Università di Genova

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Electrically driven fluids

Based on:

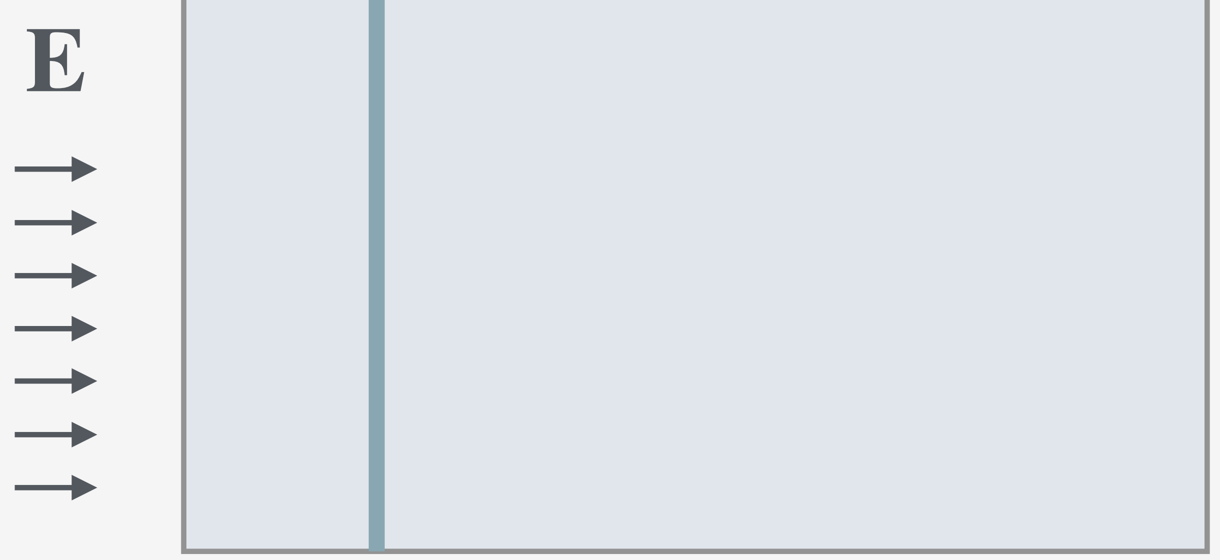
- J. High Energ. Phys. 2023, 218 (2023)
- J. High Energ. Phys. 2024, 114 (2024)

Jonas Rongen, Cortona, May 2025

Motivation

charged fluid

applied electric field



stationary configurations?

- hydrodynamics <u>without</u> relaxation:

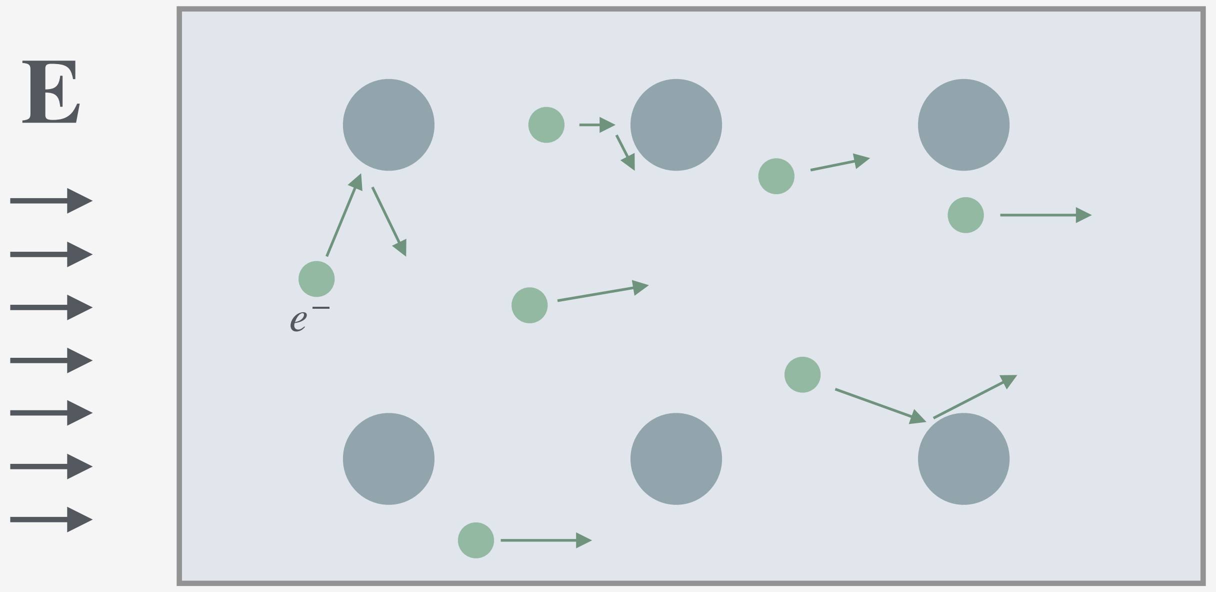
  thermal equilibrium requires $$\mathbb{E}_i - \partial_i \mu = 0$$ [Kovtun JHEP 28 (2016)]

  → velocity of fluid is unconstrained in magnitude

- in contrast with experiments

Motivation

Drude's model



- electron transport in conductor

$$\langle \frac{dp}{dt} \rangle = q\mathbf{E} - \Gamma \langle \mathbf{p} \rangle$$

Relaxation term prevents indefinite acceleration of charge carriers

- system relaxes to driven steady state

$$\langle \mathbf{v} \rangle = \frac{q\mathbf{E}}{\Gamma}$$

- conductivity

$$\mathbf{J} = nq\langle \mathbf{v} \rangle = \frac{nq^2}{\Gamma}\mathbf{E}$$

DC conductivity σ_{DC}

Lack In hydrodynamics: manifests as an infinite DC conductivity

Hydrodynamics

- effective field theoretical in the long-wavelength, long-timescale limit

1) Hydrodynamics uses that system is characterised by symmetries

$$\nabla_\mu T^{\mu\nu} = 0$$

- Can additionally have U(1) internal symmetry giving us conservation of charge current

$$\nabla_\mu J^\mu = 0$$

- These two equations are starting point of hydrodynamics

→ give us equations of motion to solve (see later!)

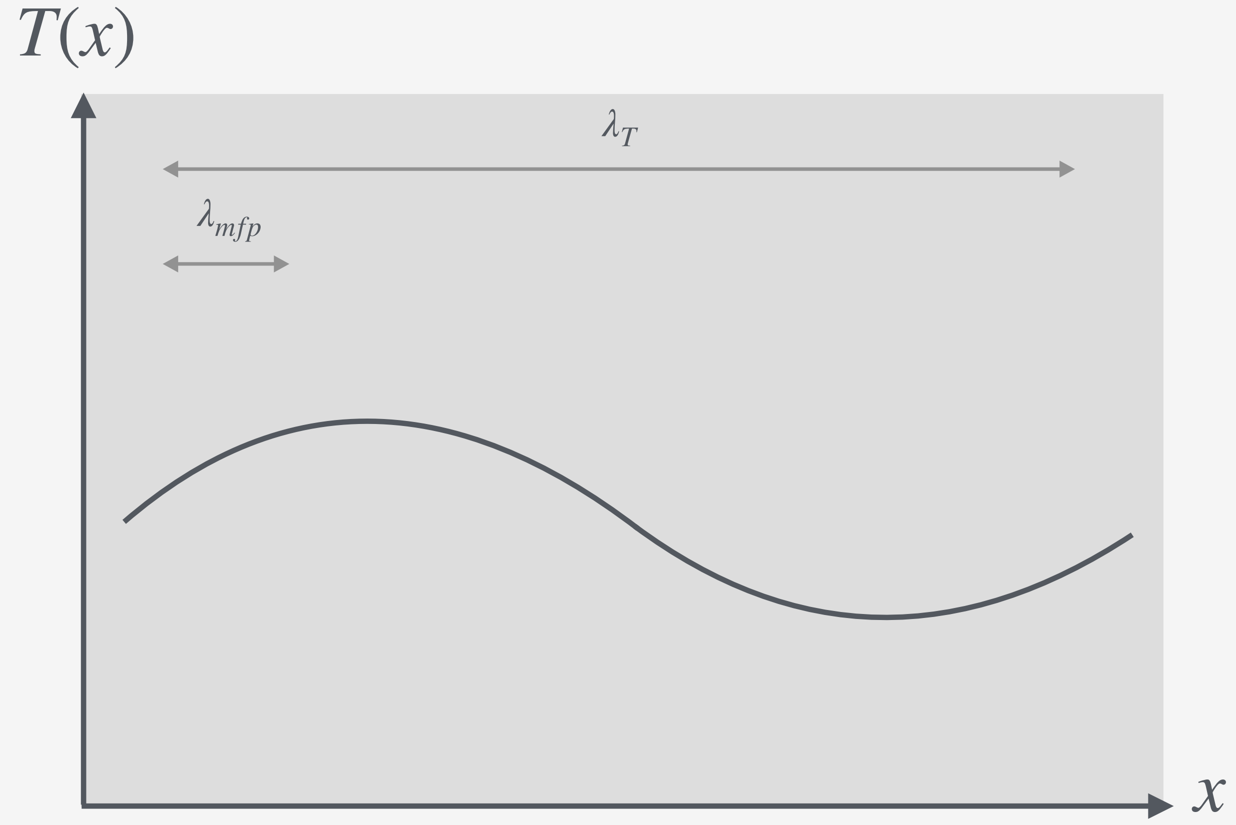
Hydrodynamics

2) thermal equilibrium described by constant thermodynamic quantities T, μ, v^μ

- account for fluctuations away from equilibrium → have non-equilibrium system

$$\boxed{T, \mu, v^\mu \rightarrow T(x), \mu(x), v^\mu(x)}$$

- assume: patches of local thermodynamic equilibrium



- Probe system on length scales much larger than typical mean free path $\lambda_{mfp}/\lambda_T \ll 1$

Hydrodynamics

- Allows for gradient expansion of $T^{\mu\nu}$ and J^{μ} in terms of the hydrodynamic variables, called **constitutive relations**

→ order zero means order zero in derivatives,...

- Example: charge current of system with Lorentz symmetry and internal U(1) symmetry

$$J^{\mu} = nu^{\mu} + \sigma_0 \left(E^{\mu} - T\nabla\left(\frac{\mu}{T}\right)\right) + \mathcal{O}\left(\partial^2\right)$$

non-dissipative

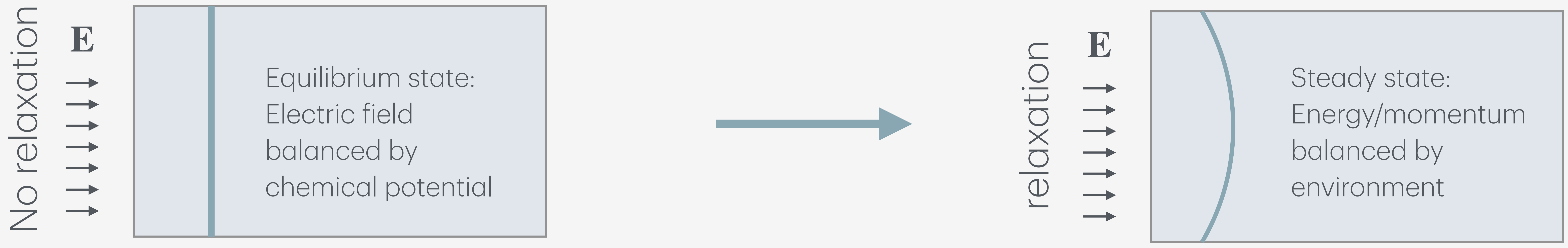
$$\nabla_{\mu} s^{\mu} = 0$$

Transport coefficients

dissipative

- generating functional: placing fluid on curved spacetime (where T, μ, v^{μ} become geometrised thermodynamic quantities)

Motivation



- proprosal: incorporate relaxation terms for energy and momentum into the definition of stationarity

- momentum relaxation and external sources constrain velocity to take a specific stationary value

  → presence of introduced sinks breaks boost invariance

- requires us to use boost agnostic hydrodynamics

Boost agnostic hydrodynamics

[Jensen et al. PRL 109 (2012),
de Boer et al., SciPostPhys. 9, 018 (2020),
Armas, Jain, SciPostPhys. 11, 054 (2021),...]

- fluid velocity becomes a thermodynamic variable (momentum conjugate)

$$\boxed{P(T, \mu, \vec{v}, \mathbb{E})}$$

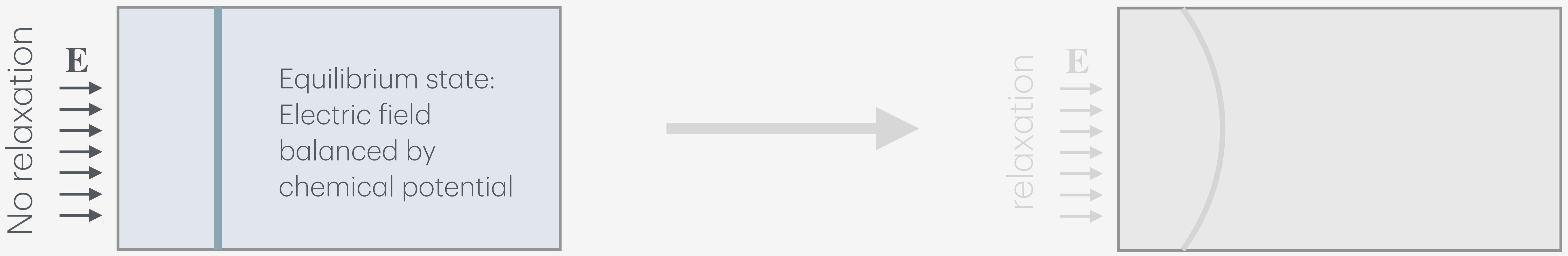
- thermodynamic densities

$$\boxed{n = \left(\frac{\partial P}{\partial \mu}\right), \quad s = \left(\frac{\partial P}{\partial T}\right), \quad \rho_{\mathrm{m}} = 2\left(\frac{\partial P}{\partial \vec{v}^2}\right), \quad \kappa_{\mathbb{E}} = 2\left(\frac{\partial P}{\partial \vec{\mathbb{E}}^2}\right)}$$

- In contrast to Galilean or Lorentz: velocity cannot be set to zero

$\rightarrow$ different inertial frames represent distinct hydrodynamic states

Stationarity



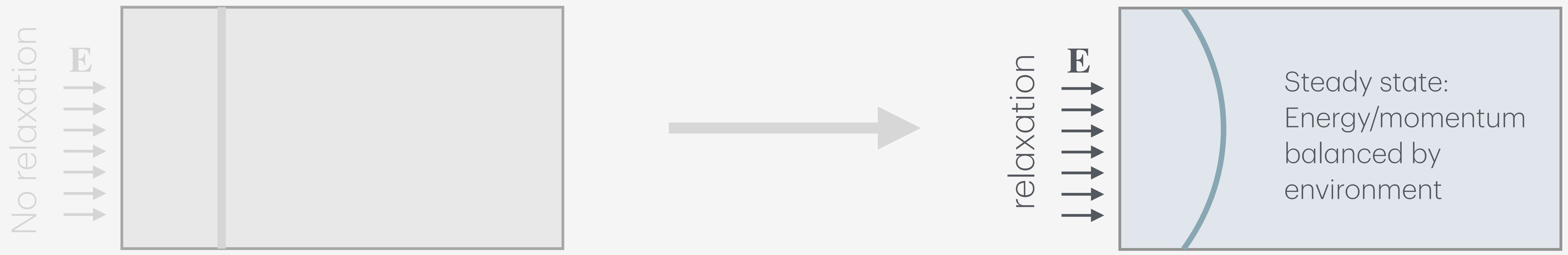
- Hydrostatic constraints: place fluid on Aristotelean geometry $(\tau_\mu,\ h_{\mu\nu},\ A_\mu)$, notion of time β^μ

$$\mathscr{L}_\beta A_\mu + \partial_\mu \Lambda = 0 \longrightarrow \mathbb{E}_\mu - \partial_\mu \mu = u^\nu \left(\mathbb{E}_\nu \tau_\mu - \mu \partial_{[\nu} \tau_{\mu]} \right) \xrightarrow{\text{flat}} \mathbb{E}_i - \partial_i \mu = 0$$

$$\mathscr{L}_\beta \tau_\mu = 0 \longrightarrow \frac{\partial_\mu T}{T} - u^\nu \left(\partial_\nu \tau_\mu - \partial_\mu \tau_\nu \right) = 0 \xrightarrow{\text{flat}} \partial_\mu T = 0$$

$$\mathscr{L}_\beta h_{\mu\nu} = 0, \partial_{[\mu} F_{\nu\rho]} = 0, \ldots \xrightarrow{\text{flat}} \{\partial_t v^i = 0, \quad \partial_i v_j + \partial_j v_i = 0, \quad \partial_t \mathbb{E}_i + v^j \partial_j \mathbb{E}_i + \mathbb{E}_j \partial_i v^j = 0\}$$

Relaxations



- Diffeomorphism and gauge invariance → conservation equations
- To move away from conservation add non-conservative forces

$$e^{-1}\partial_\mu\left(eT^\mu_\rho\right) + T^\mu\partial_\rho\tau_\mu - \frac{1}{2}T^{\mu\nu}\partial_\rho h_{\mu\nu} - F_{\rho\mu}J^\mu = \Gamma_\rho$$

$$e^{-1}\partial_\mu\left(eJ^\mu\right) = 0$$

flat →

$$\partial_t\varepsilon + \partial_i J^i_\varepsilon - \mathbb{E}_i J^i = -\hat{\Gamma}_\varepsilon$$

$$\partial_t P_i + \partial_j T^j_i - n\mathbb{E}_i = -\hat{\Gamma}^i_{\mathbf{P}}$$

$$\partial_t n + \partial_i J^i = 0$$

Relaxation at order zero

- (non-)conservation equations at $\mathcal{O}(\partial^0)$

$$nv^i\left(\mathbb{E}_i - \partial_i\mu\right) = \hat{\Gamma}_\varepsilon + \mathcal{O}(\partial) \qquad \mathbb{E} \sim \mathcal{O}(\partial^0)$$

$$n\left(\mathbb{E}_i - \partial_i\mu\right) = \Gamma_{\mathbf{P}}P_i + \mathcal{O}(\partial) \qquad \mathcal{O}(\partial^0) \quad \hat{\Gamma}^i_P = \Gamma_P P_i$$

- Assuming that neither of the sites is zero on their own we treat these expressions as conditions for hydrostaticity → modify our hydrostaticity condition by

$$\mathbb{E}_i - \partial_i\mu = 0 \rightarrow n(\mathbb{E}_i - \partial_i\mu) - \Gamma_{\mathbf{P}}P_i = 0$$

- energy and momentum relaxations related through

$$\hat{\Gamma}_\varepsilon = \Gamma_{\mathbf{P}}v_i$$

At higher order?

Relaxation at order one

- To obtain first order corrections: require fluid to locally obey second law of thermodynamics

[Boer, Hartong, Have, Obers, Sybesma, Armas, Jain,...]

$$e^{-1}\partial_\mu (eS^\mu) \geq 0$$

- What we find to satisfy 2nd law of thermodynamics:

$$\hat{\Gamma}_\varepsilon = \rho_m \Gamma v_j \left(n v^j + J^j_{(1),\text{NHS}} + J^j_{(1),\text{D}} \right) + \mathcal{O}(\partial^3)$$

Conductivities

- To compute the conductivities (needed to compare to Drude) we employ linear response theory

- study how each of the charge currents δJ^i, $\delta Q^i = \delta J^i_\epsilon - \mu\delta J^i \equiv \delta T^i_0 - \mu\delta J^i$, δP^i responds to perturbations of the $\mathbb{E}, T, v_{0j}$

- captured in the response matrix

$$\begin{pmatrix} \delta J_i \\ \delta Q_i \\ \delta P_i \end{pmatrix} = \begin{pmatrix} \sigma_{ij} & T\alpha_{ij} & \zeta^1_{ij} \\ T\bar{\alpha}_{ij} & T\kappa_{ij} & \zeta^2_{ij} \\ \zeta^3_{ij} & \zeta^4_{ij} & \zeta^5_{ij} \end{pmatrix} \begin{pmatrix} \delta E_j \\ \delta(-\partial_j T/T) \\ \delta v_{0j} \end{pmatrix}$$

- Consider small fluctuations of our fluid away from a stationary configuration with $T =$ const, $\mu =$ const

Conductivities

- The AC conductivities given by the $\mathbf{k} \to \mathbf{0}$ limit are

$$\sigma(\omega, \mathbf{0}) = \sigma_0 + \frac{n(n - \Gamma \rho_m \sigma_0)}{\rho_m(\Gamma - i\omega)}$$

(no Onsager reciprocity yet)

- Noticing that $\sigma(\omega \to 0) = \sigma_{DC} = n^2/\rho_m \Gamma$

- can write

$$\sigma(\omega) = \sigma_0 + \frac{\sigma_{\mathrm{DC}} - \sigma_0}{1 - i\omega\tau}$$

(sum of incoherent term and Drude term)

- differs from the standard hydrodynamic conductivity ⟶ appears in certain holographic models when momentum-breaking parameter becomes large enough [Z. Zhou et al., Phys. Rev. D 94 (2016); R.A. Davison et al., JHEP 09 (2015)]

Imposing time-reversal invariance

- want system to respect microscopic time reversal symmetry in effective correlates at $\omega \neq 0$ for a state at zero velocity

- In this case the conductivity becomes

$$\boxed{\sigma = \frac{\sigma_{DC}}{1 - i\omega\Gamma^{-1}}}$$

(Drude with DC conductivity)

$$\sigma_{DC} = \frac{n^2}{\rho_m \Gamma}$$

- Incoherent conductivity disappeared → can only appear if the system does not form a steady state or if we violate Onsager reciprocity

- Main result: thermo-electric conductivities of our model assume Drude form when imposing positivity of entropy production and Onsager reciprocity

Conclusion

- Considered hydrodynamic model of a charged fluid in an external electric field in the presence of impurities that relax momentum and energy.
- Looked for steady states

  → find that stationarity constraints need to be modified to incorporate relaxations
- included dissipative corrections and related energy relaxation to momentum relaxation
- allows us to consider conductivity of fluids that reach a stationary state in a driving electric field
- positivity of entropy production and Onsager reciprocity constrained transport in the fluid

  → no incoherent conductivity to make a contribution to the DC
- Further: stability of the model? hydrodynamical realisation of steady states in prope brane models?

Thank you!

More slides

Boost agnostic hydrodynamics

[Jensen et al. PRL 109 (2012),
de Boer et al., SciPostPhys. 9, 018 (2020),
Armas, Jain, SciPostPhys. 11, 054 (2021),...]

- Aristotelean spacetime: manifold equipped with two metrics

$$\tau_\mu\,, \quad h_{\mu\nu} = \delta_{ab} e^a_\mu e^b_\nu \; \left(= \mathrm{diag}(0,1,1,1)\right)$$
$$e = det(\tau, e^a_\mu)$$

- energy-momentum tensor

$$T^\mu_\nu = - T^\mu \tau_\nu + T^{\mu\rho} h_{\rho\nu}$$

- Dynamical evolution: time-like Killing vector β^μ

- Geometrization of thermal parameters in terms of $(\tau_\mu,\, h_{\mu\nu},\, A_\mu)$

$$T = \frac{1}{\tau_\mu \beta^\mu}, \quad \mu = T\left(A_\mu \beta^\mu + \Lambda\right), \quad u^\mu = T\beta^\mu, \quad F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} = \mathbb{E}_\mu \tau_\nu - \mathbb{E}_\nu \tau_\mu$$

Relaxation at order one

- simplification: assume hydrostaticity condition and constitutive relation for momentum relaxation term to be exact
- i.e. relations true at all orders in derivatives
- flat spacetime

$$\boxed{\mathbb{E}_i - \partial_i \mu - \frac{\Gamma P_i}{\mathrm{n}} = 0, \quad \hat{\Gamma}^i_{\vec{P}} = \Gamma P_i}$$

- Still: $\hat{\Gamma}_\epsilon$ receives derivative correction as it was derived as a consequence of the equations of motion on hydrostatic solutions → constitutive relation cannot be freely specified
- We choose $\mathbb{E} \sim \mathcal{O}(\partial)$

Relaxation at order one

- To obtain first order corrections: require fluid to locally obey second law of thermodynamics

[Boer, Hartong, Have, Obers, Sybesma, Armas, Jain,...]

$$\boxed{e^{-1}\partial_\mu \left(eS^\mu\right) \geq 0}$$

- The entropy current can be split into $S^\mu = S^\mu_{can} + S^\mu_{non}$

  - S^μ_{can} from covariantising Euler relation

$$\boxed{S^\mu_{can} = -T^\mu{}_\nu \beta^\nu + P\beta^\mu - \frac{\mu}{T} J^\mu - \kappa_{\mathbb{E}} \mathbb{E}^\nu \mathbb{E}_\nu \beta^\mu}$$

  - S^μ_{non} together with relaxation scalar cancel hydrostatic contributions to entropy production

Relaxation at order one

- Using (non-)conservation equation of energy-momentum tensor and charge current, divergence of canonical entropy current in terms of altered stationarity condition is

$$
\begin{aligned}
e^{-1}\partial_\mu\left(eS^\mu_{\text{can}}\right)+\left(\beta^\rho+\frac{1}{nT}\left(J^\nu-J^\nu_{(0)}\right)h_{\nu\sigma}h^{\sigma\rho}\right)\Gamma_\rho \\
=\left(T^\mu-T^\mu_{(0)}\right)\mathcal{L}_\beta\tau_\mu-\frac{1}{2}\left(T^{\mu\nu}-T^{\mu\nu}_{(0)}\right)\mathcal{L}_\beta h_{\mu\nu}-\left(J^\mu-J^\mu_{(0)}\right)\delta'_{\mathscr{B}}A_\mu
\end{aligned}
$$

where

$$\delta_{\mathscr{B}}A_\mu := \mathcal{L}_\beta A_\mu - \partial_\mu\Lambda = \mathcal{L}_\beta A_\mu - \partial_\mu\left(\frac{u^\nu A_\nu-\mu}{T}\right)$$

$$\delta'_{\mathscr{B}}A_\mu = \delta_{\mathscr{B}}A_\mu - \frac{1}{nT}h_{\mu\nu}h^{\nu\rho}\Gamma_\rho$$

- Rewriting divergence in this way allows us to isolate the order one in derivatives contributions to the constitutive relations of $T^\mu, T^{\mu\nu}, J^\mu$

Relaxation at order one

- decompose each constitutive relations into: hydrostatic, non-hydrostatic non-dissipative and dissipative corrections

$$T^{\mu} - T^{\mu}_{(0)} = T^{\mu}_{\mathrm{HS}} + T^{\mu}_{\mathrm{NHS}} + T^{\mu}_{\mathrm{D}}$$
$$T^{\mu\nu} - T^{\mu\nu}_{(0)} = T^{\mu\nu}_{\mathrm{HS}} + T^{\mu\nu}_{\mathrm{NHS}} + T^{\mu\nu}_{\mathrm{D}}$$
$$J^{\mu} - J^{\mu}_{(0)} = J^{\mu}_{\mathrm{HS}} + J^{\mu}_{\mathrm{NHS}} + J^{\mu}_{\mathrm{D}}$$

- assume: separate relaxation contributions into those that can be expressed in terms of stationary tensor structures and those that vanish at stationarity

- What we find to satisfy 2nd law of thermodynamics:

$$\hat{\Gamma}_{\varepsilon} = \rho_m \Gamma v_j \left(n v^j + J^j_{(1),\mathrm{NHS}} + J^j_{(1),\mathrm{D}} \right) + \mathcal{O}(\partial^3)$$

Even more slides

Generating functional

- generating functional $W[\tau, h, A]$: correlation functions

(leading term) $$W_{(0)}[\tau, h, A] = \int d^{d+1}x \, e \, P\left(T, \mu, \vec{\mathbb{E}}^2, \vec{v}^2, \vec{v} \cdot \vec{\mathbb{E}}\right)$$

- define one-point functions

$$T^{\mu\nu} = \frac{2}{e}\frac{\delta W}{\delta h_{\mu\nu}}, \quad T^{\mu} = -\frac{1}{e}\frac{\delta W}{\delta \tau_{\mu}}, \quad J^{\mu} = \frac{1}{e}\frac{\delta W}{\delta A_{\mu}}$$

Relaxation

$$\Gamma_\rho = -T\hat{\Gamma}_\sigma\left(\left(\beta^\sigma + \frac{1}{nT}\left(J^\mu_{\text{NHS}} + J^\mu_{\text{D}}\right)h_{\mu\nu}h^{\nu\sigma}\right)\tau_\rho - \frac{1}{T}h^{\sigma\mu}h_{\mu\rho}\right)$$
$$-\Gamma_{\rho\sigma}\left(\beta^\sigma + \frac{1}{nT}\left(J^\mu_{\text{NHS}} + J^\mu_{\text{D}}\right)h_{\mu\nu}h^{\nu\sigma}\right),$$

$$\Gamma_{\mu\nu} = \Gamma\left(c_1\tau_\mu\tau_\nu + c_2 h_{\mu\nu}\right) + \mathcal{O}\left(\partial^3\right)$$

Hydrostatic part

- Hydrostatic part has to satisfy following non-conservation equation

$$\partial_\mu {T_{\mathrm{HS}}}^\mu{}_\nu - F_{\nu\mu} J^\mu_{\mathrm{HS}} - \Gamma^{\mathrm{HS}}_\nu = 0,$$
$$\partial_\mu J^\mu_{\mathrm{HS}} = 0$$

- At order $\mathcal{O}(\partial^0)$ in constitutive relations: $\Gamma^{HS}_{(1),\nu} = \rho_{\mathrm{m}} \Gamma\left(\mathbf{v}^2, v_i\right)$

- At order $\mathcal{O}(\partial^1)$ in constitutive relations: find that using only hydrostatic conditions that do not involve relaxation term $\Gamma^{HS}_{(2),\nu} \equiv 0$

- Now considering entropy production in presence of relaxation terms

- Have freedom to define S^μ_{non} , Γ^{non} satisfying

$$e^{-1}\partial_\mu\left(eS^\mu_{\mathrm{non}}\right) + \Gamma^{\mathrm{non}} = -T^\mu_{\mathrm{HS}}\mathscr{L}_\beta \tau_\mu + \frac{1}{2}T^{\mu\nu}_{\mathrm{HS}}\mathscr{L}_\beta h_{\mu\nu} + J^\mu_{\mathrm{HS}}\delta'_{\mathscr{B}}A_\mu$$
$$\Gamma^{\mathrm{non}} = -\frac{1}{nT}J^\mu_{\mathrm{HS}} h_{\mu\sigma} h^{\sigma\rho}\Gamma_\rho$$

- In this way we eliminate all stationary configurations consistent with positivity of entropy production (by defining a relaxation scalar and non-canonical entropy current that cancels hydrostatic contributions to entropy production)

$$S^\mu = S^\mu_{can} + S^\mu_{non}$$

Non-hydrostatic, non-dissipative part

- Part that makes no contribution to entropy production but is not hydrostatic

$$T^{\mu}_{\rm NHS}\mathscr{L}_\beta\tau_\mu - T^{\mu\nu}_{\rm NHS}\frac{1}{2}\mathscr{L}_\beta h_{\mu\nu} - J^{\mu}_{\rm NHS}\delta'_{\mathscr{B}}A_\mu \equiv 0$$

- At order one: must be linear combinations of $\mathscr{L}_\beta\tau_\mu, \quad \mathscr{L}_\beta h_{\mu\nu}, \quad \delta'_{\mathscr{B}}A_\mu$

- Correspondingly equation above is quadratic form in hydrostatic constraints

- quadratic form: to fail to contribute to entropy production must be antisymmetric (in this way no entropy production)

$$\begin{pmatrix} T^{\mu}_{(1),\rm NHS} \\ T^{\mu\nu}_{(1),\rm NHS} \\ J^{\mu}_{(1),\rm NHS} \end{pmatrix} = \begin{pmatrix} 0 & N_2^{\mu(\rho\sigma)} & N_1^{\mu\rho} \\ -N_2^{\rho(\mu\nu)} & 0 & N_3^{\rho(\mu\nu)} \\ -N_1^{\rho\mu} & -N_3^{\mu(\rho\sigma)} & 0 \end{pmatrix} \begin{pmatrix} \mathscr{L}_\beta\tau_\rho \\ -\frac{1}{2}\mathscr{L}_\beta h_{\rho\sigma} \\ -\delta'_{\mathscr{B}}A_\rho \end{pmatrix}$$

- We obtained most general tensor structures consistent with our symmetries and defined 24 non-hydrostatic, non-dissipative transport coefficients

Dissipative part

- Dissipative terms lead production of entropy

- Analogously dissipative contributions can be written in quadratic form in terms of symmetric coefficient matrix, allowing for entropy production

$$\begin{pmatrix} T^{\mu}_{(1),\mathrm{D}} \\ T^{\mu\nu}_{(1),\mathrm{D}} \\ J^{\mu}_{(1),\mathrm{D}} \end{pmatrix} = \begin{pmatrix} D_1^{\mu\rho} & D_2^{\mu(\rho\sigma)} & D_3^{\mu\rho} \\ D_2^{\rho(\mu\nu)} & D_4^{(\mu\nu)(\rho\sigma)} & D_5^{\rho(\mu\nu)} \\ D_3^{\rho\mu} & D_5^{\mu(\rho\sigma)} & D_6^{\mu\rho} \end{pmatrix} \begin{pmatrix} \mathscr{L}_\beta \tau_\rho \\ -\frac{1}{2}\mathscr{L}_\beta h_{\rho\sigma} \\ -\delta'_{\mathscr{B}} A_\rho \end{pmatrix}$$

- Obtained most general structures consistent with our symmetries and defined 42 dissipative transport coefficient terms