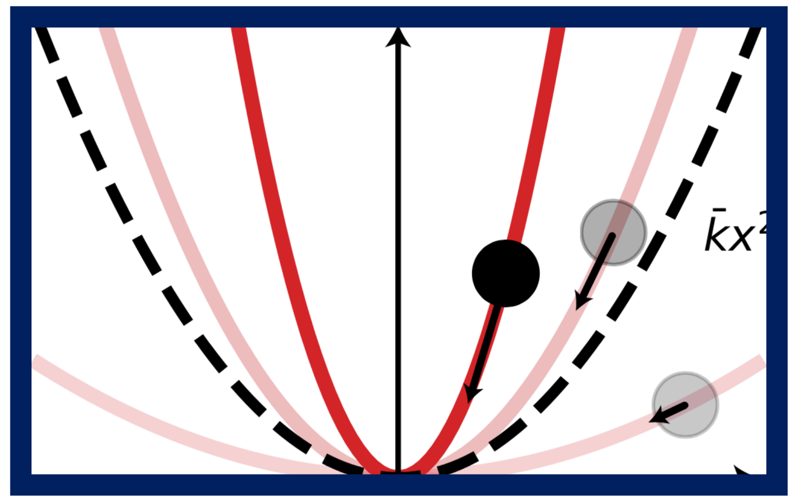

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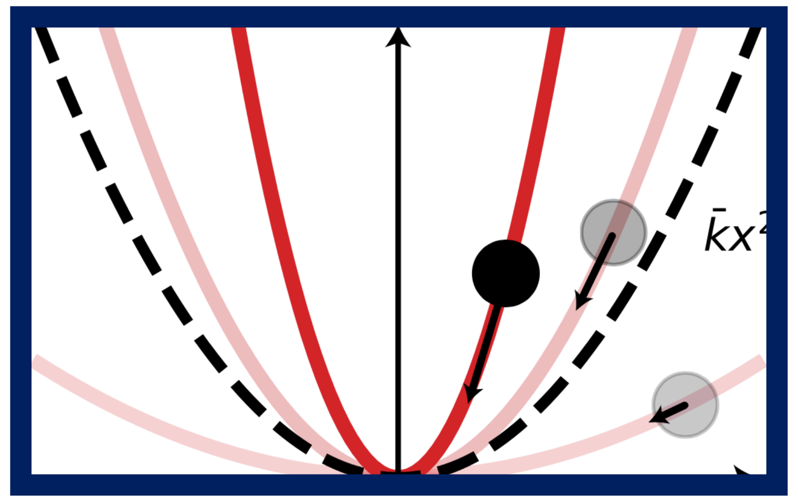

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  - Cumulant Generating Function ✗
  - Saddle-point estimation of the RF

- New Large Deviations results for **quadratic functionals** of **Gauss-Markov chains**

J Math Phys 2023, Zamparo and Semeraro

  - Time-discretization procedure
  - Evaluation of the SCGF functional form
  - Evaluation of the SCGF domain
  - Continuum limit
  - Evaluation of the RF through Legendre-Fenchel transform

LDT for quadratic functionals of Gauss-Markov chains

J Math Phys 2023, Zamparo and Semeraro

- Time-discretization procedure Langevin Equations→ Markov chain $X_{n+1} = SX_n + G_n$

$\mathscr{W}_a \cdot \tau \rightarrow$ quadratic functional $$W_N = \underbrace{\frac{1}{2} < X_0, LX_0 > + \frac{1}{2} < X_N, RX_N >}_{\text{boundary terms}} + \underbrace{\frac{1}{2}\sum_{n=1}^{N} < X_n, UX_n > + \frac{1}{2}\sum_{n=2}^{N} < X_n, VX_{n-1} >}_{\text{bulk contributions}}$$

- Evaluation of the SCGF functional form

$$\varphi(\lambda) = \lim_{N\to\infty} \frac{1}{N} \log < e^{\lambda W_N} > = -\frac{1}{4\pi}\int_0^{2\pi} \ln\det F_\lambda(\theta)\ d\theta$$

Symbol matrix

$$F_\lambda(\theta) = (I - S^T e^{\imath\theta})(I - S^T e^{-\imath\theta}) - \lambda(U + Ve^{-\imath\theta} + V^T e^{\imath\theta})$$

- Evaluation of the SCGF domain — two technical requests on positive definiteness

Primary domain $P = (\tilde{\lambda}_-, \tilde{\lambda}_+)$

$F_\lambda(\theta)$ is positive definite for all $\theta \in (0,2\pi)$

Effective domain $E = (\lambda_-, \lambda_+) \subseteq P$

the matrices $\mathscr{L}_\lambda$ and $\mathscr{R}_\lambda$ related to the initial conditions (Σ_0) and boundary terms (L, R) are positive definite

- Continuum limit

$$\phi(\lambda) = \lim_{\epsilon\to 0} \frac{\varphi(\lambda)}{\epsilon}$$

- Evaluation of the RF through Legendre-Fenchel transform

$$I(w) = \sup_{\lambda\in E}\{w\lambda - \phi(\lambda)\}$$

Singular Rate Function

PRL 2023,
Semeraro, Gonnella,
Suma and Zamparo

- SCGF $\phi(\lambda) = \frac{1+\kappa}{2} - \frac{1}{2}\sqrt{(1+\kappa)^2 - 4Pe^2\lambda(1+\lambda)}$

- Primary domain $P = (\tilde{\lambda}_-, \tilde{\lambda}_+)$

$$\tilde{\lambda}_\pm = -\frac{1}{2} \pm \sqrt{1 + \left(\frac{1+\kappa}{Pe}\right)^2} \qquad Pe = \frac{Fd}{k_BT} \qquad \kappa = \frac{kd^2}{k_BT}$$

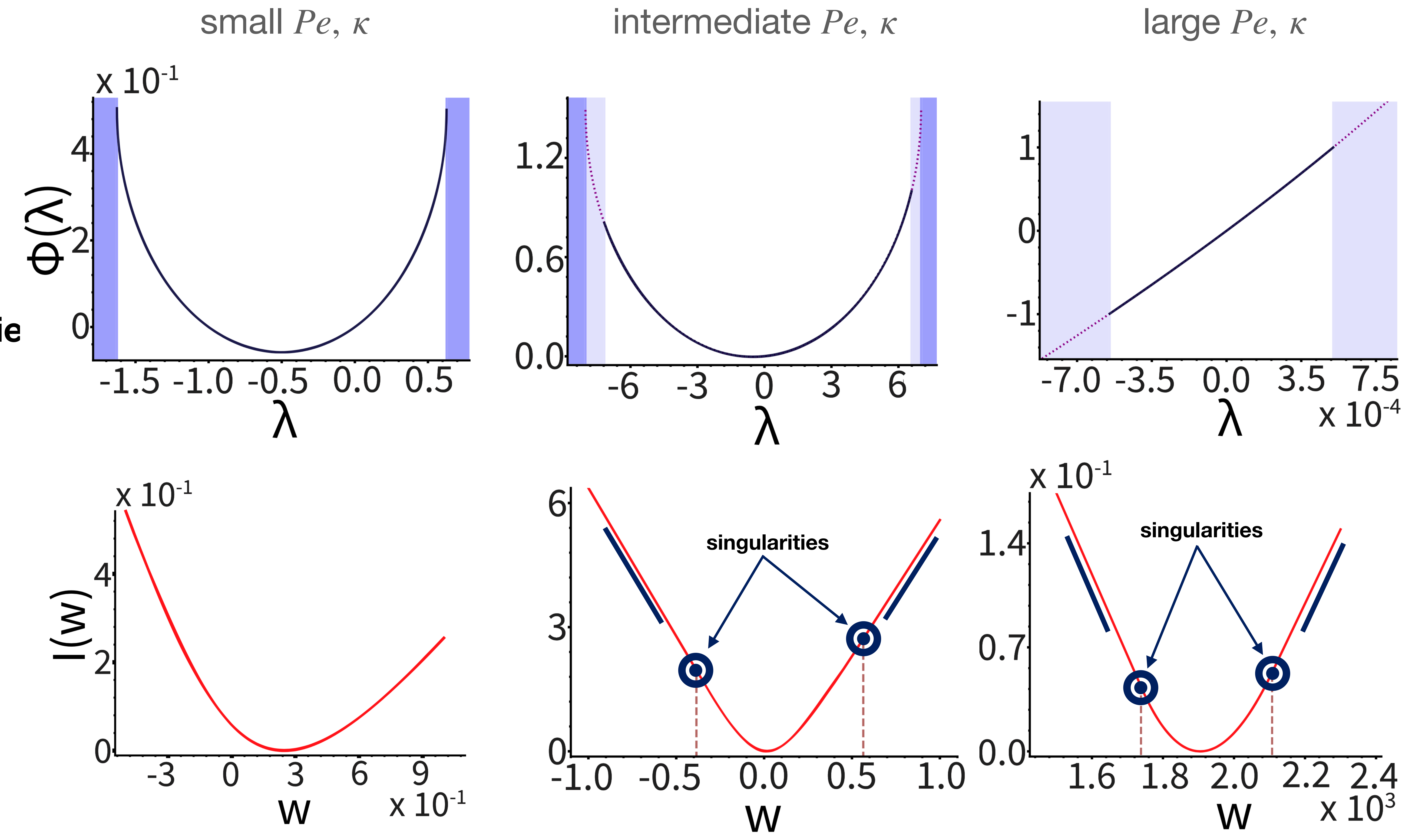
- $E = (\lambda_-, \lambda_+) \neq (\tilde{\lambda}_-, \tilde{\lambda}_+) = P \longrightarrow$ **Singularitie**

- Rate function

$$I(w) = \begin{cases} (w - w_-)\lambda_- + i(w) & w \leq w_- \\ i(w) & w_- < w < w_+ \\ (w - w_+)\lambda_+ - i(w) & w \geq w_+ \end{cases}$$

$$i(w) = \frac{1}{2}\left(\sqrt{1 + \left(\frac{w}{Pe}\right)^2} + \sqrt{(1+\kappa)^2 + Pe^2 - 1 - \kappa - w}\right)$$

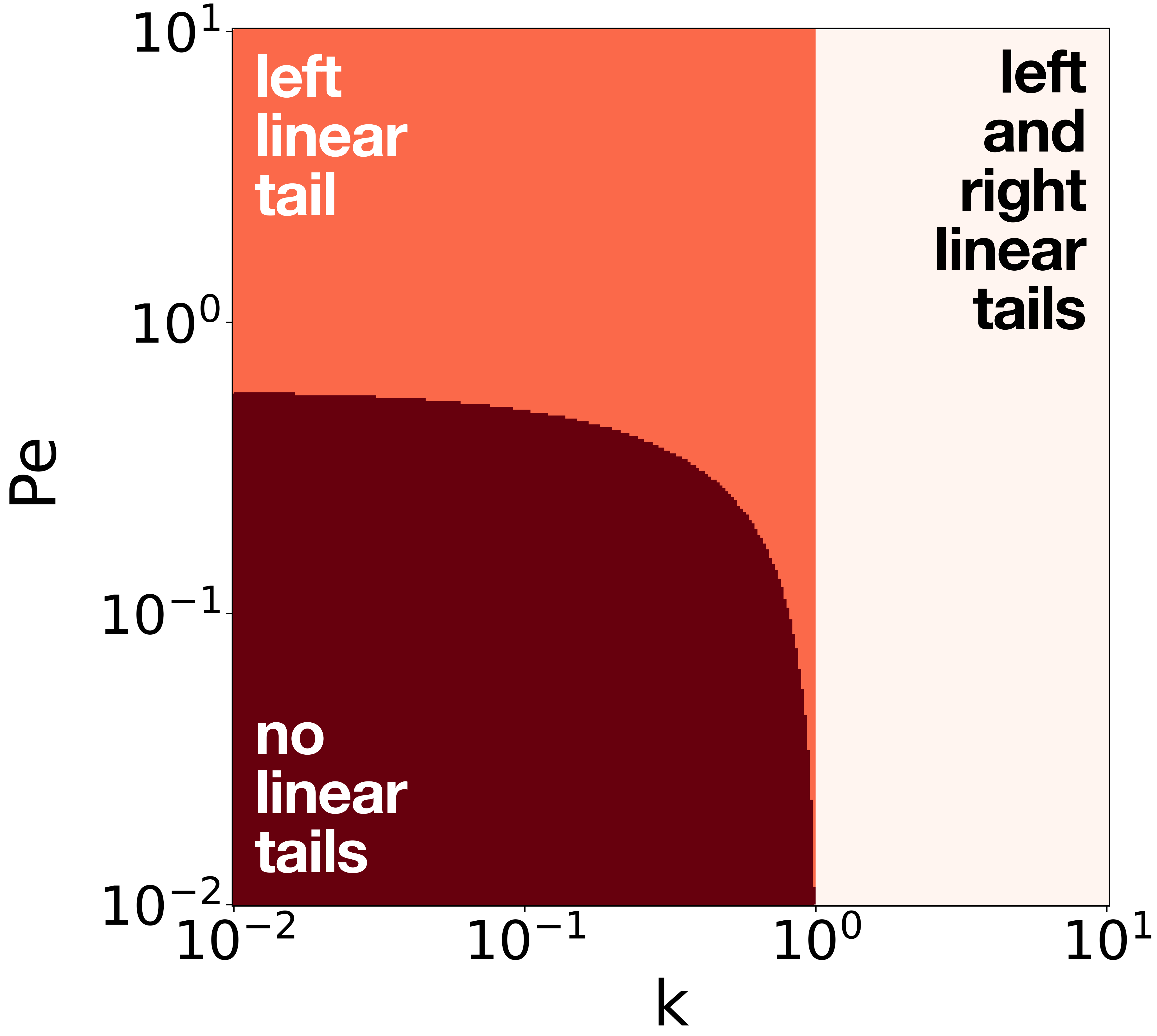
- **Fluctuation Relation** not always satisfied



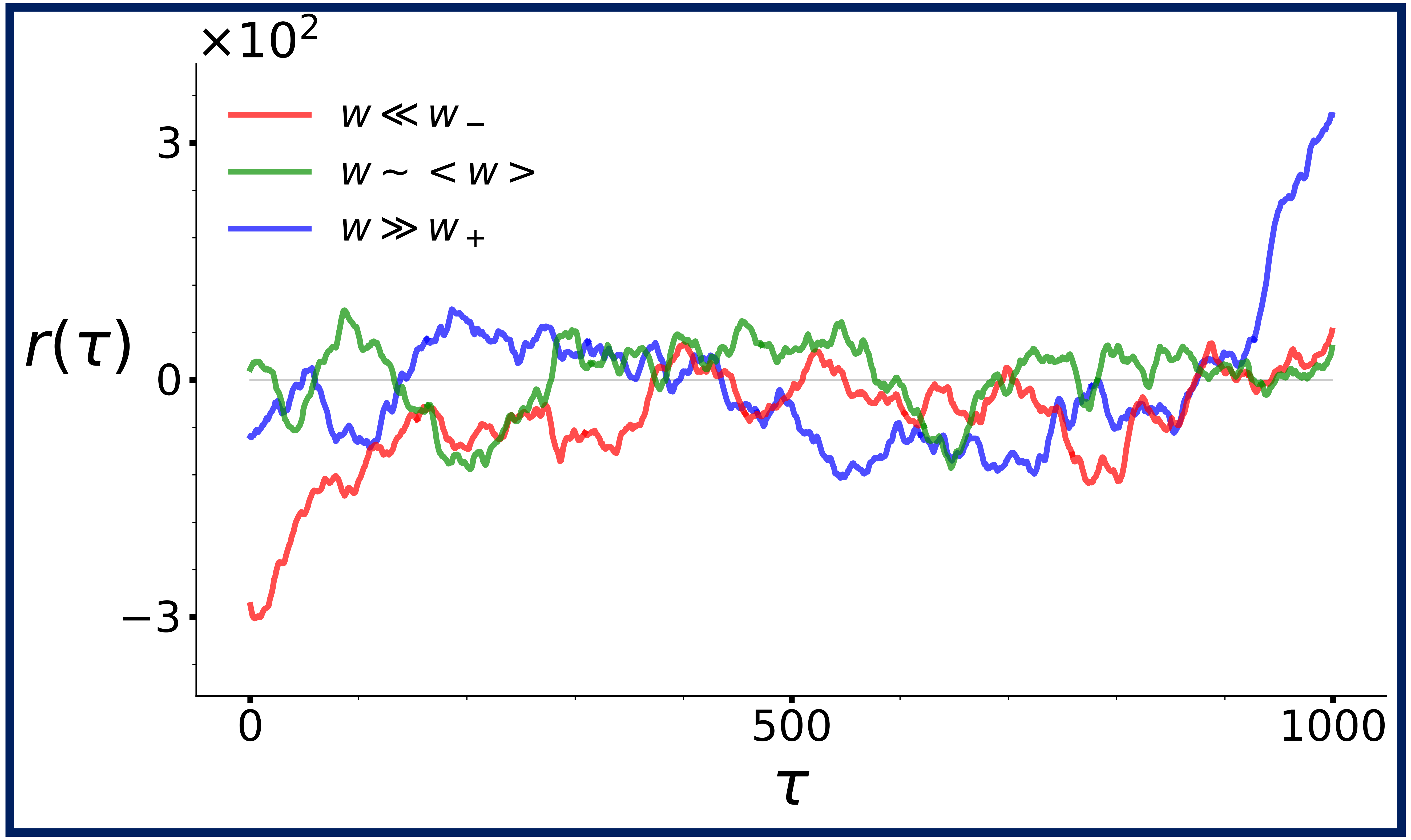
Linear Tails and Trajectory Separation

PRL 2023, Semeraro, Gonnella, Suma and Zamparo

- **Singularities phase diagram**



- **Physical Mechanism:**
  singular trajectories are characterised and selected by **big jumps** in the initial ($w \ll w_{-}$) or final ($w \gg w_{+}$) values



Fluctuations of Injected power

J Phys A,
Carollo, Semeraro,
Gonnella and Zamparo

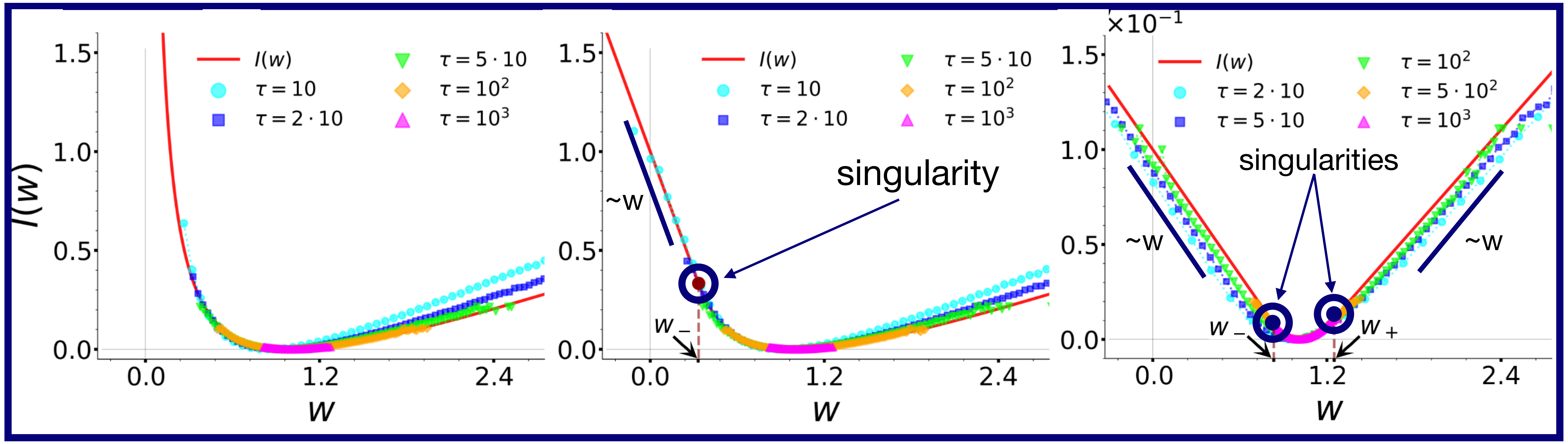
- Underdamped Brownian particle with external harmonic potential

$$\begin{cases} \dot{x} = v(t) \\ \dot{v}(t) = -\gamma v(t) - kx(t) + \sqrt{2D}\ \xi(t) \end{cases}$$

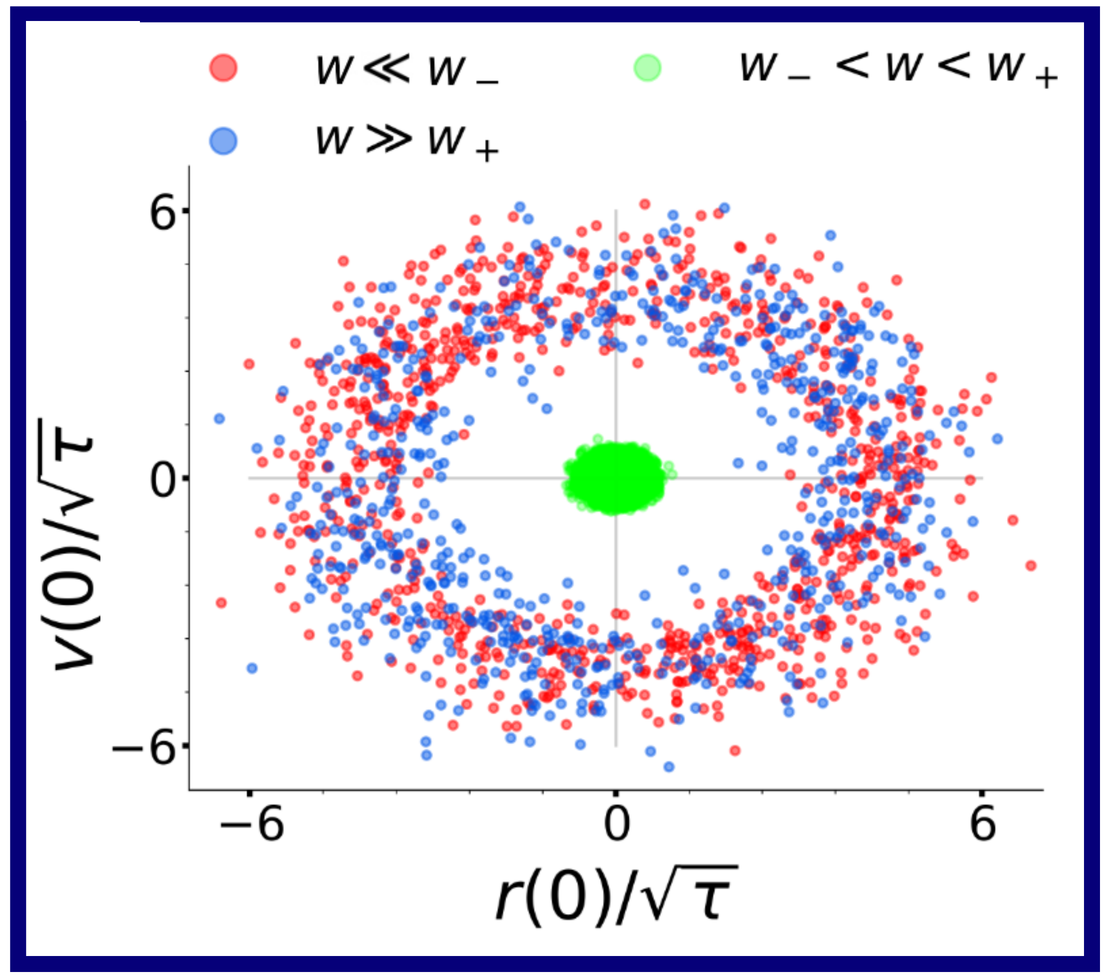
- Power injected by the random force

$$W_\tau = \frac{\sqrt{2D}}{\tau}\int_0^\tau \xi(t)\dot{x}(t)dt = \frac{1}{2\tau}[v^2(\tau) - v^2(0)] + \frac{k}{2\tau}[x^2(\tau) - x^2(0)] + \frac{\gamma}{\tau}\int_0^\tau v(t)\dot{x}(t)\ dt$$

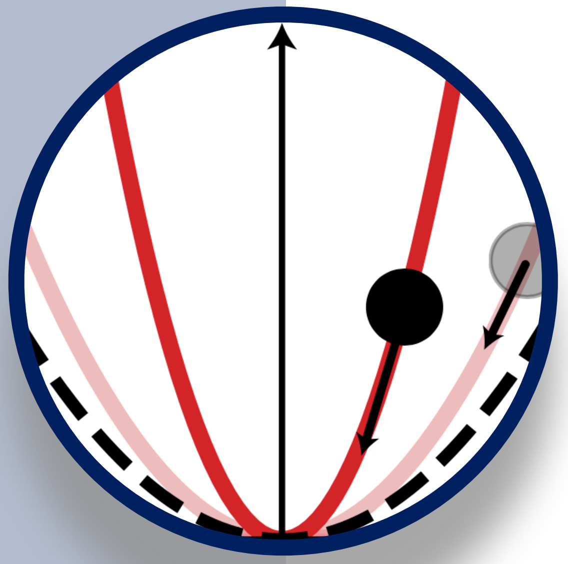
- Singular Rate Functions
  - Fixed initial conditions $x(0) = 0,\ v(0) = 0$
  - Stationary initial conditions $\sigma_x^2 = D/(k\gamma),\ \sigma_v^2 = D/\gamma$
  - Generic uncorrelated initial conditions $\sigma_x^2 = 10,\ \sigma_v^2 = 10$



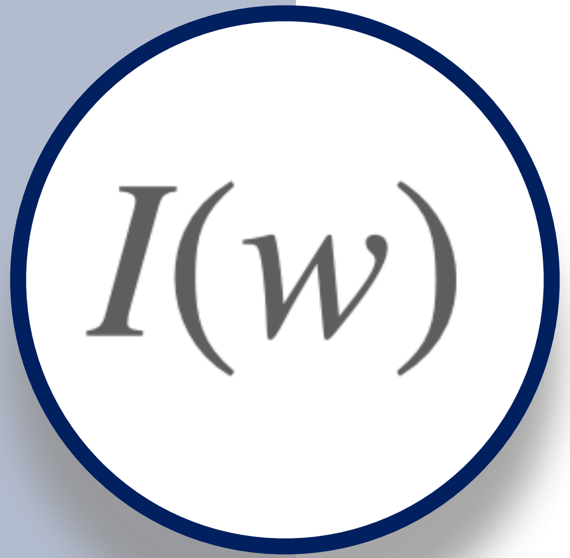
- Singular trajectories are associate to **big jumps** in the initial conditions



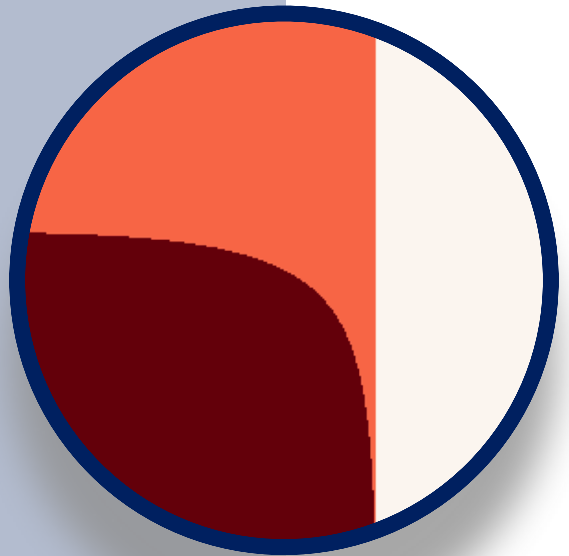
Take-home messages



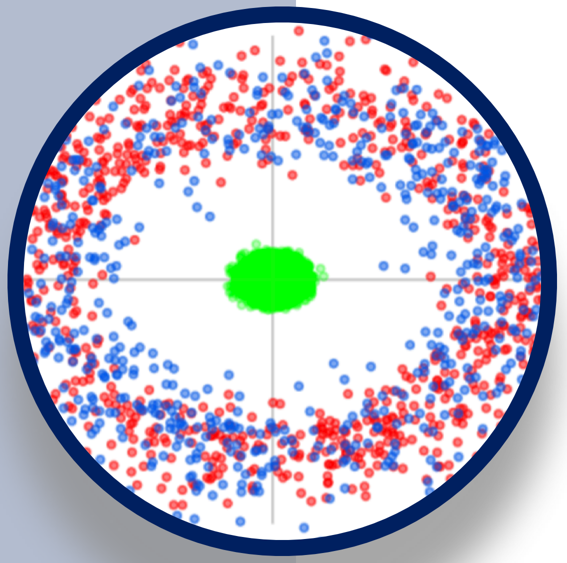
- Fluctuations of Active Work
  for a single harmonically confined AOUP



- Analysis with new Large Deviation Results
  for quadratic functionals of Gauss-Markov chains



- **Singular Rate Functions** are analytically obtained
- Initial or final **big jumps** are responsible
  for the appearance of singularities



- Big jumps behind singular rate functions and
  dynamical phase transitions in Brownian systems

Acknowledgements

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Università degli Studi di Bari and INFN Bari

Dr. Marco Zamparo

Università del Piemonte Orientale

Thank for your attention!

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- Dynamics of complex fluids
- Active Matter

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Marco Zamparo
(past member)

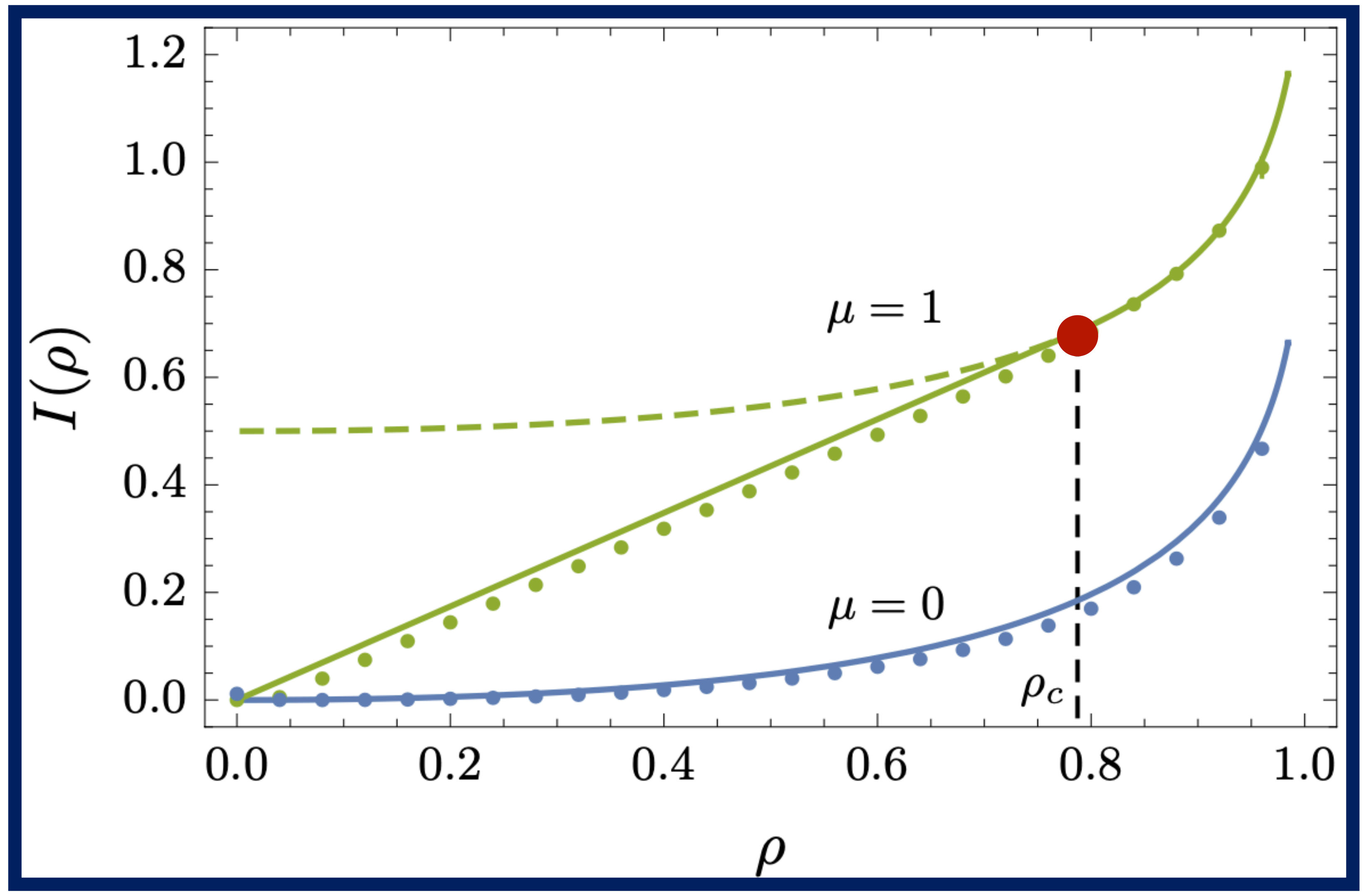
Dynamical Phase Transitions: examples

- Driven Brownian particle $\dot{x}(t) = \mu \cdot t + \sqrt{2D}\ \xi(t)$
  - Empirical measure:
  - fraction of time spent by the particle in the interval $[a, b]$

$$\rho = \frac{1}{\tau}\int_0^\tau 1_{[a,b]}(x(t))\ dt$$

  - Singular trajectories:
    particles rapidly escaping from $[a, b]$
    give rise to the RF left linear branch

EPL 2017, Nyawo and Touchette



- System of many interacting ABPs
  - Active Work $\mathscr{W}_a = \frac{1}{\tau}\int_0^\tau a(s)v(s)\ ds$
  - Singular trajectories:
  - particles dragged against their active force
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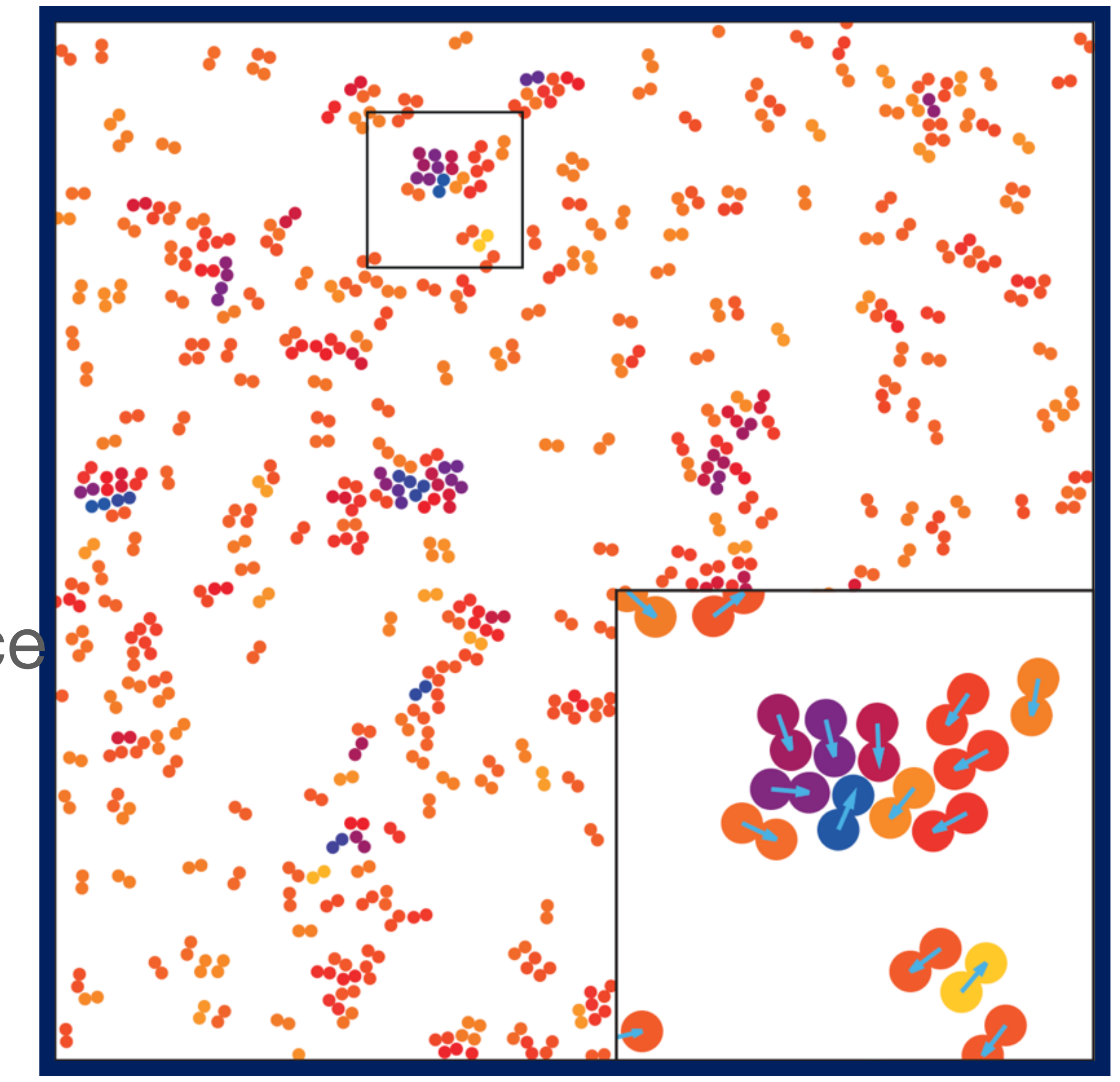
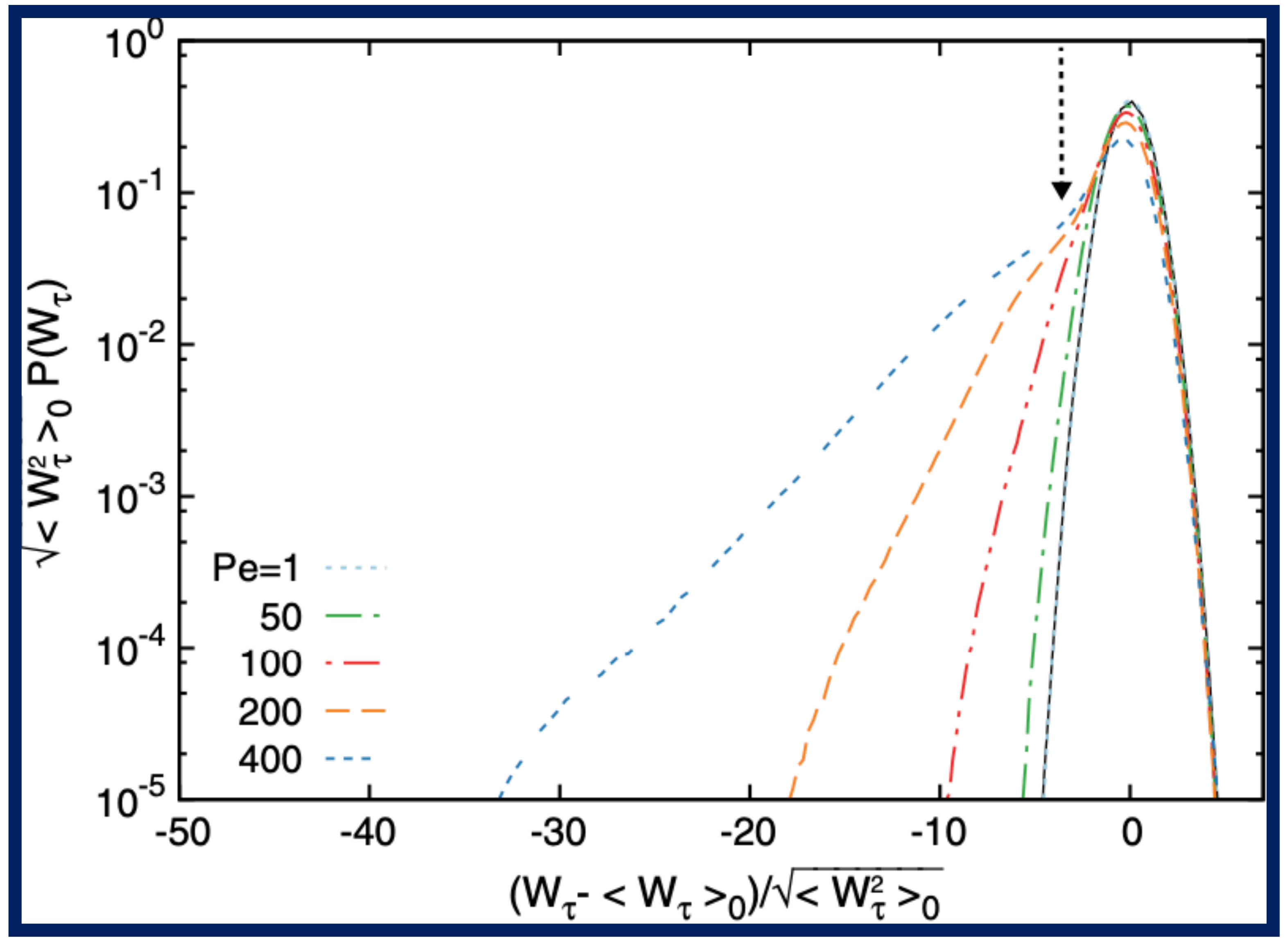
PRL 2017, Cagnetta, Corberi, Gonnella and Suma

PRE 2021, Keta, Fodor, van Wijland and Cates





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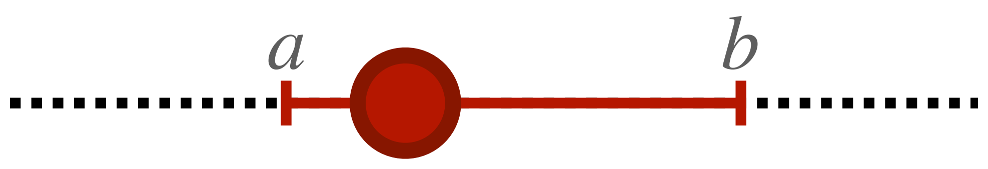

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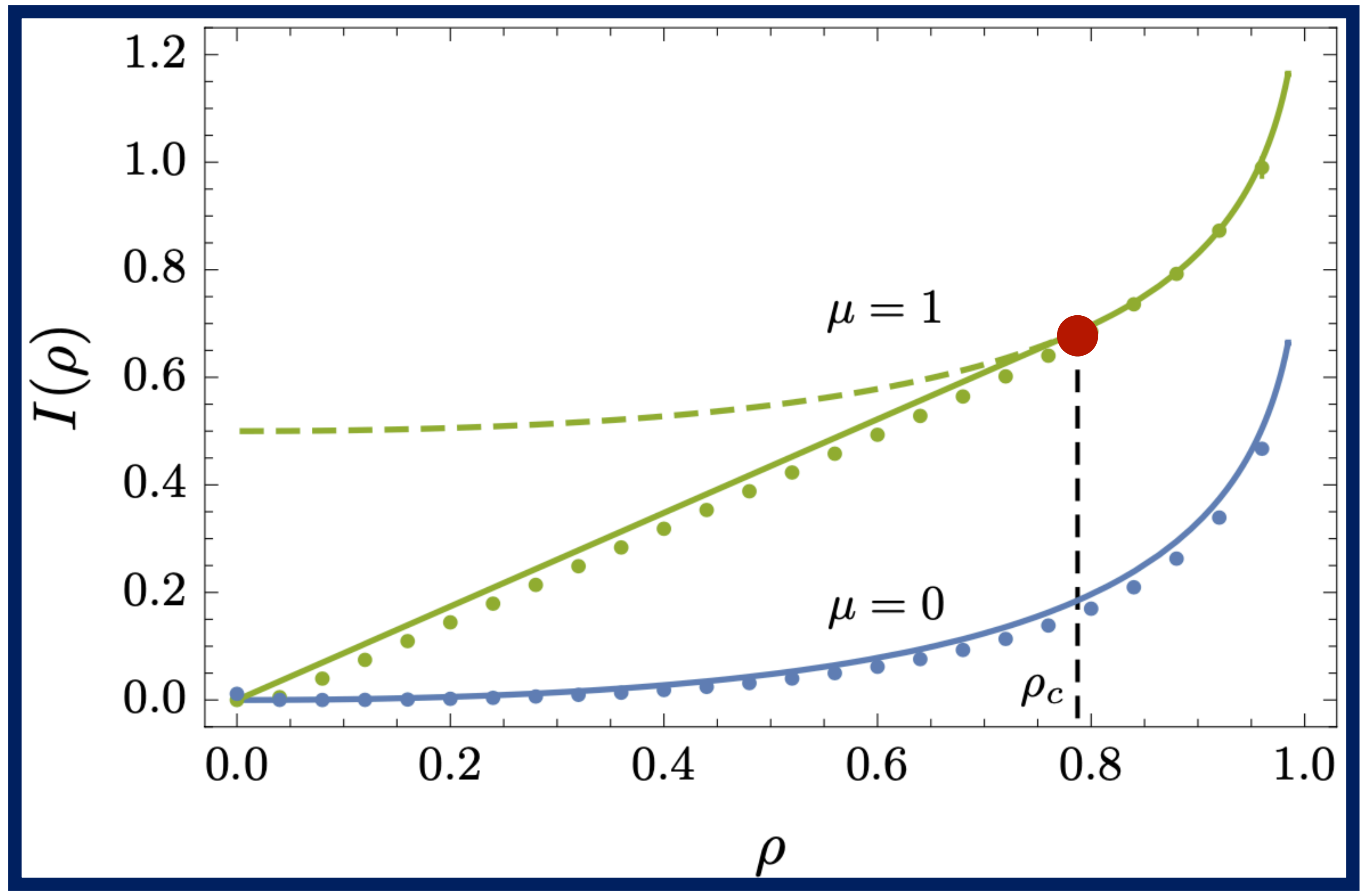

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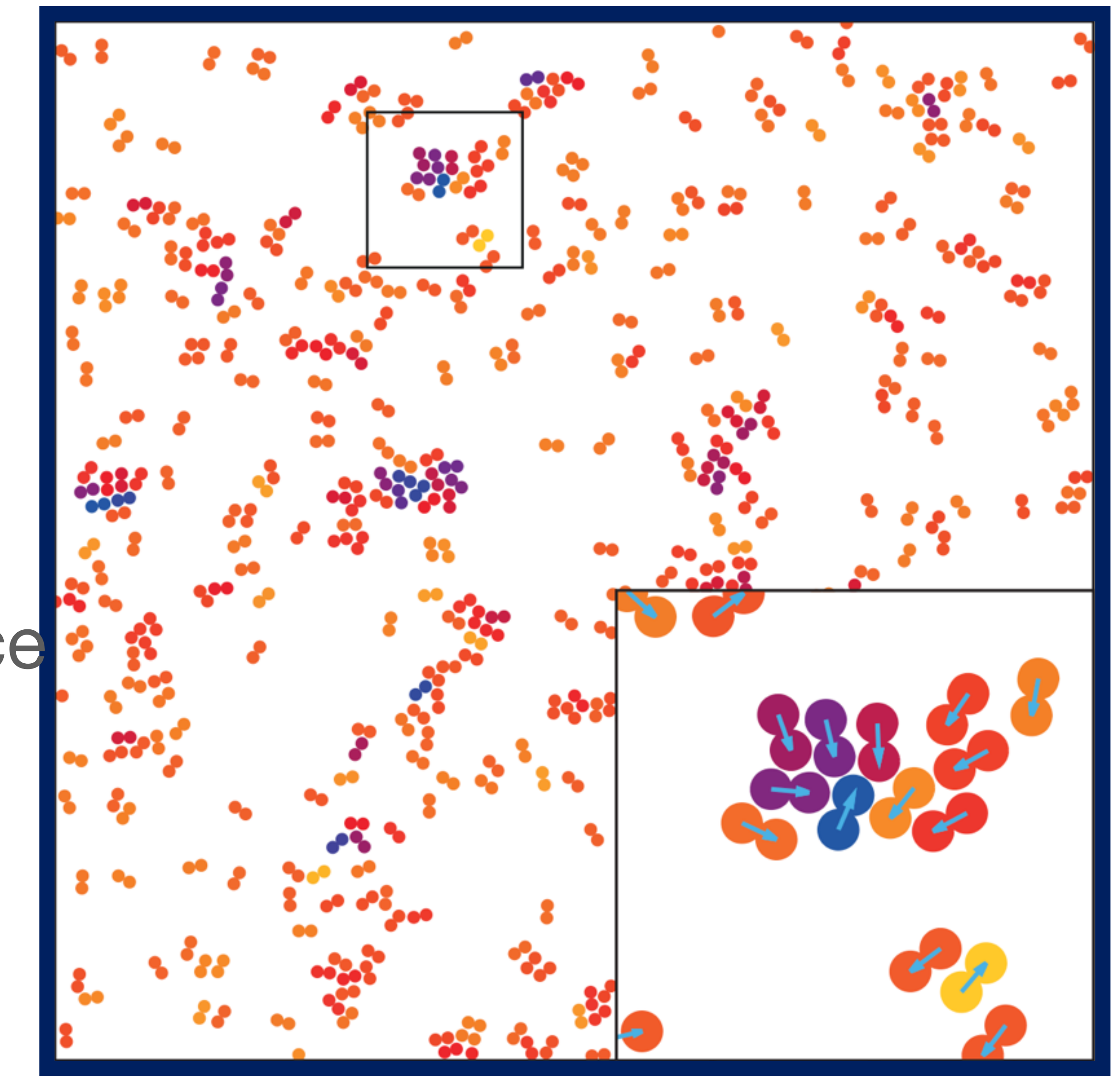

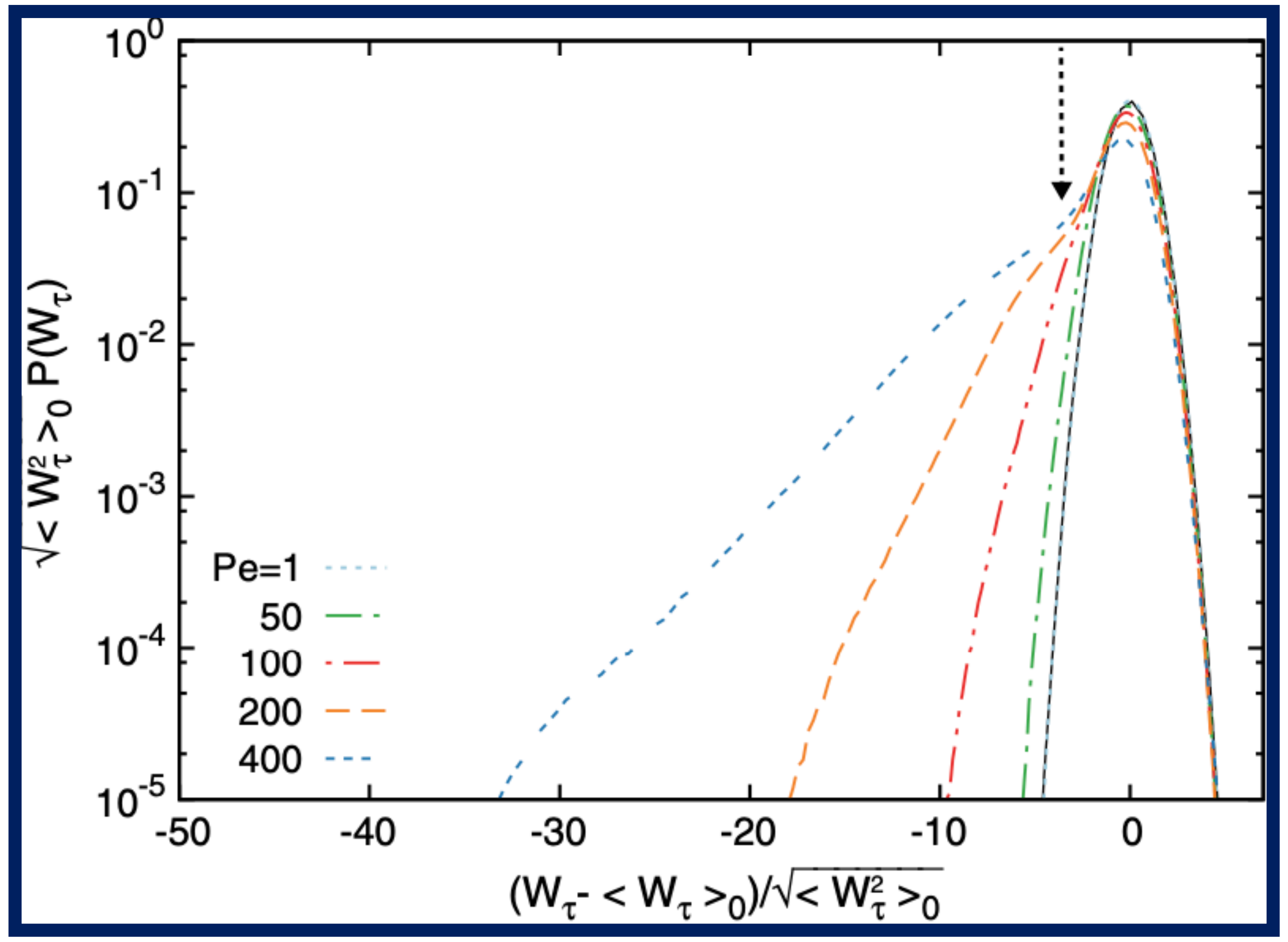

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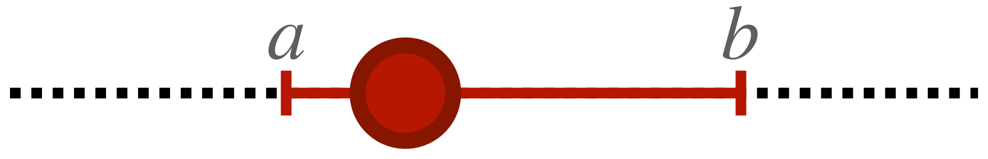

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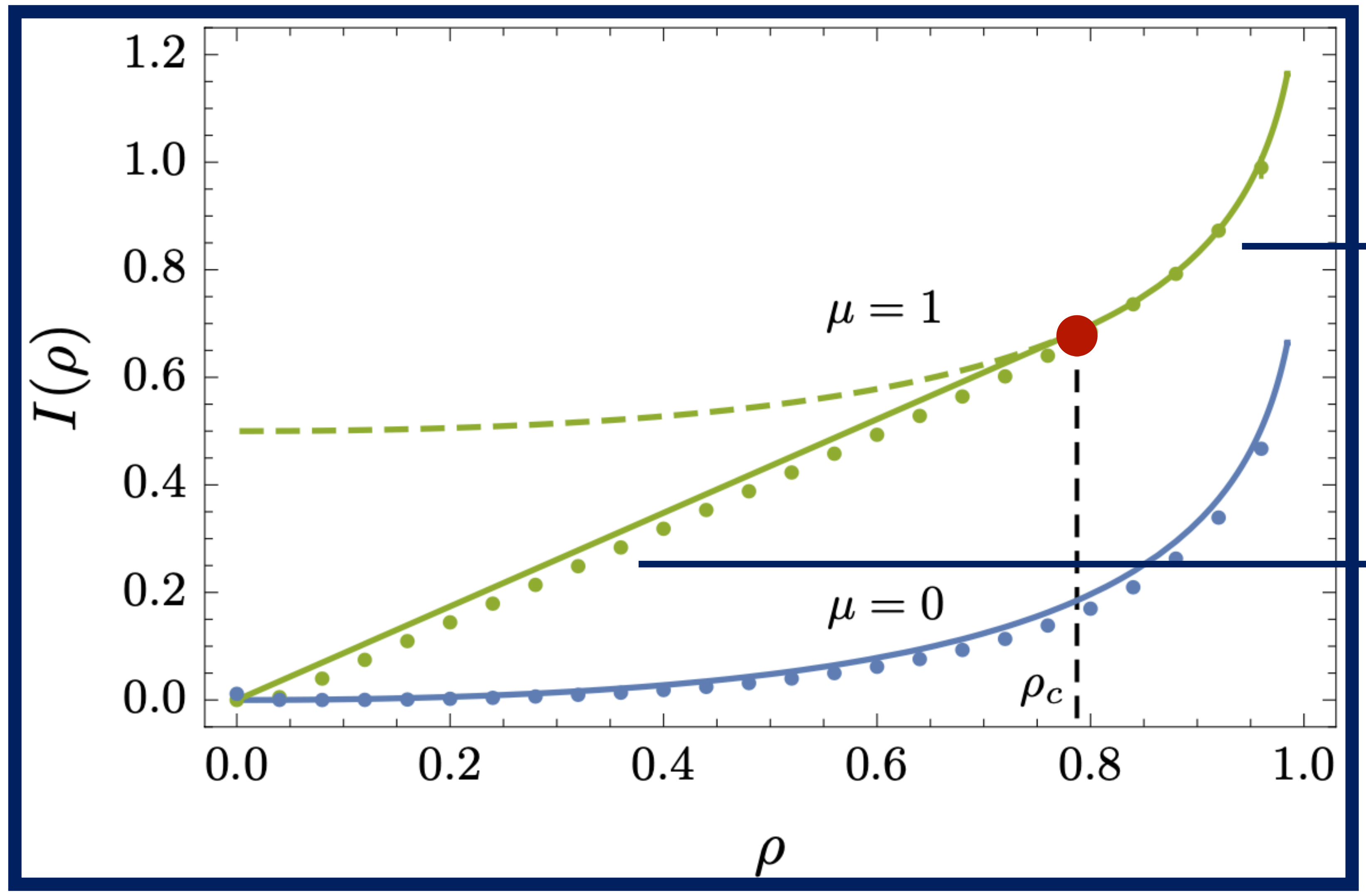

particle mostly confined in the interval $[a, b]$

particle rapidly escaping from the interval $[a, b]$

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PRL 2017, Cagnetta, Corberi, Gonnella and Suma

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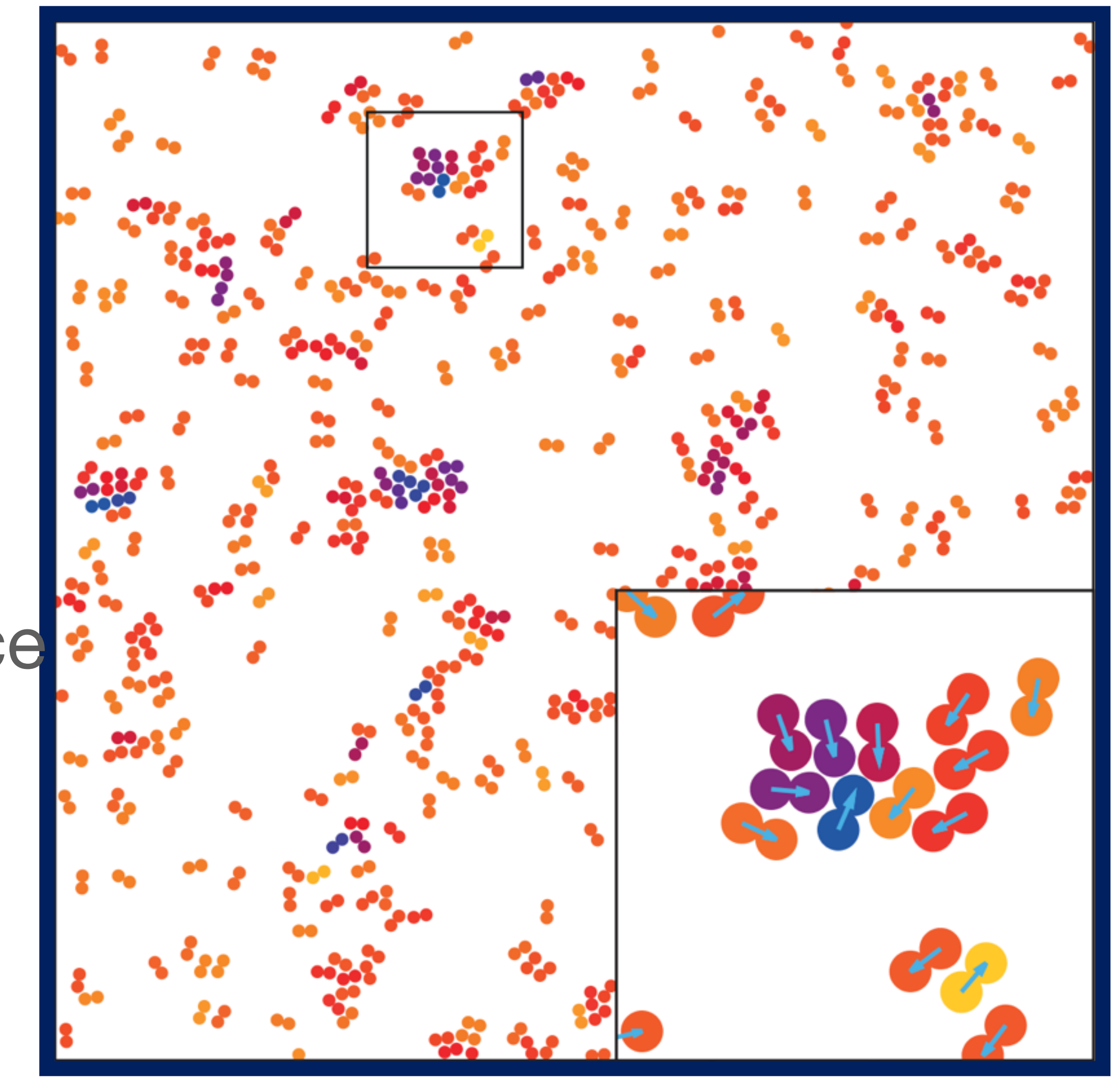
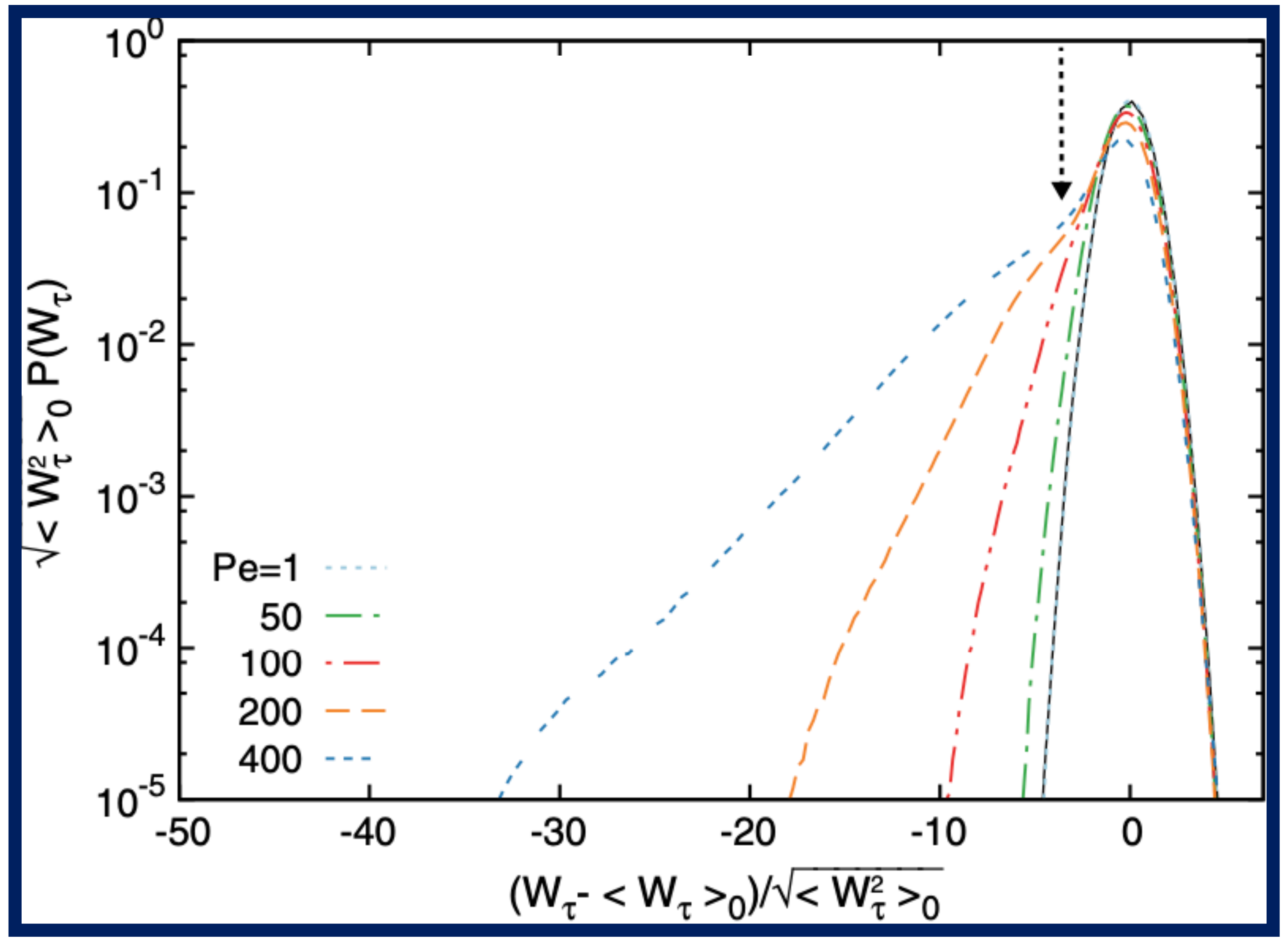




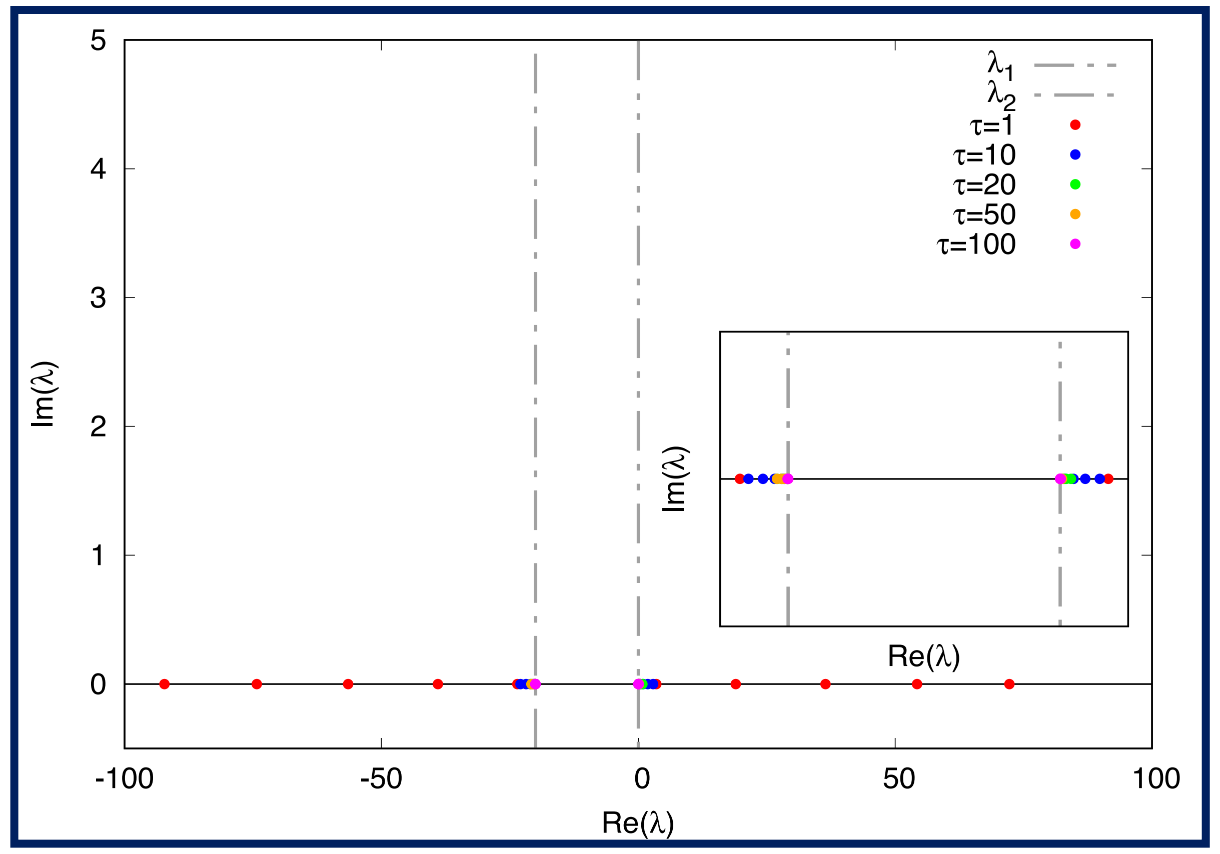
Free AOUP: saddle point

J Stat Mech 2021,
Semeraro, Suma, Petrelli,
Cagnetta and Gonnella

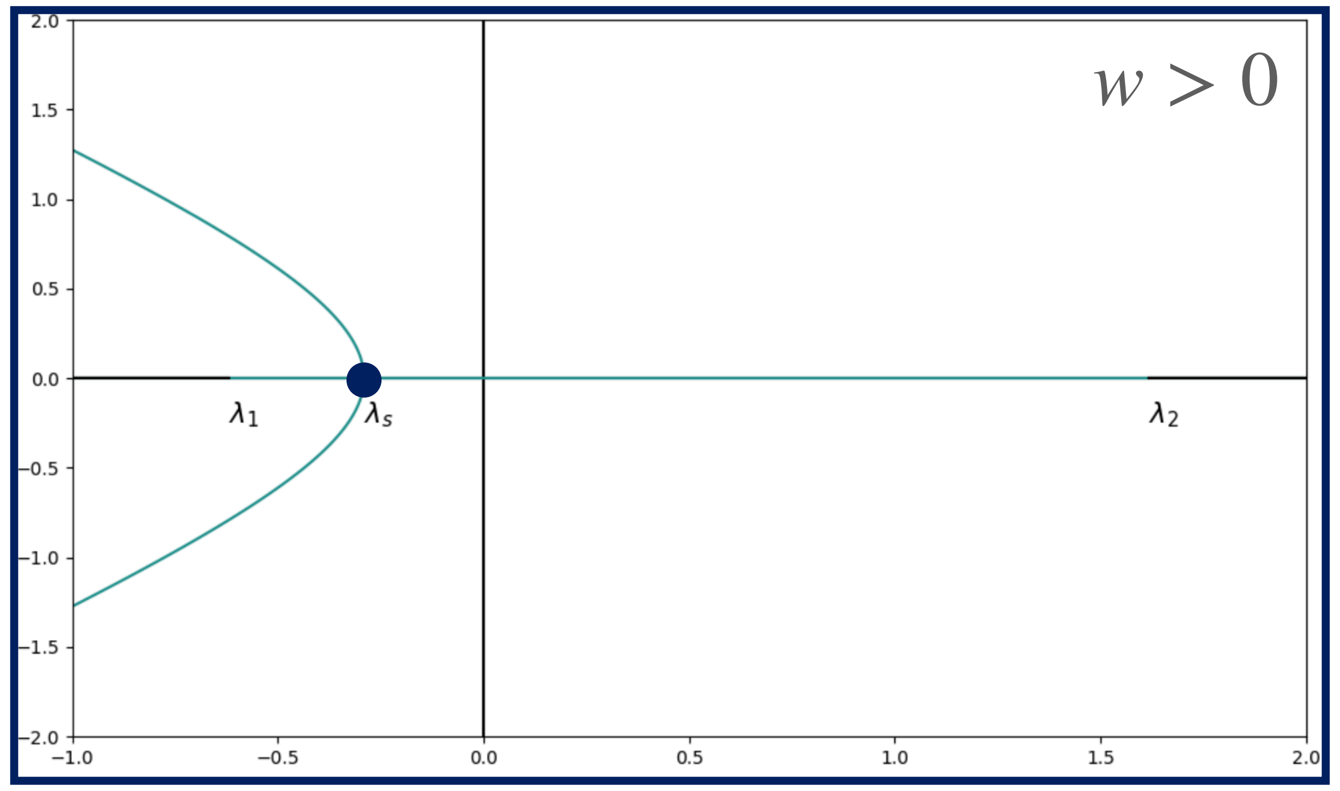
- Cumulant Generating Function $\langle e^{\mu \mathscr{W}_a} \rangle = \dfrac{e^{\tau \frac{d}{2}(\gamma_R - \alpha)}}{\left(\frac{1+e^{-2\tau\alpha}}{2}\right)^{d/2} \left(2 + \frac{\gamma_R^2+\alpha^2}{\gamma_R \alpha}\tanh(\tau_\alpha)\right)^{d/2}} = F(\mu) e^{\tau \frac{d}{2}(\gamma_R - \alpha)}$ $\quad \alpha = \sqrt{\gamma_R^2 - 4D_R \mu \gamma (1 + D_T \gamma \mu)}$

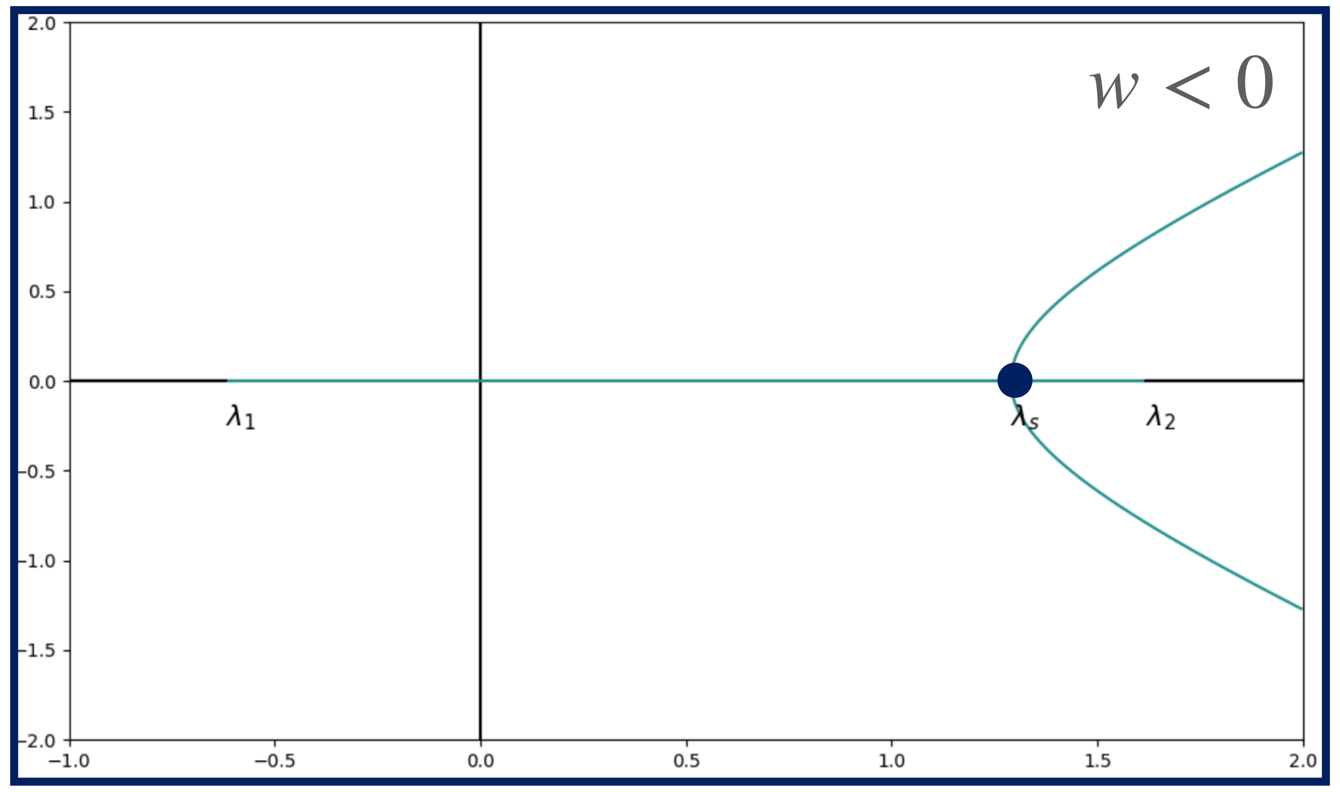
Introduction to Path-Integral Methods in Physics and Polymer Science, Wiegel 1986, Work Scientific

- Sources of singularities
  - Branch points $\tilde{\mu}_{1/2} = \dfrac{\gamma_R^2}{4D_R\gamma}\left(-\dfrac{1 \pm \sqrt{1+4A}}{2A}\right)$ $\quad A = \dfrac{\gamma_R^2 D_T}{4D_R}$
  - Poles of $F(\mu)$ $\quad \tilde{\mu}_{1/2}(y) = \dfrac{\gamma_R^2}{4D_R\gamma}\left(-\dfrac{1 \pm \sqrt{1+4A(1+y^2)}}{2A}\right)$



- Saddle-point estimation of $p(w) = \dfrac{1}{2\pi\imath}\displaystyle\int_{-\imath\infty}^{+\imath\infty} d\mu \; F(\mu) e^{\tau \frac{d}{2}(\gamma_R - \alpha)}$
  - Steepest descent paths
    $Im[\mu w - \phi(\mu)] = 0$
  - Saddle-points $\mu w - \phi(\mu) = 0 \rightarrow \tilde{\mu}_{\pm}^{(s)} = \dfrac{\gamma_R^2}{4D_R\gamma}\left(-\dfrac{1 \pm \sqrt{1+4A\left(\frac{4\tilde{w}^2-1}{4(A+\tilde{w})^2}\right)}}{2A}\right)$





  - Integration along steepest dissent paths deformed to pass by $\tilde{\mu}_{\pm}^{(s)}$ and avoid non-analicities of the integrand

$$p(w) \asymp \frac{F(\tilde{\mu}^{(s)})}{2\pi}\left(\frac{2\pi}{\mathscr{C}}\right)^{1/2} e^{-\tau I(w)} \longrightarrow$$ Extraction of the Rate Function $I(w)$

LDT for quadratic functionals of Gauss-Markov chains

- Continuous model ($\gamma, T, d = 1$)

$$\begin{cases} \dot{x}(t) = a(t) - \kappa x(t) + \sqrt{2}\ \xi(t) \\ \dot{a}(t) = -a(t) + Pe\sqrt{2}\ \eta(t) \end{cases} \longrightarrow$$

- Discrete model as a Gauss-Markov chain
  $\tau = N \cdot dt,\ x_n, a_x \equiv x(n \cdot dt), a(n \cdot dt), \{\xi_n\}, \{\eta_n\}$ sequence of normal rv

$$\begin{cases} r_{n+1} = (1 - \kappa dt)\ r_n + a_n\ dt + \sqrt{2dt}\ \xi_n \\ a_{n+1} = (1 - dt)\ a_n + Pe\sqrt{2dt}\ \eta_n \end{cases} \longrightarrow \quad X_{n+1} = SX_n + D\zeta_n \qquad X_n = (x_n, a_n)^T$$

initial conditions covariance matrix

$$\Sigma_0 = \begin{pmatrix} \frac{1+\kappa+Pe^2}{\kappa(1+\kappa)} & \frac{Pe^2}{1+\kappa} \\ \frac{Pe^2}{1+\kappa} & Pe^2 \end{pmatrix} \qquad \Sigma_0 = \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_a^2 \end{pmatrix}$$

$$S = \begin{pmatrix} 1 - \kappa dt & dt \\ 0 & 1 - dt \end{pmatrix} \qquad D = \begin{pmatrix} \sqrt{2dt}dt & dt \\ 0 & Pe\sqrt{2dt} \end{pmatrix} \qquad \zeta_n = \begin{pmatrix} \xi_n & 0 \\ 0 & \eta_n \end{pmatrix}$$

- Entire trajectory $\{(x_0, a_0), \ldots, (a_N, x_N)\}$ is Gaussian distributed with zero mean and covariance matrix

$$\Sigma_N = \begin{pmatrix} \Sigma_0^{-1} + S^\top D^{-2} S & -S^\top D^{-2} & & & \\ -D^{-2}S & D^{-2} + S^\top D^{-2} S & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & D^{-2} + S^\top D^{-2} S & -S^\top D^{-2} \\ & & & -D^{-2}S & D^{-2} \end{pmatrix}^{-1}$$

- Discretisation of Active Work as a quadratic functional

$$\mathscr{W}_a \cdot \tau = \int_0^\tau a(t)\dot{r}(t)\ dt \longrightarrow \quad W_N = \frac{1}{2}\sum_{n=1}^{N} (a_n + a_{n-1})(r_n - r_{n-1}) = \frac{1}{2}(r_0 \quad a_0 \quad \ldots \quad r_N \quad a_N)\, \mathsf{M}_N \begin{pmatrix} r_0 \\ a_0 \\ \vdots \\ r_N \\ a_N \end{pmatrix}$$

quasi-Toeplitz block matrix $\longrightarrow$

$$\mathsf{M}_N \equiv \begin{pmatrix} -E_+ & E_-^\top & & & \\ E_- & 0 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 0 & E_-^\top \\ & & & E_- & E_+ \end{pmatrix} \qquad E_\pm \equiv \frac{1}{2}\begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}$$

LDT for quadratic functionals of Gauss-Markov chains

- Evaluation of the SCGF (generalization of Szegö theorem)

$$\varphi(\lambda) = \lim_{N\to\infty} \frac{1}{N} \boxed{\log < e^{\lambda W_N} >} = -\frac{1}{4\pi}\int_0^{2\pi} \ln\det F_\lambda(\theta)\ d\theta$$

Symbol matrix

$$F_\lambda(\theta) \equiv Ve^{-i\theta} + U + V^\top e^{i\theta}$$
$$-(D^{-2}S + \lambda E_-)e^{-i\theta} + D^{-2} + S^\top D^{-2}S - (D^{-2}S + \lambda E_-)^\top e^{i\theta}$$

- Positive definiteness

$$\log < e^{\lambda W_N} > \underset{\substack{\uparrow \\ \text{Gaussian integral}}}{=} -\frac{1}{2}\ln\det\boxed{(\Sigma_N^{-1} - \lambda M_N)} - N\ln(2\ dt\ Pe) - \frac{1}{2}\ln\det\Sigma_0$$

bulk block Toeplitz

$$\Sigma_N^{-1} - \lambda M_N = \begin{pmatrix} L & V^\top & & \mathsf{T}_N & \\ V & U & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & U & V^\top \\ & & & V & R \end{pmatrix}$$

$$L = \Sigma_0^{-1} + S^\top D^{-2}S + \lambda E_+$$
$$R = D^{-2} - \lambda E_+$$
$$U = D^{-2} + S^\top D^{-2}S$$
$$V = -D^{-2}S - \lambda E_-$$

- bulk block Toeplitz matrix T_N is positive definite $\longrightarrow$ symbol matrix $F_\lambda(\theta)$ positive definite for all $(0,2\pi)$ $\longrightarrow$ **Primary domain** $P = (\tilde{\lambda}_-, \tilde{\lambda}_+)$

- Schur complement $\mathsf{S}_N \equiv \begin{pmatrix} L - V^\top(\mathsf{T}_N^{-1})_{11}V & -V^\top(\mathsf{T}_N^{-1})_{1N}V^\top \\ -V(\mathsf{T}_N^{-1})_{N1}V & R - V(\mathsf{T}_N^{-1})_{NN}V^\top \end{pmatrix} \xrightarrow{N\to\infty} \begin{pmatrix} \mathscr{L}_\lambda & 0 \\ 0 & \mathscr{R}_\lambda \end{pmatrix}$ positive definite

Hermitian

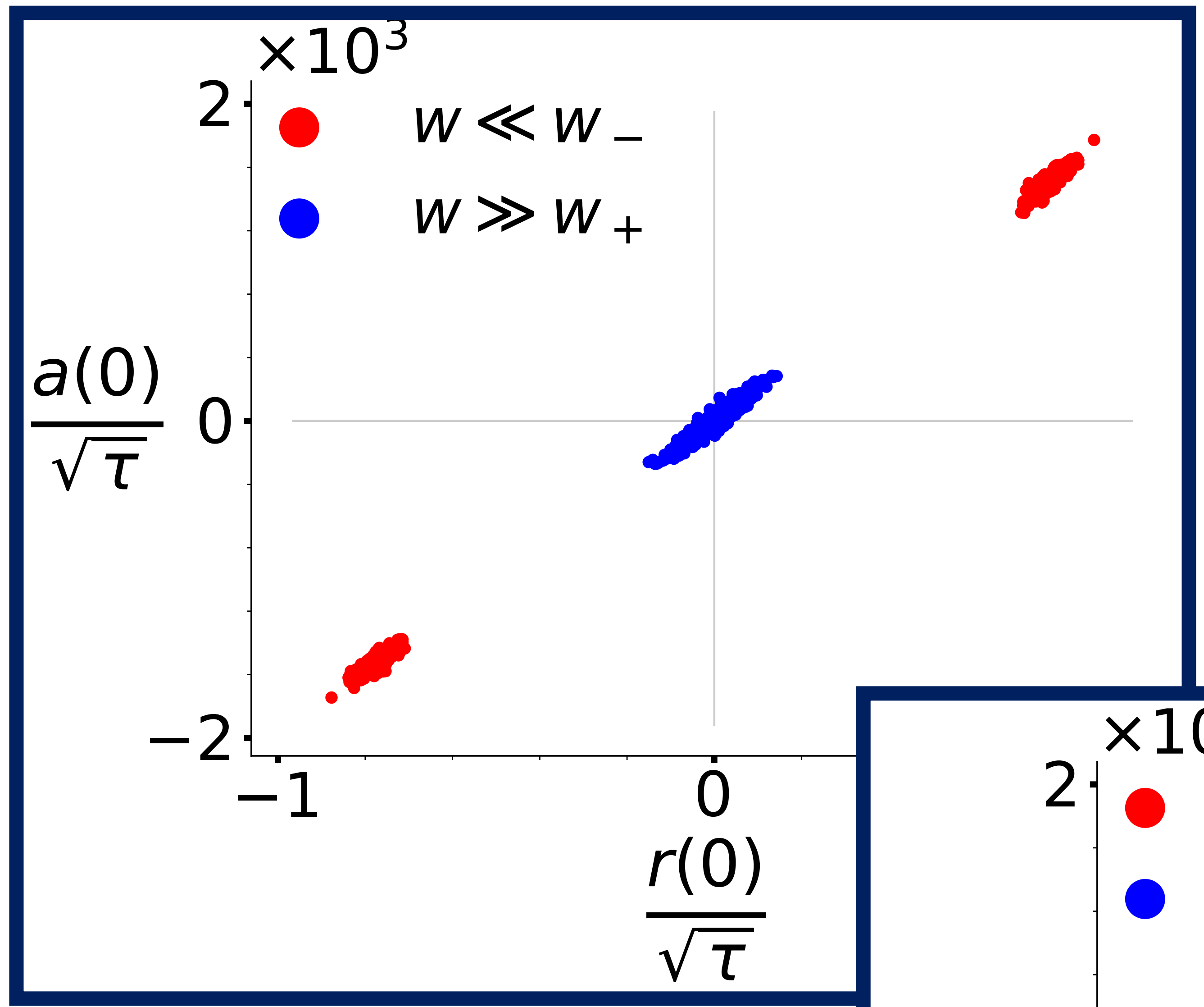
$\xrightarrow{\text{positive definiteness of}}$
$$\left[\begin{aligned} &\mathscr{L}_\lambda \equiv \Sigma_0^{-1} + S^\top D^{-2}S + \lambda E_+ - (D^{-2}S + \lambda E_-)^\top \Phi_\lambda(0) H_\lambda^{-1}(D^{-2}S + \lambda E_-) \\ &\mathscr{R}_\lambda \equiv D^{-2} - \lambda E_+ - (D^{-2}S + \lambda E_-) K_\lambda^{-1}\Phi_\lambda(0)(D^{-2}S + \lambda E_-)^\top \end{aligned}\right.$$
$\longrightarrow$ **Effective domain** $E = (\lambda_-, \lambda_+)$

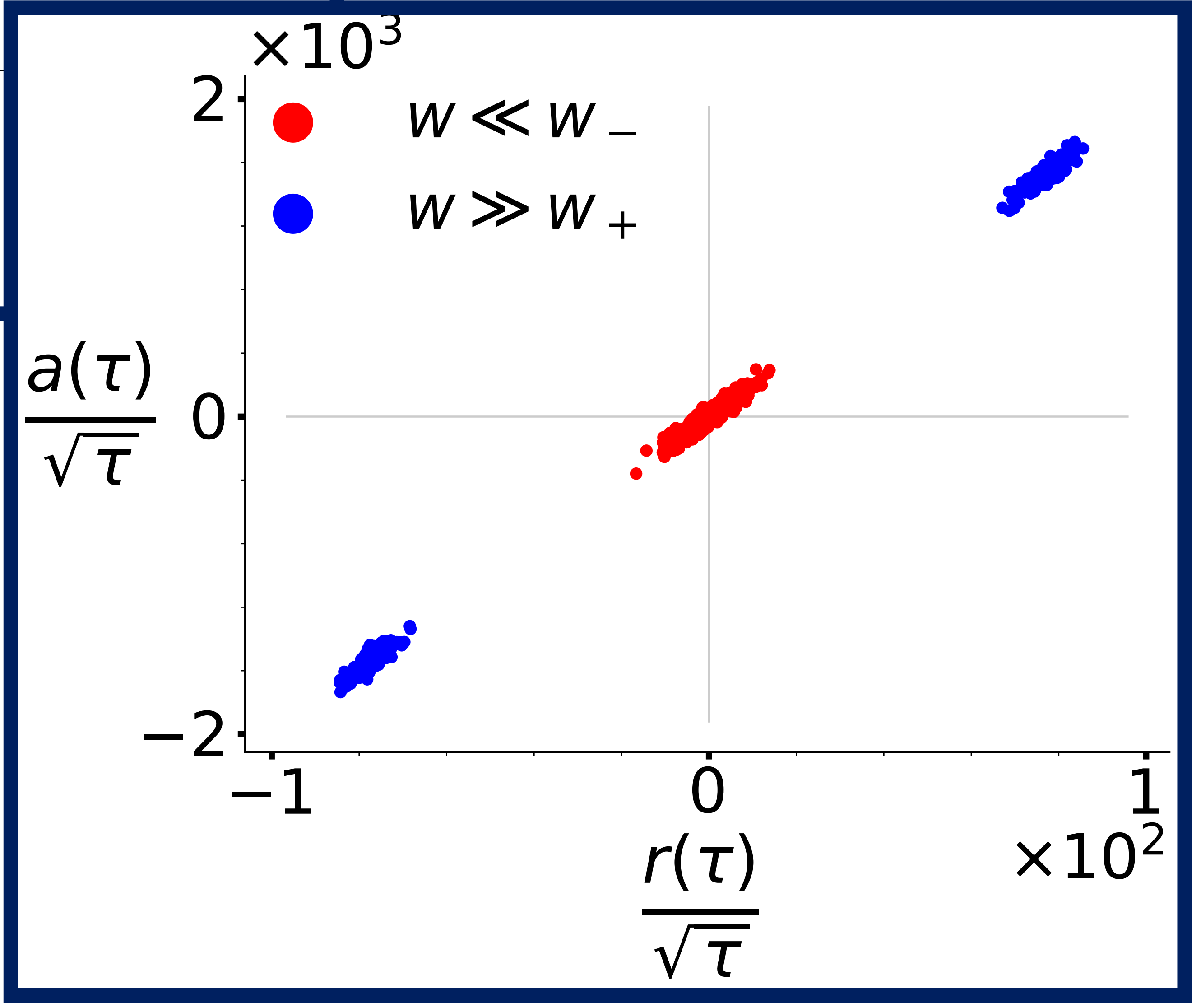
$$\Phi_\lambda(n) \equiv \frac{1}{2\pi}\int_0^{2\pi} F_\lambda^{-1}(\theta)e^{-in\theta}d\theta \qquad \underbrace{H_\lambda \equiv I + (D^{-2}S + \lambda E_-)\Phi_\lambda(1) \qquad K_\lambda \equiv I + \Phi_\lambda(1)(D^{-2}S + \lambda E_-)}_{\text{invertible}}$$

Harmonically confined AOUP: condensation

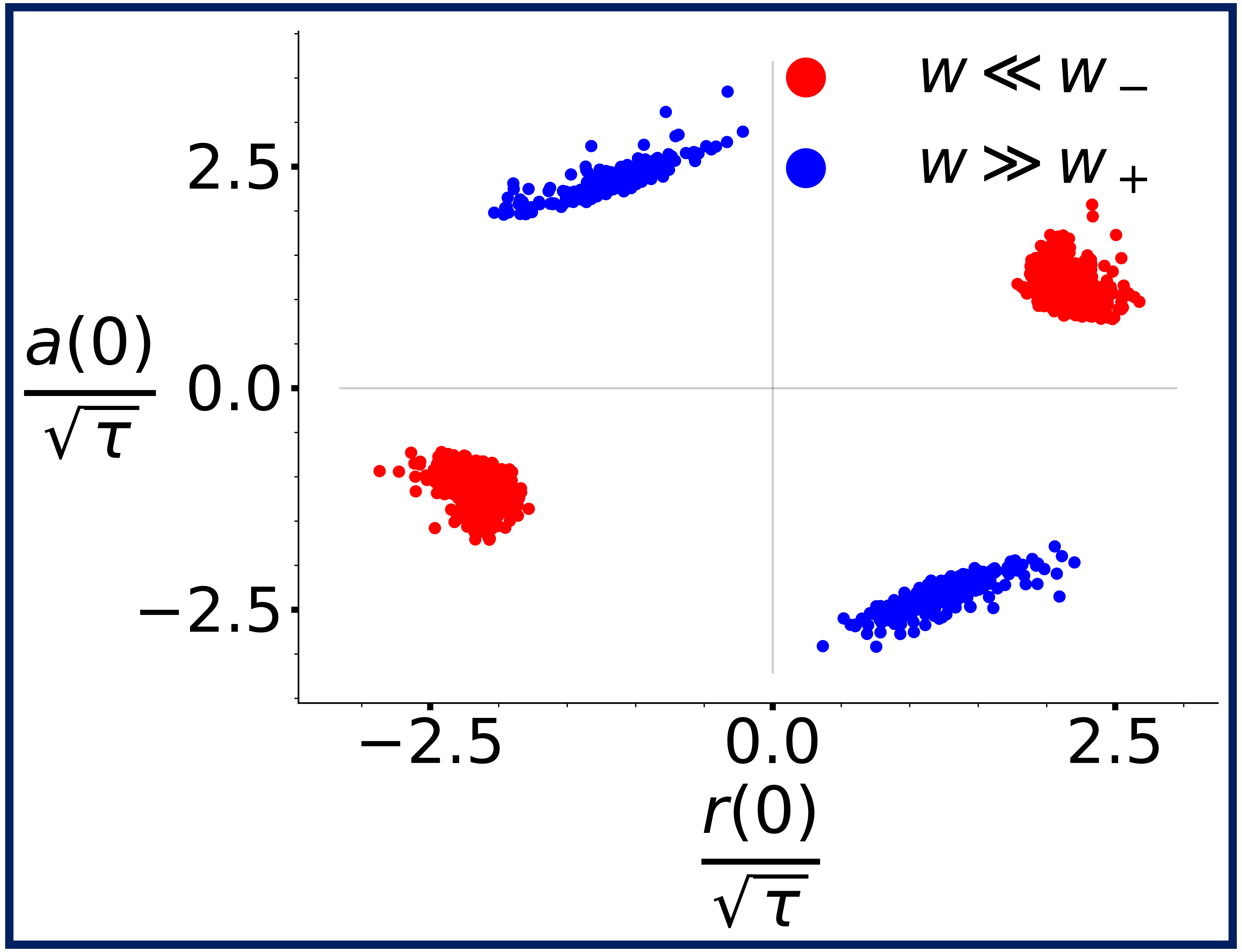
- Trajectories corresponding to $\mathscr{W}_a$ values in the linear tails are characterized by *big jumps* in the initial or final conditions → **Condensation mechanism**

- At stationarity





- Fixed initial conditions



Entropy Production at stationarity

PRL 2023,
Semeraro, Gonnella,
Suma and Zamparo

- **Entropy Production** $\mathscr{S}$ is closely related to the Active Work $\mathscr{W}_a$

$$\mathscr{S} \equiv \ln \frac{P_\tau(r(t), a(t))}{P_\tau(r(\tau - t), a(\tau - t))} = \mathscr{W}_a + \frac{1}{2}\begin{pmatrix} r(\tau) & v(\tau) \end{pmatrix} \Omega \begin{pmatrix} r(\tau) \\ v(\tau) \end{pmatrix} - \begin{pmatrix} r(0) & v(0) \end{pmatrix} \Omega \begin{pmatrix} r(0) \\ v(0) \end{pmatrix} \qquad \Omega = \Sigma_0^{-1} - \begin{pmatrix} \kappa & 0 \\ 0 & Pe^{-2} \end{pmatrix}$$

- SCGF with $P = (\lambda_-, \lambda_+) = E$ $\quad \phi(\lambda) = \frac{1+\kappa}{2} - \frac{1}{2}\sqrt{(1+\kappa)^2 - 4Pe^2\lambda(1+\lambda)} \qquad \tilde{\lambda}_\pm = \lambda_\pm = -\frac{1}{2} \pm \sqrt{1 + \left(\frac{1+\kappa}{Pe}\right)^2}$

- Rate function. No linear tails. $\quad I(w) = i(w) = \frac{1}{2}\left(\sqrt{1 + \left(\frac{w}{Pe}\right)^2} + \sqrt{(1+\kappa)^2 + Pe^2} - 1 - \kappa - w\right)$

- **No** singularities and Dynamical Phase Transitions !

- **Fluctuation Relation** satisfied !

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Fluctuations of Injected power

J Phys A,
Carollo, Semeraro,
Gonnella and Zamparo

- Trajectories corresponding to $\mathscr{W}_\tau$ values in the linear tails are characterized by *big jumps* in the initial conditions with similar patterns

- Positive- and negative-tail trajectories are discerned by looking the the white noise realisations