Feynman Integrals & Scattering Amplitudes
in Particle Physics, Gravitation & Cosmology

Pierpaolo Mastrolia

New Frontiers in Theoretical Physics
Cortona, 23.05.2025

In collaboration with: P. Benincasa, **G. Brunello**, S. Cacciatori, **V. Chestnov, G. Crisanti**, B. Eden, **W. Flieger, M. Giroux, M. Gottwald, H. Frellesvig**, S. Laporta, **M.K. Mandal,** *S. Matsubara-Heo*, S. Mizera, **R. Patil**, J. Steinhoff, **T. Scherdin**, **S. Smith**, **WJ Torres Bobadilla, F. Vazao,** *N. Takayama*

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Outline

- **Feynman Calculus**
  - Integral relations, Differential Equations and Special Functions
- **Amplitudes and Diagrams for Gravitational Wave Physics**
  - Post-Minkowskian Corrections
  - Post-Newtonian Corrections
- **Vector Space Structure of *Twisted Period Integrals*** (Feynman, GKZ, Euler-Mellin, A-hypergeometric)
  - De Rahm co-homology groups
  - Intersection Numbers
- **Intersection Theory Applications**
  - Feynman Integrals
  - Beyond Feynman Integrals
  - Cosmological correlators and Wave Functions
- **Differential Space Structure of *Twisted Period Integrals***
  - Annihilators and D-modules
- **Conclusions**

Differential Equations

Theoretical Physics goals: *modelling* Nature by *modelling* changes: Systems' Evolution

Describe how promptly a quantity changes with respect to the change in one or more other quantities

- **Differential Equations**

$$\partial_x^{(n)} f(x) + p_{n-1}(x)\, \partial_x^{(n-1)} f(x) + \ldots + p_1(x)\, \partial_x^{(1)} f(x) + p_0(x) f(x) = 0$$

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- **Linear relations**

$$f_n(x) + a_{n-1}\, f_{n-1}(x) + \ldots + a_1\, f_1(x) + a_0\, f_0(x) = 0$$

Differential Equations

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$$\partial_x^{(n)} f(x) = -\,p_{n-1}(x)\,\partial_x^{(n-1)} f(x) - \ldots - p_1(x)\,\partial_x^{(1)} f(x) - p_0(x) f(x)$$

- **Linear relations**

$$f_n(x) + a_{n-1} f_{n-1}(x) + \ldots + a_1 f_1(x) + a_0 f_0(x) = 0$$

$$f_n(x) = -\,a_{n-1} f_{n-1}(x) - \ldots - a_1 f_1(x) - a_0 f_0(x)$$

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$$\partial_x^{(n)} f(x) + p_{n-1}(x)\, \partial_x^{(n-1)} f(x) + \ldots + p_1(x)\, \partial_x^{(1)} f(x) + p_0(x) f(x) = 0$$

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- **Linear relations**

$$f_n(x) + a_{n-1}\, f_{n-1}(x) + \ldots + a_1\, f_1(x) + a_0\, f_0(x) = 0$$

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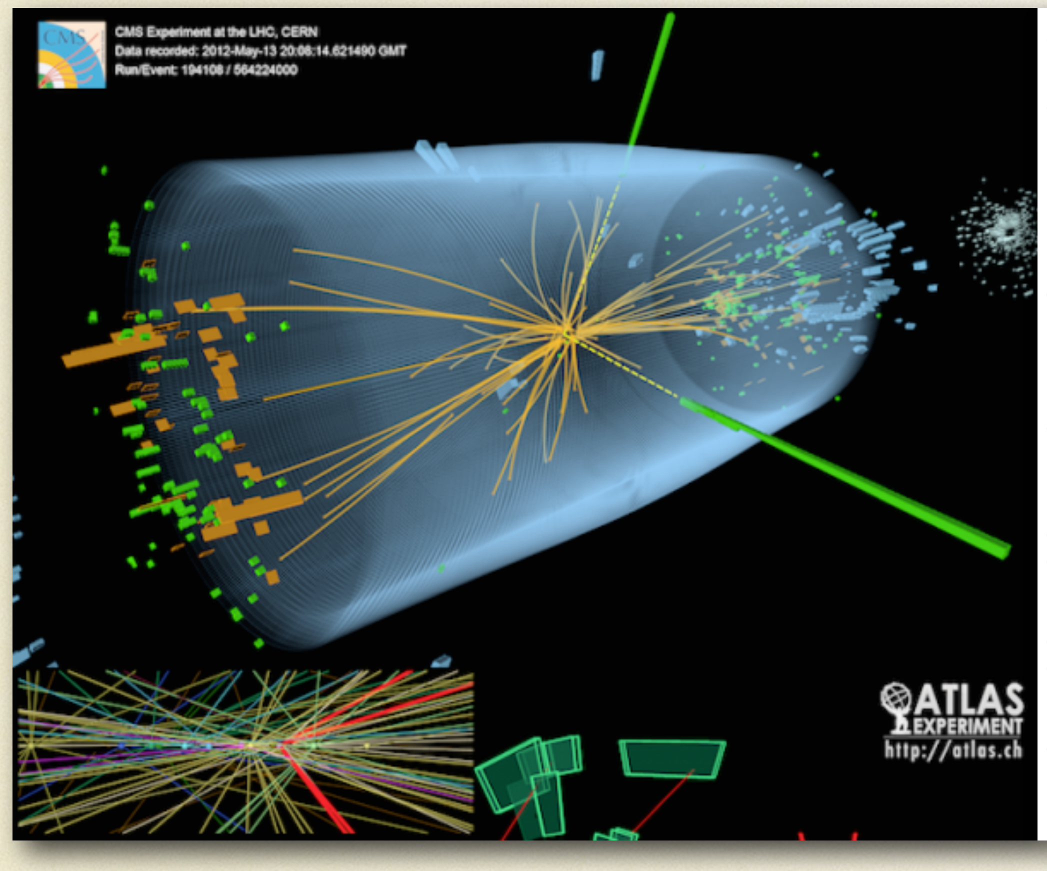
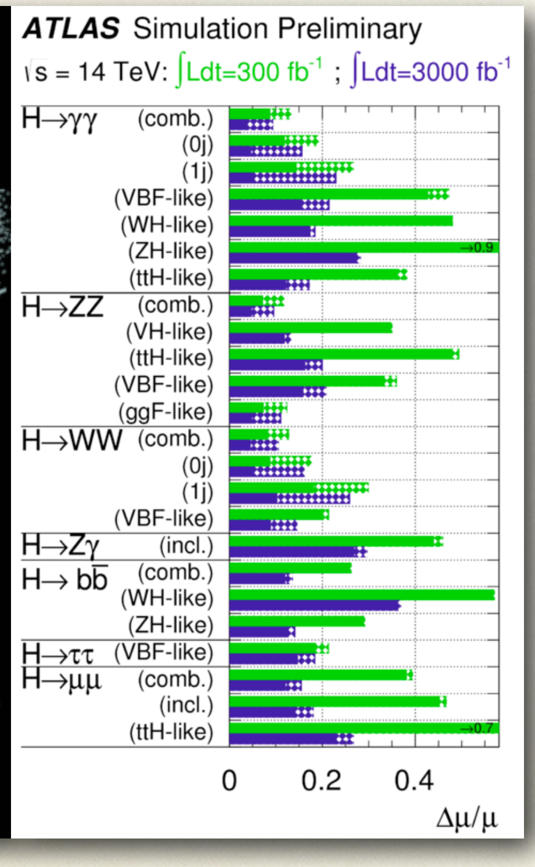
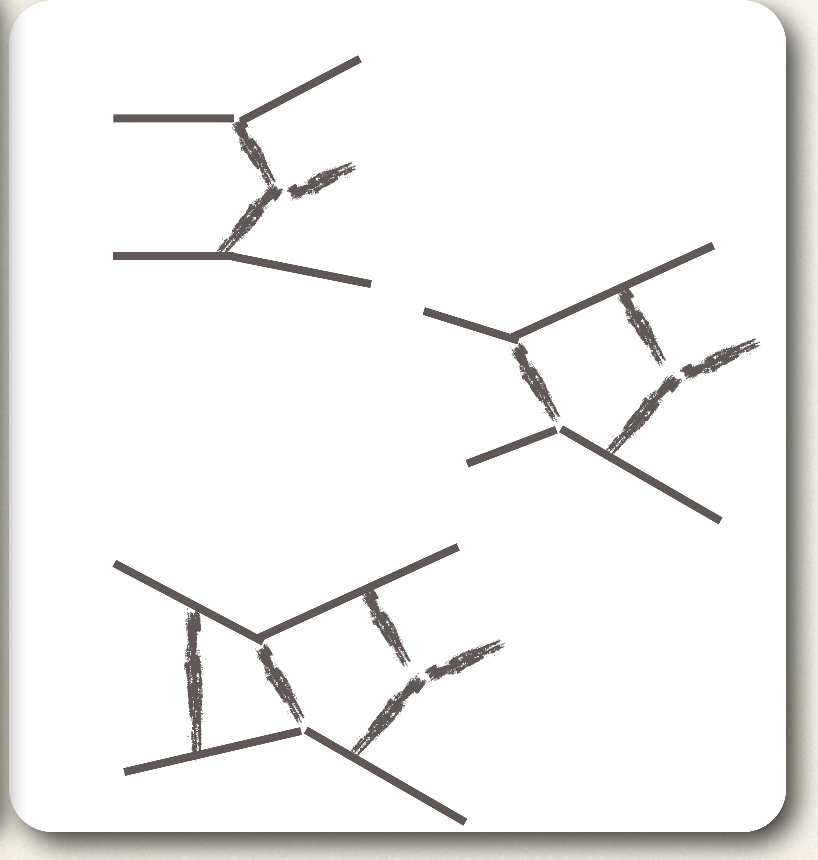
Decomposition formulas
in terms
of **n *independent* elements**

Impact of Scattering Amplitudes & Multiloop Calculus / Frontier of Theoretical Physics

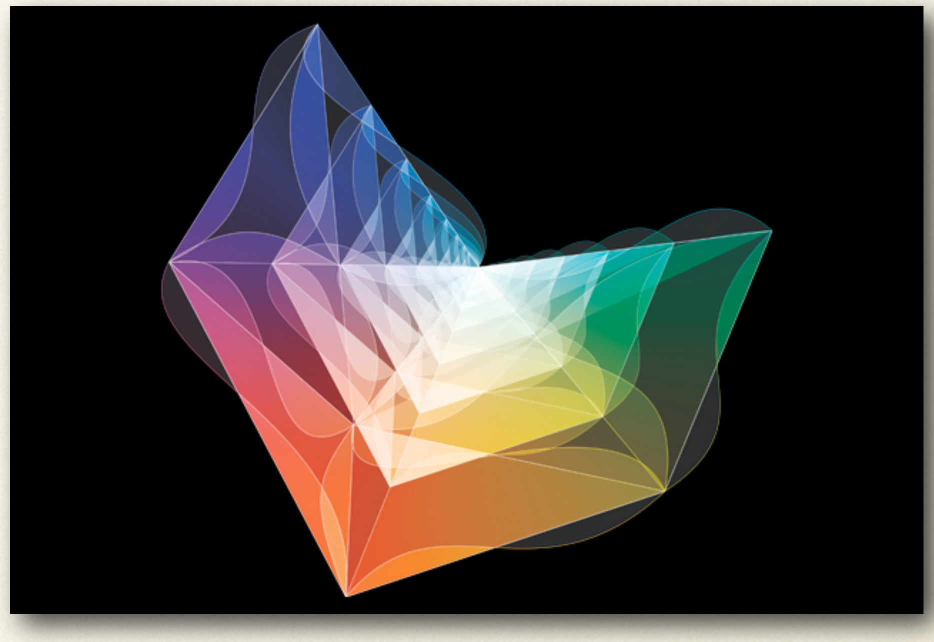
- **Collider Phenomenology**

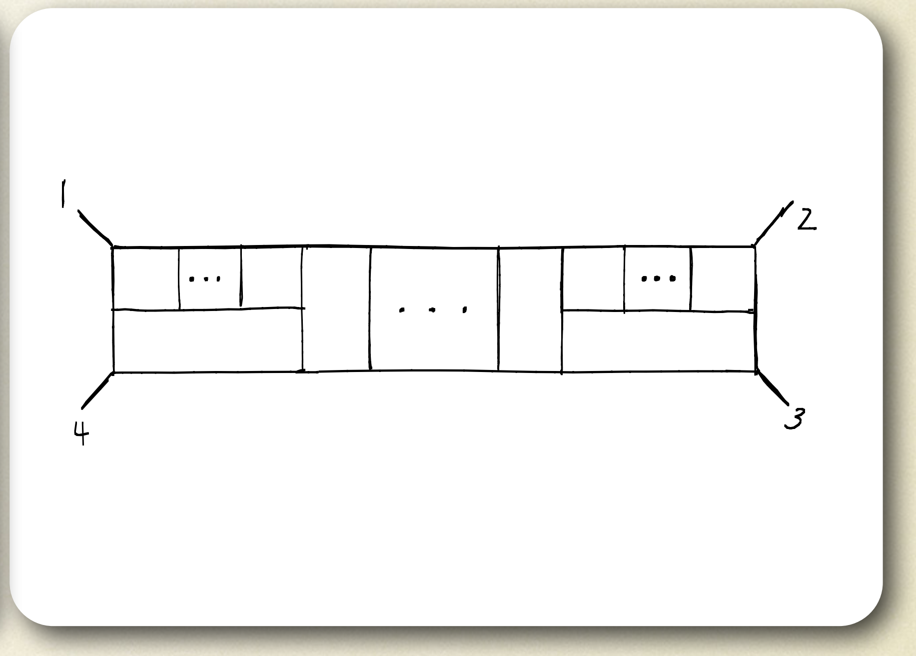




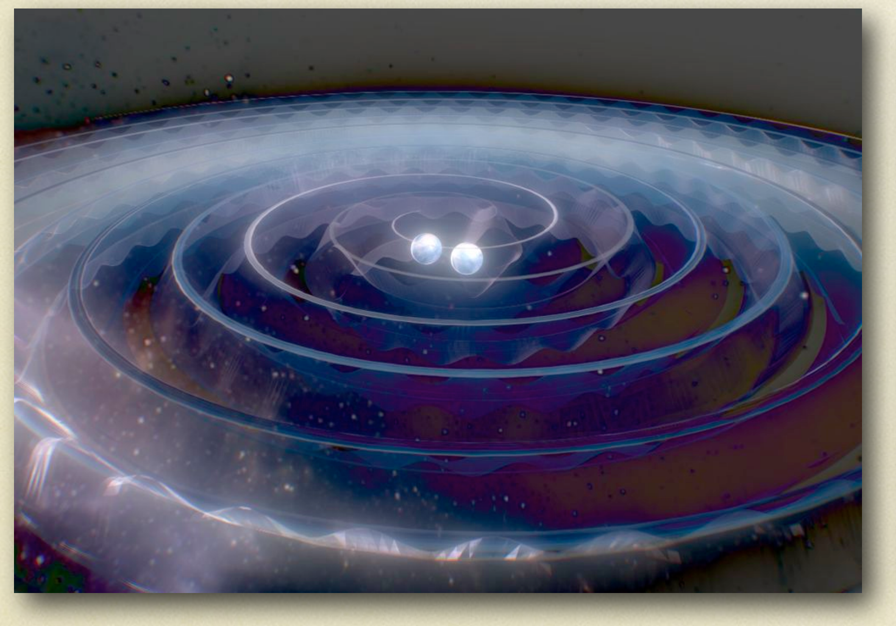


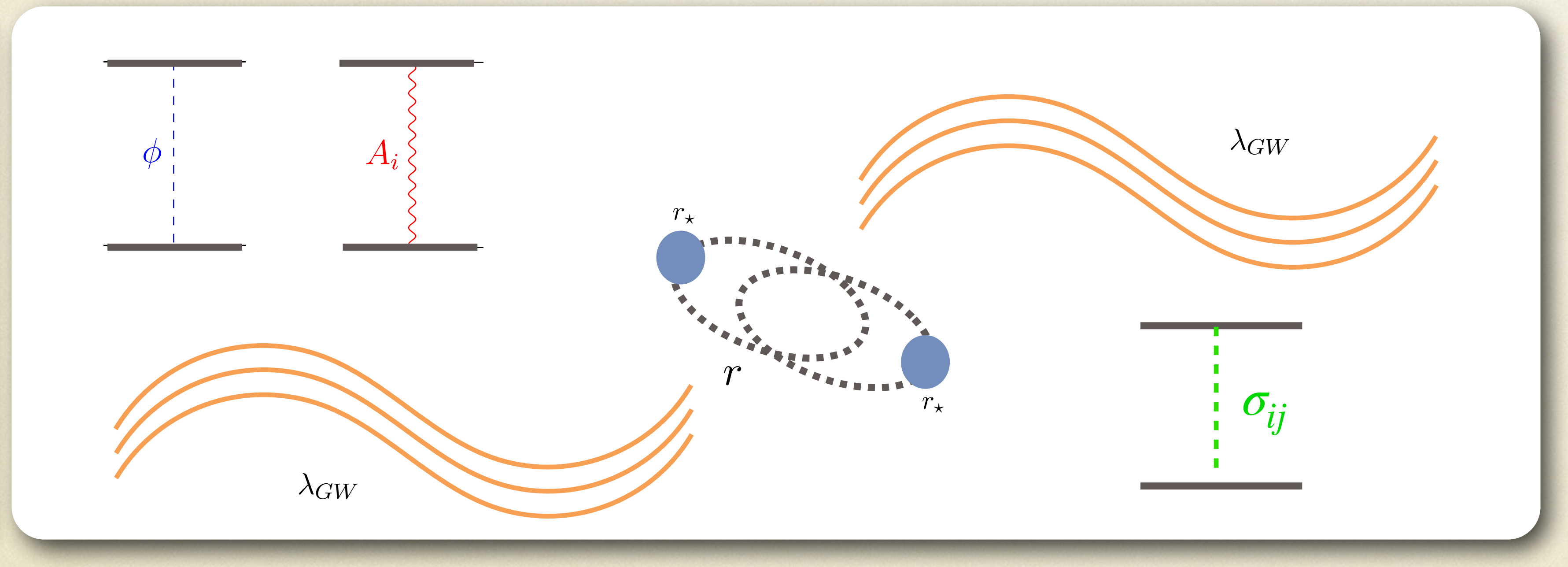
- **Geometry of Quantum Field Theory**



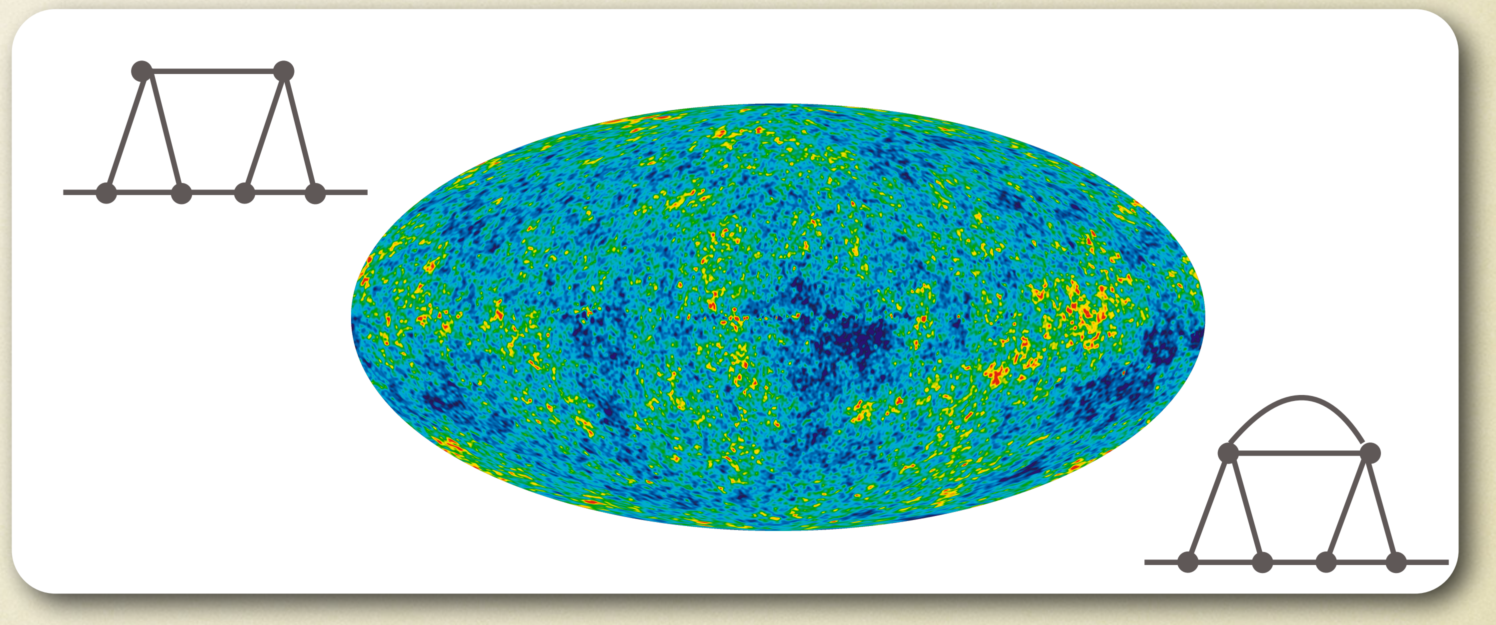


- **EFT Classical General Relativity**



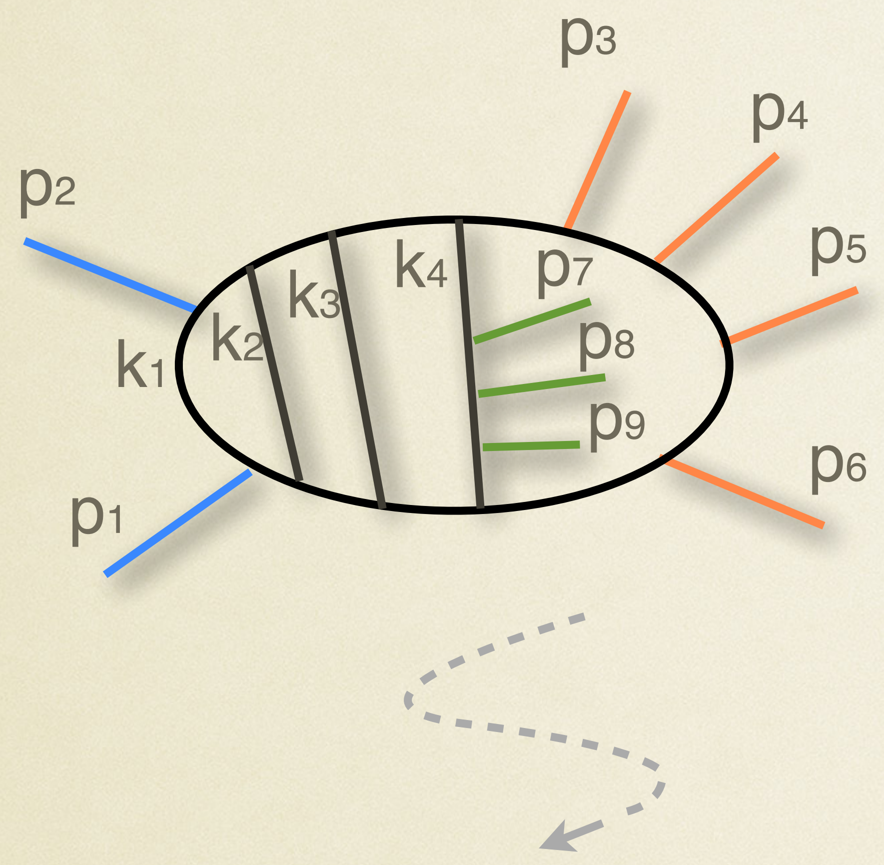


- **Cosmology**



Feynman Integrals

- **Momentum-space Representation**



$$= I^{[d]}_{a_1,\ldots,a_N} = \int \prod_{i=1}^{L} d^d k_i \left(\prod_{n=1}^{N} \frac{1}{D_n^{a_n}} \right)$$

N-denominator
generic Integral

L loops, $E{+}1$ external momenta,

$N = LE + \frac{1}{2}L(L{+}1)$ (generalised) denominators

total number of *reducible* and *irreducible* scalar products

't Hooft & Veltman

$$D_n = (p_1 \pm p_2 \pm \ldots \pm k_1 \pm k_2 \pm \ldots)^2 - m_n^2$$

Feynman Integrals / (a few) Evaluation Methods

Integration-by-parts identities

Chetyrkin, Tkachov
Remiddi, Laporta; Laporta
Zhang, Larssen
……………….

Direct Integral Evaluation

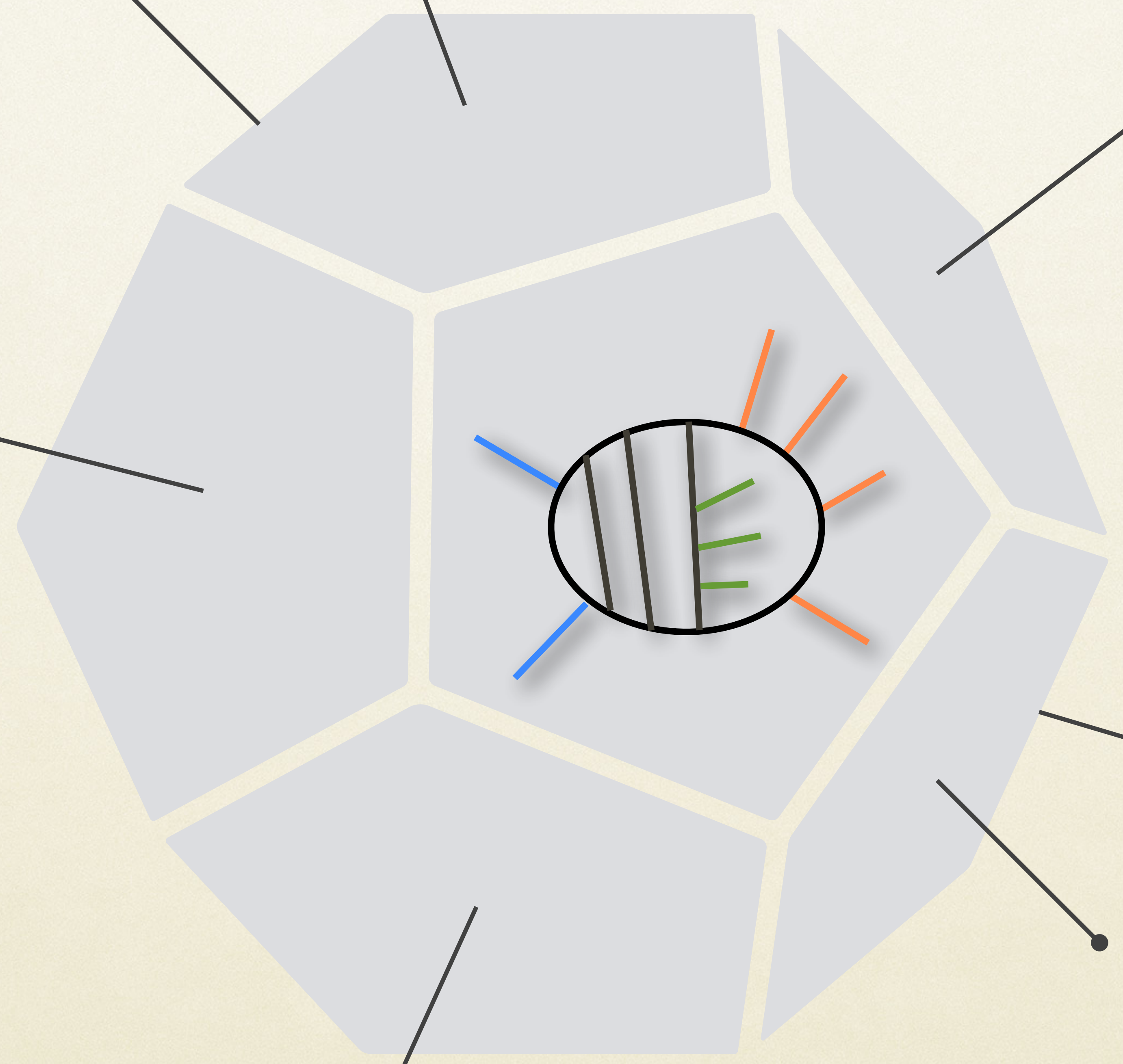
Passarino, Veltman, t'Hooft, Veltman
Beneke, Smirnov, Tausk
Binoth, Heinrich,
Anastasiou, Melnikov
Weinzierl
Czakon
……………….

Unitarity-based and on-shell methods

Bern, Dixon, Dunbar, Kosower
Britto, Cachazo, Feng, Witten
Brandhuber, Spence, Travaglini
Britto, Buchbinder, Cachzao, Feng + P.M.
Glover, Badger & P.M.
Anastasiou, Britto, Kunszt & P.M.
Forde; Badger
Beger, Bern, Dixon, Forde, Kosower, Ita, Maitre
Ellis, Giele, Kunszt, Melnikov, Zanderighi
……………….

Integrand Decomposition & Polynomial division

Ossola Papadopoulos Pittau
Ossola Papadopoulos Pittau & P.M.
Bevilacqua, Garzelli, Papadopoulos, Pittau, Worek
Ossola, Reiter, Tramontano, & P.M.
Mirabella, Ossola, Peraro & P.M.
Ossola & P.M.
Gluza, Kajda, Kosower
Badger, Frellsvig, Zhang
Zhang
Peraro, Primo & P.M.
Abreu, Febres Cordero, Ita, De Laurentis
……………….



Difference Equations

Tarasov
Laporta
Lee
……………….

Differential Equations

Barucchi, Ponzano, Regge
Bern, Dixon, Kosower
Kotikov, Remiddi,
Gehrmann Remiddi
Bonciani, Remiddi, & P.M.
Czakon
Henn,
Papadopoulos,
Liu, Ma, Peking
……………….

What's next?

Feynman Integrals

- **Integration-by-parts Identities (IBPs)**

$$\int \prod_{i=1}^{L} d^d k_i \, \frac{\partial}{\partial k_j^\mu} \left(v_\mu \prod_{n=1}^{N} \frac{1}{D_n^{a_n}} \right) = 0$$

$$v_\mu = v_\mu(p_i, k_j)$$

arbitrary

- IBP equations
- Contiguity relations

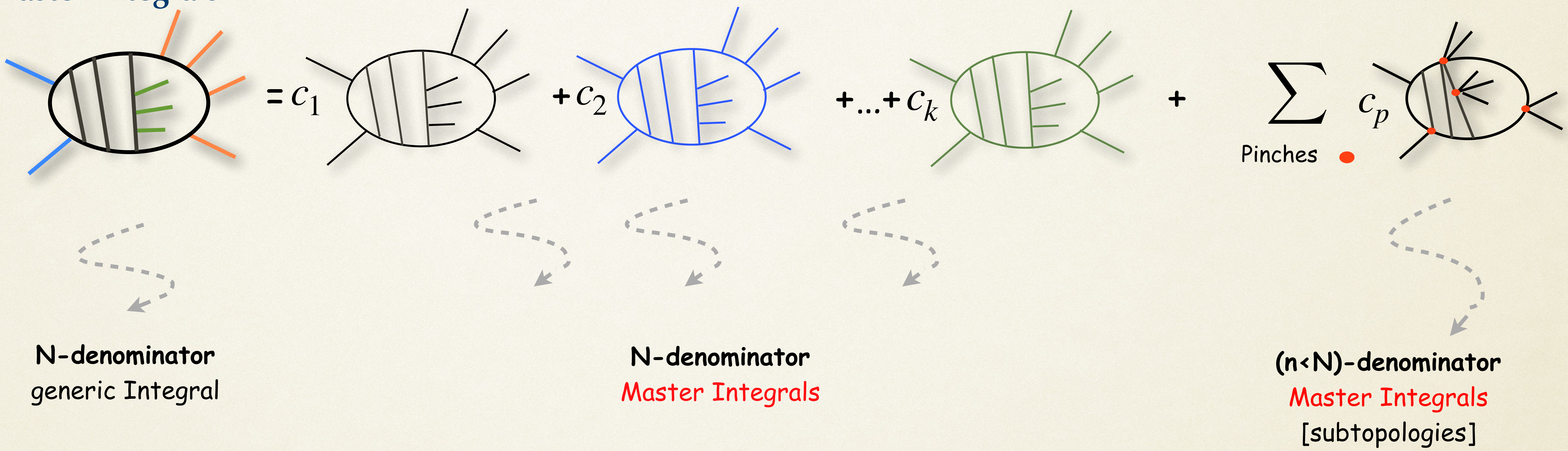
$$\sum_i b_i \, I^{[d]}_{a_1,\ldots,a_i \pm 1,\ldots,a_N} = 0$$

- **Generating an *overdimensioned (sparse) systems of linear equations***
- **Solutions:**

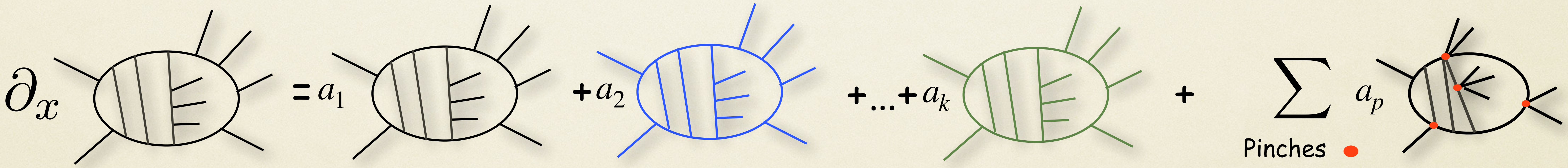
☑ Gauss' Elimination
☑ Groebner Bases
☑ Syzygy Equations
☑ **Finite Fields + Chinese Remainder Theorem + Rational Functions Reconstruction**

Linear relations for Feynman Integrals

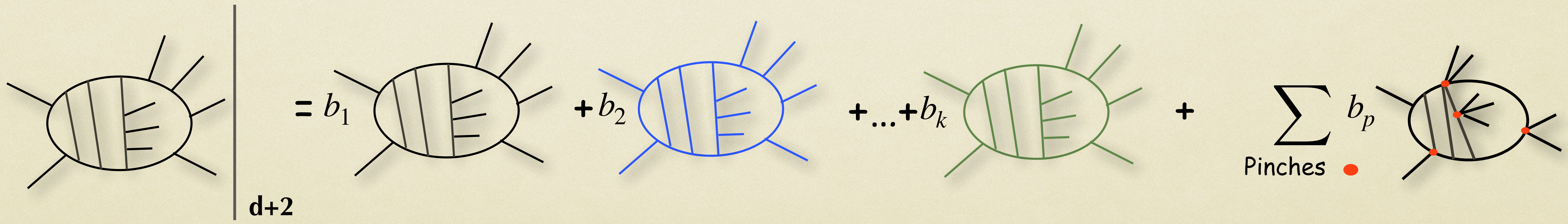
- Decomposition in terms of independent Master Integrals



- 1st order Differential Equations for MIs

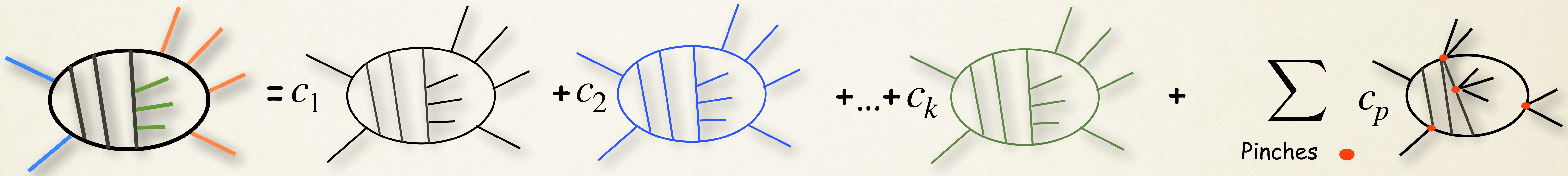


- Dimension-Shift relations and Gram determinant relations

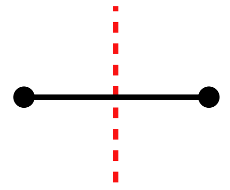


Linear relations for Feynman Integrals

- **Relations among Integrals in dim. reg.**



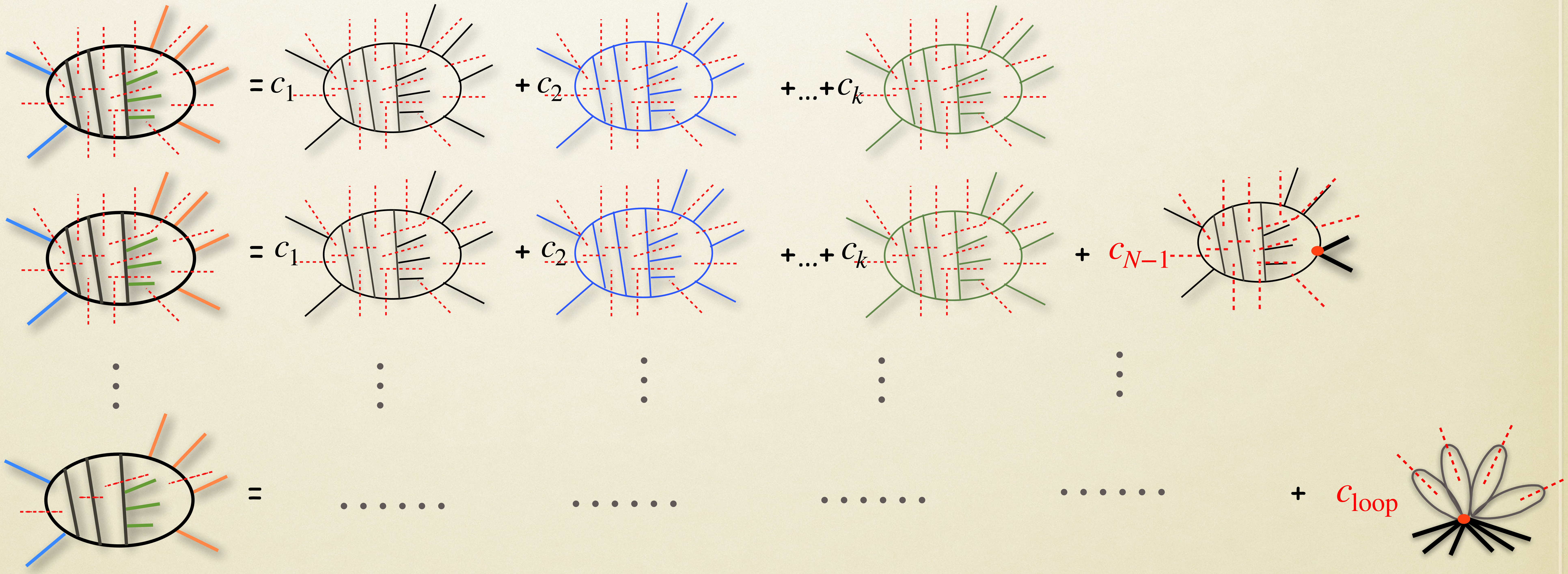
- **Unitarity-based and on-shell methods** $\frac{1}{D_i} \to \delta(D_i)$



N-denominator cut

(N-1)—denominator cut

(#loop)—denominator cut

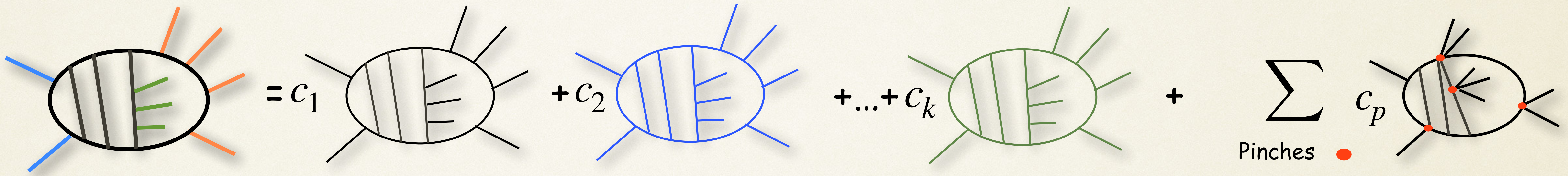


☑ Novel **integrand generation**: product of tree-amplitudes/diagrams; **complex momenta** across the cut

☑ Novel **complex-integration** techniques: (see 1loop 4ple-cut, 3ple-cut, 2ple-cut, …)

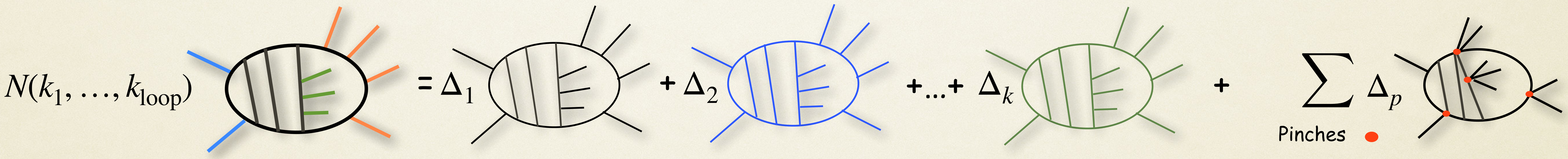
Linear relations for Feynman Integrands

- **Relations among Integrals in dim. reg.**



- **OPP Integrand Decomposition**

[integrand identity]



☑ Δ_i are polynomials and $c_i \in \Delta_i$

☑ c_i determined by **polynomial fitting**

- (Block)-triangular system of linear equations:
  principle of polynomial identity: integration NOT required

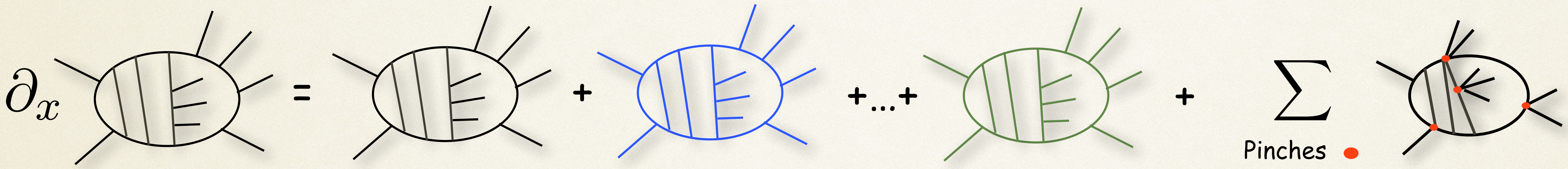
- **Cuts vs Residues vs Remainders**

☑ Δ_i , therefore c_i , determined by **polynomial division** (Δ_i are the **remainders**)

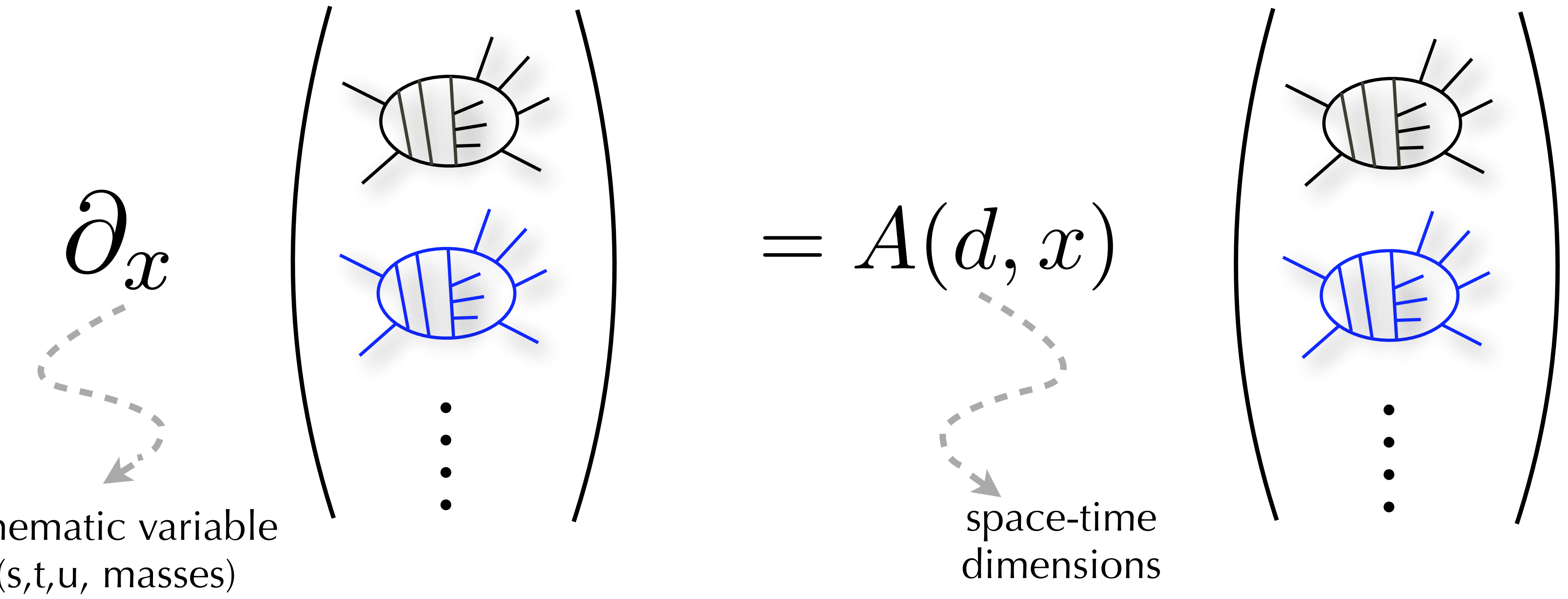
Evaluating Master Integrals / Differential Equations and Theory Special Functions

- **1st order Differential Equations for MIs** Barucchi, Ponzano; Kotikov; Remiddi, & Gerhmann; … Bern, Dixon, Kosower, …, Anastasiou, Melnikov, Steinhauser, Weinzierl,… Henn, Plefka; Lee;

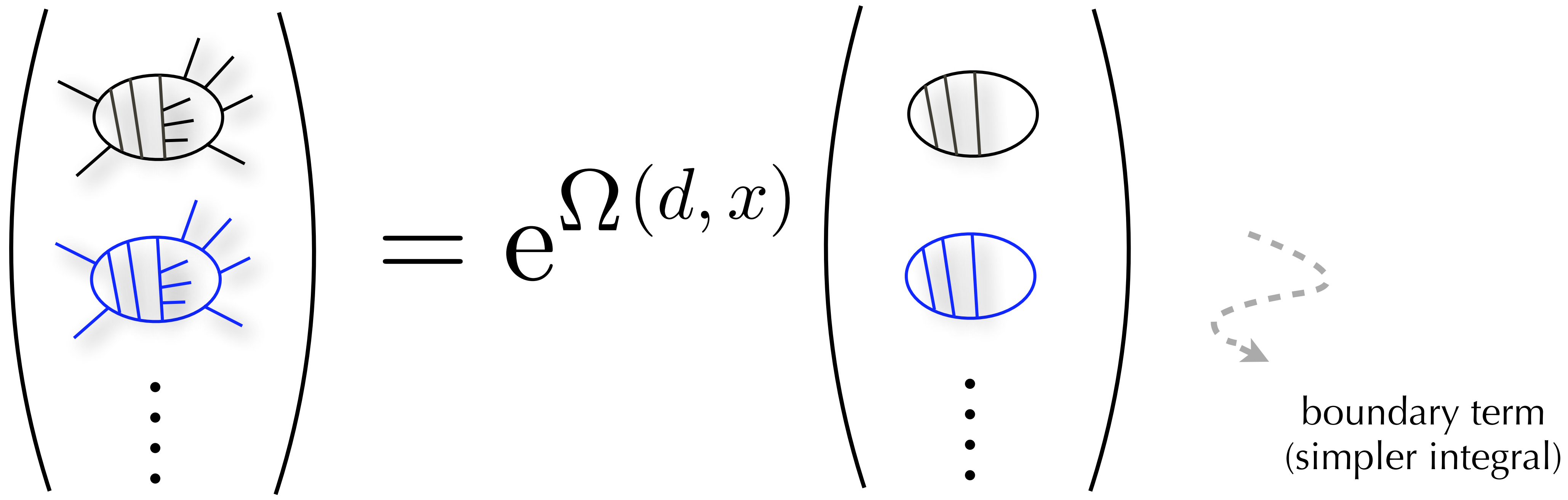
Bonciani, Remiddi **& P.M.** Argeri, diVita, Mirabella, Schubert, Tancredi, Schlenck **& P.M.**; …



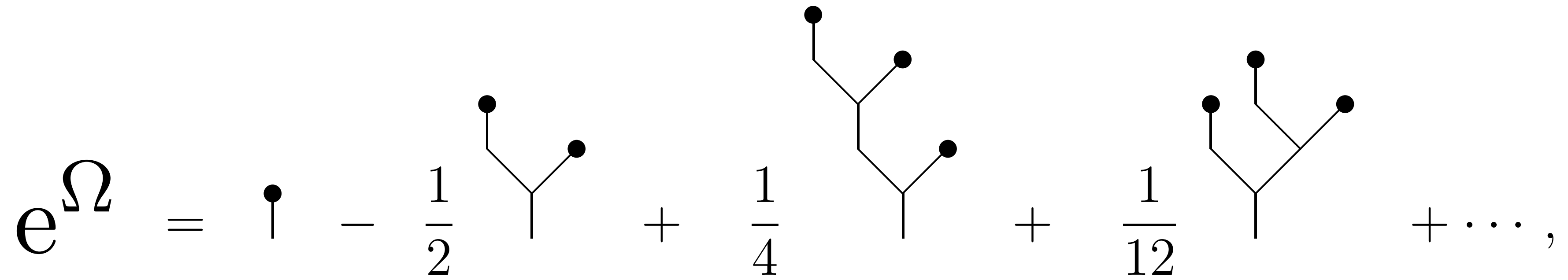
- **System of 1st ODE**

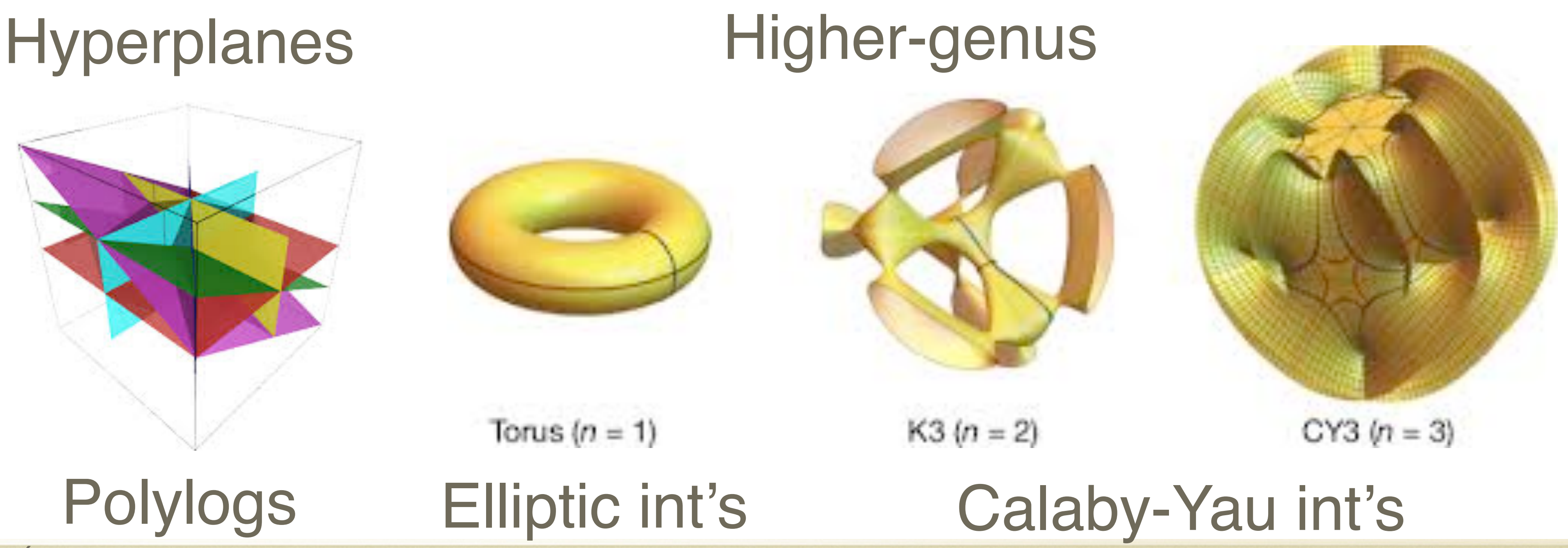


- **Solution**



- **Dyson/Magnus Series: Iterated Integrals and Geometry**

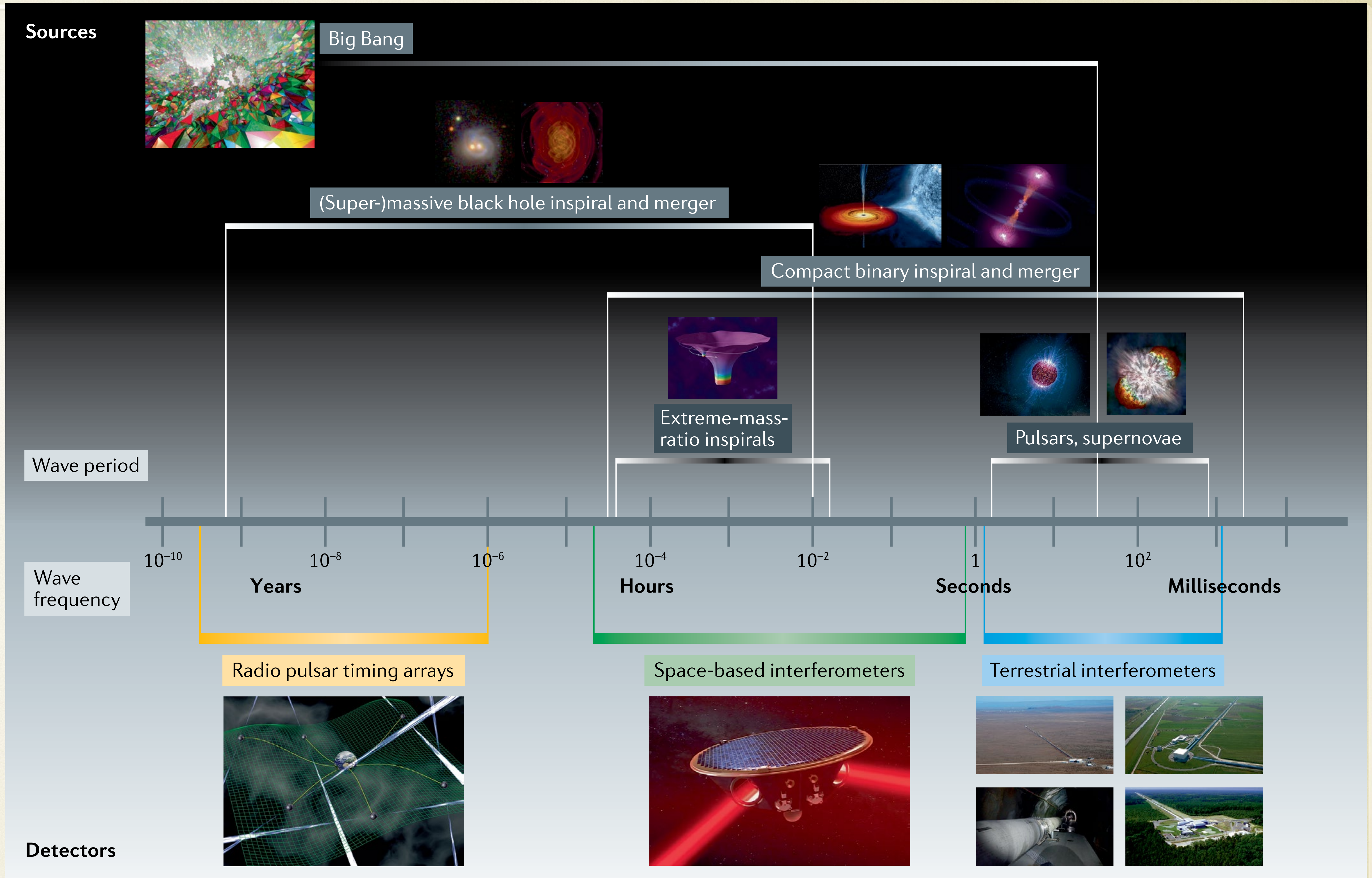




GR EFT and Amplitudes/Diagrammatic approach

Motivation

- Gravitational Waves a new window on the Universe
- Two-body dynamics and radiative effects to exploring the most extreme conditions of spacetime and matter
- Next generation detectors, ground-based and in space, need of accurate waveform templates
- Precision Physics vs Precision Calculations: Multi-Loop Calculus, Scattering Amplitudes and General Relativity



[Bailes et al. 2021]

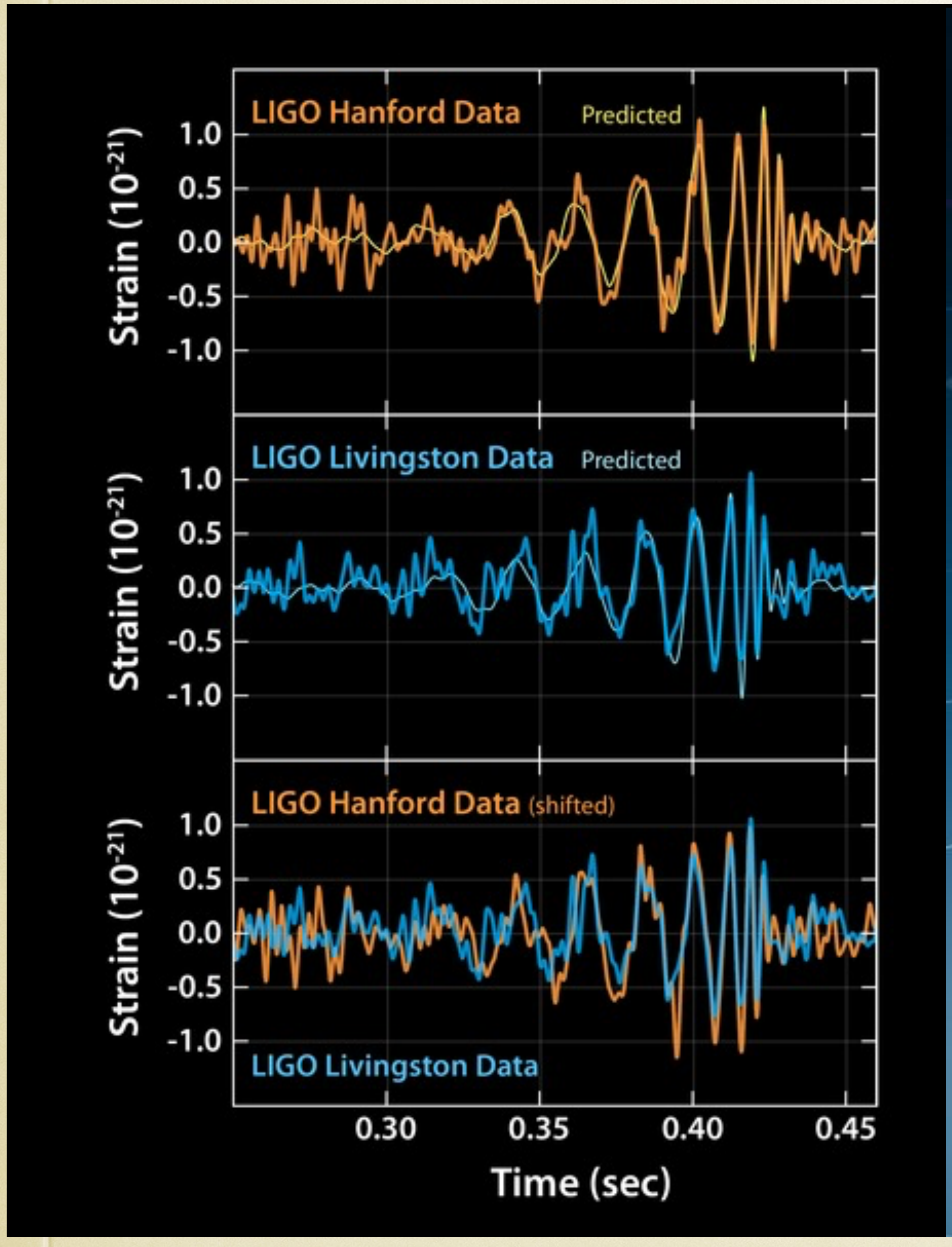
Based on collaborations with:

G. **Brunello**, J. Steinhoff, M.K. **Mandal**, R. **Patil**

D. Bini, T. Damour, A. **Geralico**, S. Laporta

S. Foffa, R. Sturani, C. Sturm, W.J. **Torres Bobadilla**

GW Detection

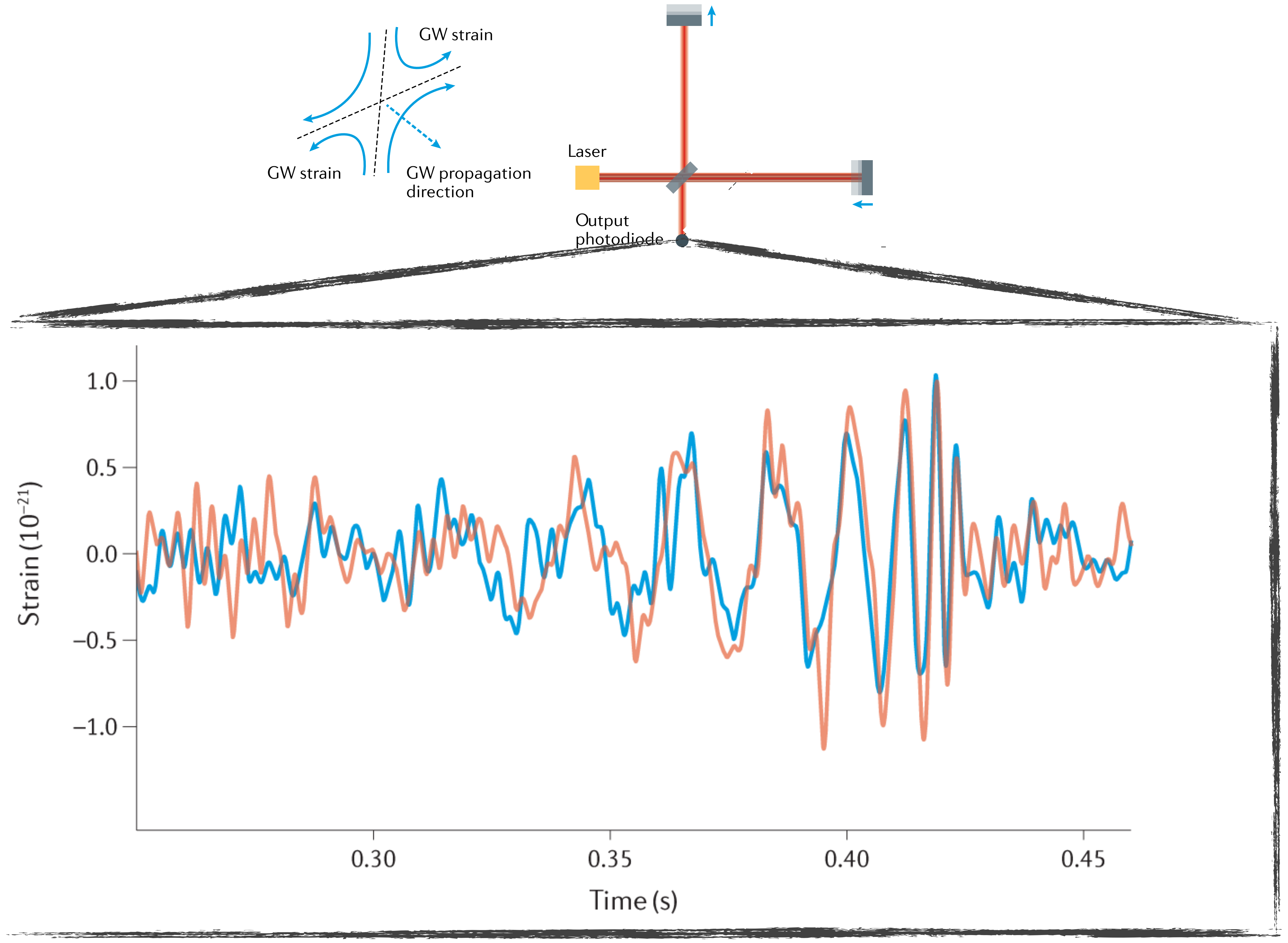


LIGO-Virgo Detection: **GW150914**

Cortona is here

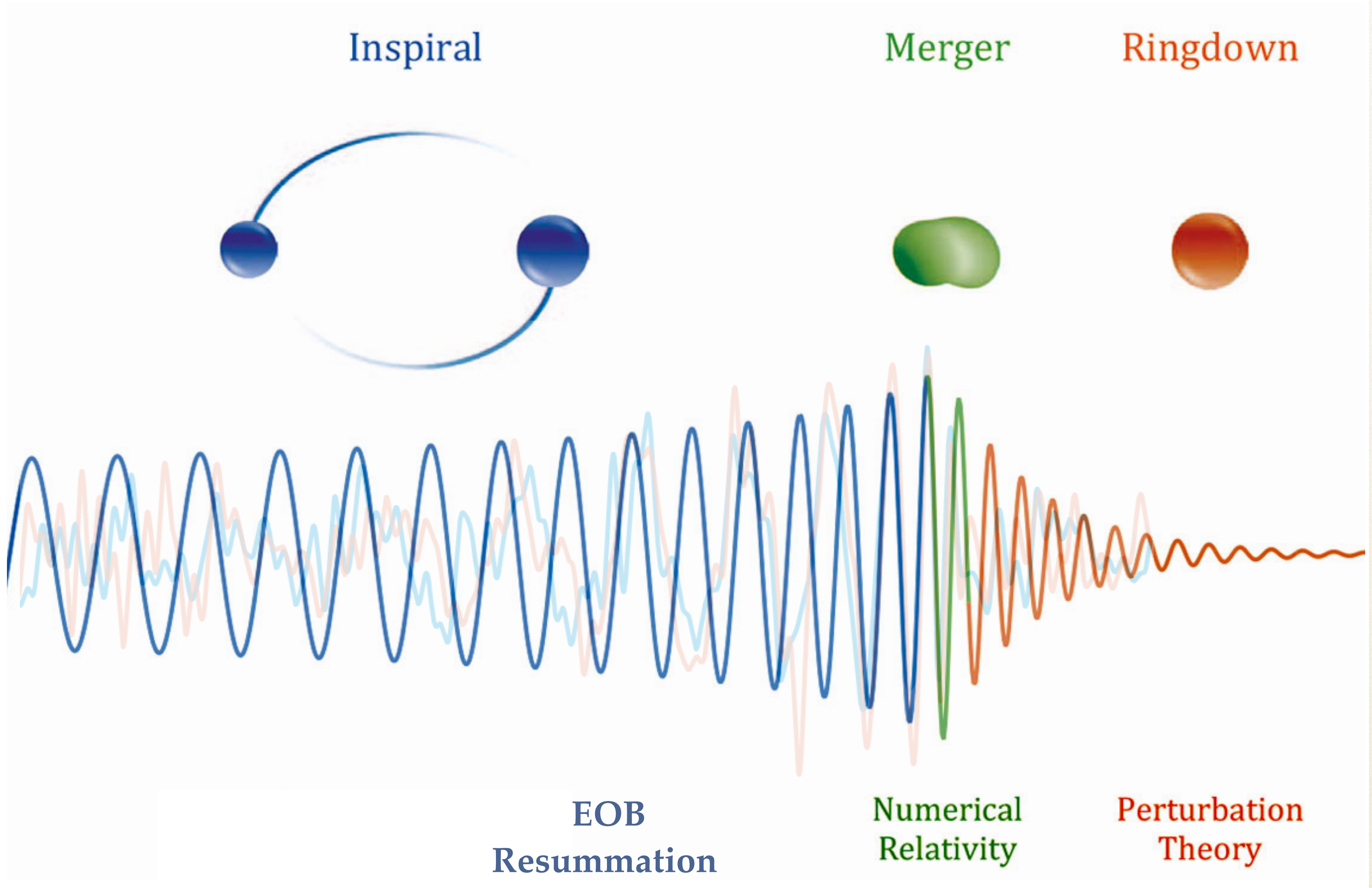
Two-body dynamics and GW signal

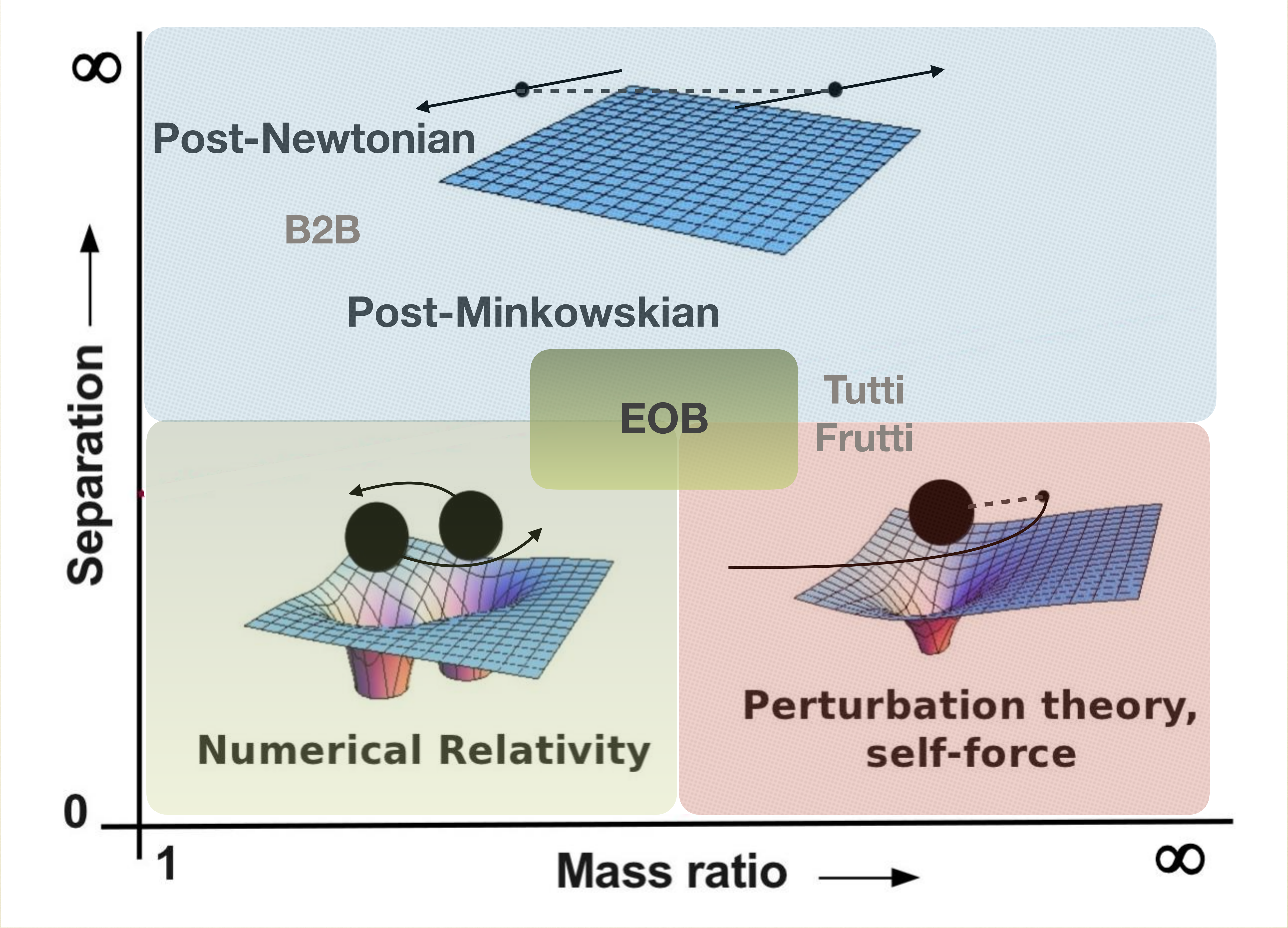
- **Real Event**



Two-body dynamics and GW signal

- Waveform Model and Computing Techniques

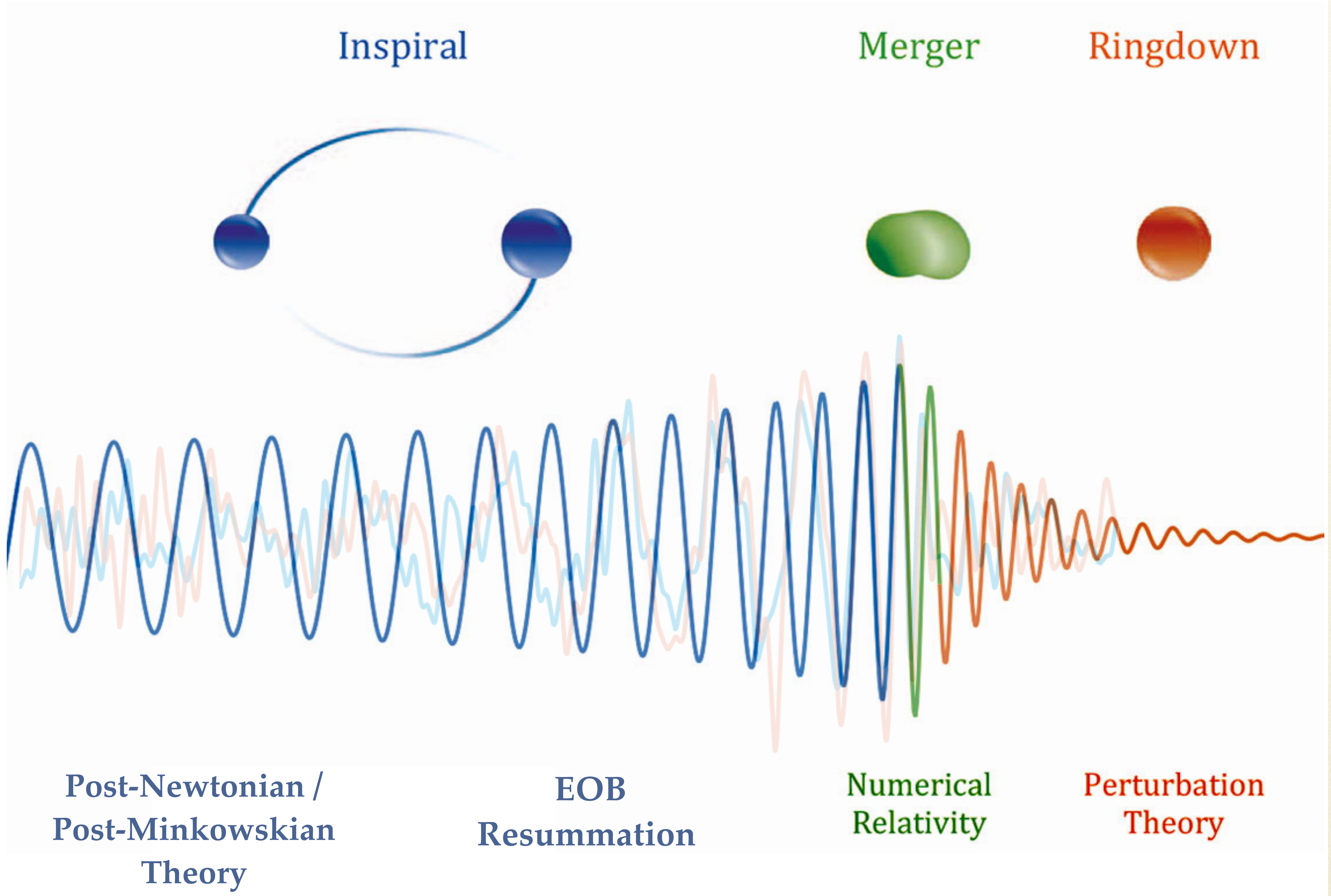




[adapted from: Barak]

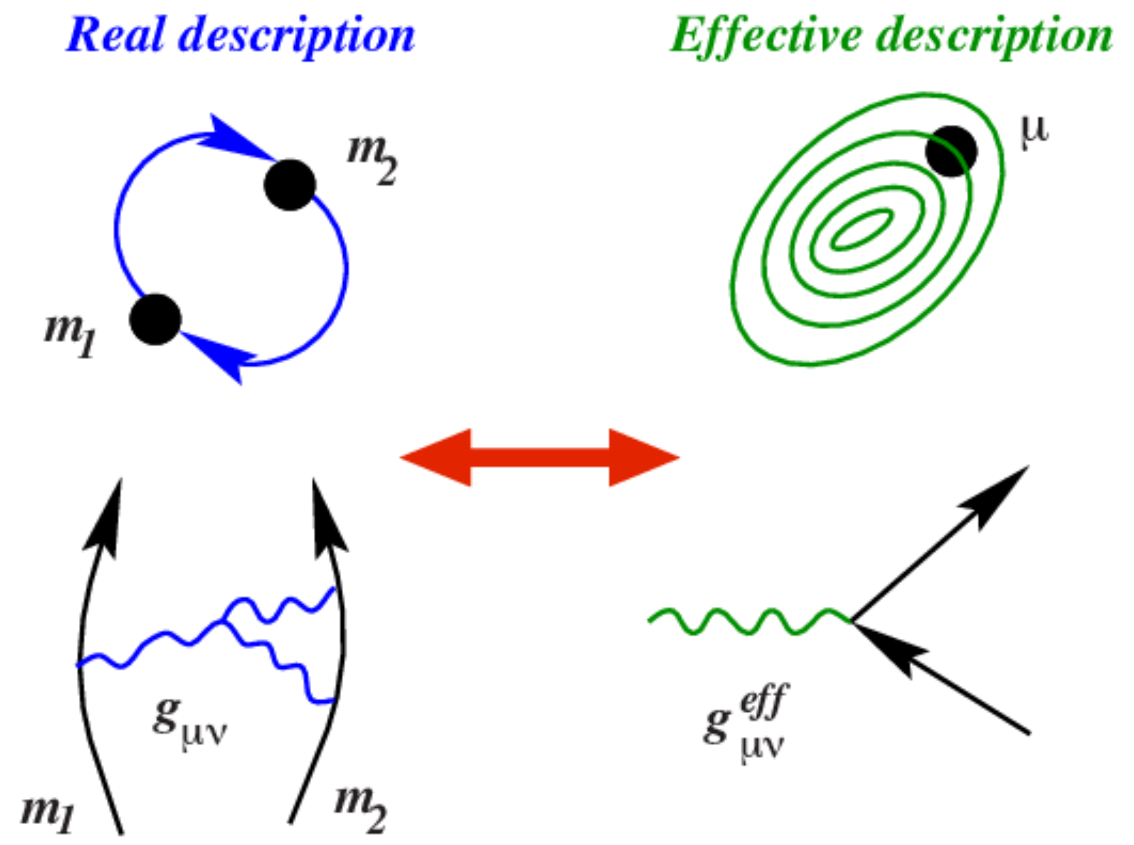
Two-body dynamics and GW signal

▸ **Waveform Model and Computing Techniques**



Effective One Body (EOB) Formalism [Buonanno Damour]

the contributions coming from different kinematic regions for combined and calibrated with Numerical Relativity



- Post-Minkowskian Expansion [relativistic scattering]

  $G_N \frac{m}{r} << v^2 \sim 1$

  Expansion in powers of G_N

- Post-Newtonian Expansion [non relativistic system]

  $G_N \frac{m}{r} \sim v^2 << 1$

  Expansion in powers of v/c

- BH perturbation theory / self force

  Expansion for small metric deformation

  $\delta g_{\mu\nu} \sim \epsilon = m_2/m_1 << 1$

GR Effective Field Theory (GREFT)

- **GREFT Action** $$S_{tot}[x_a, g] = S_{GR}[g] + S_{m_a}[x_a, g]$$

[Goldberger, Rothstein]

- **Einstein Hilbert + gauge fixing**

$$S_{GR}[g] = 2\Lambda^2 \int d^{d+1}x \sqrt{-g}\left(R - \frac{1}{2}\Gamma^{\mu}\Gamma_{\mu}\right)$$

$$\Lambda^{-1} = \sqrt{32\pi G_N}$$

- **Source/Worldline**

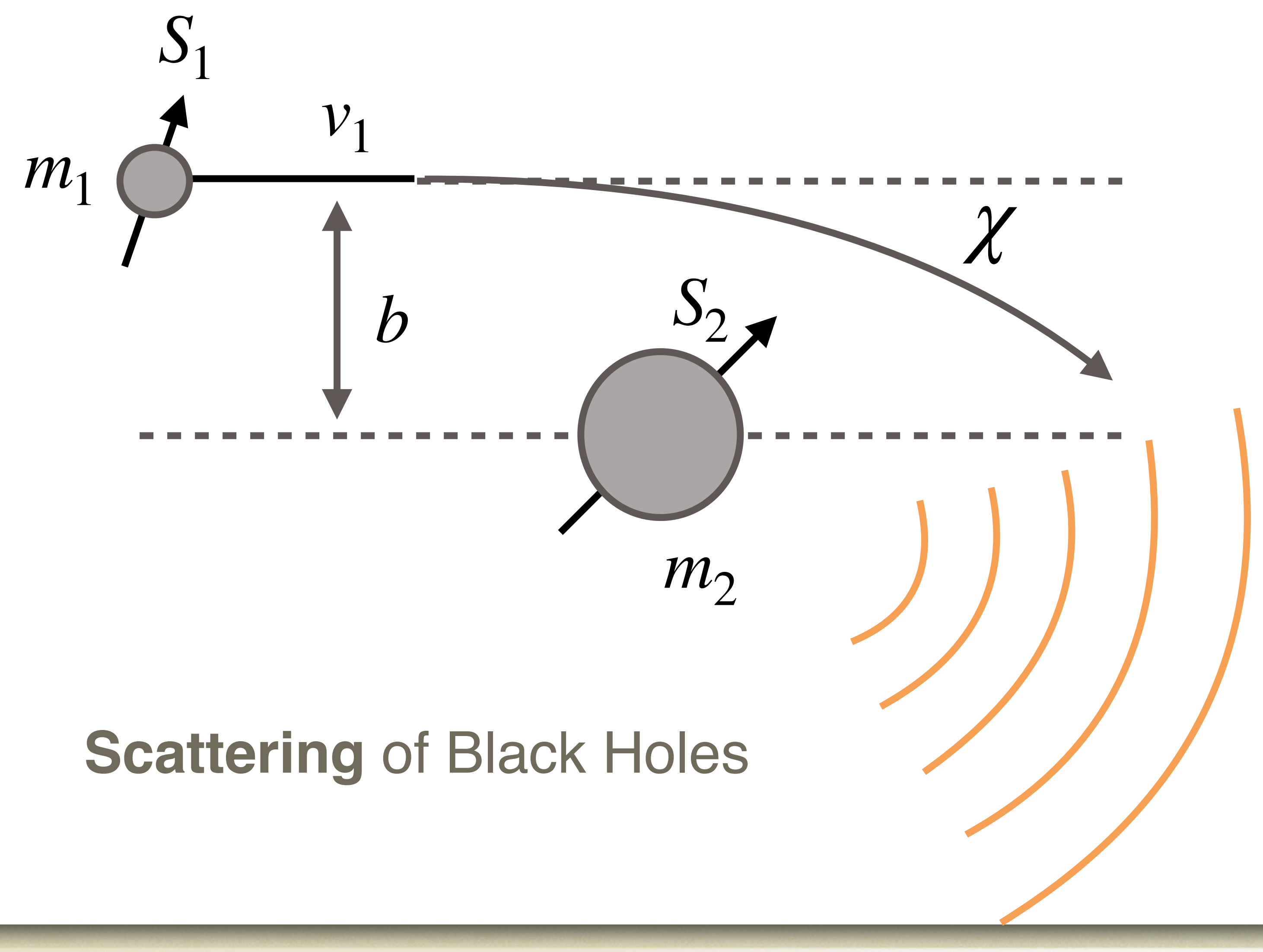
[internal structure: spin, tidal, …]

$$S_{m_a}[x_a, g] = S_{pp}[x_a, g] + \delta S_{m_a}[x_a, g]$$

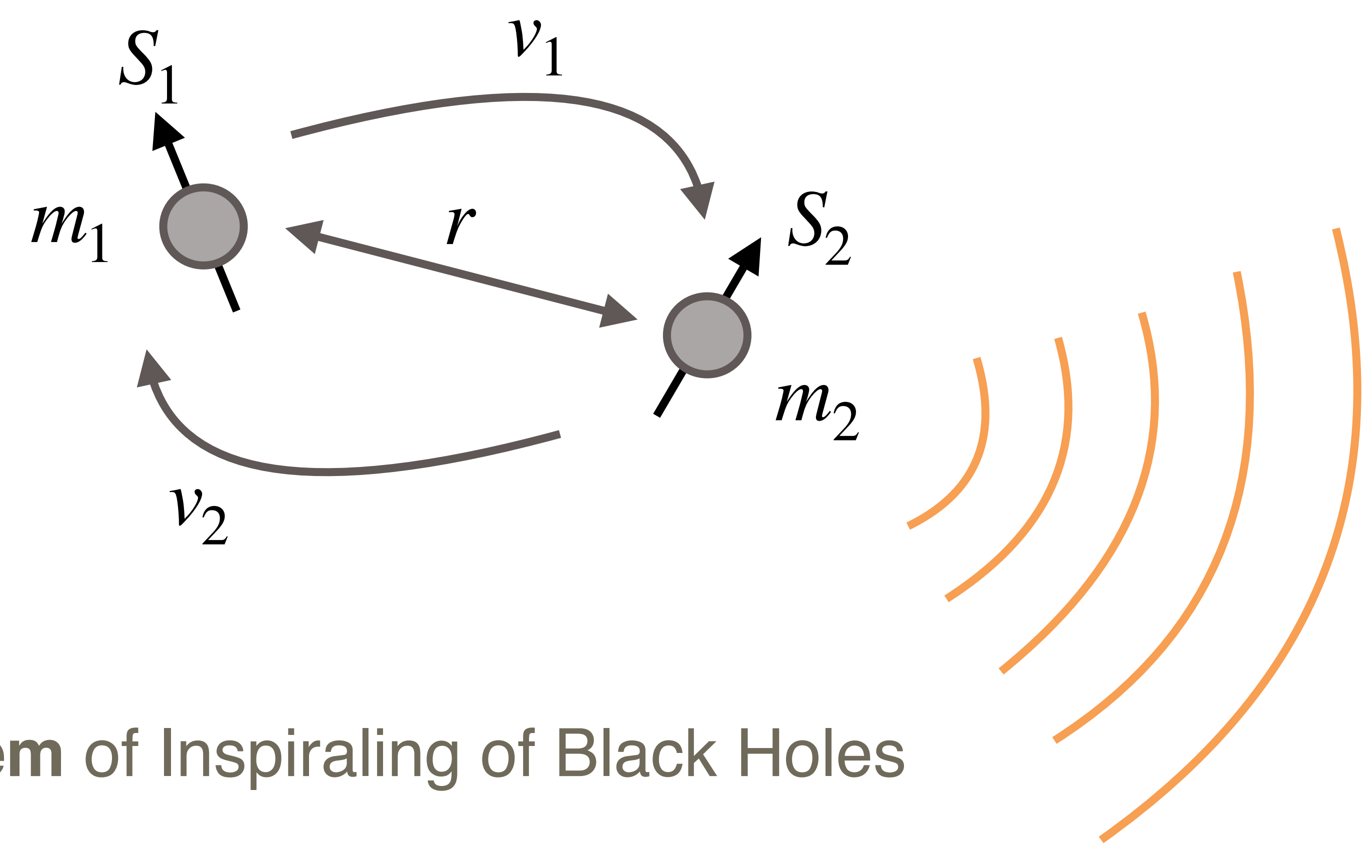
$$= -m_a \int d\tau_a = -m_a \int dt \sqrt{-g_{\mu\nu}(x_a)\dot{x}_a^{\mu}\dot{x}_a^{\nu}}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{G}\, h_{\mu\nu}$$

GREFT@Two-body problem

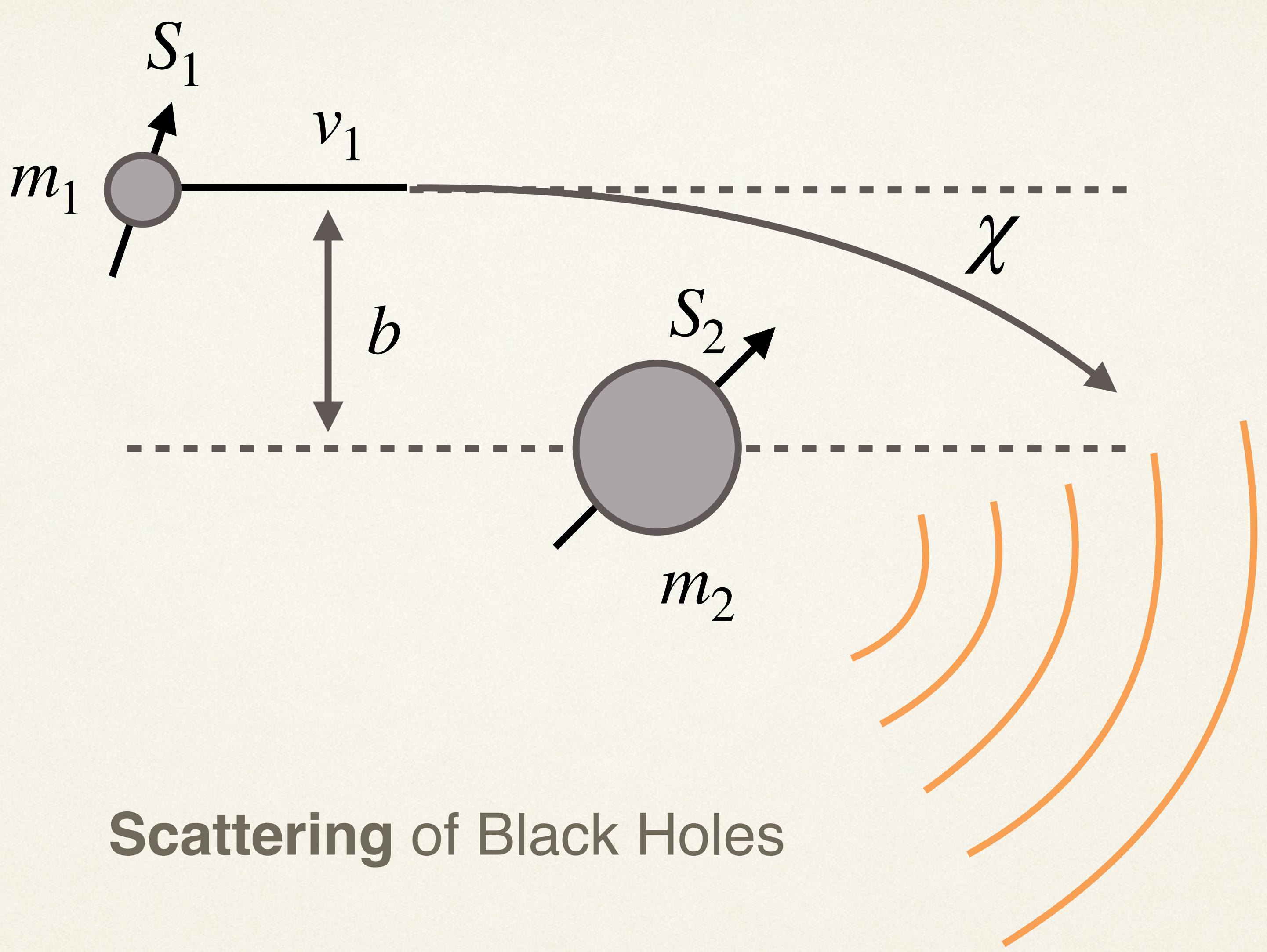


Scattering of Black Holes



Bound system of Inspiraling of Black Holes

GREFT / PM Corrections



Scattering of Black Holes

Corrections to the Newtonian potential:

- **relativistic velocities**:

$$G_N \frac{m}{r} << v^2 \sim 1$$

Expansion in powers of G_N

- Dynamics in **Post-Minkowskian perturbative scheme**

- At **nPM** order: G_N^n

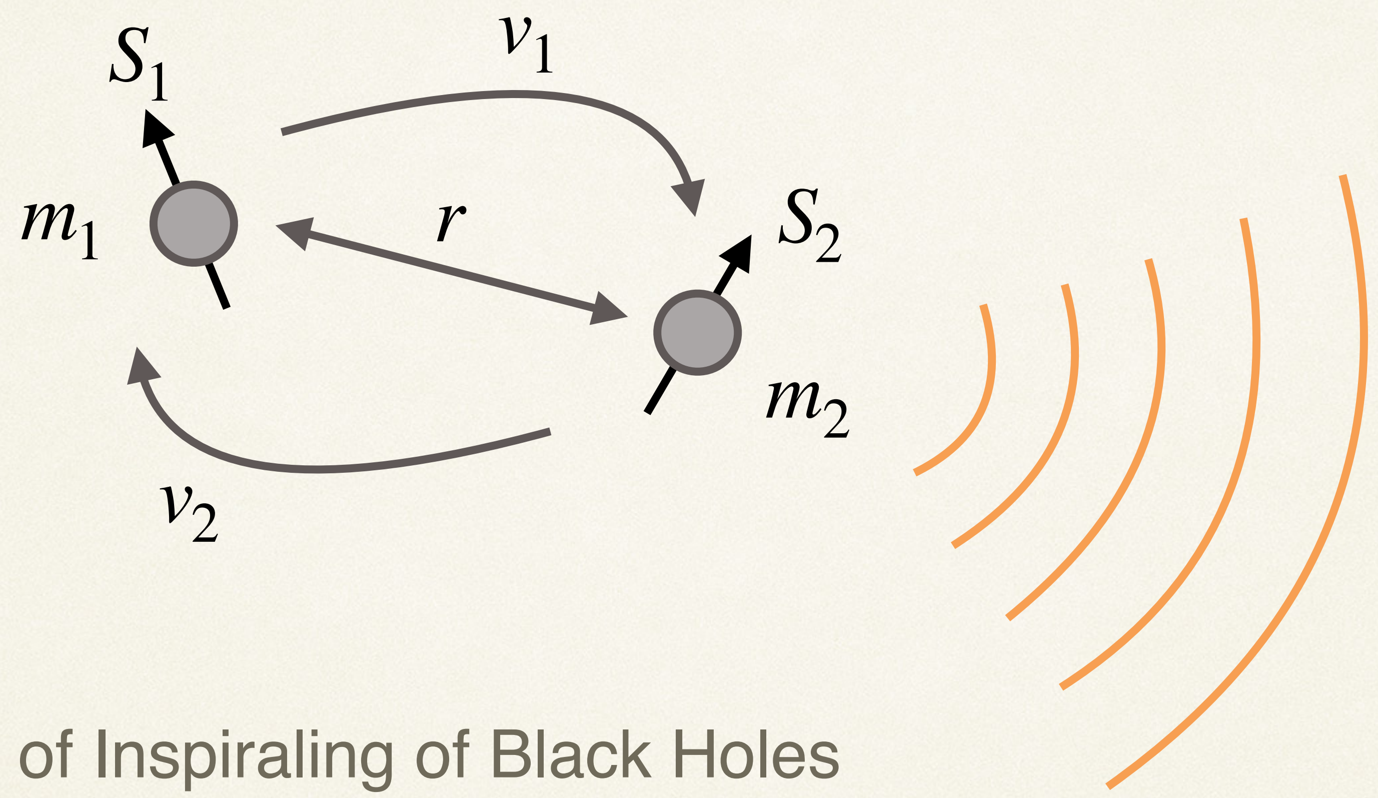
Astrophysicists/Cosmologists' whishlist

		0PN	1PN	2PN	3PN	4PN	5PN	6PN		
	G	(1	+ v^2	+ v^4	+ v^6	+ v^8	+ v^{10}	+ v^{12}	+ …)	1PM
1979-81 →	G^2	(1	+ v^2	+ v^4	+ v^6	+ v^8	+ v^{10}	+ v^{12}	+ …)	2PM
2019 →	G^3	(1	+ v^2	+ v^4	+ v^6	+ v^8	+ v^{10}	+ v^{12}	+ …)	3PM
2021 →	G^4	(1	+ v^2	+ v^4	+ v^6	+ v^8	+ v^{10}	+ v^{12}	+ …)	4PM
2024 →	G^5	(1	+ v^2	+ v^4	+ v^6	+ v^8	+ v^{10}	+ v^{12}	+ …)	5PM
	G^6	(1	+ v^2	+ v^4	+ v^6	+ v^8	+ v^{10}	+ v^{12}	+ …)	6PM
	G^7	(1	+ v^2	+ v^4	+ v^6	+ v^8	+ v^{10}	+ v^{12}	+ …)	7PM

[credit: Bern et al.]

…Westphal, Damour, Cheung, Rothstein, Solon, Bern, Roiban, Shen, Zeng, Parra-Martinez, Ruf, Hermann, Buonanno, Porto, Dlapa, Kalin, Liu, Neef, Bjerrum-Bohr, Vanhove, Plante, Cristofoli, Damgaard, Guevara, Ochirov, Vines, Di Vecchia, Veneziano, Heisenberg, Russo, Plefka, Jakobsen, Mogull, Brandhuber, Travaglini, De Angelis, Accetulli-Huber, Luna, Kosmopoulos, and collaborators…

GREFT / PN Corrections



Bound system of Inspiraling of Black Holes

Corrections to the Newtonian potential:

- Non-relativistic velocities:
- Virial theorem: $G_N \frac{m}{r} \sim v^2 << 1$

  Expansion in powers of v/c

- Dynamics in Post-Newtonian perturbative scheme
- At nPN order: $G_N^{n-\ell} v^{2\ell}$

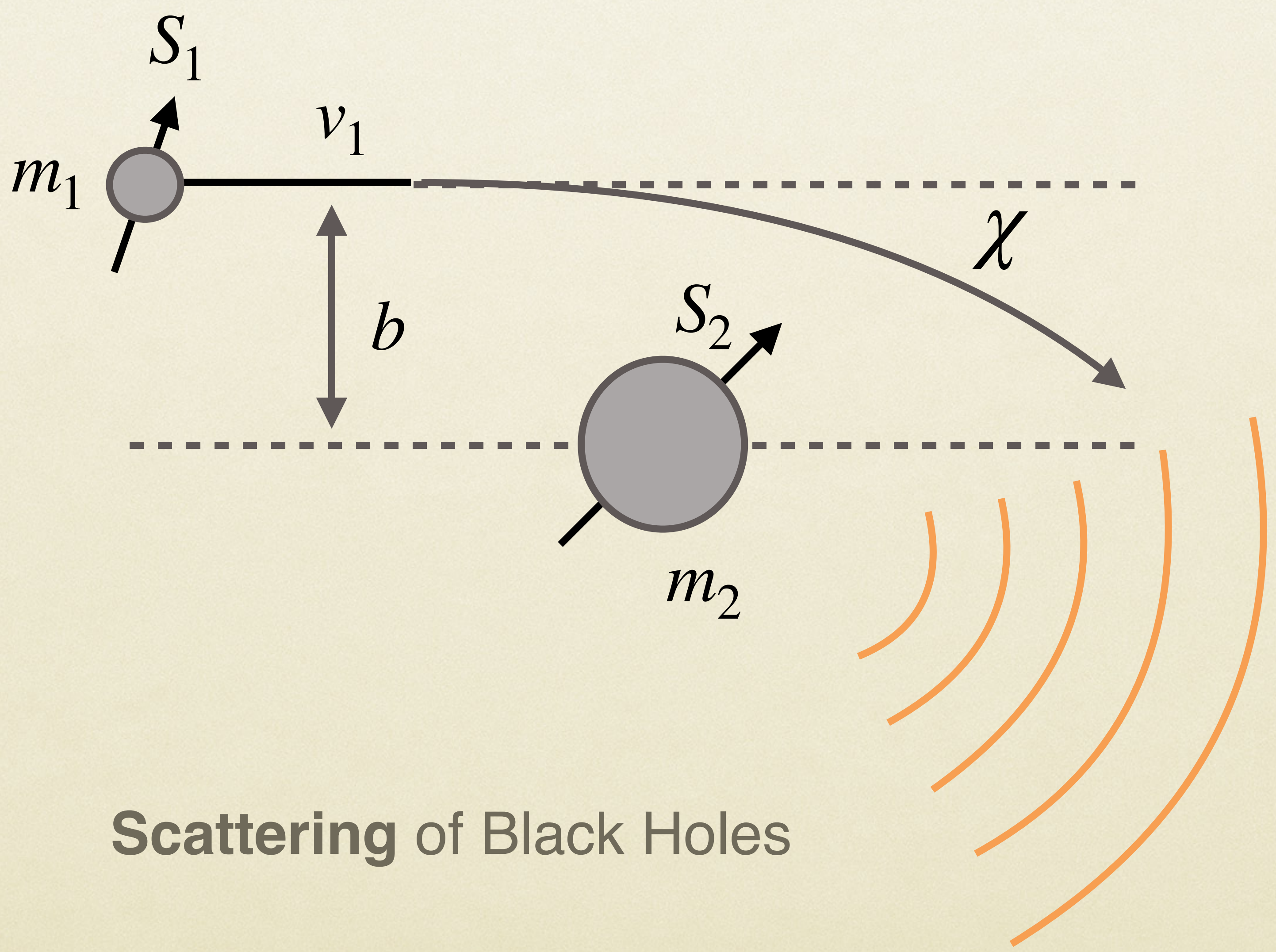
Astrophysicists/Cosmologists' whishlist

	0PN		1PN		2PN		3PN		4PN		5PN		6PN		7PN			
G	(1	+	v^2	+	v^4	+	v^6	+	v^8	+	v^{10}	+	v^{12}	+	v^{14}	+	$\cdots$)	1PM
G^2			(1	+	v^2	+	v^4	+	v^6	+	v^8	+	v^{10}	+	v^{12}	+	$\cdots$)	2PM
G^3					(1	+	v^2	+	v^4	+	v^6	+	v^8	+	v^{10}	+	$\cdots$)	3PM
G^4							(1	+	v^2	+	v^4	+	v^6	+	v^8	+	$\cdots$)	4PM
G^5									(1	+	v^2	+	v^4	+	v^6	+	$\cdots$)	5PM
G^6											(1	+	v^2	+	v^4	+	$\cdots$)	6PM
G^7													(1	+	v^2	+	$\cdots$)	7PM
	1687		1938		1980		2000		2014		2021							

[credit: Bern et al.]

...Jaranowski, Schaefer, Damour, Blanchet, Faye, Porto, Rothstein, Goldberger, Foffa, Sturani, Bini, Buonanno, Geralico, Sturm, Torres Bobadilla, Bluemlein, Maier, Marquard, Levi, Steinhoff, Vines, Antonelli, Kavanagh, Khalil, Galley, von Hippel, McLeod, Edison, Kim, Morales, Yin, Mandal, Patil, Teng, P.M. ...and collaborators ....

GR EFT for PM corrections / Diagrammatic approach

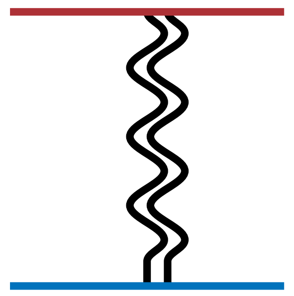


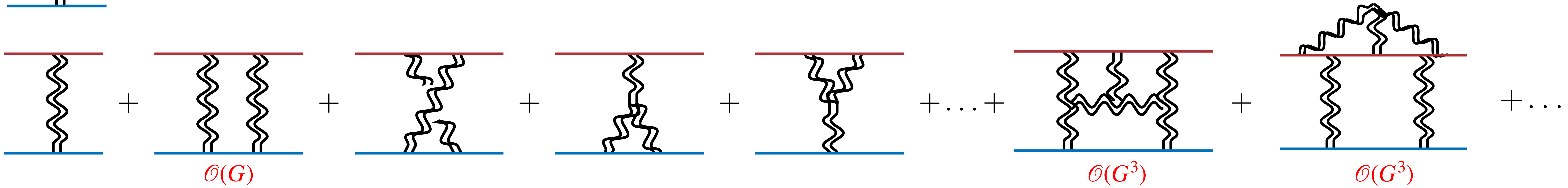
Scattering of Black Holes

PM Corrections / Scattering Amplitudes based approach

- **GR EFT Feynman rules / on-shell / double-copy / spinor-formalism / recurrence relations**
- **Amplitudes based approach**

Cheung, Rothstein, Solon
Bern, Cheung, Hermannn, Parra Martinez, Roiban, Schen Solon, Zeng…
(see Correia et al.'s review)

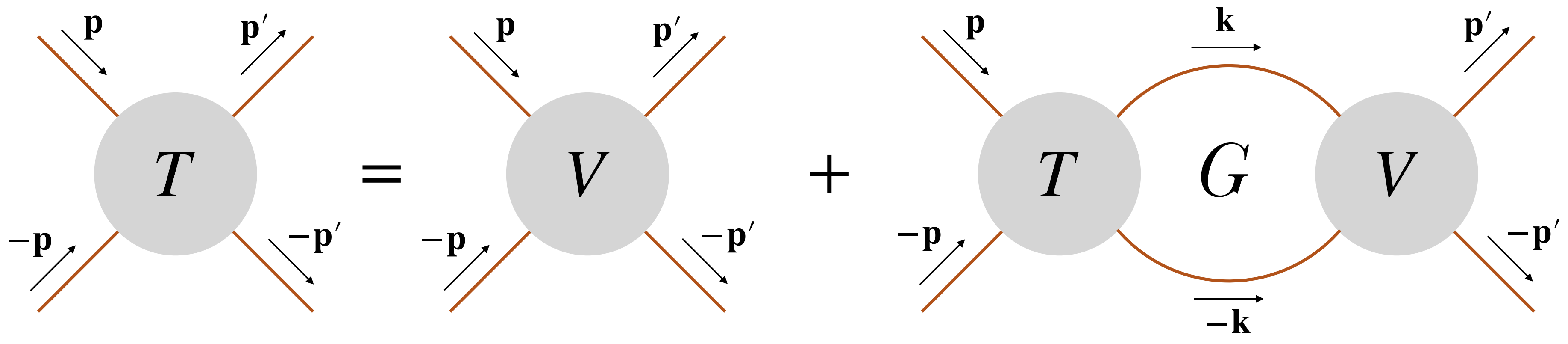
$i\mathcal{M}_0 =$  $= 16\pi G \dfrac{(2(p_1\cdot p_2)^2 - m_1^2 m_2^2)}{t} \equiv 16\pi G \dfrac{m_1^2 m_2^2(2\sigma^2-1)}{t}\,, \qquad \sigma \equiv \dfrac{p_1\cdot p_2}{m_1 m_2}$

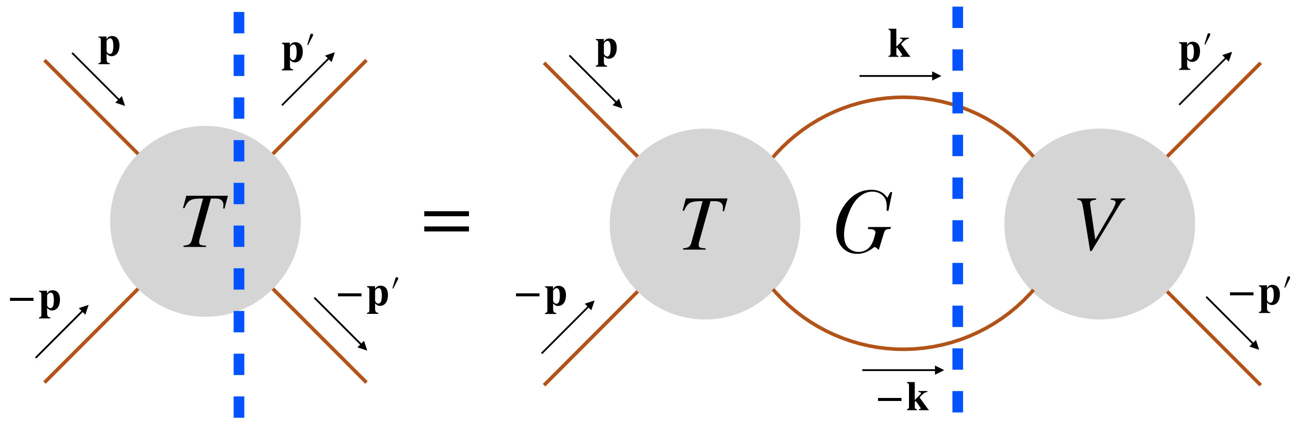
$i\mathcal{M} =$ 

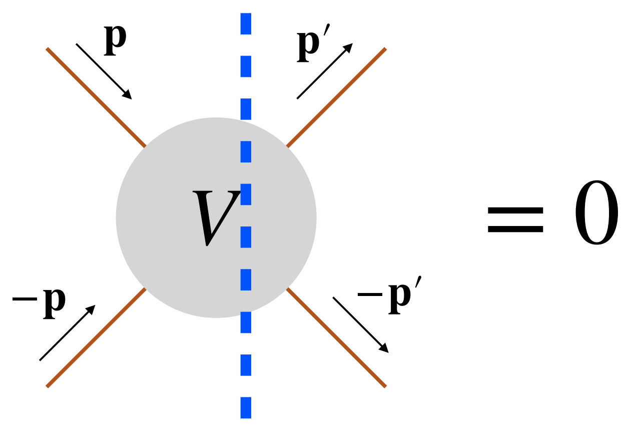
- **Potential V from Lippman-Schwinger equation**

Correia, Isabella

$$T = V + VGV + VGVGV + \cdots = V\frac{1}{1-GV}$$







$$\mathrm{Im}\,G(\mathbf{p},\mathbf{k}) = \delta(|\mathbf{k}|^2 - |\mathbf{p}|^2)$$

$$G(\mathbf{p},\mathbf{k}) = \frac{1}{|\mathbf{k}|^2 - |\mathbf{p}|^2 - i\epsilon}$$

PM Corrections / Classical Observables based approach (KMOC)

Kosower, Maybee,o'Connel

- **Asymptotic states**

$$|\psi\rangle_{in} = \int d\phi(p_1)d\phi(p_2)\ \phi_1(p_1)\phi_2(p_2)e^{i(b_1\cdot p_1+b_2\cdot p_2)}\ |p_1,p_2\rangle_{in} \qquad |\psi\rangle_{out} = S|\psi\rangle_{in} \qquad S = 1 + i\ T$$

On-shell phase space integral — wavefunction — Two particle momentum eigenstates

- **Expectation value of a Physical Observable:**

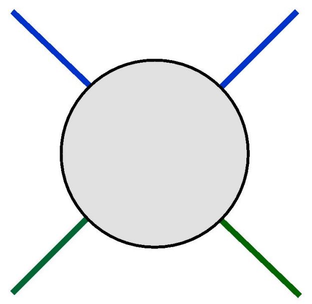
$$\Delta\langle\mathcal{O}\rangle = \langle\mathcal{O}\rangle_{out} - \langle\mathcal{O}\rangle_{in} = {}_{out}\langle\psi|\mathcal{O}|\psi\rangle_{out} - {}_{in}\langle\psi|\mathcal{O}|\psi\rangle_{in} = {}_{in}\langle\psi|S^\dagger[\mathcal{O},S]|\psi\rangle_{in}$$

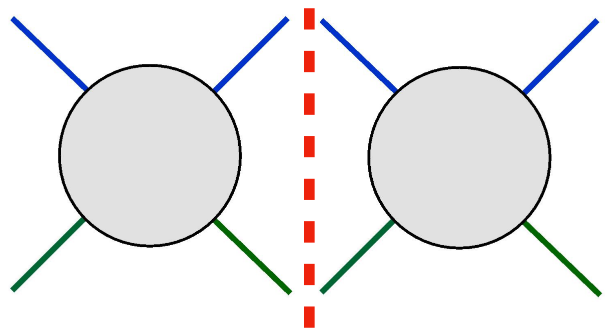
>> **[in-in** formalims]

- **Impulse** as ***Fourier Transform*** of Scattering Amplitudes

$$d\mu = \prod_{i=1}^{2} \hat{d}^D q_i\, \delta(2p_i\cdot q_i + q_i^2)\, e^{ib_i\cdot q_i}$$

$$\Delta\langle p_1^\mu\rangle = \int d\mu\, \hat{\delta}^D(q_1+q_2)\Big(\ q^\mu \quad [\text{4-point amplitude}] \quad - i \int d(LIPS)\ \ell^\mu \quad [\text{Iteration terms}] \quad \Big).$$





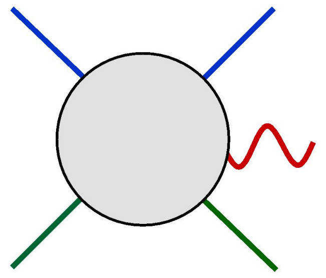
Fourier Transform — 4-point amplitude — Iteration terms

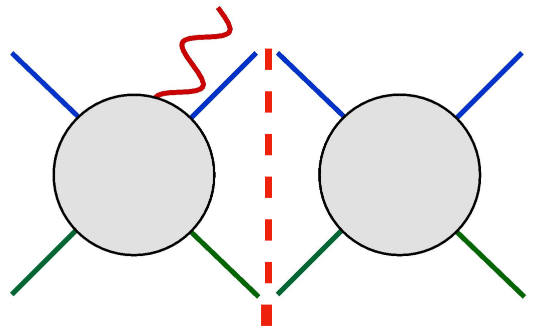
Caron Huot, Giroux, Hanesdottir, Mizera

- **Waveform** as ***Fourier Transform*** of Scattering Amplitudes

Brunello, De Angelis

$$\Delta\langle\mathcal{W}_h\rangle(u,\vec{n}) = \frac{1}{4\pi r}\int_0^\infty \hat{d}\omega \int d\mu\Big\{\hat{\delta}^D(q_1+q_2-k)e^{-i\omega u}\Big(\quad [\text{5-point amplitude}] \quad - i\int d(LIPS) \quad [\text{Iteration terms}] \quad \Big) + c.c.\Big\}$$





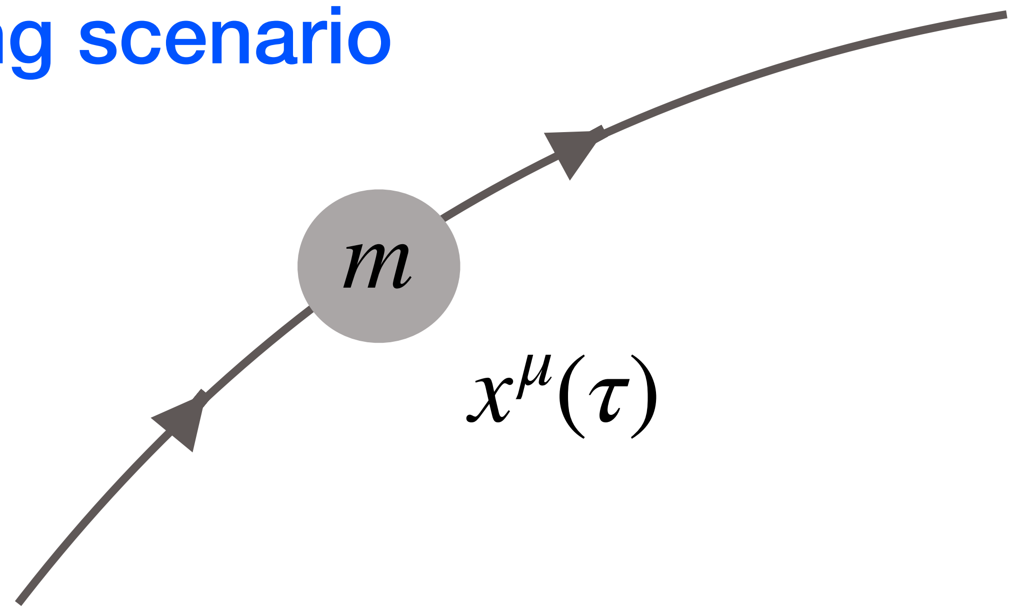
Fourier Transform — 5-point amplitude — Iteration terms

[credit: Brunello]

PM Corrections / QFT(EFT) Worldlines formalism

Driesse, Mogull, Plefka, Jakobsen, Sauer, Steinhoff, Usovitsch
Dlapa, Kallin, Liu, Porto

- **Scattering scenario**



$$x_i^\mu(\tau) = b_i^\mu + v_i^\mu\,\tau + \sum_{n=1}^{\infty} G^n z_i^{(n)\,\mu}(\tau)$$

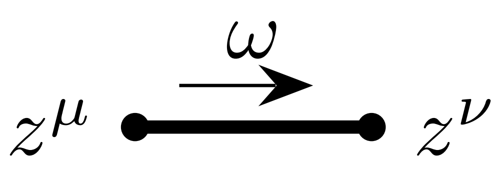
straight line: „in“ state — deflections

$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{G}\sum_{n=0}^{\infty} G^n h_{\mu\nu}^{(n)}(x)$$

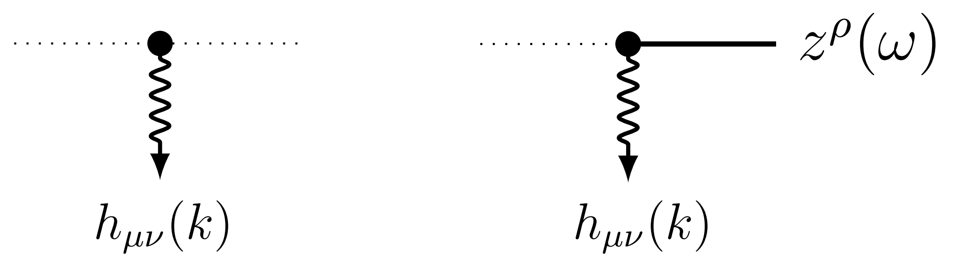
emitted radiation

>> [in-in formalims]

- **Feynman rules**

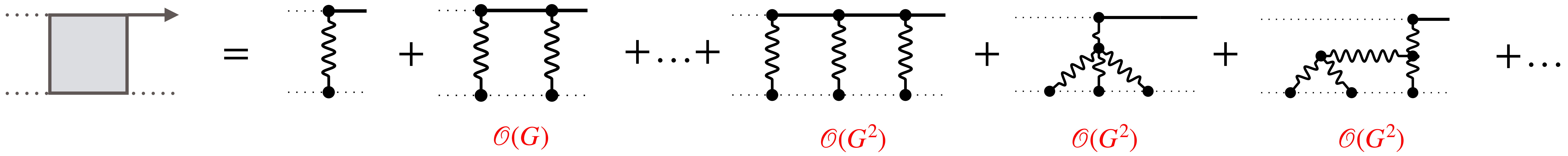


$$\langle z^\mu(\omega) z^\nu(-\omega)\rangle = -\frac{i}{m}\frac{\eta_{\mu\nu}}{(\omega+i0)^2}$$



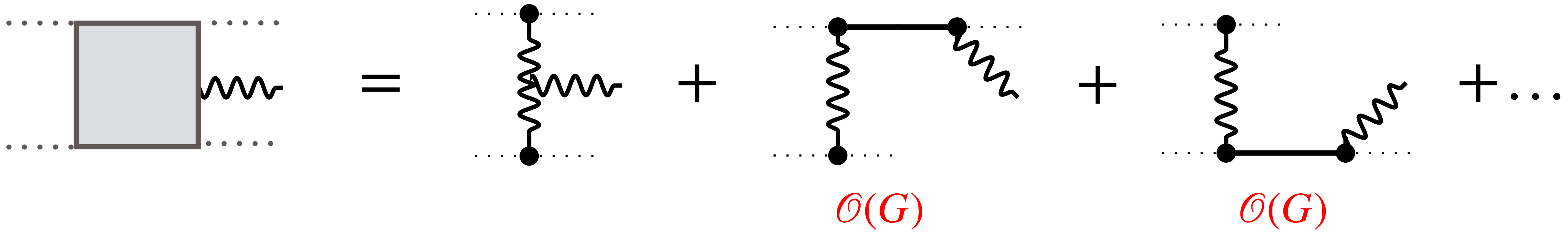
- **Impulse**

$\langle \Delta p_1^\mu \rangle =$



- **Waveform**

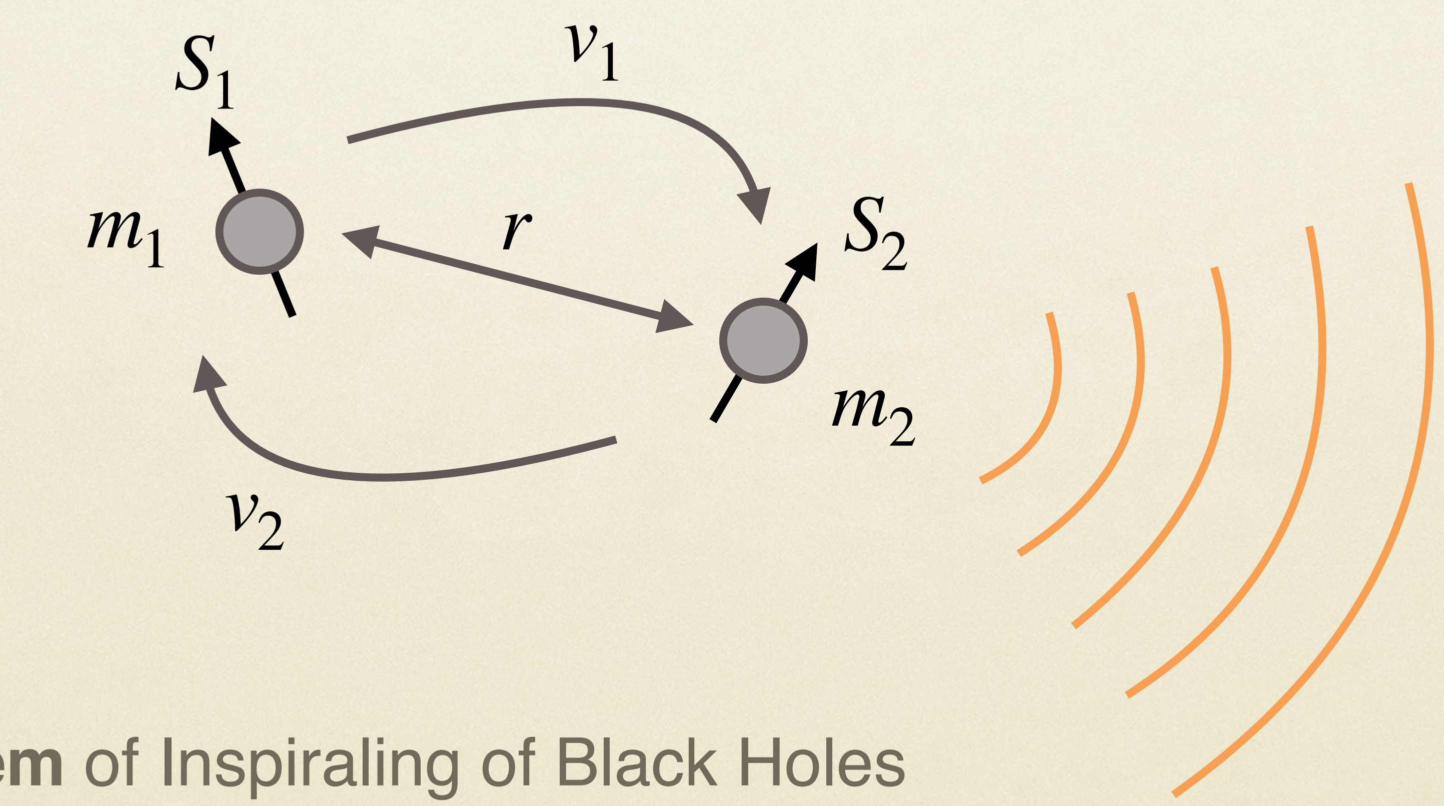
$\langle h_{\mu\nu} \rangle =$



- **Pushed to 5PM !!!**

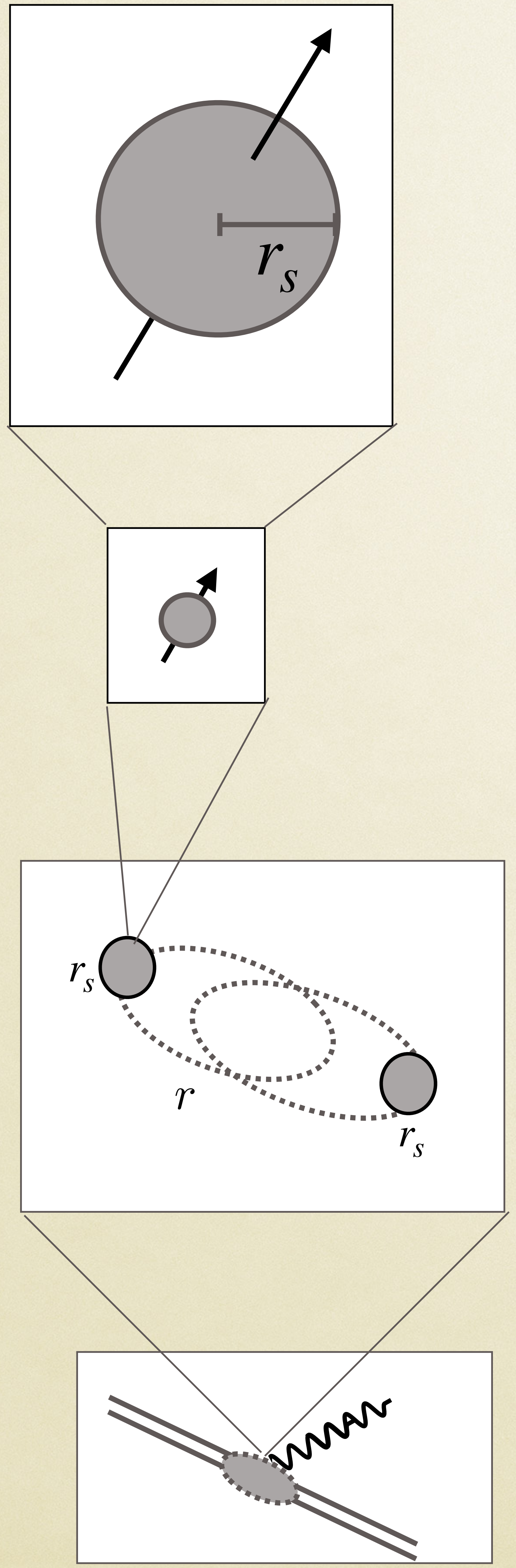
Driesse, Mogull, Plefka, Jakobsen, Sauer, Steinhoff, Usovitsch

GR EFT for PN corrections / Diagrammatic approach



Bound system of Inspiraling of Black Holes

Scales Hierarchies



$\sim r_s$ Black hole / Neutron star

$\gg r_s$ Point particle + internal structure

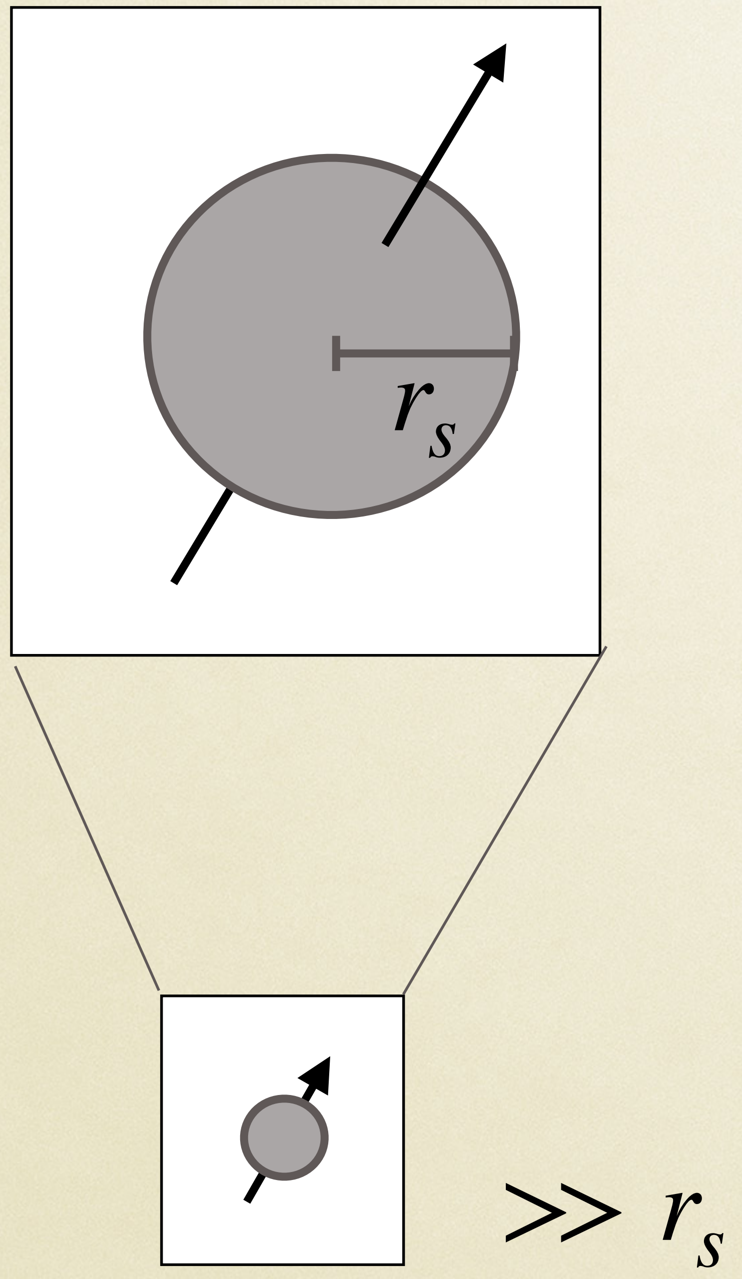
$\sim r$ Binary + Near zone + Far zone

$\gg r$ GW emission + Multipoles

[credit: Patil]

Point particle modelling

compact object as a **point particle** with internal structure parametrised **Wilson coefficients**



Mass: $$\int dt\; m\sqrt{g^{L}_{\mu\nu}u^{\mu}u^{\nu}}$$

Spin induced quadrupole

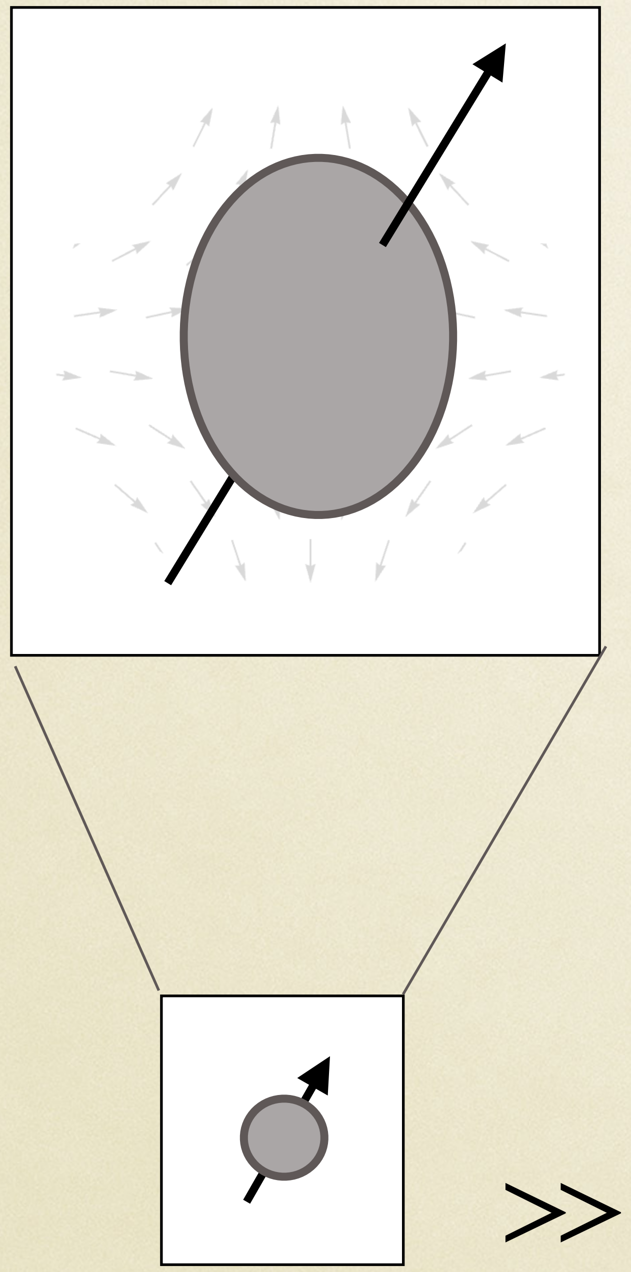
Spin: $$\int dt\,\left\{-\frac{1}{2}S_{\mu\nu}\Omega^{\mu\nu}-\frac{1}{2mc}\left(C^{(0)}_{\mathrm{ES}^2}\right)\frac{E_{\mu\nu}}{u}\left[S^{\mu}S^{\nu}\right]_{\mathrm{STF}}+\cdots\right\}$$

[Porto, Levi, Steinhoff, …]

[credit: Patil]

Point particle modelling

compact object as a **point particle** with internal structure parametrised **Wilson coefficients**



$\gg r_s$

Mass:

$$\int dt\; m\sqrt{g^{L}_{\mu\nu}u^{\mu}u^{\nu}}$$

Spin:

$$\int dt \left\{ -\frac{1}{2}S_{\mu\nu}\Omega^{\mu\nu} - \frac{1}{2mc}\left(C^{(0)}_{\mathrm{ES}^2}\right)\frac{E_{\mu\nu}}{u}\left[S^{\mu}S^{\nu}\right]_{\mathrm{STF}} + \cdots \right\}$$

Spin induced quadrupole

[Porto, Levi, Steinhoff, …]

Adiabatic Tides:

$$\int dt \left\{ \lambda\,\frac{z}{4}E_{\mu\nu}E^{\mu\nu} + \lambda\kappa\frac{G_N^2 m^2}{c^6}\frac{z}{2}\frac{dE_{\mu\nu}}{dt}\frac{dE^{\mu\nu}}{dt} + \cdots \right\}$$

Post-adiabatic Love number

Static Love number

[Flanagan, Hinderer, Damour, …]

Dynamic Tides:

$$\int dt \left\{ \frac{z}{4\lambda\omega_f^2}\left[\frac{c^2}{z^2}\dot{Q}_{\mu\nu}\dot{Q}^{\mu\nu} - \omega_f^2 Q_{\mu\nu}Q^{\mu\nu}\right] - \frac{z}{2}Q^{\mu\nu}E_{\mu\nu} + \cdots \right\}$$

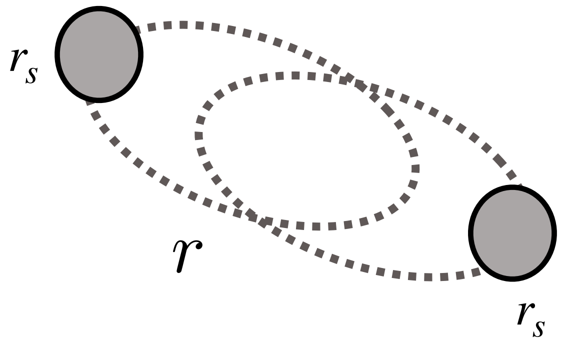
[Hinderer, Steinhoff, …]

Fundamental mode frequency

[credit: Patil]

Coalescing Binary System

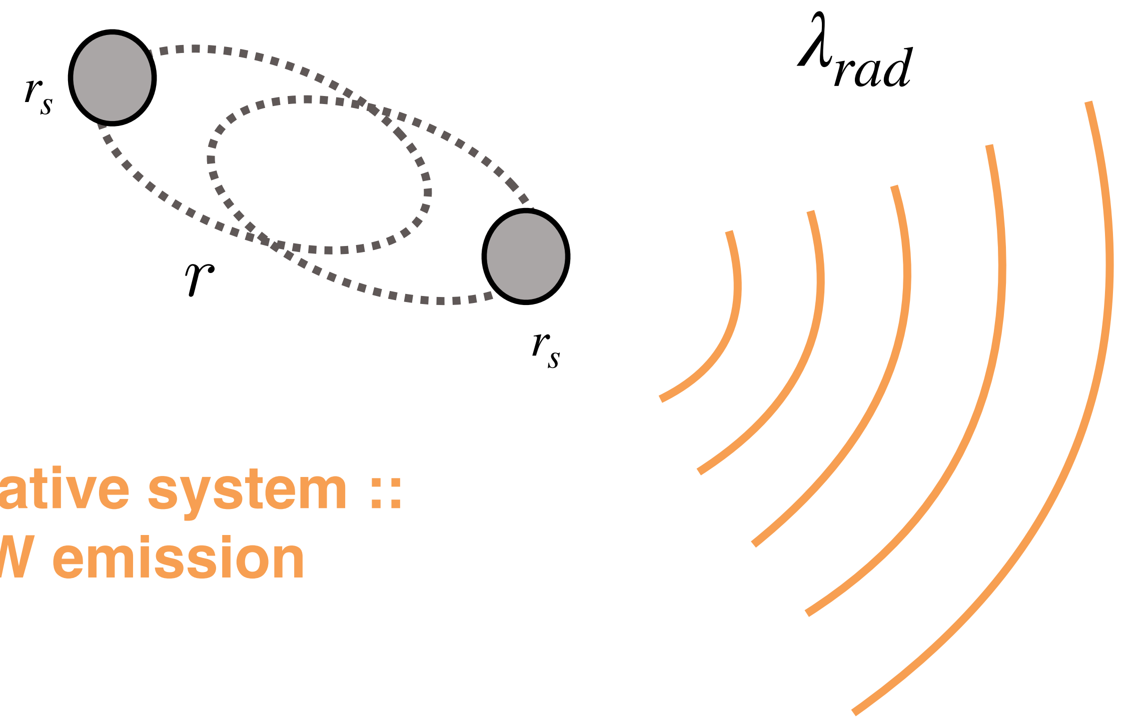
- **Double Hierarchy**



Conservative system ::
~~GW emission~~

$$\boxed{r_s \ll r} \ll \lambda_{rad}$$

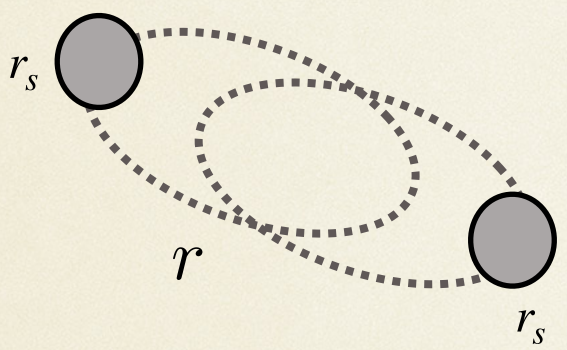
$$r_s \ll \boxed{r \ll \lambda_{rad}}$$



Dissipative system ::
GW emission

Coalescing Binary System

- **Double Hierarchy**

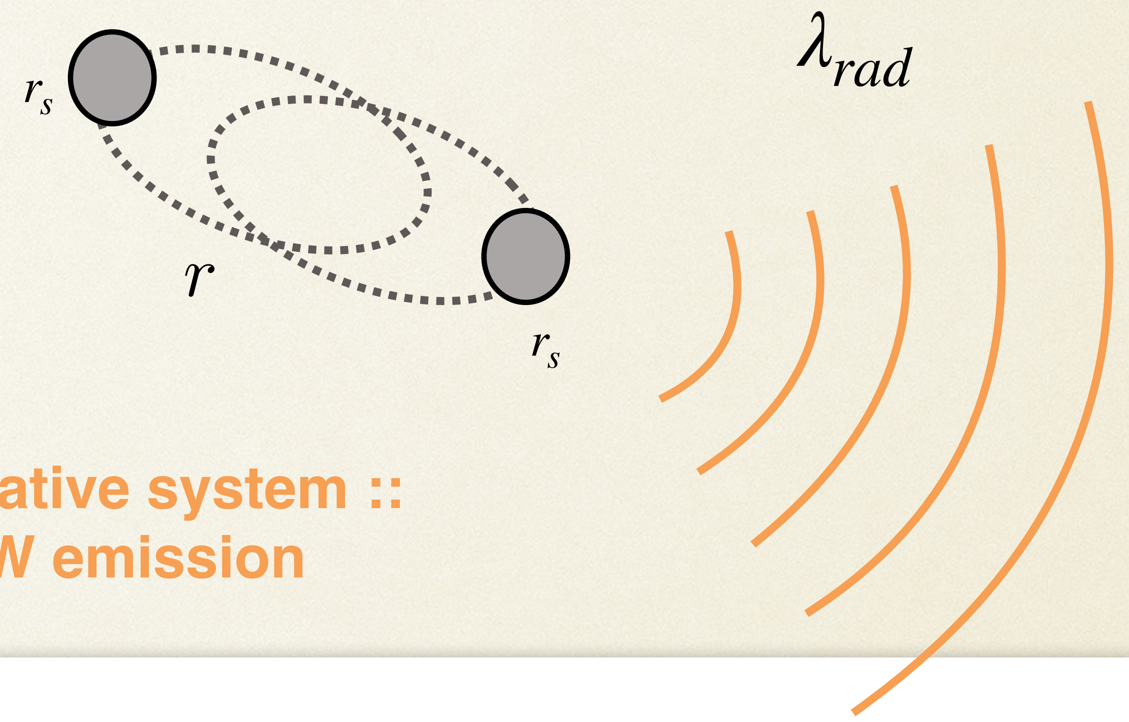


Conservative system :: ~~GW emission~~

$$r_s \ll r \ll \lambda_{rad}$$

$$r_s \ll r \ll \lambda_{rad}$$

Dissipative system :: GW emission



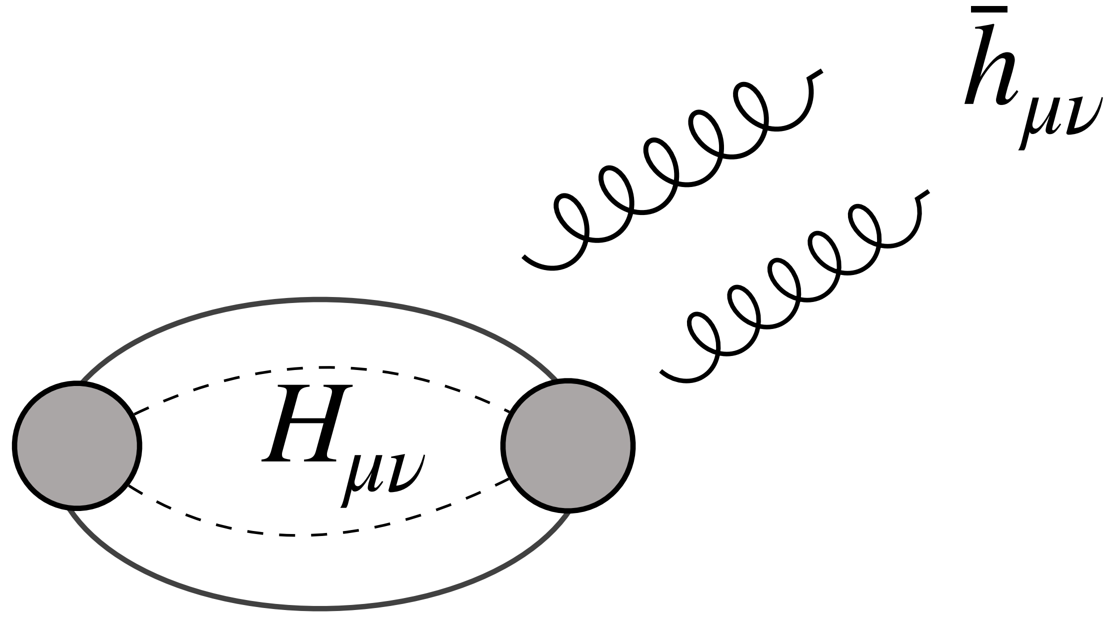
- **Non-relativistic approximation [method of regions]:** [Beneke Smirnov]
  - Weak field expansion:

$$v \ll 1 \qquad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad h_{\mu\nu} = H_{\mu\nu} + \bar{h}_{\mu\nu}$$

- **Potential gravitons** $H_{\mu\nu}$: $(k_0, \mathbf{k}) \sim \left(\frac{v}{r}, \frac{1}{r}\right)$ - - - - - -
- **Radiation gravitons** $\bar{h}_{\mu\nu}$: $(k_0, \mathbf{k}) \sim \left(\frac{v}{r}, \frac{v}{r}\right)$
- **Worldline/BH** x_a :

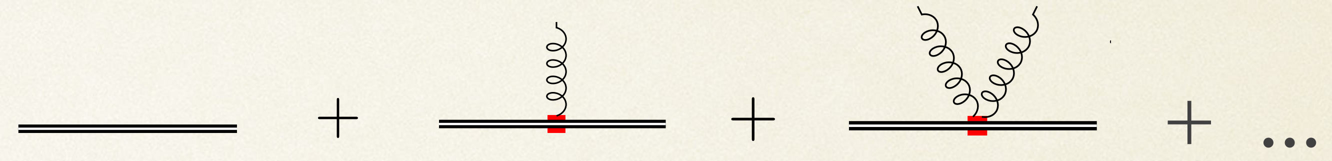
- **Effective action** by **integrating out gravitons:**

$$e^{iS_{eff}[x_a]} = \int D\bar{h} \int DH \, e^{iS_{tot}[x_a, H, \bar{h}]}$$



GREFT Action / Near & Far zones

$$e^{iS_{eff}[x_a]} = \int D\bar{h} \int DH\ e^{iS_{tot}[x_a,H,\bar{h}]} = \int D\bar{h}\ e^{\left\{ iS_{bulk}[\bar{h}] + \cdots \right\}}$$

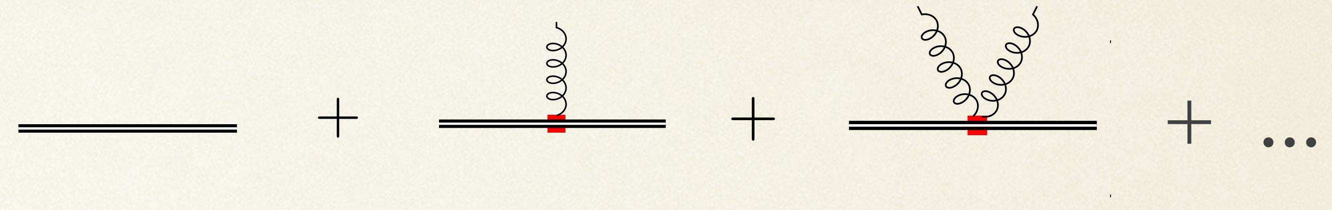


- **Near zone (r)**

$$S_{pot}[x_a, g] = S_{GR}[g] + S_{m_a}[x_a, g] \qquad g_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu}$$

GREFT Action / Near & Far zones

$$e^{iS_{eff}[x_a]} = \int D\bar{h} \int DH\, e^{iS_{tot}[x_a,H,\bar{h}]} = \int D\bar{h}\, e^{\left\{ iS_{bulk}[\bar{h}] + \cdots \right\}}$$

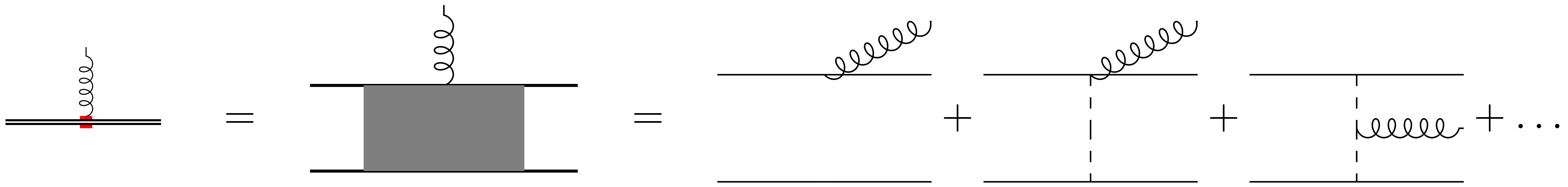


- **Near zone (r)**

$$S_{pot}[x_a, g] = S_{GR}[g] + S_{m_a}[x_a, g] \qquad g_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu}$$

- **Far zone (λ_{rad})**

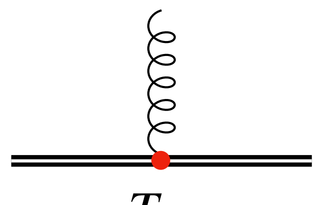
$$S_{rad}[g, \{Q_i\}] = S_{GR}[g] + S_{mult}[g, \{Q_i\}] \qquad g_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu}$$



- **Multipole Action EFT *matching* Far zone**

Binary system as a linear source $T_{\mu\nu}$ of size r emitting $\bar{h}_{\mu\nu}$:

$$S_{mult} = -\frac{1}{2}\int d^4x T^{\mu\nu}\bar{h}_{\mu\nu}$$



$$S_{mult}[\bar{h}, \{Q_i\}] = \int dt \left[\frac{1}{2}E\bar{h}_{00} - \frac{1}{2}\epsilon_{ijk}L^i\bar{h}_{0j,k} - \frac{1}{2}Q^{ij}\mathscr{E}_{ij} - \frac{1}{6}O^{ijk}\mathscr{E}_{ij,k} - \frac{2}{3}J^{ij}B_{ij} + \dots \right]$$

- $\mathscr{E}_{ij}$, B_{ij} are the electric and magnetic components of the Riemann tensor
- $\{Q_i\}$: multipole moments $E, L^i, Q^{ij}, O^{ijk}, J^{ij}$

$$\mathscr{E}_{ij} = R_{0i0j} \approx -\frac{1}{2}\left(\bar{h}_{00,ij} + \ddot{\bar{h}}_{ij} - \dot{\bar{h}}_{0i,j} - \dot{\bar{h}}_{0j,i} + \mathscr{O}(\bar{h}^2)\right)$$

$$B_{ij} = \frac{1}{2}\epsilon_{ikl}R_{0jkl} \approx \frac{1}{4}\epsilon_{ikl}\left(\dot{\bar{h}}_{jk,l} - \dot{\bar{h}}_{jl,k} + \bar{h}_{0l,jk} - \bar{h}_{0k,jl} + \mathscr{O}(\bar{h}^2)\right)$$

PN-GREFT Diagrammar

[Goldberger, Rothstein]
[Gilmore, Ross]
[Foffa, Sturani]

- **Action**

$$S_{pot}[x_a, g] = S_{GR}[g] + S_{m_a}[x_a, g] \qquad S_{m_a}[x_a, g] = S_{pp}[x_a, g] + \delta S_{m_a}[x_a, g]$$

- **Kaluza-Klein parametrization:**

[Kol Smolkin] [Blanchet Damour]

$$\gamma_{ij} = \delta_{ij} + \frac{\sigma_{ij}}{\Lambda} \qquad c_d = 2\frac{d-1}{d-2}$$

$$g_{\mu\nu} = e^{2\phi/\Lambda}\begin{pmatrix} -1 & A_j/\Lambda \\ A_i/\Lambda & e^{-c_d\frac{\phi}{\Lambda}}\gamma_{ij} - A_iA_j/\Lambda^2 \end{pmatrix}$$

Graviton = Scalar + Vector + Sym. Tensor

10 = 1 + 3 + 6

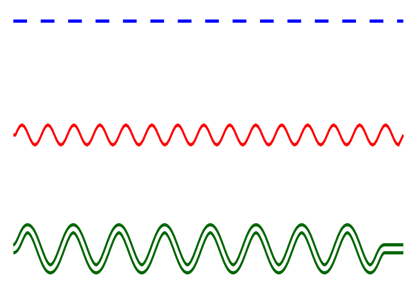
$$g_{\mu\nu} \Rightarrow \phi \quad A^i \quad \sigma^{ij}$$

- **Feynman rules for:** x_a ϕ A^i σ^{ij}

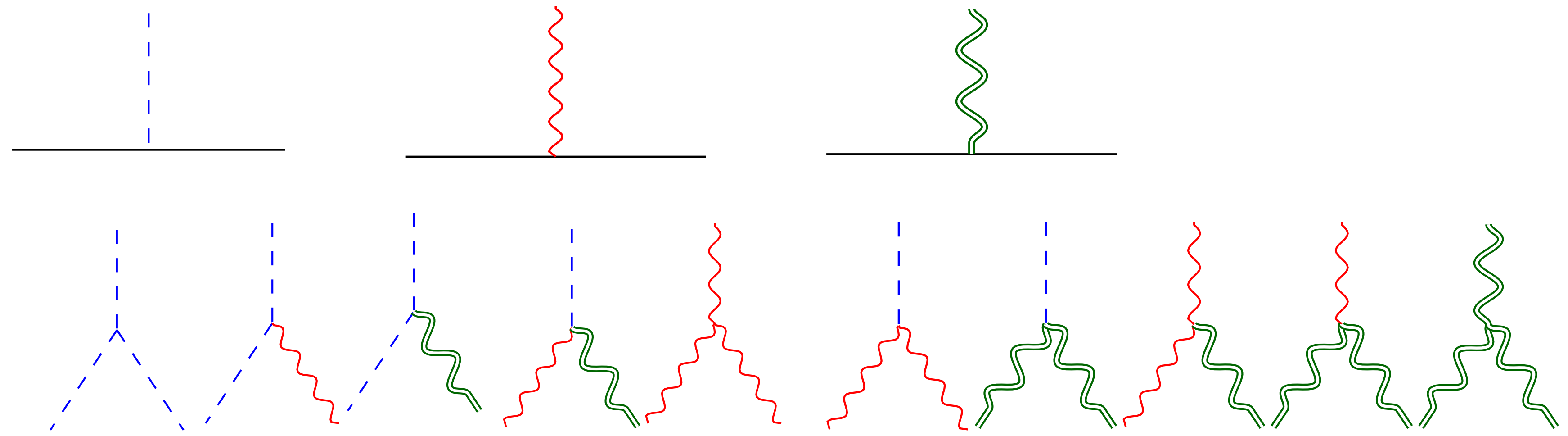
x_a ——

Static / non-propagating source

ϕ
A^i
σ^{ij}



Propagating fields



PN-GREFT Diagrammar

[Goldberger, Rothstein]
[Gilmore, Ross]
[Foffa, Sturani]

- Action

$$S_{pot}[x_a, g] = S_{GR}[g] + S_{m_a}[x_a, g] \qquad S_{m_a}[x_a, g] = S_{pp}[x_a, g] + \delta S_{m_a}[x_a, g]$$

- Kaluza-Klein parametrization:

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$$\gamma_{ij} = \delta_{ij} + \frac{\sigma_{ij}}{\Lambda} \qquad c_d = 2\frac{d-1}{d-2}$$

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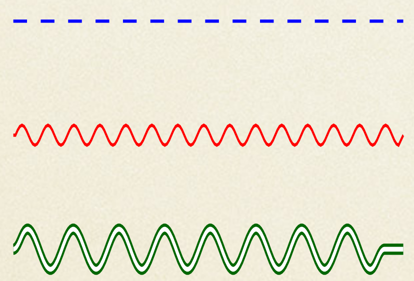
Graviton = Scalar + Vector + Sym. Tensor
10 = 1 + 3 + 6

$$g_{\mu\nu} \Rightarrow \phi \quad A^i \quad \sigma^{ij}$$

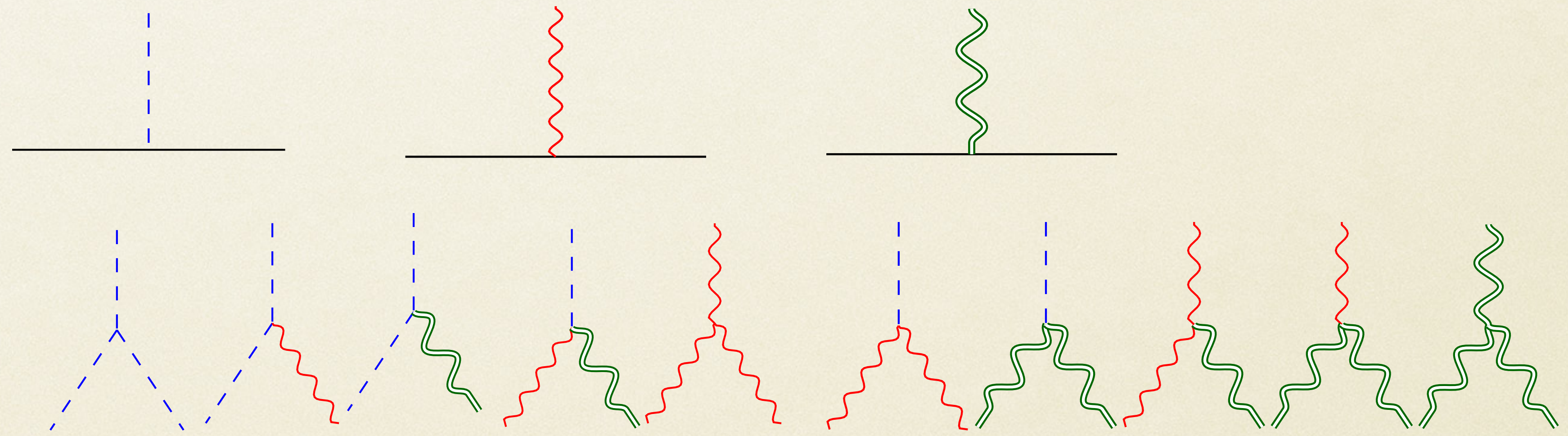
- Feynman rules for: x_a ϕ A^i σ^{ij}

x_a

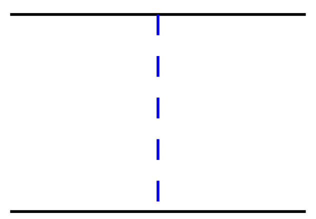
Static / non-propagating source

ϕ, A^i, σ^{ij} 

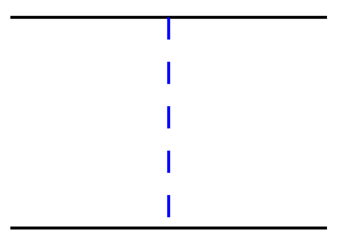
Propagating fields



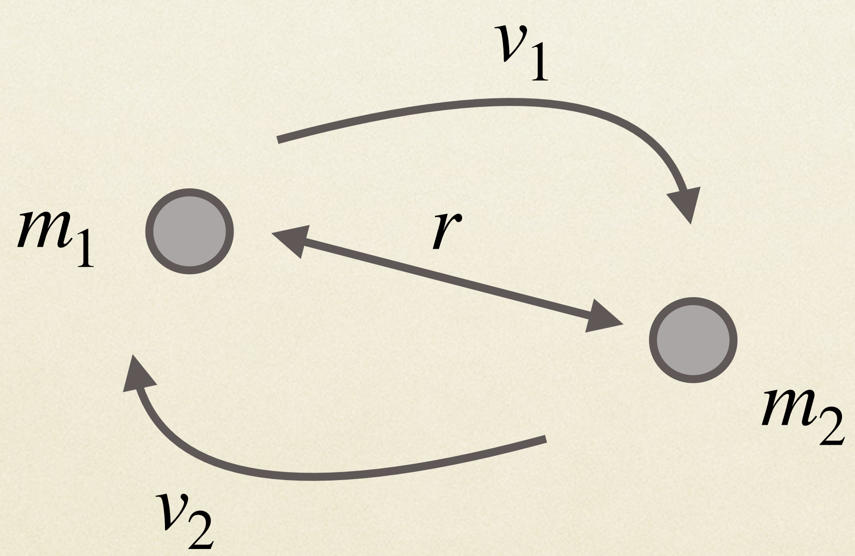
- Newton Potential

$$\mathcal{M}_{0PN} = $$  $$= \frac{im_1m_2}{2c_d\Lambda^2}\frac{1}{\mathbf{p}^2}$$

- Fourier transform: from amplitude to the effective action:

$$\mathcal{L}_{0PN} = -i\lim_{d\to 3}\int\frac{d^d\mathbf{p}}{(2\pi)^d}e^{i\mathbf{p}(x_1-x_2)}\Bigg($$  $$\Bigg) = \frac{G_Nm_1m_2}{r}$$

GR EFT for PN corrections / near zone spineless

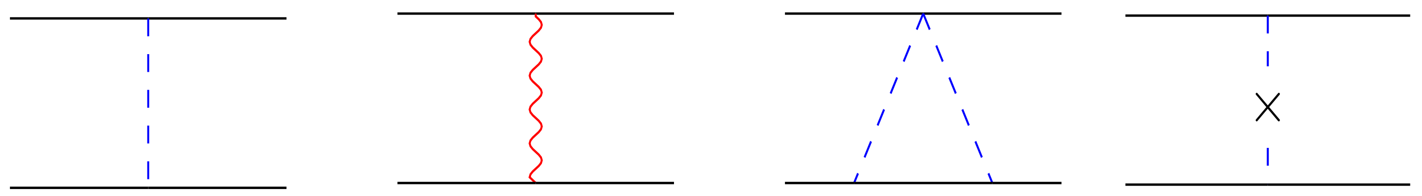


Bound system of Inspiraling of Black Holes

PN-Corrections / GREFT Potential

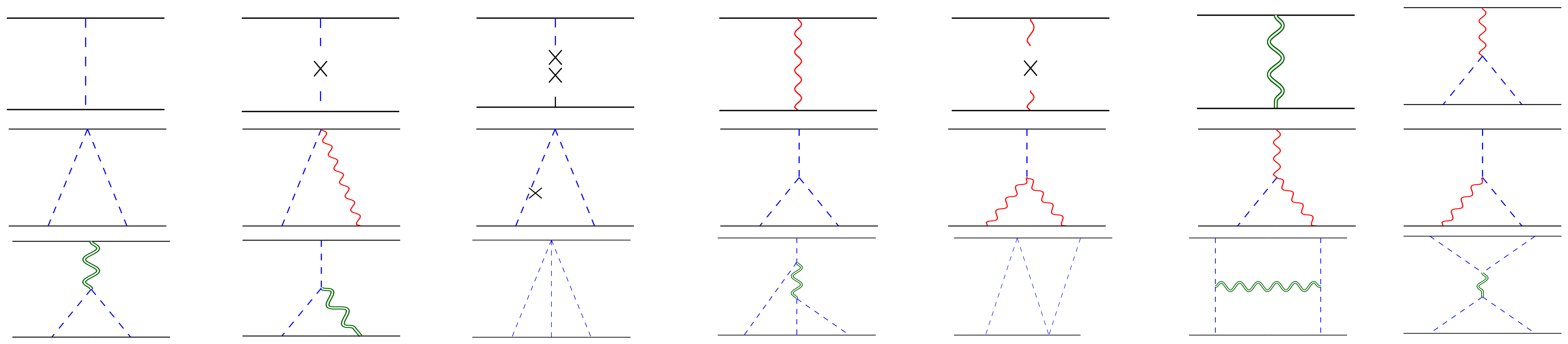
- **1PN corrections:**

  Einstein, Infeld, Hoffman (1938)



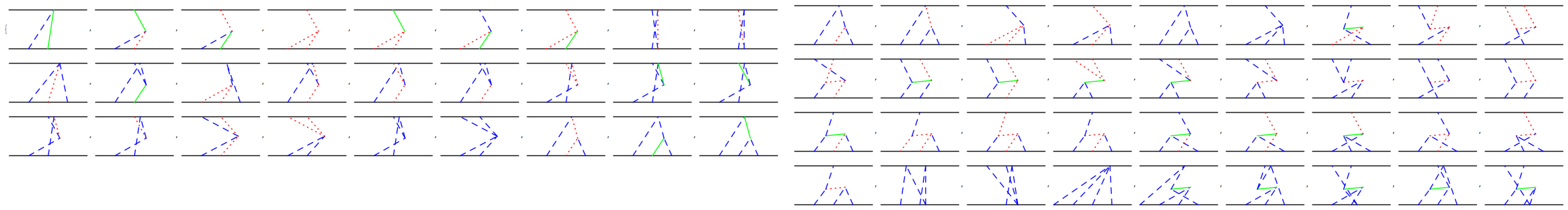
- **2PN corrections:**

  Ohta-Okamura-Kimura-Hiida (1974)
  Gilmore, Ross (2008)



- **3PN corrections:**

  Jaranowski,Schaefer (1997); Damour, Jaranowski, Schaefer (1997); Blachę, Faye (2000); Damour, Jaranowski Schaefer (2001); Foffa Sturani (2011)



- **4PN: corrections:**

  Damour, Jaranowski, Schaefer (2014);
  Bernard, Blanchet, Bohe, Faye, Marsa (2016);
  Foffa, Sturani, Sturm & P.M. (2016);
  Foffa, Porto, Rothstein, Sturani (2019)
  Blumlein, Maier, Marquard,Schaefer (2020)

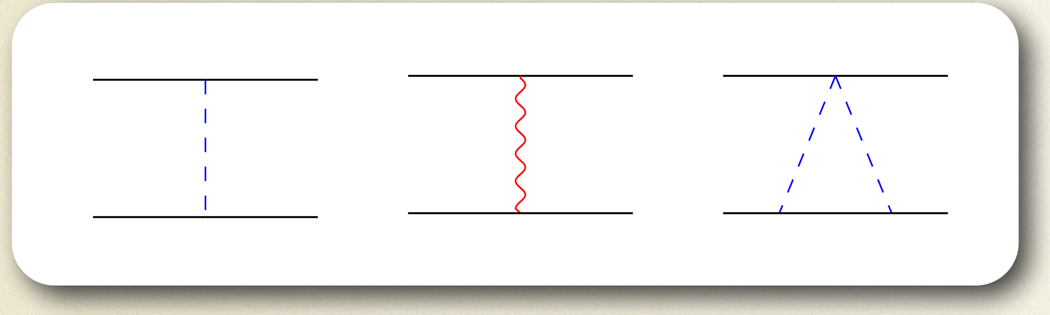
- **5PN: corrections:**

  Bini, Damour, Geralico (2019);
  Foffa, Sturani, Sturm, Torres Bobadilla & P.M. (2019);
  Blumlein, Maier, Marquard, Schaefer (2020,2021)

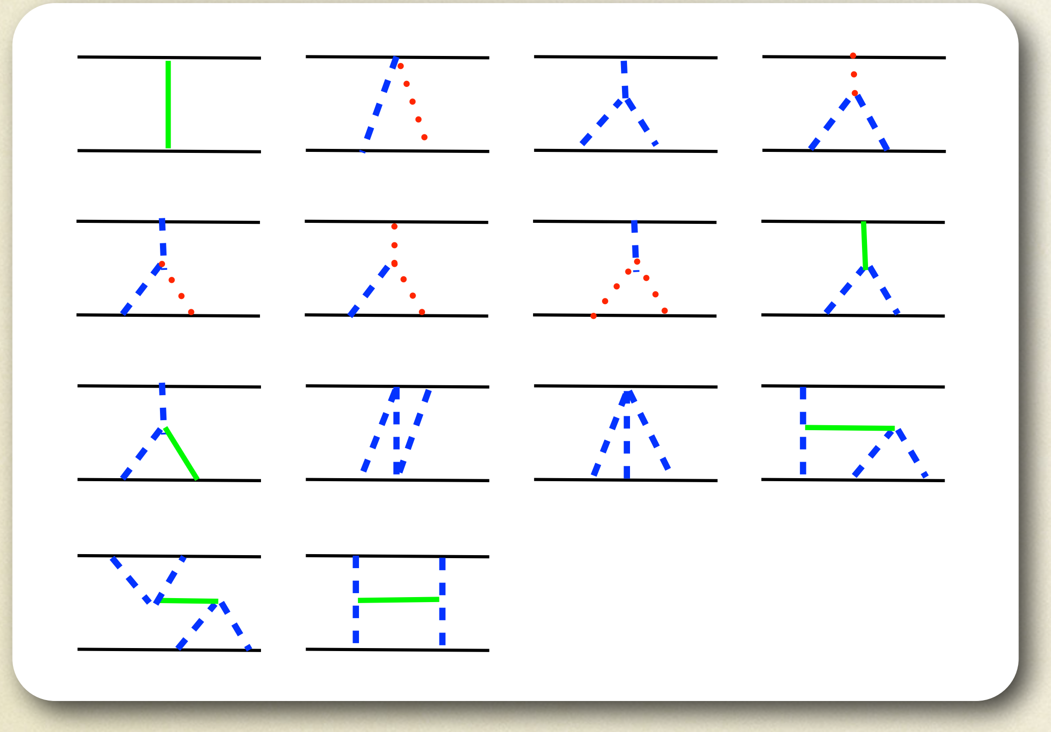
A closer look at the 4PN-Corrections

- Loop nr. $0 \leq \ell \leq n - 1$

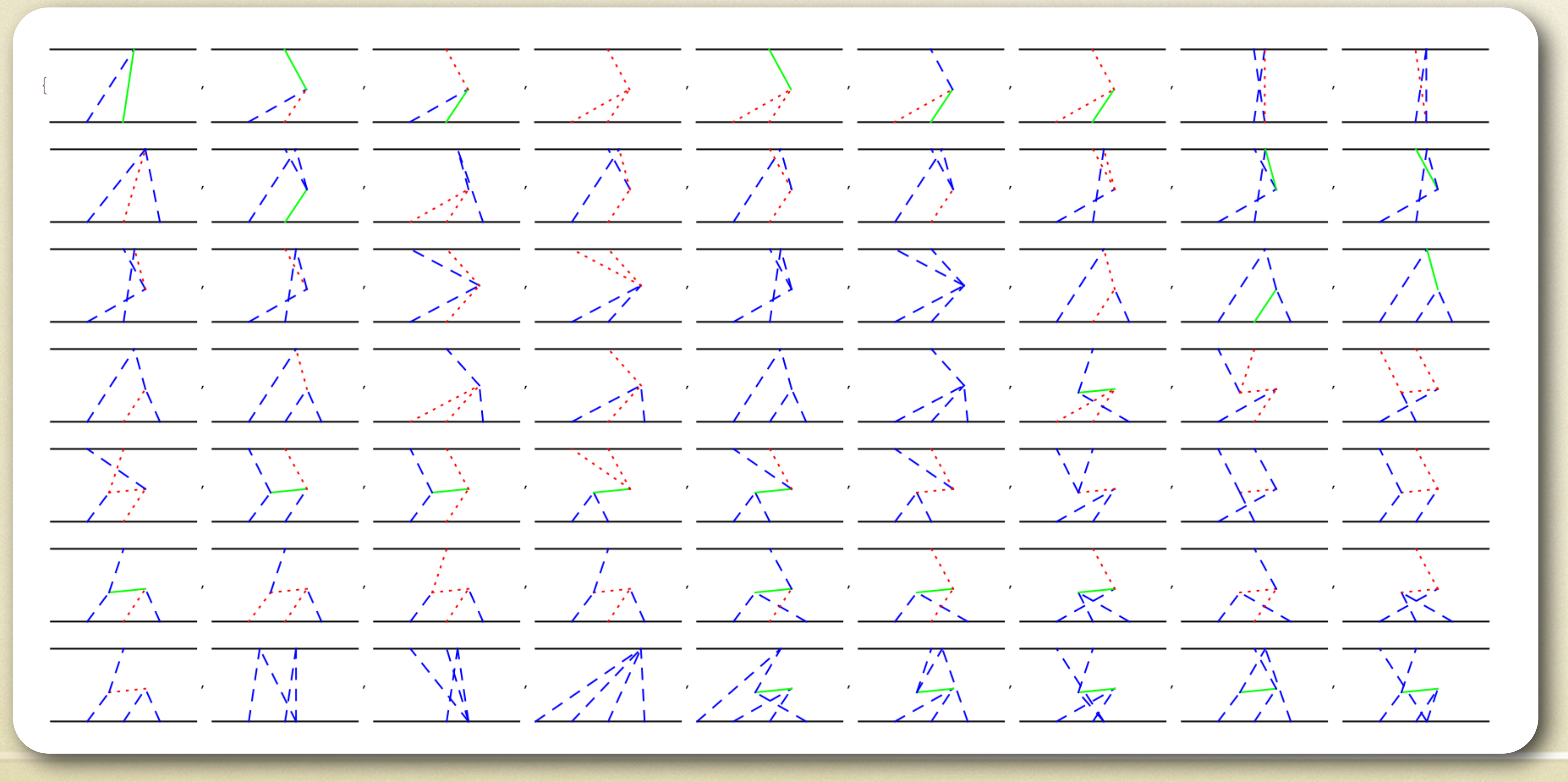
1PN



2PN

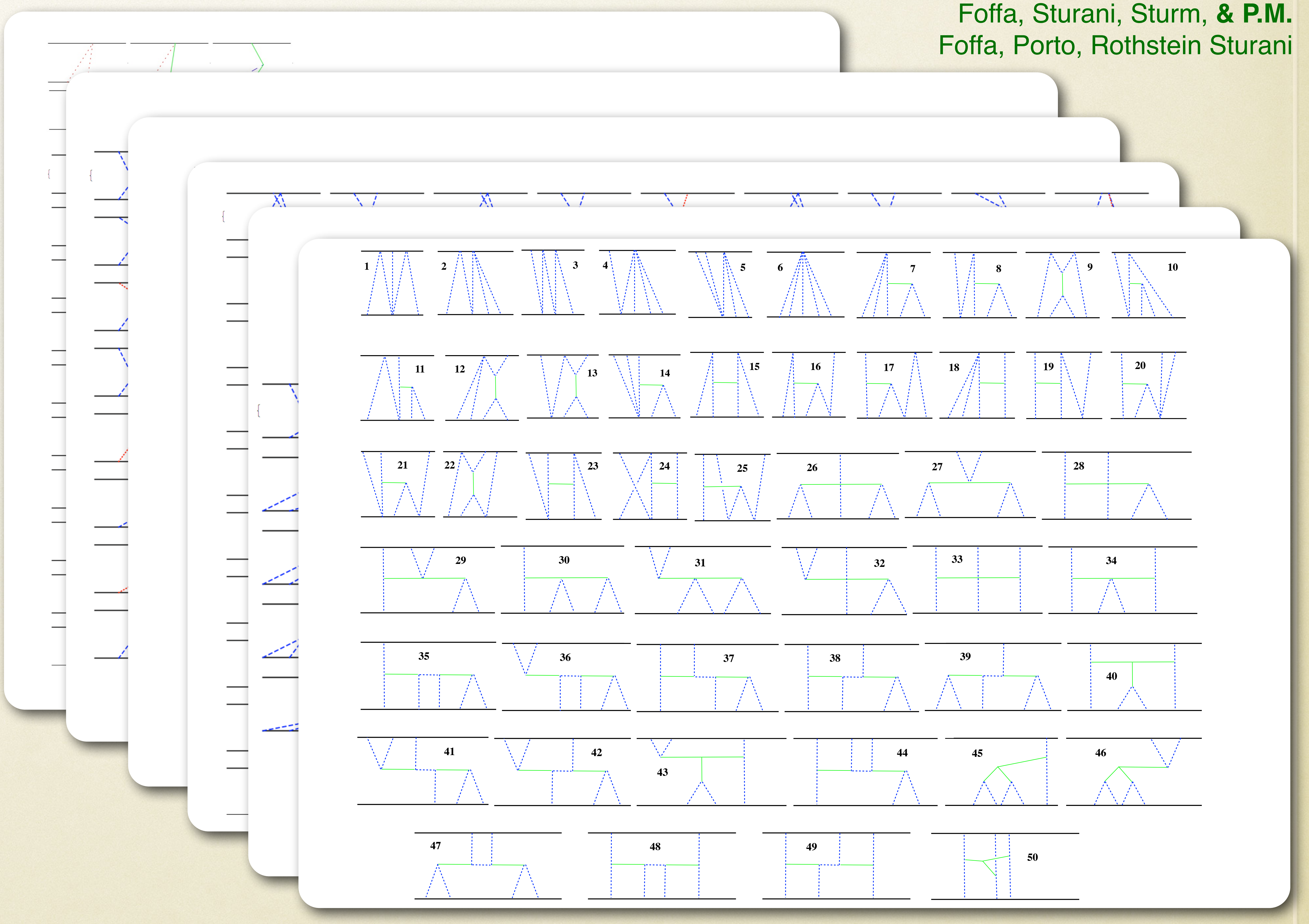


3PN



4PN : 605 GREFT diagrams (up-to 4-loops)

Foffa & Sturani
Foffa, Sturani, Sturm, **& P.M.**
Foffa, Porto, Rothstein Sturani

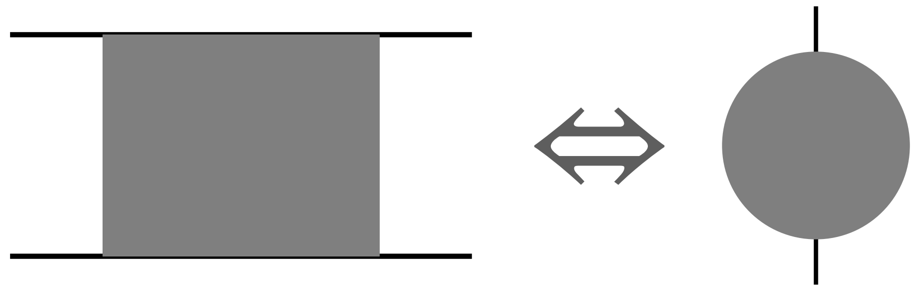


GREFT Diagrams vs 2-point QFT Integrals / a key observation

Foffa, Sturani, Sturm, & **P.M.** (2016)

Computational techniques:

- From Effective diagrams to QFT Amplitudes:
- World-lines are not propagating



- EFTGravity Amplitudes of order G_N^{ℓ} mapped into $(\ell - 1)$—loop 2-point functions with massless internal lines:

$$\mathscr{M} = \sum_i c_i \, I_i^{MI}$$

- Dimensional Regularization $d = 3 + \epsilon$
- Integration-by-parts (IBP) decomposition
- Master Integrals evaluation

- Amplitudes evaluation with QFT multi-loop techniques

- From QFT Amplitudes to Effective Lagrangians:

$$\mathscr{L}_{eff}[x_a] = -\, i \lim_{d \to 3} \int \frac{d^d \mathbf{p}}{(2\pi)^d} e^{i \mathbf{p} \cdot \mathbf{r}} \Bigg(\quad \Bigg)$$

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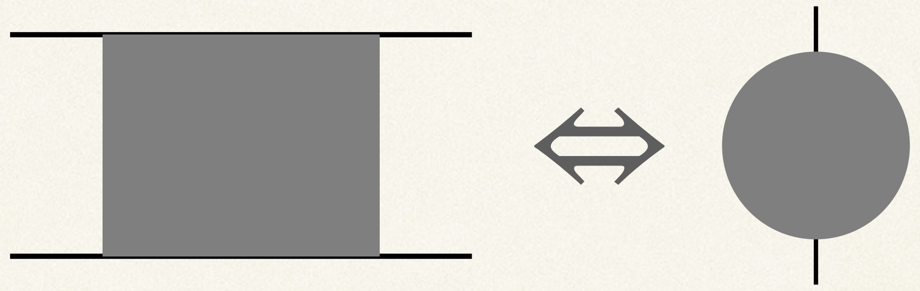

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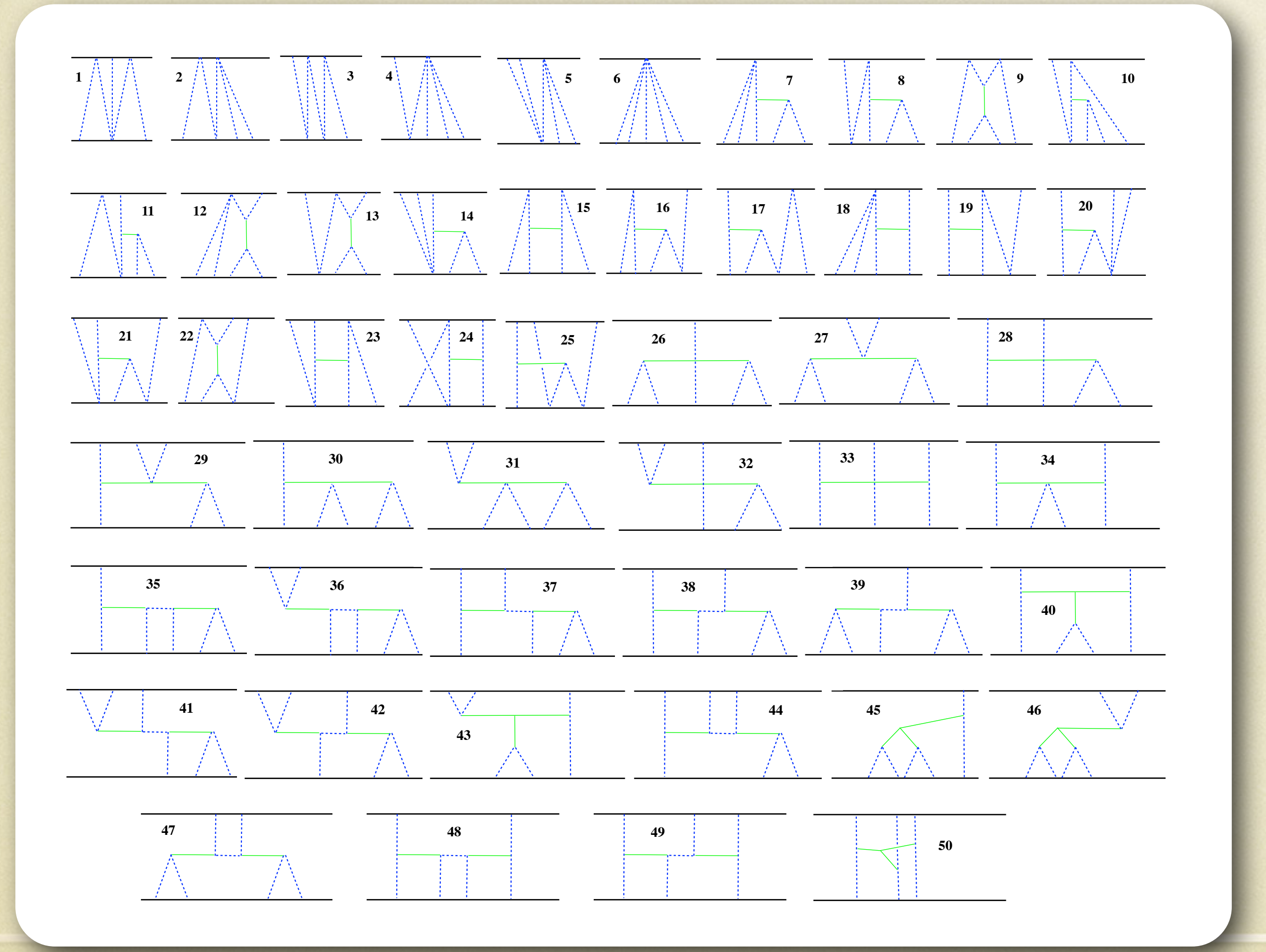
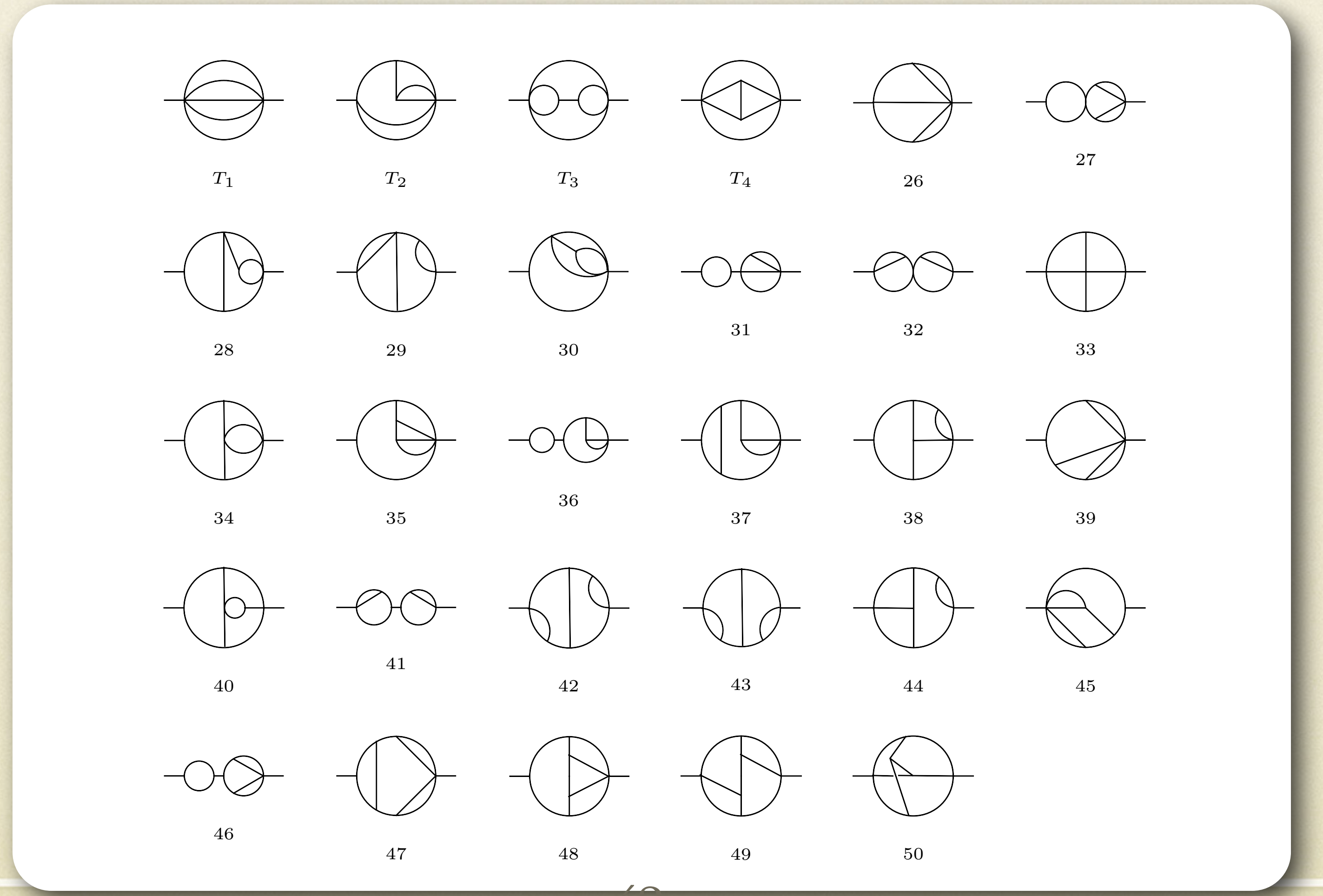
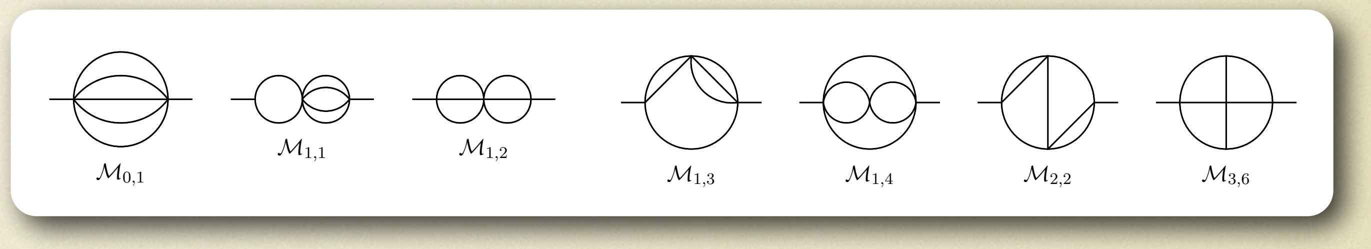
4PN static O(G^5): 50 4-loop GREFT diagrams



29 4-loop QFT diagrams

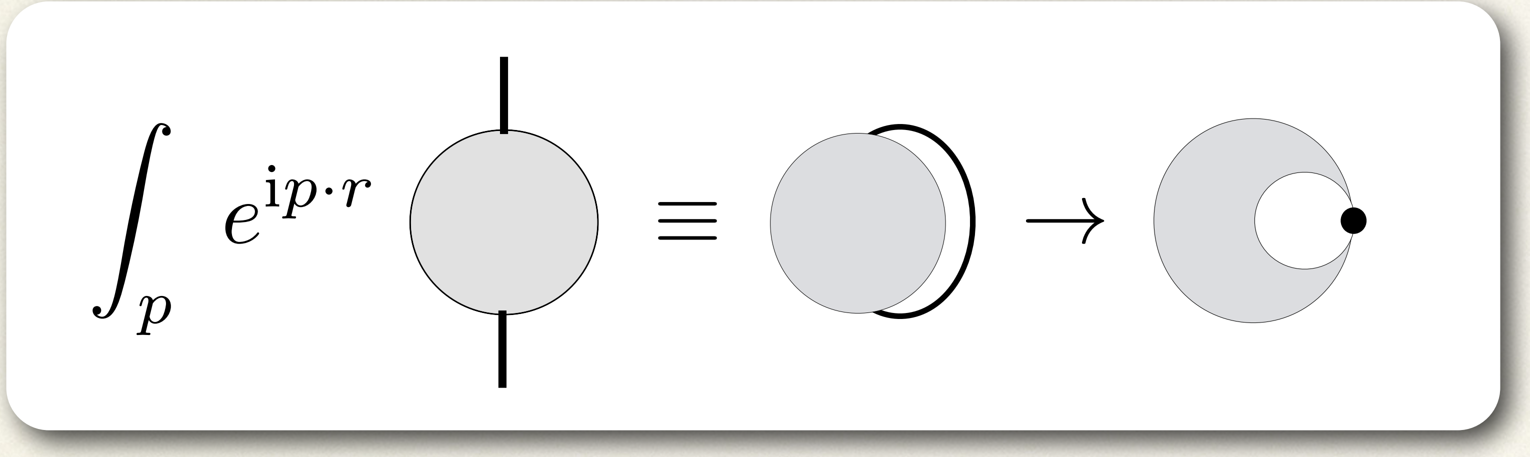


7 4-loop Master Integrals

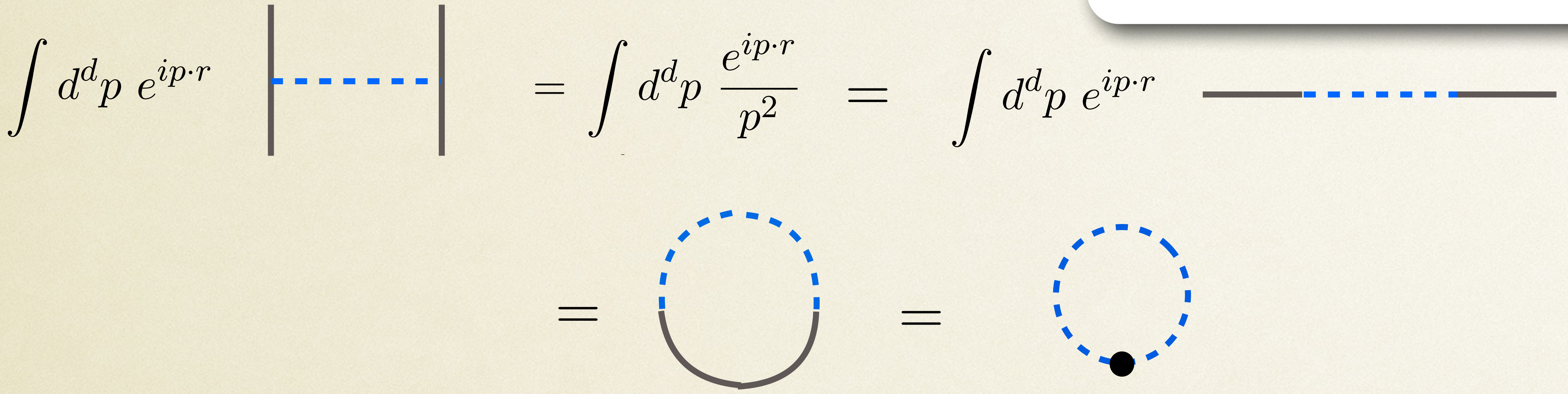


GREFT Diagrams vs 2-point QFT Integrals / Factorization Theorem

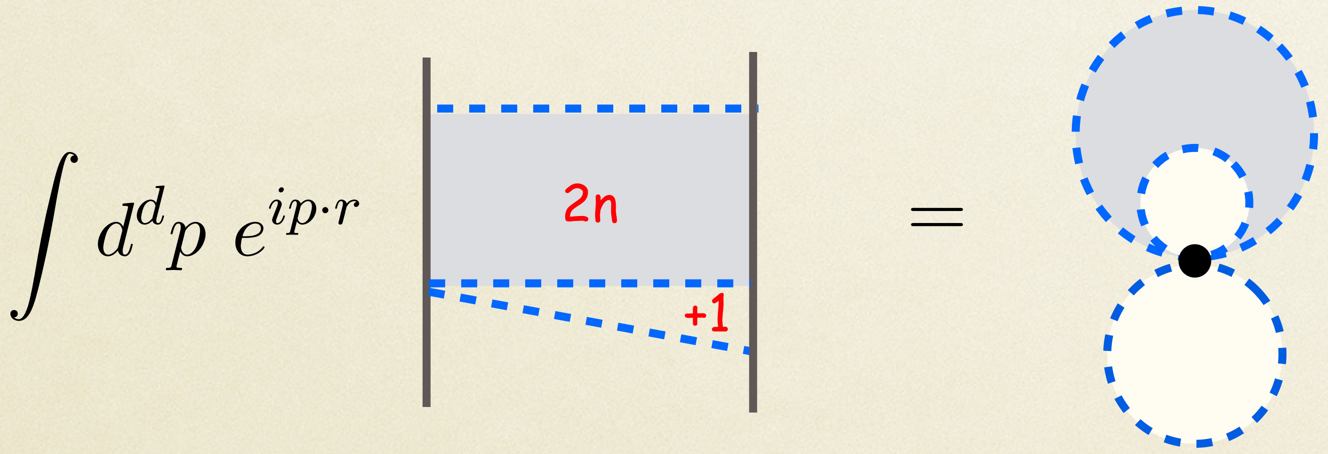
Foffa, Sturani, Sturm, Torres-Bobadilla & **P.M.** (2019)



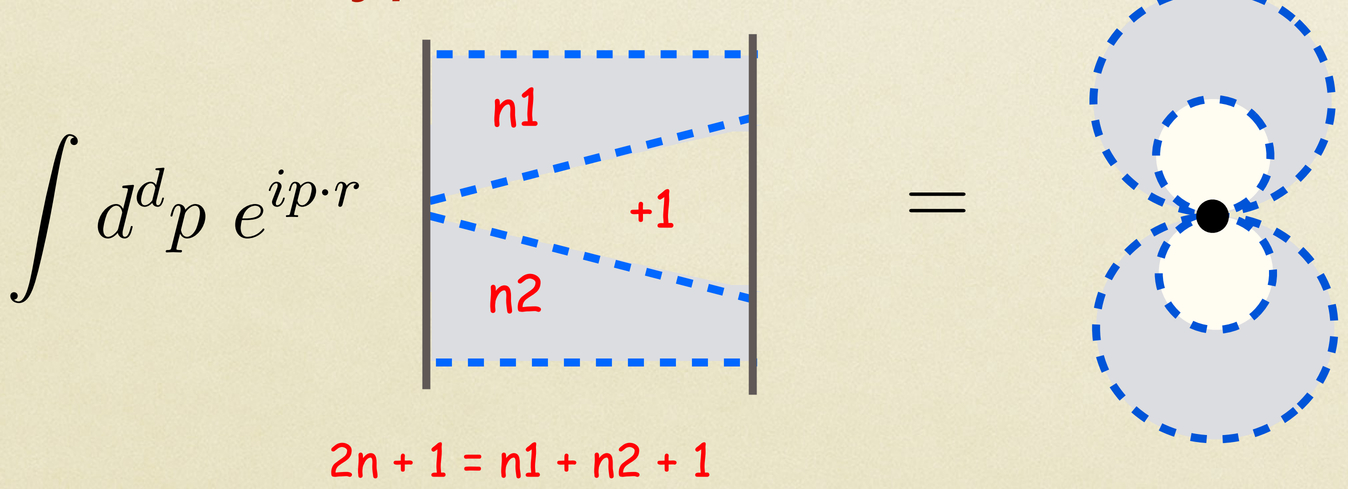
Newton Potential (reloaded):



(2n+1)-PN corrections: *Type-A*



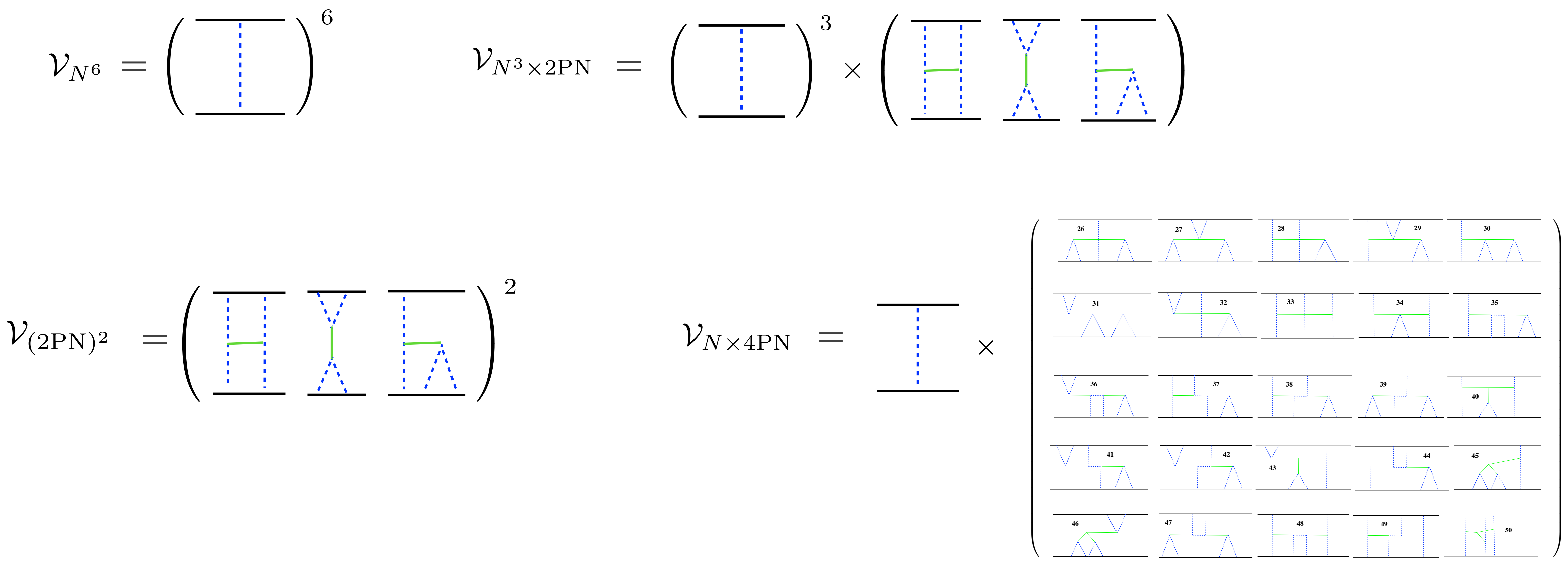
(2n+1)-PN corrections: *Type-B*



- static (2n+1)-PN Potential as product of lower-PN Potential terms

5PN static O(G^6): 154 5-loop GREFT diagrams

Foffa, Sturani, Sturm, Torres-Bobadilla & **P.M.**



$$\mathcal{V}^{(5\text{PN})}_{\text{static}} = \mathcal{V}_{N^6} + \mathcal{V}_{N^3\times 2\text{PN}} + \mathcal{V}_{N\times 4\text{PN}} + \mathcal{V}_{(2\text{PN})^2}$$

5PN O(G^5 v^2): 1220 4-loop GREFT diagrams

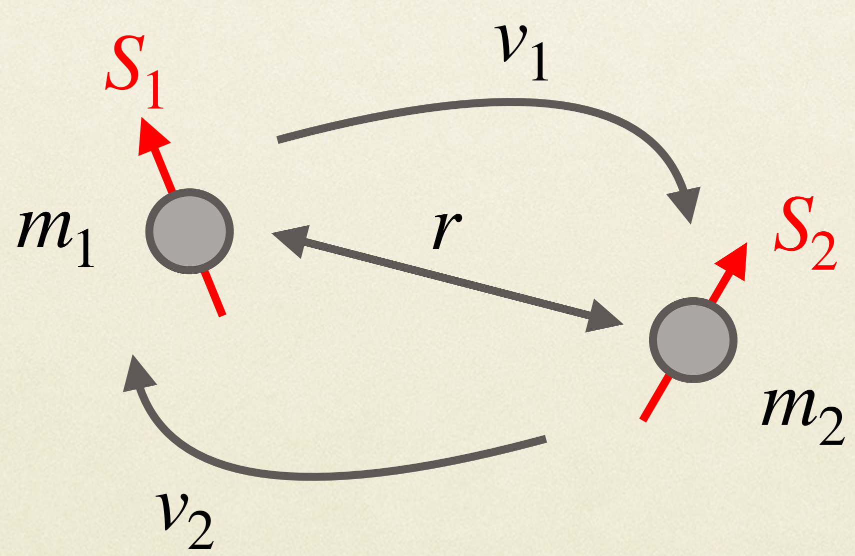
Foffa, Sturani,, Torres-Bobadilla (2020)

- **Factorization Th'm: NO 5-loop diagram explicitly computed**

- Results confirmed and completed by explicit evaluation of 2pt-QFT 5-loop Integrals

Blümelein, Maier, Marquard, Schäfer (2019-21)

GR EFT for PN corrections / near zone **spinning**



Bound system of Inspiraling of Black Holes

Near Zone with Spin / PN Corrections

Porto (2013)
Levi, Steinhoff (2015)
......
Kim, Levi, Yin (2022)
Mandal, Patil, Steinhoff & P.M. (2022)
Levi, Morales, Yin (2022)
Levi, Yin (2022)

Action for Spinning compact object

$$S_{pot}[x_a, g] = S_{GR}[g] + S_{m_a}[x_a, g] \qquad S_{m_a}[x_a, g] = S_{pp}[x_a, g] + \delta S_{m_a}[x_a, g]$$

$$S_{m_a}[x_a, g] = \sum_{a=1,2} \int d\tau \left(-m_{(a)} c \sqrt{u_{(a)}^2} - \frac{1}{2} S_{(a)\mu\nu} \Omega^{\mu\nu}_{(a)} - \frac{S_{(a)\mu\nu} u^\nu_{(a)}}{u^2_{(a)}} \frac{du^\mu_{(a)}}{d\tau} + \mathcal{L}^{(R)}_{(a)} + \mathcal{L}^{(R^2)}_{(a)} + \dots \right)$$

$$u^\mu_{(a)} \equiv \dot{x}^\mu_a$$

Wilson coefficients that describe the internal structure

$$\mathcal{L}^{(R)}_{(a)} = -\frac{1}{2m_{(a)}c} \left(C^{(0)}_{\mathrm{ES}^2} \right)_{(a)} \frac{E_{\mu\nu}}{u_{(a)}} \left[S^\mu_{(a)} S^\nu_{(a)} \right]_{\mathrm{STF}} + \dots$$

$$\mathcal{L}^{(R^2, S^0)}_{(a)} = \frac{1}{2} \left(C^{(2)}_{\mathrm{E}^2} \right)_{(a)} \frac{G_N^2 m_{(a)}}{c^5} \frac{E_{\mu\nu} E^{\mu\nu}}{u^3_{(a)}} S^2_{(a)} + \dots$$

$$\mathcal{L}^{(R^2, S^2)}_{(a)} = \frac{1}{2} \left(C^{(0)}_{\mathrm{E}^2\mathrm{S}^2} \right)_{(a)} \frac{G_N^2 m_{(a)}}{c^5} \frac{E_{\mu\alpha} E_\nu{}^\alpha}{u^3_{(a)}} \left[S^\mu_{(a)} S^\nu_{(a)} \right]_{\mathrm{STF}} + \dots$$

Electric and Magnetic components of Riemann tensor

$$E_{\mu\nu} \equiv R_{\mu\alpha\nu\beta} u^\alpha u^\beta$$

$$B_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\alpha\beta\gamma\mu} R^{\alpha\beta}{}_{\delta\nu} u^\gamma u^\delta$$

STF = Symmetrized Trace-Free

PN order	0	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6	6.5
S^0	0PN	1PN		2PN		3PN		4PN		5PN		6PN	
S^1			LO		NLO		N2LO		N3LO		N4LO		
S^2				LO		NLO		N2LO		N3LO			
S^3							LO		NLO				
S^4								LO		NLO			
S^5											LO		NLO
S^6												LO	

(L+1)PM/loop order
- tree
- 1-loop
- 2-loop
- 3-loop
- 4-loop
- 5-loop
- 6-loop
- 7-loop

[credit: Vines/Roiban]

GREFT Diagrams vs 2-point QFT Integrals

Kim, Levi, Yin (2022)
Mandal, Patil, Steinhoff & P.M. (2022)

S^0

Order	Diagrams	Loops	Diagrams
0PN	1	0	1
1PN	4	1	1
		0	3
2PN	21	2	5
		1	10
		0	6
3PN	130	3	8
		2	75
		1	38
		0	9

(a) Non-spinning sector

S^1

Order	Diagrams	Loops	Diagrams
LO	2	0	2
NLO	13	1	8
		0	5
N^2LO	100	2	56
		1	36
		0	8
N^3LO	894	3	288
		2	495
		1	100
		0	11

(b) Spin-orbit sector

S^2

Order	Diagrams	Loops	Diagrams
LO	1	0	1
NLO	7	1	3
		0	4
N^2LO	58	2	27
		1	24
		0	7
N^3LO	553	3	125
		2	342
		1	76
		0	10

(a) Spin1-Spin2 and Spin1^2 (Spin2^2) sector

Order	Diagrams	Loops	Diagrams
LO	1	0	1
NLO	4	1	1
		0	3
N^2LO	25	2	7
		1	12
		0	6
N^3LO	168	3	15
		2	101
		1	43
		0	9

(b) ES^2 sector

Order	Loops	Diagrams
LO	1	1

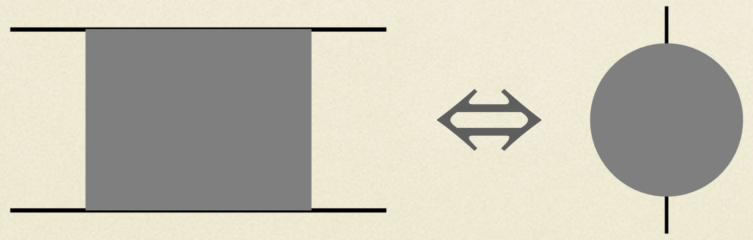
(c) E^2 sector

Order	Loops	Diagrams
LO	1	1

(d) E^2S^2 sector

- **Mapping** to 2-point Functions

$$\mathscr{L}_{eff}[x_a, \dot{x}_a, \ddot{x}_a, \ldots, S_a, \dot{S}_a, \ldots] = -i \lim_{d\to 3} \int \frac{d^d\mathbf{p}}{(2\pi)^d} e^{i\mathbf{p}\cdot\mathbf{r}} \Bigg(\quad \Bigg)$$



$$\mathscr{M} = \sum_i c_i\, I_i^{MI}$$

- Dimensional Regularization $d = 3 + \epsilon$
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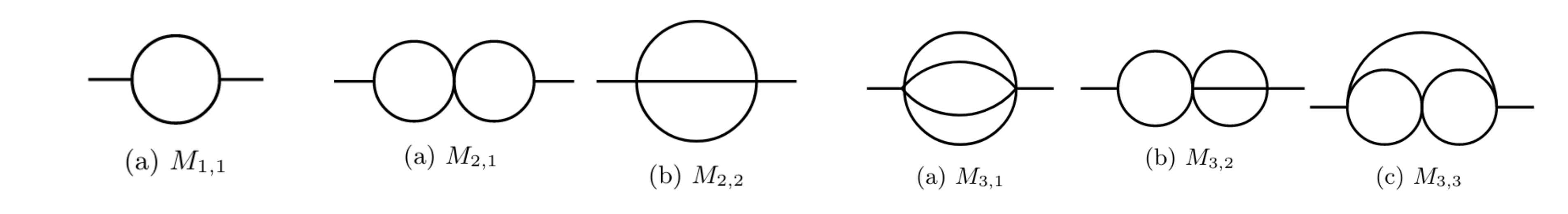
- Master Integrals evaluation



GREFT Diagrams vs 2-point QFT Integrals

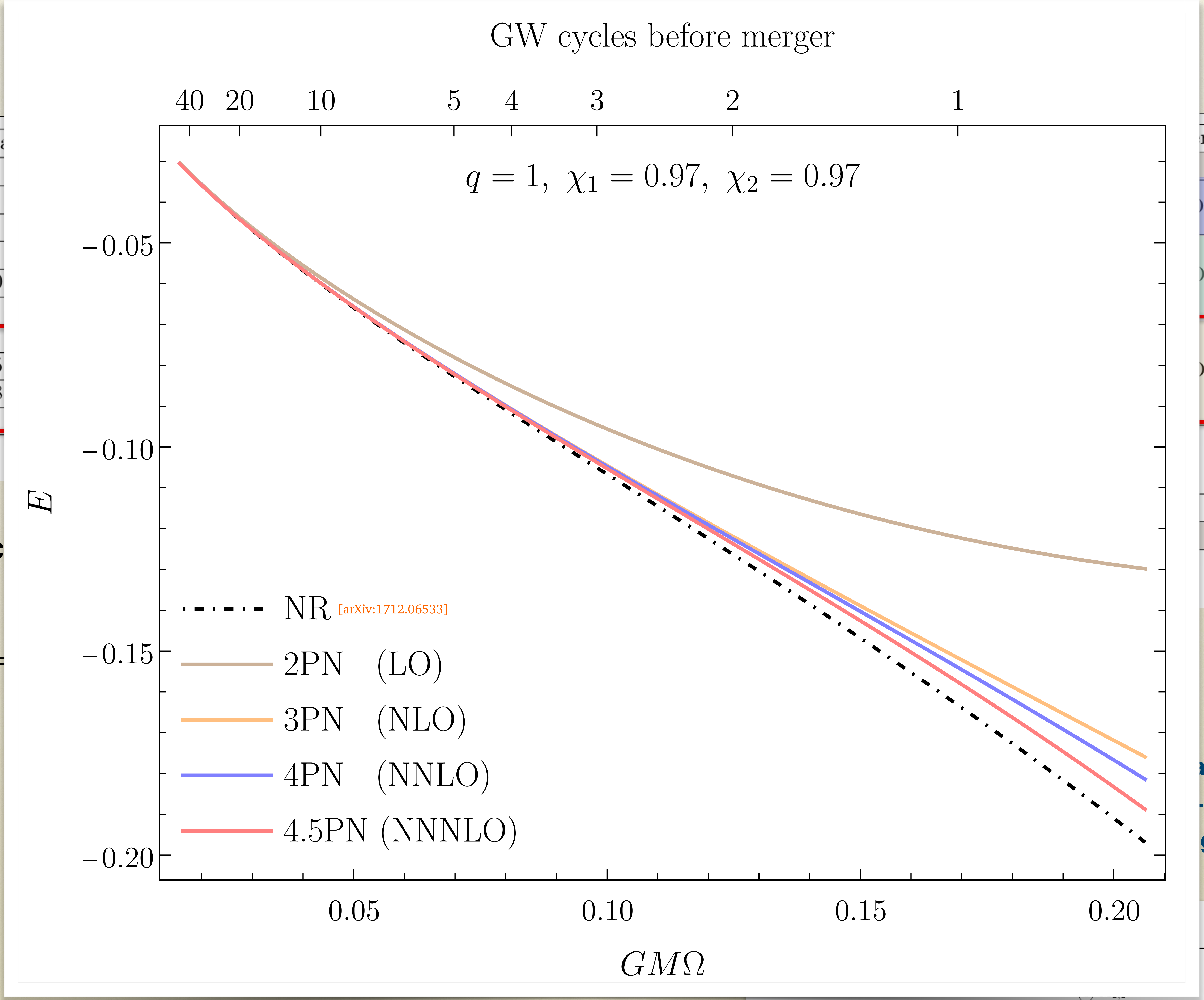
Kim, Levi, Yin (2022)

Mandal, Patil, Steinhoff & P.M. (2022)

s^0

Order	Diagrams	Loops	Diagra
0PN	1	0	1
1PN	4	1	1
		0	3
2PN	21	2	5
		1	10
		0	6
3PN	130	3	8
		2	75
		1	38
		0	9

(a) Non-spinning sector



r	Diagrams	Loops	Diagrams
	1	0	1
	4	1	1
		0	3
	25	2	7
		1	12
		0	6
	168	3	15
		2	101
		1	43
		0	9

(b) ES^2 sector

Order	Loops	Diagrams
LO	1	1

(d) E^2S^2 sector

- **Mapping** to 2-point Func

$$\mathscr{L}_{eff}[x_a, \dot{x}_a, \ddot{x}_a, \ldots, S_a, \dot{S}_a, \ldots] =$$

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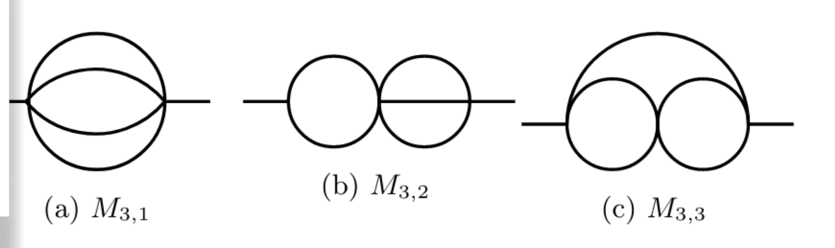
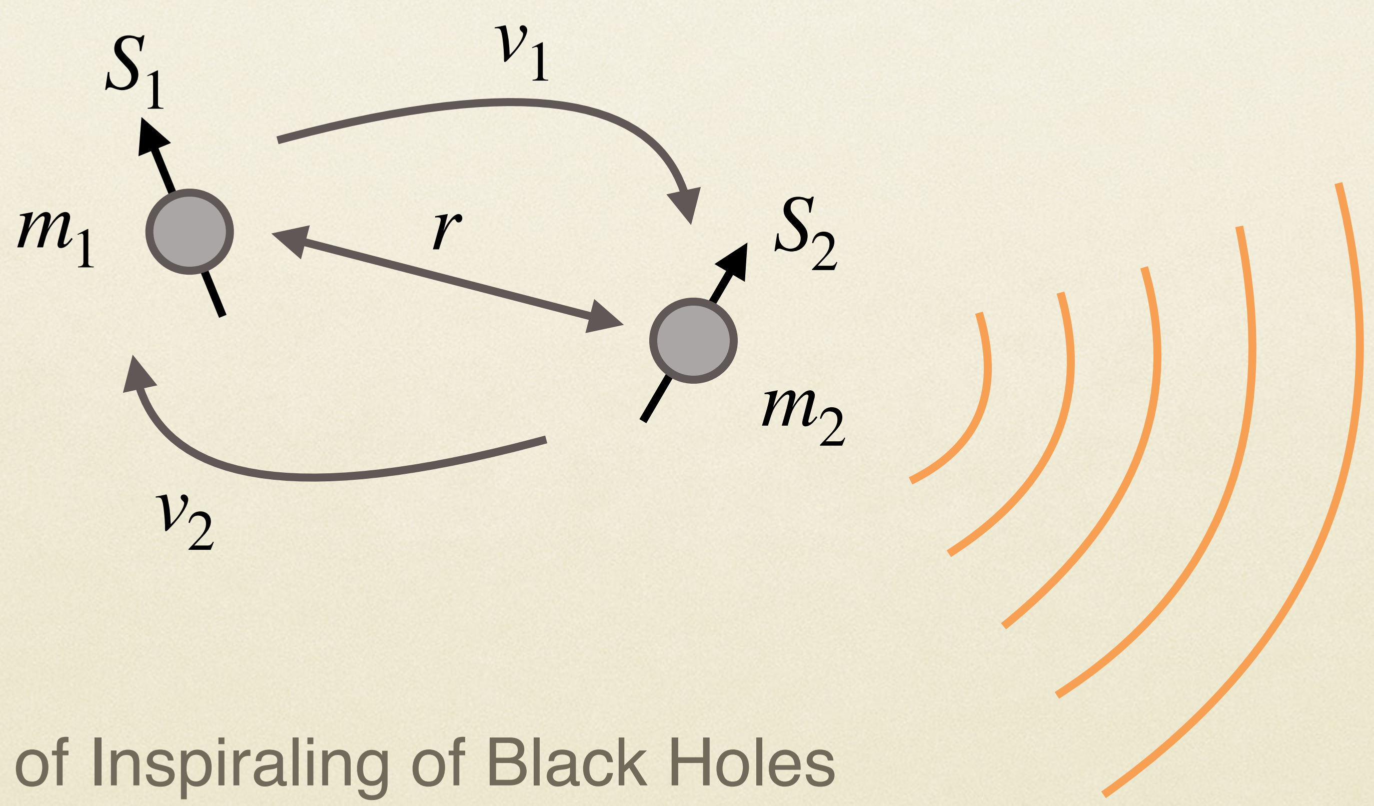
-by-parts (IBP) decomposition

grals evaluation



GR EFT for PN corrections / far zone spinning



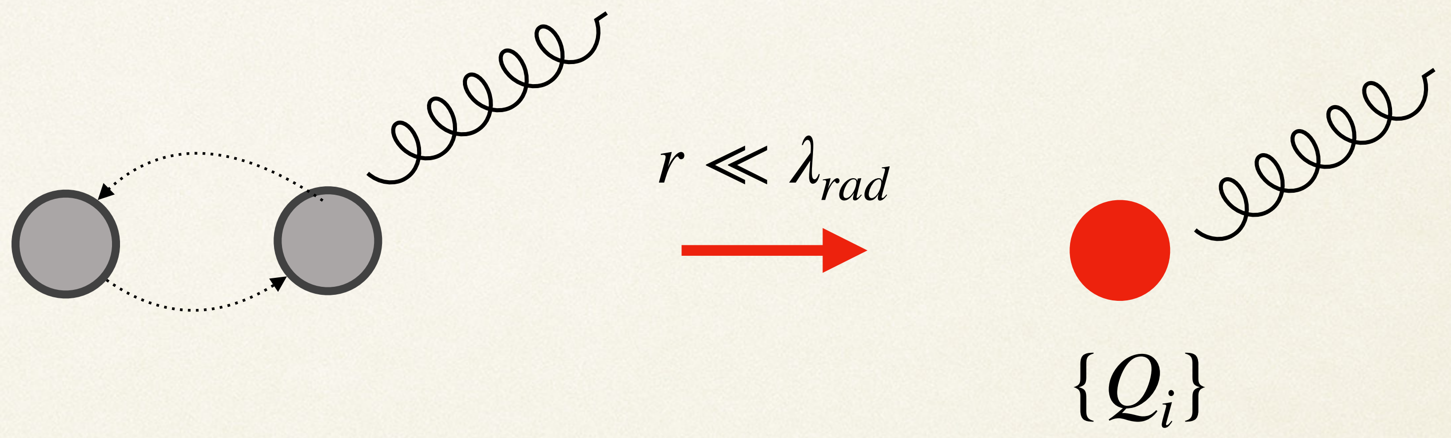
Bound system of Inspiraling of Black Holes

Far Zone / GREFT diagrammar

Thorne (1980)
Goldberger, Rothstein (2005)
Goldberger, Ross (2009)
Galley, Tiglio (2009,2012)
Foffa, Sturani (2012); Ross (2012)
Galley, Leibovich, Porto, Ross (2015)
Leibovich, Maia, Rothstein, Yang (2019)
Blanchet et al.(2021)
....

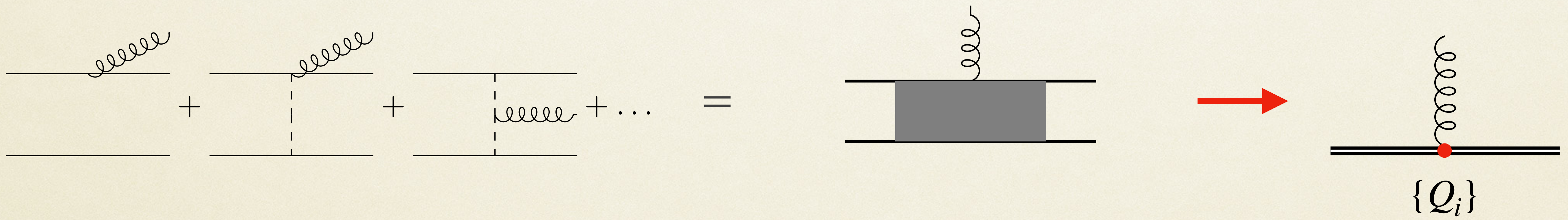
$$S_{rad}[g, \{Q_i\}] = S_{GR}[g] + S_{mult}[g, \{Q_i\}]$$

- Far zone contributions to the conservative dynamics are needed, starting at $4PN$ order



Multipole source emitting gravitons

- Far zone long-wavelength EFT:

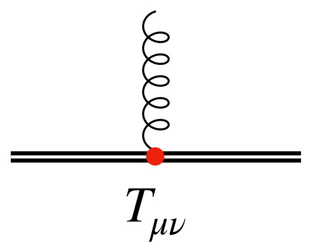


EFT matching

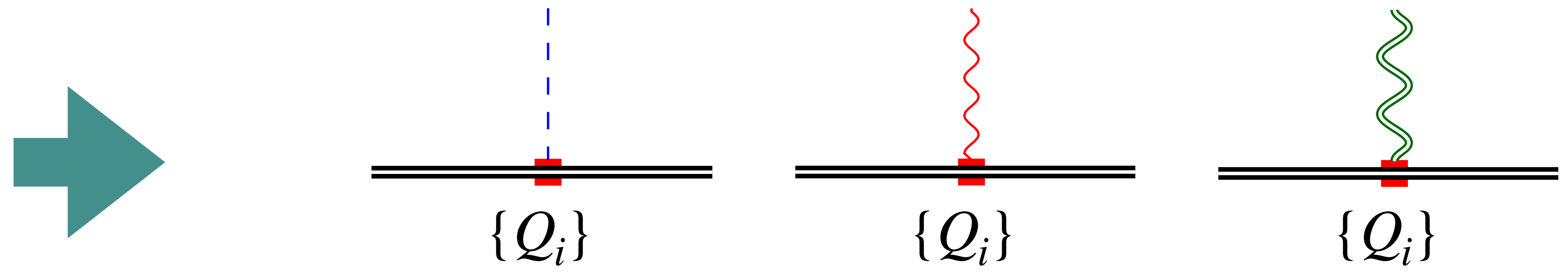
- Multipole Action EFT matched to the Far zone

Binary system as a linear source $T_{\mu\nu}$ of size r emitting $\bar{h}_{\mu\nu}$:

$$S_{mult} = -\frac{1}{2}\int d^4x T^{\mu\nu}\bar{h}_{\mu\nu}$$



$$S_{mult}[\bar{h}, \{Q_i\}] = \int dt\left[\frac{1}{2}E\bar{h}_{00} - \frac{1}{2}\epsilon_{ijk}L^i\bar{h}_{0j,k} - \frac{1}{2}Q^{ij}\mathscr{E}_{ij} - \frac{1}{6}O^{ijk}\mathscr{E}_{ij,k} - \frac{2}{3}J^{ij}B_{ij} + \dots\right]$$



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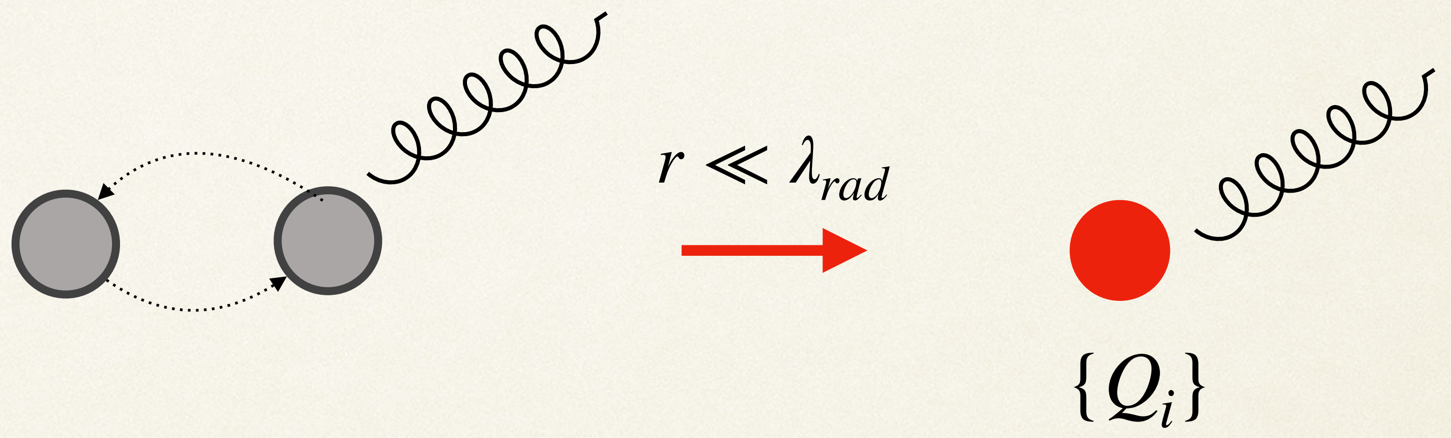

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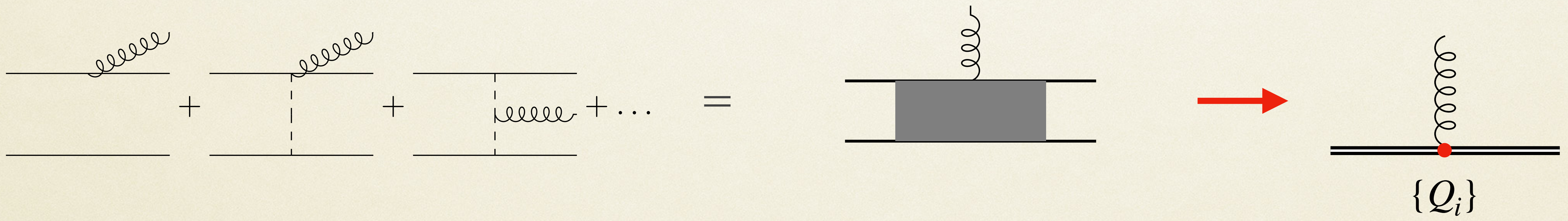
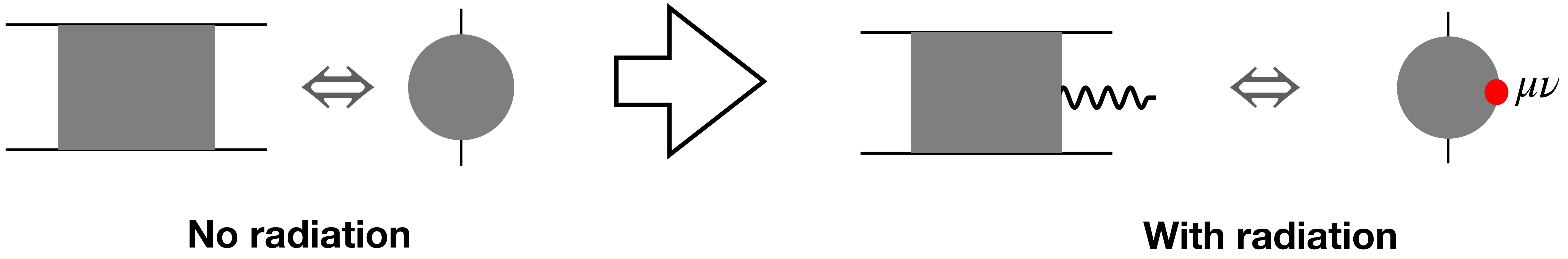


EFT matching

- **Multipole Action EFT matched to the Far zone**

- **Mapping to 2-point function** Mandal, Patil, Steinhoff & P.M. (2024)



$$\bar{h}_{\mu\nu}\Big|_{k\to 0}$$

$$\mathscr{M} = \sum_i c_i \, I_i^{MI}$$

- Dimensional Regularization $d = 3 + \epsilon$
- Integration-by-parts (IBP) decomposition
- Master Integrals evaluation

Far Zone / Flux

Mandal, **Patil**, Steinhoff, & PM

PN orders	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	...
Point Particle	**LO**		**NLO**	**LO (Her)**	**NNLO**	**NLO (Her)**	**N^3LO +NNLO (Her)**	**N^3LO (Her)**	**N^4LO +N^4LO (Her)**	**N^5LO (Her)**	
Spin-orbiit				**LO**		**NLO**	**LO (Her)**	**NNLO**	**NLO (Her)**	N^3LO +NNLO (Her)	
Spin^2					**LO**		**NLO**	**LO (Her)**	**NNLO**	NLO (Her)	
Spin^3								**LO**		NLO	
Spin^4									LO		

Disconnected
Tree
1 Loop
2 Loop
3 Loop

$$
\begin{aligned}
\mathscr{F}^{\text{pp}}_{cir} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ & 1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right)x + 4\pi x^{3/2} + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right)x^2 + \left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2} \\
& + \left[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_E - \frac{856}{105}\ln(16\,x) + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\right]x^3 \\
& + \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2} \\
& + \Bigg[-\frac{323105549467}{3178375200} + \frac{232597}{4410}\gamma_E - \frac{1369}{126}\pi^2 + \frac{39931}{294}\ln 2 - \frac{47385}{1568}\ln 3 + \frac{232597}{8820}\ln x \\
& \quad + \left(-\frac{1452202403629}{1466942400} + \frac{41478}{245}\gamma_E - \frac{267127}{4608}\pi^2 + \frac{479062}{2205}\ln 2 + \frac{47385}{392}\ln 3 + \frac{20739}{245}\ln x\right)\nu \\
& \quad + \left(\frac{1607125}{6804} - \frac{3157}{384}\pi^2\right)\nu^2 + \frac{6875}{504}\nu^3 + \frac{5}{6}\nu^4\Bigg]x^4 \\
& + \left[\frac{265978667519}{745113600} - \frac{6848}{105}\gamma_E - \frac{3424}{105}\ln(16\,x) + \left(\frac{2062241}{22176} + \frac{41}{12}\pi^2\right)\nu - \frac{133112905}{290304}\nu^2 - \frac{3719141}{38016}\nu^3\right]\pi x^{9/2} + \mathcal{O}(x^5)\Bigg\}
\end{aligned}
$$

[Blanchet, Faye, Henry, Larrouturou, Trestini (2023)]

$$x = \left(GM\,\omega_{\text{orb}}\right)^{2/3}$$

[credit: Patil]

Far Zone / Flux

Mandal, **Patil**, Steinhoff, & PM

PN orders	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	...
Point Particl	**LO**		**NLO**	**LO (Her)**	**NNLO**	**NLO (Her)**	**N^3LO +NNLO (Her)**	**N^3LO (Her)**	**N^4LO +N^4LO (Her)**	**N^5LO (Her)**	
Spin-orbiit				**LO**		**NLO**	**LO (Her)**	**NNLO**	**NLO (Her)**	N^3LO +NNLO (Her)	
Spin^2											
Spin^3											
Spin^4											

$$\mathcal{F}^{\rm pp}_{cir} = \frac{32c^5}{5G}\nu^2 x \ldots$$

PN orders	0	0.5	1	1.5	2	2.5	3
$E_{\mu\nu}E^{\mu\nu}$	**LO**		**NLO**	**LO (Her)**	**NNLO**	**NLO (Her)**	N^3LO +NNLO (Her)
$B_{\mu\nu}B^{\mu\nu}$			**LO**		**NLO**	**LO (Her)**	NNLO
$\nabla_\alpha E_{\mu\nu}\nabla^\alpha E^{\mu\nu}$					**LO**		NLO
$\dot{E}_{\mu\nu}\dot{E}^{\mu\nu}$							LO

$$\begin{aligned}
\mathcal{F}^{\rm AT}_{E} = &\left\{\left(\frac{768\nu^2}{5}+\frac{192\nu}{5}\right)\widetilde{\lambda}_{(+)}+\frac{192}{5}\delta\widetilde{\lambda}_{(-)}\right\}x^{10}\\
&+\left\{\left(-992\nu^3-\frac{9736\nu^2}{35}-\frac{1408\nu}{35}\right)\widetilde{\lambda}_{(+)}+\left(-\frac{184\nu^2}{5}-\frac{1408\nu}{35}\right)\delta\widetilde{\lambda}_{(-)}\right\}x^{11}\\
&+4\pi\left\{\left(\frac{768\nu^2}{5}+\frac{192\nu}{5}\right)\widetilde{\lambda}_{(+)}+\frac{192}{5}\delta\widetilde{\lambda}_{(-)}\right\}x^{23/2}\\
&+\left\{\left(3088\nu^4+\frac{63692\nu^3}{35}-\frac{1299706\nu^2}{945}+\frac{5344\nu}{45}\right)\widetilde{\lambda}_{(+)}\right.\\
&\quad\left.+\left(-\frac{11116\nu^3}{15}+\frac{149566\nu^2}{105}+\frac{5344\nu}{45}\right)\delta\widetilde{\lambda}_{(-)}\right\}x^{12}+\mathcal{O}\left(x^{25/2}\right).
\end{aligned}$$

[Henry, Faye, Blanchet(2020), Mandal, Patil, **PM**, Steinhoff (2024)]

$$x = \left(GM\,\omega_{\rm orb}\right)^{2/3}$$

…urou, Trestini (2023)]

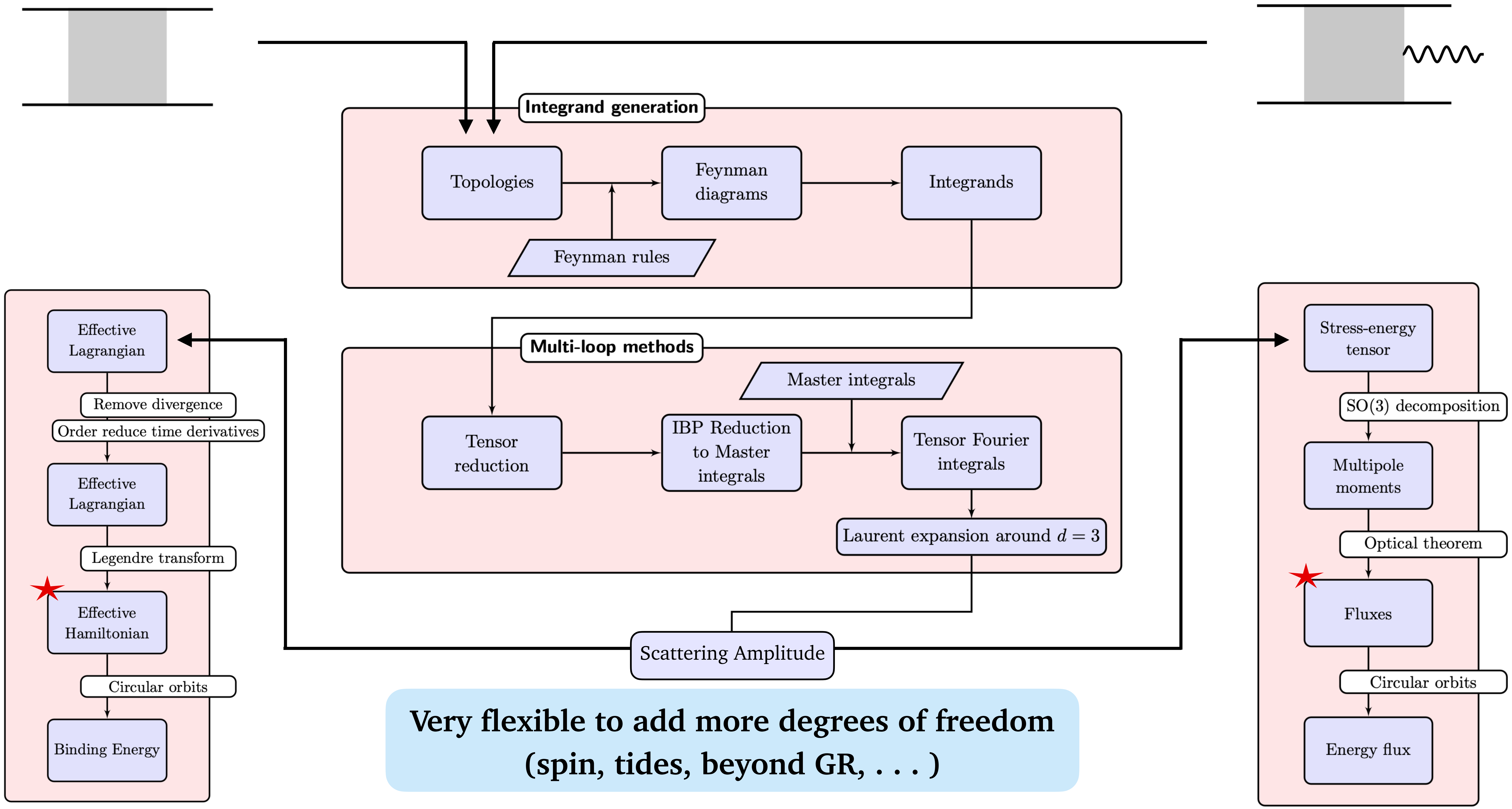
$$\ldots\omega_{\rm orb}\Big)^{2/3}$$

$$+\left[\frac{265978667519}{745113600}-\frac{6848}{105}\gamma_{\rm E}-\frac{3424}{105}\ln(16\,x)+\left(\frac{2062241}{22176}+\frac{41}{12}\pi^2\right)\nu-\frac{133112905}{290304}\nu^2-\frac{3719141}{38016}\nu^3\right]\pi x^{9/2}+\mathcal{O}(x^5)\Big\}$$

[credit: Patil]

PN-GREFT Diagammar & Multi-Loop Automation

Mandal, **Patil**, Steinhoff, & PM



$$E_{\rm pp}(x) = -x\frac{1}{2} + x^2\left\{\frac{3}{8}+\frac{\nu}{24}\right\} + x^3\left\{\frac{27}{16}-\frac{19}{16}\nu+\frac{1}{48}\nu^2\right\} + x^4\left\{\frac{675}{128}+\left(-\frac{34445}{1152}+\frac{205\pi^2}{192}\right)\nu+\frac{155}{192}\nu^2+\frac{35}{10368}\nu^3\right\}+\mathcal{O}(x^5)$$

$$\mathcal{F}_E^{\rm pp} = \left\{\frac{32\nu^2}{5}\right\}x^5 + \left\{-\frac{56\nu^3}{3}-\frac{2494\nu^2}{105}\right\}x^6 + 4\pi\left\{\frac{32\nu^2}{5}\right\}x^{3/2} + \left\{\frac{208\nu^4}{9}+\frac{37084\nu^3}{315}-\frac{89422\nu^2}{2835}\right\}x^7 + \mathcal{O}\left(x^{15/2}\right),$$

Novel Perspective on (Feynman) **Calculus**

What does a (twisted) integral represent?

$$I(n, \mathbf{x}) \equiv \int_{\Gamma} d^n\mathbf{z}\, f(\mathbf{x}, \mathbf{z})$$

vanishing condition
at the boundary of the integration domain

$$f(\mathbf{x}, \mathbf{z})\Big|_{\partial\Gamma} = 0$$

$\Rightarrow$

[Stokes' theorem]

$$\int_{\Gamma} df(\mathbf{x}, \mathbf{z}) = 0 = \int_{\partial\Gamma} f(\mathbf{x}, \mathbf{z})$$

What does a (twisted) integral represent?

Space of **Integrals**
and their (linear and quadratic) relations

$$I(n, \mathbf{x}) \equiv \int_\Gamma d^n\mathbf{z}\, f(\mathbf{x}, \mathbf{z})$$

What does a (twisted) integral represent?

Space of **Integrands**
that differ by a total differential
$g = f + df$

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Space of **Integration Contours**
that differ by boundary terms
$\Sigma = \Gamma + \partial\Gamma$

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Space of (partial) **Differential Operators**
w.r.t. to $\mathbf{z}$ variables
that *annihilate* the **integral**
$L_\mathbf{z}\, f(\mathbf{x}, \mathbf{z}) = 0$

Space of **Integration Contours**
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What does a (twisted) integral represent?

De-Rham Cohomology Group

Space of **Integrands**
that differ by a total differential
$g = f + df$

Integrals Vector Space

Space of **Integrals**
and their (linear and quadratic) relations

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D-Module
(internal variables)

[see Gaia Fontana's talk]

Space of (partial) **Differential Operators**
w.r.t. to $\mathbf{x}$ variable
that *annihilate* the **integral**
$D_{\mathbf{x}}\, I(n, \mathbf{x}) = 0$

D-Module
(external variables)

Space of **Integration Contours**
that differ by boundary terms
$\Sigma = \Gamma + \partial\Gamma$

De-Rham Homology Group

What are the properties of these "spaces" ?

Space Generators / basis **dimensions = # of basis elements**

Integrals Vector Space

Space of **Integrals**
and their (linear and quadratic) relations

Master Integrals (MIs)

# of MIs

De-Rham Cohomology Group H^n_{dR}

Space of **Integrands**
that differ by a total differential
$g = f + df$

Differential forms

$\dim H^n_{\mathrm{dR}} =$ **# of independent forms**

ν

D-Module
(external variables)

Space of (partial) **Differential Operators**
w.r.t. to $\mathbf{x}$ variable
that *annihilate* the **integral**
$D_{\mathbf{x}}\, I(n, \mathbf{x}) = 0$

Standard (Std) monomials

$\mathrm{rank} =$ **# of independent Std**

De-Rham Homology Group $H_{n,\mathrm{dR}}$

Space of **Integration Contours**
that differ by boundary terms
$\Sigma = \Gamma + \partial\Gamma$

Cycles / n-chains

$\dim H_{n,\mathrm{dR}} =$ **# of independent cycles**

What we have found

Vector Space Structure of *Feynman* [- *Euler-Mellin - GKZ - A-hypegeometric*] Integrals

- **Vector decomposition**

$$I = \sum_{i=1}^{\nu} c_i \, J_i$$

Master Integral = basis

$\nu =$ dimension of the vector space

- **Projections**

$$c_i = I \cdot J_i \ , \qquad J_i \cdot J_j = \delta_{ij}$$

- **Completeness**

$$\sum_i J_i \, J_i = \mathbb{I}_{\nu \times \nu}$$

Vector Space Structure of *Feynman* [- *Euler-Mellin - GKZ - A-hypegeometric*] Integrals

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The two questions:

1) what is the vector space dimension ν ?

2) what is the *scalar product* "$\cdot$" between integrals ?

Basics of Intersection Theory

Basics of Intersection Theory / De Rham Twisted Co-Homology Groups

Aomoto, Brown, Cho, Goto, Kita, Matsubara-Heo, Mazumoto, Mimachi, Mizera, Ohara, Yoshida,…

Consider an integral I over the variables $\mathbf{z} = (z_1, z_2, \ldots, z_m)$

$$I = \underbrace{\int_{\mathcal{C}} u(\mathbf{z})}_{\text{twisted cycle}} \; \underbrace{\varphi_m(\mathbf{z})}_{\text{twisted cocycle}}$$

$u(\mathbf{z})$ is a multivalued function

$u(\partial\mathcal{C}) = 0$

$\varphi_m(\mathbf{z})$ is a differential m-form

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- **Gamma function**: $u = z^{s-1}\, e^{-z}\,, \qquad \mathcal{C} = (0, \infty) \qquad \varphi_1 = \dfrac{dz}{z^{n_1}}$
- **Euler Beta function**: $u = z^{a}\,(z-1)^{b}\,, \qquad \mathcal{C} = (0, 1) \qquad \varphi_1 = \dfrac{dz}{z^{n_1}\,(z-1)^{n_2}}$
- **$_2F_1$ Hypergeometric**: $u = z^{a}\,(z-1)^{b}\,(z-1/x)^{c}\,, \qquad \mathcal{C} = (0, 1) \qquad \varphi_1 = \dfrac{dz}{z^{n_1}\,(z-1)^{n_2}\,(z-1/x)^{n_3}}$
- **$_pF_q$ and many more**: …… …… ……

$n_i \in \mathbb{Z}$

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$u(\partial\mathcal{C}) = 0$

$\varphi_m(\mathbf{z})$ is a differential m-form

- **The dawn of Integration by parts identities:**
  - **Equivalence Classes of DIFFERENTIAL FORMS**

    There could exist many forms φ_m that upon integration give the same result I
  - **Equivalence Classes of INTEGRATION CONTOURS**

    There could exist many contours $\mathcal{C}$ that do not alter the the result of I

Vector Space Structure of Twisted Period Integrals

Basics of Intersection Theory / De Rham Twisted Co-Homology Groups

- **Integral invariance** from the **vanishing of total differential**
- **Stokes' theorem** relating the invariance upon shifting the differential forms to the invariance upon contour deformation!

$$0 = \int_C d(u\,\varphi) = \int_{\partial C} u\,\varphi$$

$$\int_C u\,\varphi = \int_C u\,(\varphi + \nabla_\omega \phi) = \int_{C+\partial\Gamma} u\,\varphi$$

- **Covariant Derivative** $\nabla_\omega \equiv d + \omega\wedge \ \equiv u^{-1}\cdot d\cdot u$ $\quad \omega \equiv d\log u$

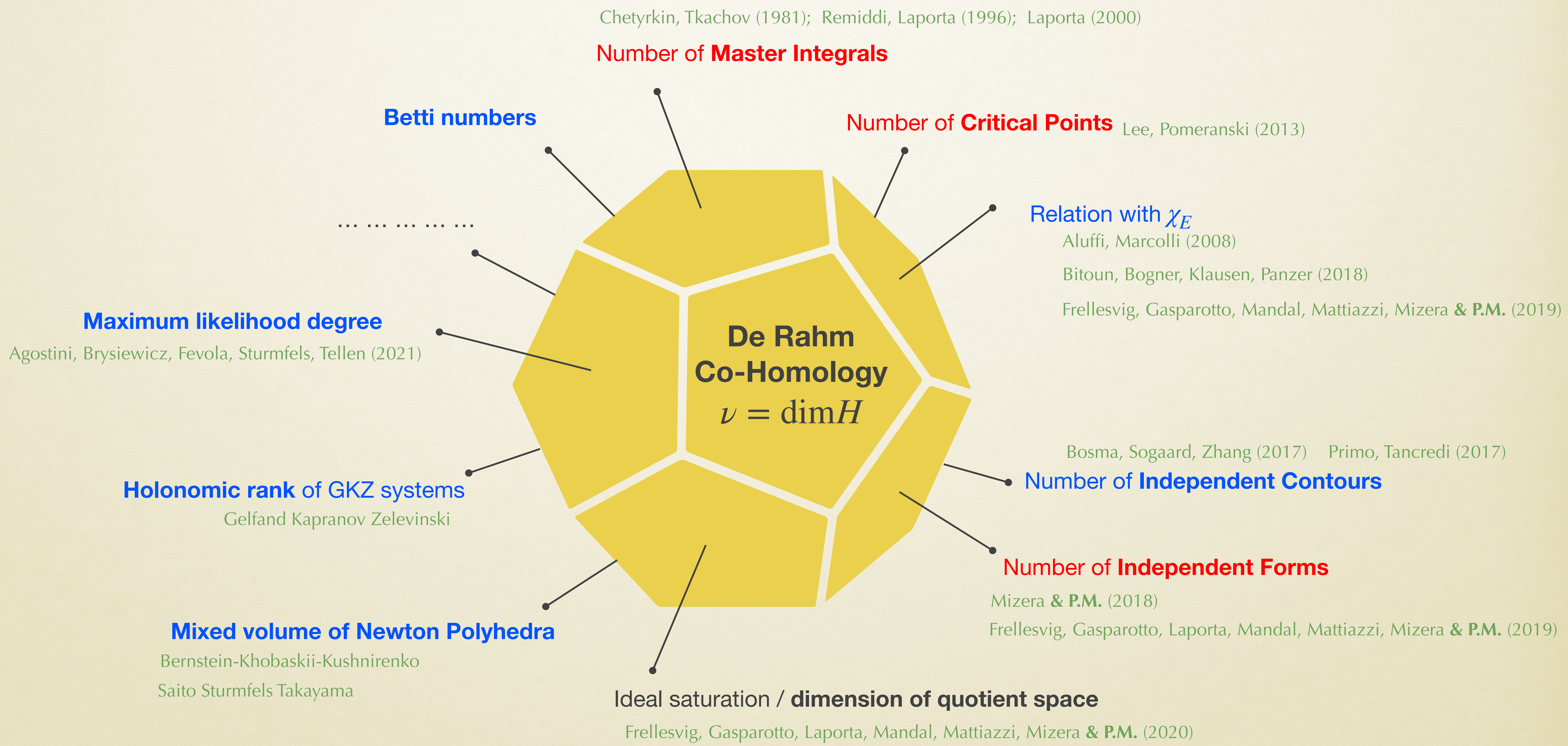
$$u \to u^{-1}$$

$$0 = \int_C d(u^{-1}\varphi) = \int_{\partial C} u^{-1}\varphi$$

$$\int_C u^{-1}\varphi = \int_C u^{-1}(\varphi + \nabla_{-\omega}\phi) = \int_{C+\partial\Gamma} u^{-1}\varphi$$

- **Dual Covariant Derivative** $\nabla_{-\omega} \equiv d - \omega\wedge \ \equiv u\cdot d\cdot u^{-1}$

Vector Space Dimensions / counting “holes”



Identity Resolution

$$\dim H^n_{\pm\omega} = \dim H_n^{\pm\omega} \equiv \nu$$

- **Cohomology Space**

[vector space of differential forms]

Cohomology basis

$\langle e_i| \in H^n_\omega$

Dual Cohomology basis

$|h_i\rangle \in H^n_{-\omega}$

$i = 1, \ldots, \nu$

Identity resolution

$$\mathbb{I}_c = \sum_{i,j=1}^{\nu} |h_i\rangle \left(\mathbf{C}^{-1}\right)_{ij} \langle e_j|$$

Metric matrix for Forms

$$\mathbf{C}_{ij} \equiv \langle e_i | h_j \rangle$$

Identity Resolution

$$\dim H^n_{\pm\omega} = \dim H_n^{\pm\omega} \equiv \nu$$

- **Cohomology Space**

[vector space of differential forms]

Cohomology basis

$\langle e_i| \in H^n_{\omega}$

Dual Cohomology basis

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Metric matrix for Forms

$$\mathbf{C}_{ij} \equiv \langle e_i|h_j\rangle$$

- **Homology Space**

[vector space of integration contours]

Homology basis

$|\gamma_i] \in H_n^{\omega}$

Dual Homology basis

$[\eta_i| \in H_n^{-\omega}$

$i = 1, \ldots, \nu$

Identity resolution

$$\mathbb{I}_h = \sum_{i,j=1}^{\nu} |\gamma_i] \left(\mathbf{H}^{-1}\right)_{ij} [\eta_j|$$

Metric Matrix for Contours

$$\mathbf{H}_{ij} \equiv [\eta_i|\gamma_j]$$

Linear Relations

Linear Relations / IBPs identity

Mizera & P.M. (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2019)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera & P.M. (2019)

- **Master Integrals from Master Forms**

Consider a set of ν MIs,

$$J_i = \int_{\mathcal{C}_R} u(\mathbf{z})\, e_i(\mathbf{z}) = \langle e_i | \mathcal{C}_R] \,, \qquad i = 1, \ldots, \nu \,,$$

- **Integral Decomposition**

$$I = \int_{\mathcal{C}_R} u(\mathbf{z})\ \varphi_L(\mathbf{z}) = \langle \varphi_L | \mathcal{C}_R] = \sum_{i=1}^{\nu} c_i\, J_i$$

- **Decomposition of Differential Forms**

$$\langle \varphi_L | \;=\; \langle \varphi_L | \mathbb{I}_c = \langle \varphi_L | \underbrace{\sum_{i,j=1}^{\nu} | h_i \rangle \left(\mathbf{C}^{-1} \right)_{ij} \langle e_j |}_{=1}$$

Linear Relations / IBPs identity

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$$I = \int_{\mathcal{C}_R} u(\mathbf{z})\; \varphi_L(\mathbf{z}) = \langle \varphi_L | \mathcal{C}_R] = \sum_{i=1}^{\nu} \boxed{c_i} J_i$$

- **Decomposition of Differential Forms**

- **Master Decomposition Formula**

$$\langle \varphi_L | \quad = \quad \langle \varphi_L | \mathbb{I}_c = \sum_{i=1}^{\nu} c_i \, \langle e_i | \,, \quad \text{with} \qquad \boxed{c_i} = \sum_{j=1}^{\nu} \langle \varphi_L | h_j \rangle \left(\mathbf{C}^{-1} \right)_{ji}$$

coefficients depend on the basis choice
but **do not depend** on the dual basis choice

Quadratic Relations

Riemann Bilinear Relations

Riemann bilinear relations for periods of closed holomorphic (non-twisted) differentials forms

$$\langle \phi_L | \phi_R \rangle = \int_\Sigma \phi_L \wedge \phi_R \;\; = \;\; \sum_{i=1}^{g} \left(\int_{a_i} \phi_L \int_{b_i} \phi_R - \int_{b_i} \phi_L \int_{a_i} \phi_R \right)$$

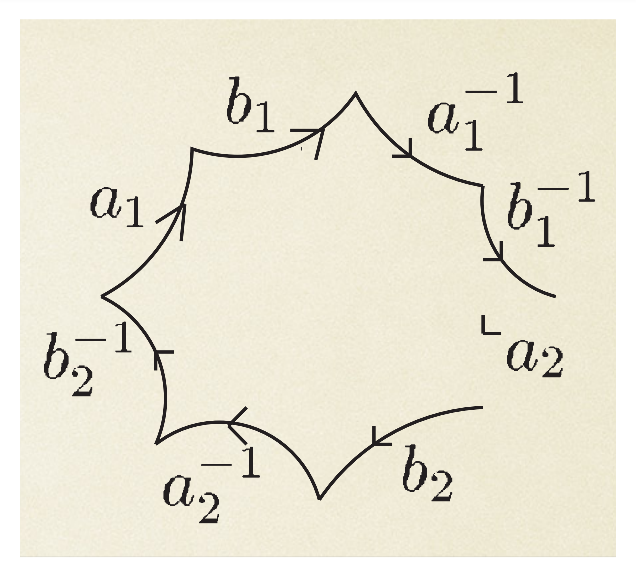
where Σ is an oriented Riemann surface of genus $g > 0$, built out of a $4g$-gon with edges $\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1}$ (where the exponent ± 1 stands for clock/anticlockwise orientation) and gluing each edge with its inverse. The integration contours a_i and b_i, for $i = 1, \ldots g$, are a canonical bases of cycles, hence intersect *transversally*, *i.e.* their pairwise intersection numbers are: $a_i \cdot a_j = b_i \cdot b_j = 0$, and $a_i \cdot b_j = -b_j \cdot a_i = \delta_{ij}$. Riemann bilinear relation can be cast as,

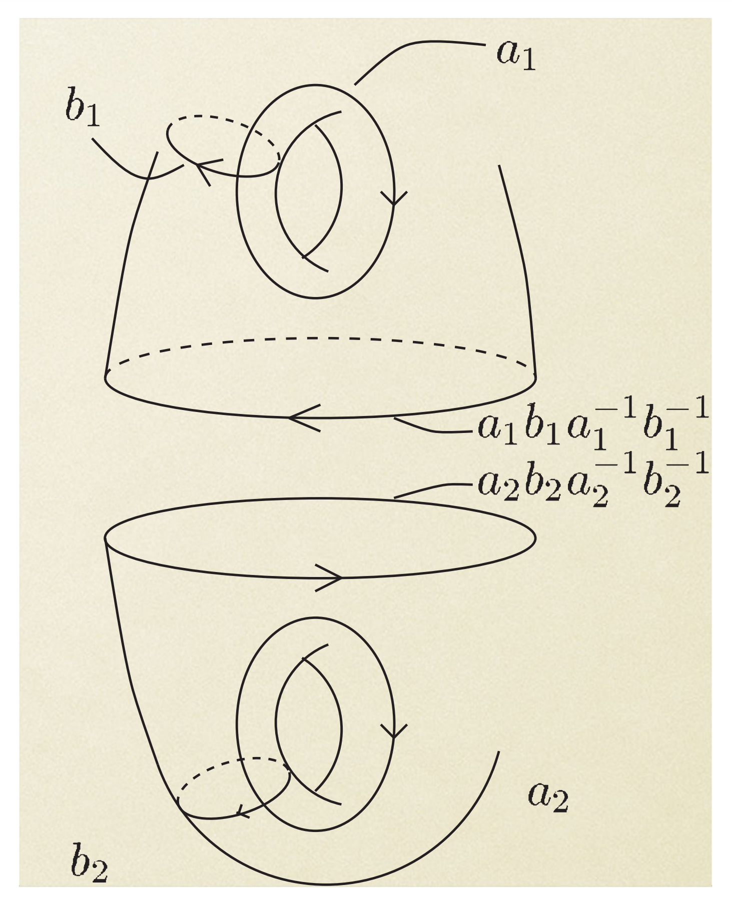
$$\langle \phi_L | \phi_R \rangle = \sum_{i,j}^{2g} \int_{\gamma_i} \phi_L \;\; (\mathbf{H}^{-1})_{ij} \int_{\gamma_j} \phi_R \; ,$$

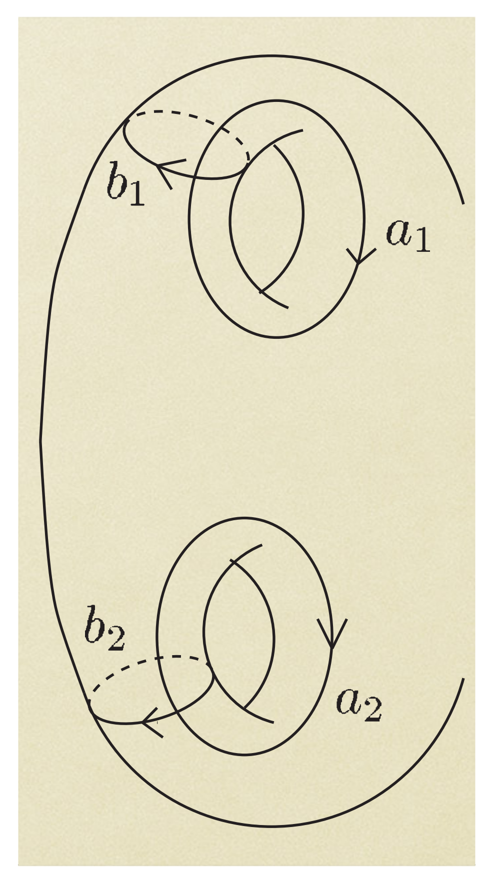
where $\{\gamma_i\}_{i=1,\ldots,g} = a_i$ and $\{\gamma_i\}_{i=g+1,\ldots,2g} = b_i$, and $\mathbf{H}_{ij} = [\gamma_i | \gamma_j]$, namely

$$\mathbf{H} = \begin{pmatrix} 0 & \mathbb{I}_{g\times g} \\ -\mathbb{I}_{g\times g} & 0 \end{pmatrix} , \qquad \text{yielding} \quad \mathbf{H}^{-1} = \begin{pmatrix} 0 & -\mathbb{I}_{g\times g} \\ \mathbb{I}_{g\times g} & 0 \end{pmatrix} ,$$

and $\mathbb{I}_{g\times g}$ is the identity matrix in the $(g \times g)$-space.







Pics: Jost, *Compact Riemann Surfaces*

Twisted Riemann Periods Relations (TRPR)

Cho, Matsumoto (1995)

[Integrals] [(dual) Integrals]

$$\langle \varphi_L | \varphi_R \rangle \;=\; \langle \varphi_L | \mathbb{I}_h | \varphi_R \rangle = \sum_{i,j=1}^{\nu} \langle \varphi_L | \gamma_i] \left(\mathbf{H}^{-1} \right)_{ij} [\eta_j | \phi_R \rangle = \sum_{i,j}^{\nu} \int_{\gamma_i} u\, \varphi_L \left(\mathbf{H}^{-1} \right)_{ij} \int_{\eta_j} u^{-1}\, \varphi_R$$

$=1$

$$[\mathcal{C}_L | \mathcal{C}_R] \;=\; [\mathcal{C}_L | \mathbb{I}_c | \mathcal{C}_R] = \sum_{i,j=1}^{\nu} [\mathcal{C}_L | h_i \rangle \left(\mathbf{C}^{-1} \right)_{ij} \langle e_j | \mathcal{C}_R] = \sum_{i,j}^{\nu} \int_{\mathcal{C}_L} u^{-1} h_i \left(\mathbf{C}^{-1} \right)_{ij} \int_{\mathcal{C}_R} u\, e_j$$

$=1$

[(dual) Integrals] [Integrals]

Generalising Riemann Bilinear Relations

A unique framework for:

Linear Relations

- Gauss contiguity relations (Twisted Period Integrals)
- Integration by parts identities (Feynman Integrals)
- Differential Equations
- Dimension-shift relations (Feynman Integrals)
- Finite difference Equations (Twisted Period Integrals)

Quadratic Relations

- Riemann Twisted Periods Relations
- KLT relations (Gravity vs Gauge-theory Amplitudes)
- Relations for Closed- vs Open-String Theory Amplitudes

…& more

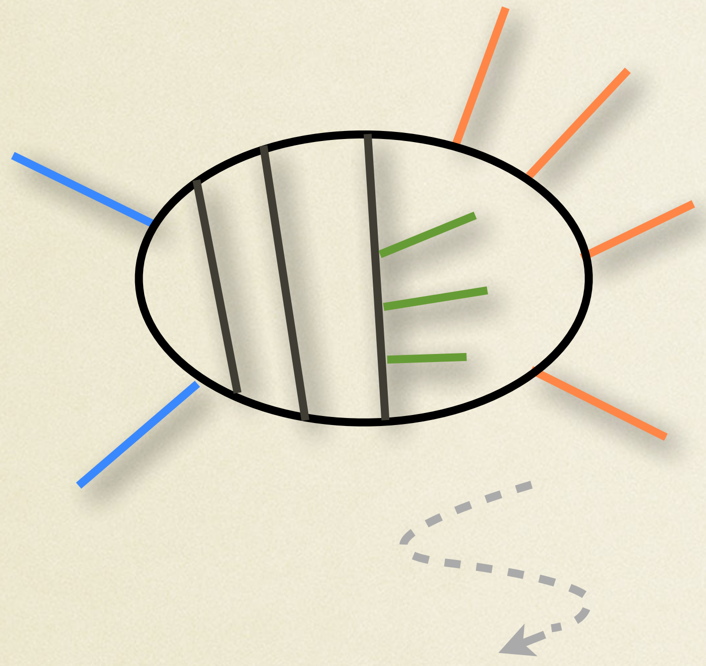
Vector Space Structure of Feynman Integrals

Parametric Representation(s)

Mizera **& P.M.** (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera **& P.M.** (2019, 2020)

- **Upon a change of integration variables**



$$I^{[d]}_{a_1,\ldots,a_N} = \int_{\mathcal{C}} u(\mathbf{z}) \;\; \varphi_N(\mathbf{z})$$

$\varphi_N(\mathbf{z}) = \hat{\varphi}(\mathbf{z}) d^N\mathbf{z}$ **differential N-form**

$$d^N\mathbf{z} = dz_1 \wedge \ldots \wedge dz_N$$

$$\hat{\varphi}_N(\mathbf{z}) = f(\mathbf{z}) \prod_i z_i^{-a_i}$$

$$u(\mathbf{z}) = \mathcal{P}(\mathbf{z})^\gamma$$

$\mathcal{P}(\mathbf{z}) =$ **graph-Polynomial**

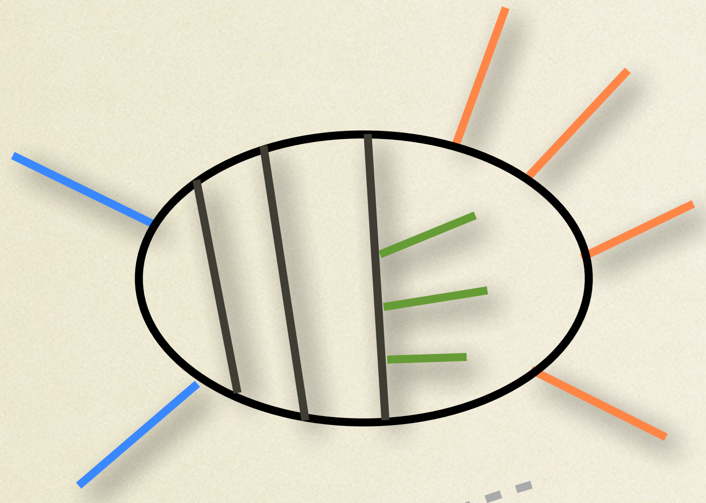
$\gamma(d) =$ **generic exponent**

Feynman Integrals :: *Baikov Representation*

Mizera & P.M. (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & P.M. (2019, 2020)

- **Denominators as integration variables** Baikov (1996) Frellesvig and Papadopoulos (2017)



N-denominator
generic Integral

$$\{D_1, \ldots, D_N\} \to \{z_1, \ldots, z_N\} \equiv \mathbf{z}$$

$$I^{[d]}_{a_1,\ldots,a_N} = \int_{\mathcal{C}} B(\mathbf{z})^{\gamma} \frac{d^n \mathbf{z}}{z_1^{a_1} z_2^{a_2} \cdots z_N^{a_N}}$$

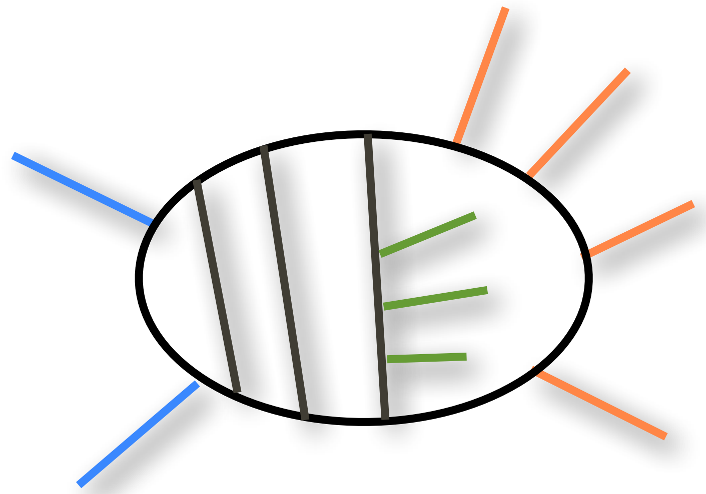
$$B(p_i, k_j) = \begin{vmatrix} k_1^2 & \ldots & (k_1 \cdot p_{E-1}) \\ \vdots & \ddots & \vdots \\ (p_{E-1} \cdot k_1) & \ldots & p_{E-1}^2 \end{vmatrix} = B(\mathbf{z}) \qquad \gamma \equiv (d - E - L - 1)/2$$

Gram determinant

Vector Space of Feynman Integrals

Mizera & **P.M.** (2018)

Frellesvig, Gasparotto, Laporta, Mandal, Mattiazzi, Mizera & **P.M.** (2019, 2020)



$$I^{[d]}_{a_1,\ldots,a_N} = \int_{\mathcal{C}} B(\mathbf{z})^{\gamma} \frac{d^n \mathbf{z}}{z_1^{a_1} z_2^{a_2} \cdots z_N^{a_N}}$$

$$u = B^{\gamma},$$ twist

$$\gamma \equiv (d - E - L - 1)/2$$

- **Vector space dimension**

$$\omega \equiv \sum_{i=1}^{n} \hat{\omega}_i \, \mathrm{d}z_i = \mathrm{d}\log(u) \qquad \mathbb{Z}_\omega = \{\text{zeroes of } \omega\} \qquad \mathbb{P}_\omega = \{\text{poles of } \omega\} \cup \{\infty\}$$

$$\nu \equiv \dim(H^n_{\pm\omega}) = \dim(\mathbb{Z}_\omega) = (-1)^n (n + 1 - \chi(\mathbb{P}_\omega)) = \text{number of solutions of the system} \begin{cases} \omega_1 = 0 \\ \vdots \\ \omega_n = 0 \end{cases}$$

(Zero-dimensional)

- **(dual) bases choices:** Master Forms for Master Integrals

$$\langle e_i | \in H^n_{\omega} \qquad | h_i \rangle \in H^n_{-\omega} \qquad i = 1, \ldots, \nu$$

- **Decomposing Forms for Decomposing Integrals**

$$\langle \varphi | = c_1 \langle e_1 | + c_2 \langle e_2 | + c_3 \langle e_3 | + \ldots + c_\nu \langle e_\nu | \qquad c_i = \sum_{j=1}^{\nu} \langle \varphi_L | h_j \rangle \left(\mathbf{C}^{-1} \right)_{ji}$$

Intersection Numbers for 1-forms

Intersection Numbers for 1-forms

- **Calculus and Differential Forms** two closed forms $\varphi_1 \,, \varphi_2$

$$\langle \varphi_1 | \varphi_2 \rangle = \frac{1}{2\pi i} \int_X \varphi_1 \wedge \varphi_2 = \frac{1}{2\pi i} \int_X d\Omega = \frac{1}{2\pi i} \int_{\partial X} d\Omega = \sum_{p \in \text{Poles}} \text{Res}_{z=p}\big(\Omega\big) \qquad d\psi_1 = \varphi_1 \qquad \Omega \equiv \psi_1 \varphi_2$$

- **Intersection Number for twisted cocycles (1-form)** Cho, Matsumoto (1996)

Zeroes and Poles of $\omega \equiv d\log(u) = \gamma d\log(B)$ $\quad \nu = \text{number of critical points} \in Z(\omega)$ $\quad P(\omega) = \{\text{poles of } \omega, \text{including } \infty\}$

$$\varphi_1 \,, \varphi_2 \in H^n_{\text{dR}}$$

$$\varphi_1 \equiv u\,\varphi_L \,, \qquad \varphi_2 \equiv u^{-1}\,\varphi_R \qquad \psi_1 \equiv u\,\psi_L$$

$$\langle \varphi_L | \varphi_R \rangle = \frac{1}{2\pi i} \int_X (u\varphi_L) \wedge (u^{-1}\varphi_R) = \frac{1}{2\pi i} \sum_{z_i \in P(\omega)} \oint_{\gamma_i} \psi_i\, \varphi_R = \sum_{z_i \in P(\omega)} \text{Res}_{z=z_i}\big(\psi_i\, \varphi_R\big)$$

$$\nabla_\omega \psi_i = \varphi_L \,, \quad \text{for } z \to z_i \in P(\omega)$$

Intersection Numbers for n-forms :: Methods

Intersection Numbers for n-forms (I)

Ohara (1998) Mizera (2019)

Frellesvig, Gasparotto, Mandal, Mattiazzi, Mizera **& P.M.** (2019)

Intersection Numbers for **n**-forms :: Recursive Formula

$$\langle \varphi_L^{(\mathbf{n})} | \varphi_R^{(\mathbf{n})} \rangle = \sum_{i,j} \langle \varphi_L^{(\mathbf{n})} | \underbrace{h_j^{(\mathbf{n-1})} \rangle (C_{(\mathbf{n-1})})^{-1}_{ji} \langle e_i^{(\mathbf{n-1})}}_{=1} | \varphi_R^{(\mathbf{n})} \rangle = \sum_{i,j} \langle \varphi_{L,i}^{(n)} | (C_{(\mathbf{n-1})})_{ij} \, \varphi_{R,j}^{(n)} \rangle$$

- **+ Global Residue Thm** Weinzierl (2020)
- **+ Polynomial Division** **Fontana** Peraro (2023)
- **+ special cohomology basis** **Crisanti, Smith** (2024)
- **+ Relative cohomology** CaronHuot Pokraka (2019-2021) **Fontana** Peraro (2023) **Brunello, Chestnov, Crisanti**, Frellesvig, Mandal **& P.M.** (2023)

Intersection Numbers for n-forms (II)

Matsumoto (1998)

Chestnov, Frellesvig, Gasparotto, Mandal **& P.M.** (2022)

$$\langle \varphi_L^{(\mathbf{n})} \,|\, \varphi_R^{(\mathbf{n})} \rangle = (2\pi \mathrm{i})^{-n} \int_X (u\, \varphi_{L,c}^{(\mathbf{n})}) \wedge (u^{-1} \varphi_R^{(\mathbf{n})}) = \sum_{p \in \mathbb{P}_\omega} \mathrm{Res}_{z=p}(\psi\, \varphi_R^{(\mathbf{n})})$$

- **nPDE**

$$\boxed{\nabla_{\omega_1} \nabla_{\omega_2} \ldots \nabla_{\omega_n} \psi = \varphi_L^{(\mathbf{n})}}$$

Intersection Numbers for n-forms (III)

Brunello, Chestnov, & P.M. (2024)

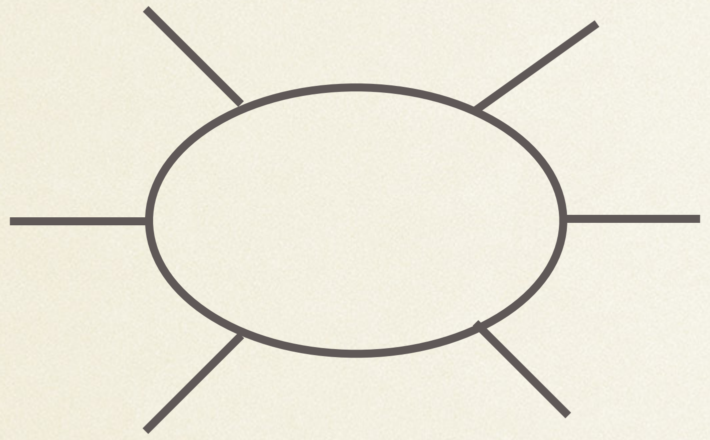
- **Rescursive formula + *Companion Tensor Algebra***

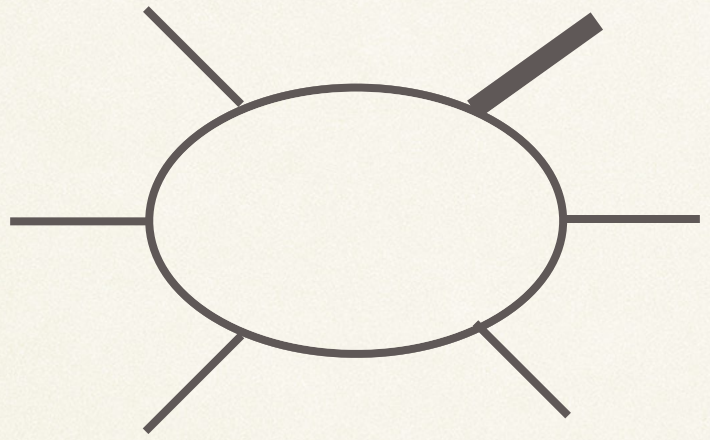
Feynman Integrals Decomposition / Projection

Complete decomposition @ 1- & 2-Loop

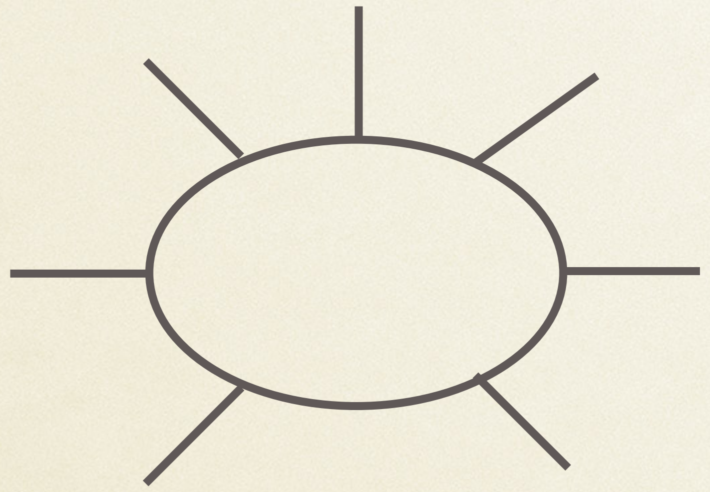
Brunello, Chestnov, **Crisanti**, Frellesvig, Gasparotto, Mandal & P.M. (2023)

☑ **1-Loop 6-point**

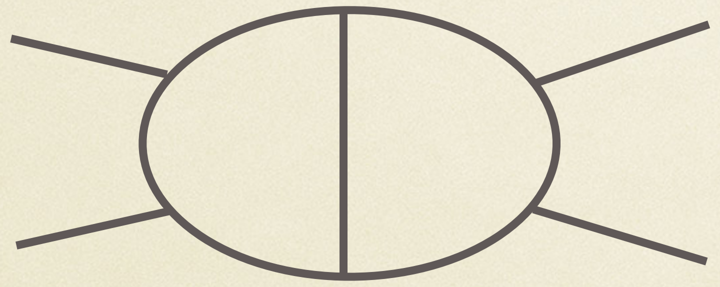




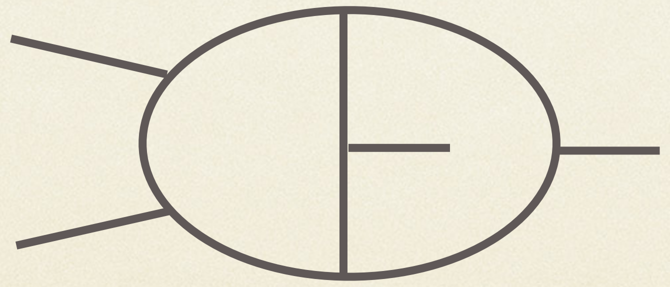
☑ **1-Loop 7-point**



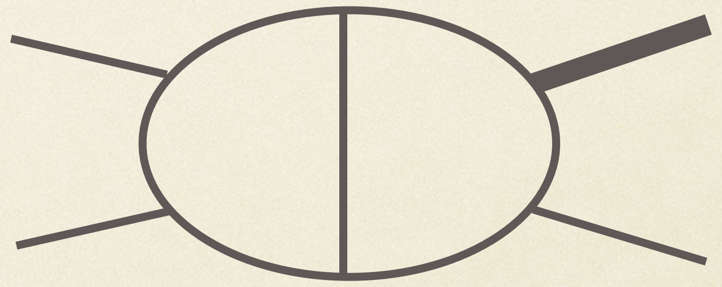
☑ **2-loop 4-point**



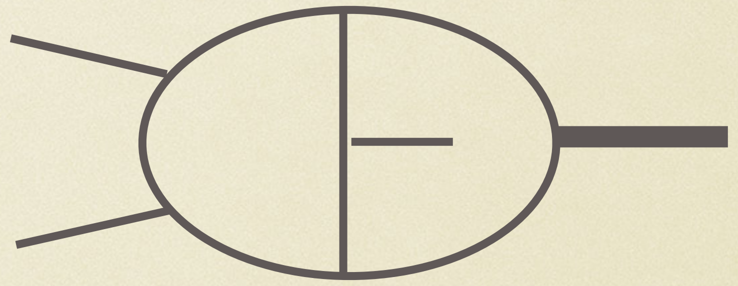
planar diagram



non-planar diagram

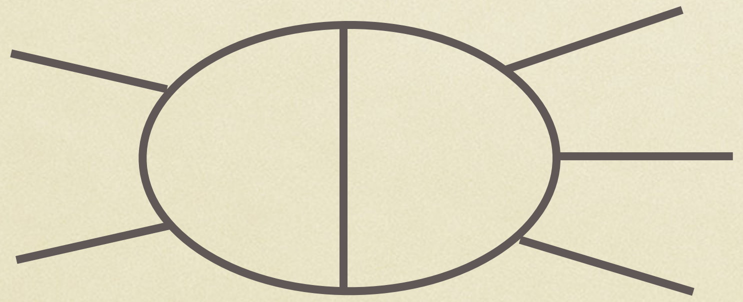


planar diagram



non-planar diagram

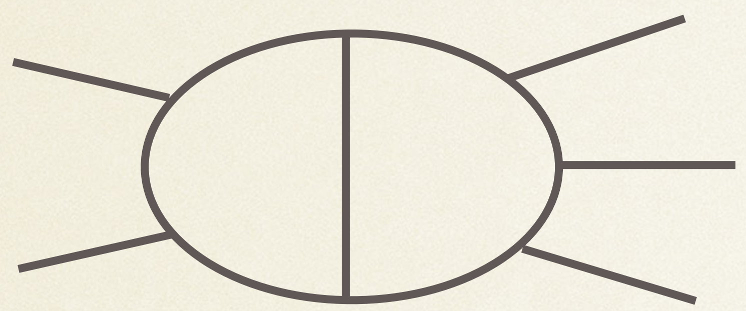
☑ **2-loop 5-point**



planar diagram

Complete decomposition @ 1- & 2-Loop

☑ **2-loop 5-point**

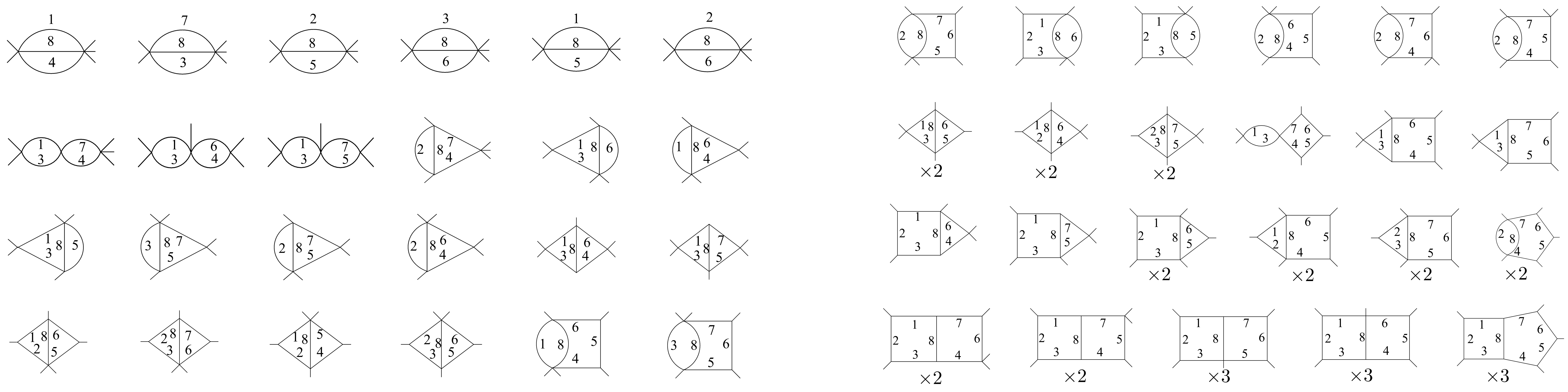


Brunello, **Chestnov**, **Crisanti**, Frellesvig, Gasparotto, Mandal & P.M. (2023)

Brunello, **Chestnov**, & P.M. (2024)

$$I_{a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10} a_{11}} = \int \mathrm{d}^{11} z \; u(\mathbf{z}) \frac{z_9^{-a_9} \, z_{10}^{-a_{10}} \, z_{11}^{-a_{11}}}{z_1^{a_1} \, z_2^{a_2} \, z_3^{a_3} \, z_4^{a_4} \, z_5^{a_5} \, z_6^{a_6} \, z_7^{a_7} \, z_8^{a_8}}$$

- **62 MIs and 47 sectors**



☑ (Numerical) decomposition **up to degree-20**

Intersections Numbers beyond Feynman Integrals

Intersections Numbers @ QM and QFT

Cacciatori **& P.M.** (2022)

- **Mapping integrals to Twisted Period Integrals**
- **Applying Intersection Theory**

Orthogonal Polynomials and Matrix Elements in QM

Case i) $$I_{nm} \equiv \int_\Gamma P_n(z)\, P_m(z)\, f(z)\, dz\,,$$

Laguerre, Legendre, Tchebishev, Gegenbauer, Hermite

Case ii) $$I_{nm} \equiv \langle n|\mathscr{O}|m\rangle = \int_\Gamma \psi_n^*(z)\, \mathscr{O}(z)\, \psi_m(z)\, f(z)\, dz$$

Harmonic oscillator, H-atom

- **Master Decomposition formula**

For the considered cases, we obtain:

$$\varphi = c_1\, e_1\,,$$ in terms of just one basic form, $e_1 = dz$

corresponding to:

$$I_{nm} = c_1\, E_1$$ (one master integral)

Orthogonality-like integrals and **matrix elements in QM** belong to a **finite dimensional vector space**

Green's Function and Kontsevich-Witten tau-function

Case iii) $$G_n \equiv \frac{\int \mathscr{D}\phi\, \phi(x_1)\cdots\phi(x_n)\, \exp[-S_E]}{\int \mathscr{D}\phi\, \exp[-S_E]}$$

Weinzierl (2020)
Gasparotto, Weinzierl (2022)

Case iv) $$Z_{KW} \equiv \frac{\int d\Phi \exp\left[-\mathrm{Tr}\left(-\frac{i}{3!}\Phi^3 + \frac{\Lambda}{2}\Phi^2\right)\right]}{\int d\Phi \exp\left[-\mathrm{Tr}\left(\frac{\Lambda}{2}\Phi^2\right)\right]}$$

Cacciatori & **P.M.**(2022)

$$c_1 = \frac{\int_\Gamma \mu\, \varphi}{\int_\Gamma \mu\, e_1}\,,$$ equivalently rewritten as $$\int_\Gamma \mu\, \varphi = c_1\, E_1$$

- **Master Decomposition formula**

- **Toy models univariate integrals**

Green's functions and correlators in QFT are determined by **intersection numbers**

"Path integrals" belong to a **finite dimensional vector space**

Intersection Numbers @ Fourier Integrals

Brunello, Crisanti, Giroux, Smith & P.M. (2023)

Fourier integrals from Intersection Theory

Brunello, Crisanti, Giroux, Smith & P.M. (2023)

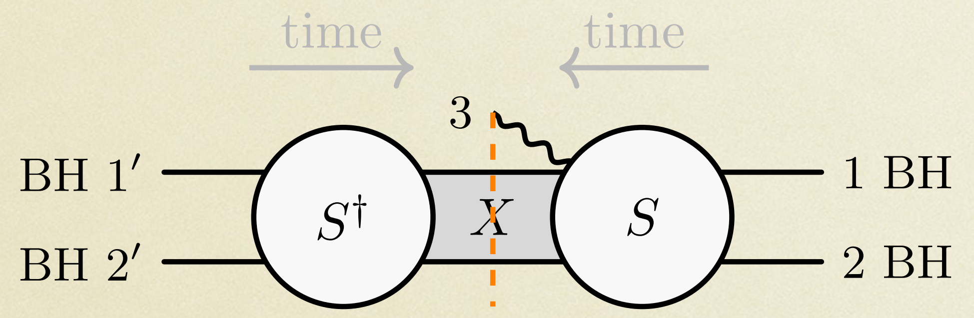
- **Fourier integrals in Baikov representation as twisted periods**

$$\tilde{f}(\{x_i\}) = \int f(\{q_i\}) \prod_{j=1}^{L} \mathrm{e}^{i q_j \cdot x_j} \frac{\mathrm{d}^{\mathrm{D}} q_j}{(2\pi)^{\mathrm{D}/2}} = \int_{C_R} u(\mathbf{z})\, \varphi_L(\mathbf{z}) \qquad u(\mathbf{z}) = \kappa\, \mathrm{e}^{i g(\mathbf{z})} B(\mathbf{z})^{\frac{\mathrm{D}-L-E-1}{2}}$$

- **Application-1: Feynman propagator in position-space**

$$I_n = \int_{\mathcal{M}} \mathrm{d\!\!{}^-}^{\mathrm{D}} q \frac{\mathrm{e}^{i q \cdot x}}{(q^2 + m^2 - i\varepsilon)^n}$$

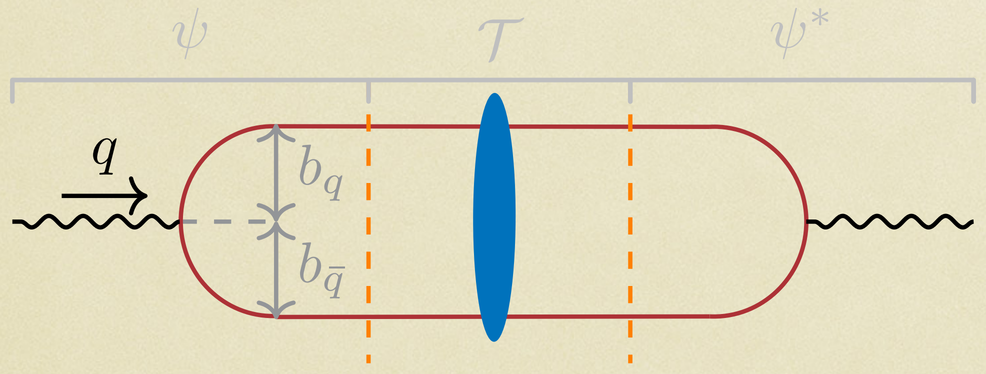
- **Application-2: Spectral gravitation wave form in KMOC formalism**

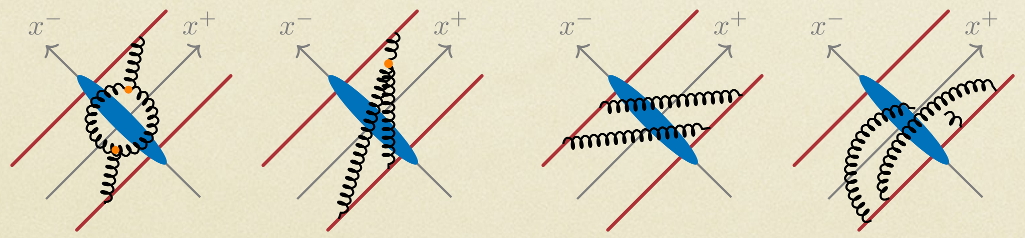


$$\mathrm{Exp}_3 = {}_{\mathrm{in}}\langle 2'1'|S^\dagger a_3 S|12\rangle_{\mathrm{in}}$$

$$\mathcal{I}^{\nu_{2m}}_{\beta_1\beta_2} = \int_{\mathcal{M}} \mathrm{d\!\!{}^-}^{\mathrm{D}} q \frac{\delta(u_1\cdot q)\delta(u_2\cdot(q-k))q^{\nu_1}\ldots q^{\nu_{2m}}\mathrm{e}^{-iq\cdot b}}{[q^2 - i\varepsilon]^{\beta_1}[(q-k)^2 - i\varepsilon]^{\beta_2}}$$

- **Application-3: QCD Color Dipole Scattering and Balitski-Kovchegov Equations**





$$I^{ij} = \int_{\mathbb{R}^{2\mathrm{D}}} \mathrm{d\!\!{}^-}^{\mathrm{D}} q_1 \mathrm{d\!\!{}^-}^{\mathrm{D}} q_2 \frac{N_I^{ij}(q_1,q_2)\mathrm{e}^{i(q_1\cdot x_1 + q_2\cdot x_2)}}{q_1^2(q_1^2\tau + q_2^2)}$$

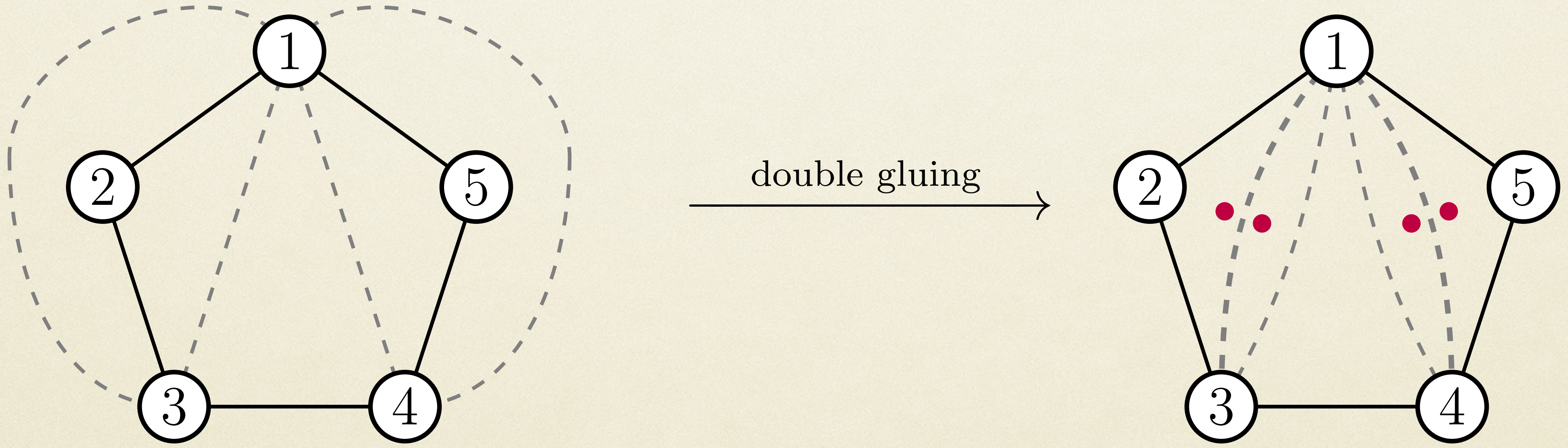
$$G^{ij} = \int_{\mathbb{R}^{2\mathrm{D}}} \mathrm{d\!\!{}^-}^{\mathrm{D}} q_1 \mathrm{d\!\!{}^-}^{\mathrm{D}} q_2 \frac{N_G^{ij}(q_1,q_2)\mathrm{e}^{i(q_1\cdot x_1 + q_2\cdot x_2)}}{(q_1+q_2)^2(q_1^2\tau + q_2^2)}$$

$$N_I^{ij} = q_1^i q_2^j\,,$$

$$N_G^{ij} = \delta^{ij}(q_1^2 - q_2^2) - \frac{2q_1^i(q_1+q_2)^j}{u} + \frac{2(q_1+q_2)^i q_2^j}{u\tau}$$

Intersection Numbers @ Gluing Method in N=4 SYM

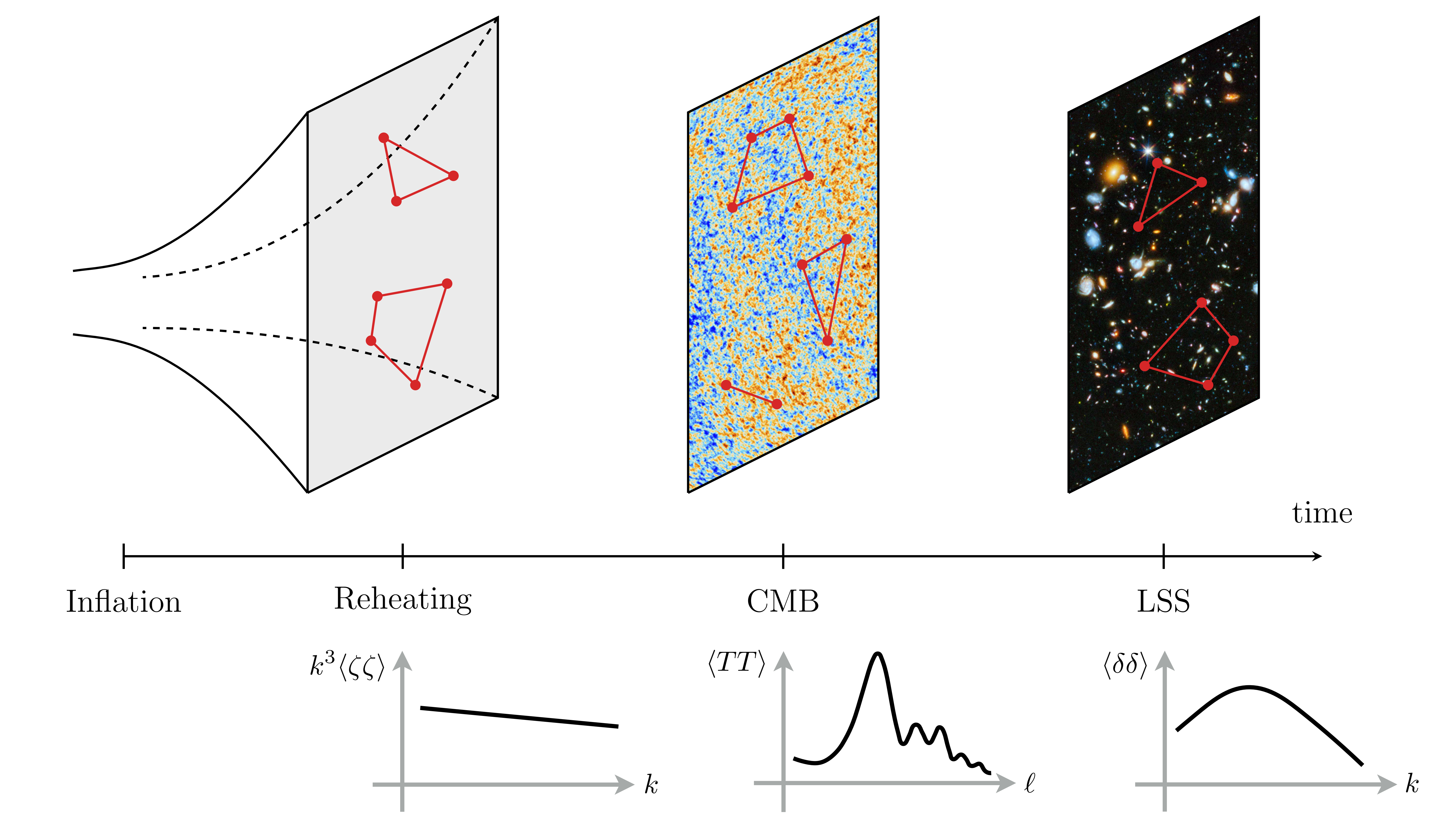
Crisanti, Eden, Gtottwald, Scherdin & P.M. (2024)



Intersection Numbers @ Cosmological Integrals

Benincasa, Brunello, Mandal, Vazão, & PM (2024)

Cosmological Correlators and Wavefunctions



[credit: Joyce]

- Initial conditions for structures in our universe
- Physics of Inflation
- Quantum Field Theory in Curved Spacetime

Cosmological Correlators and Wavefunctions

Arkani-Hamed, Benincasa, Postnikov
Arkani-Hamed, Baumann, Hillmann, Joyce, Lee, Pimentel
Benincasa, Vazao
(see Beinincasa et al.'s review)

- **Toy-model:** conformally coupled scalar field (with polynomial self-interactions),

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}(\partial\phi)^2 - \frac{1}{12}R\phi^2 - \sum_{p>2}\frac{\lambda_p}{p!}\phi^p\right]$$

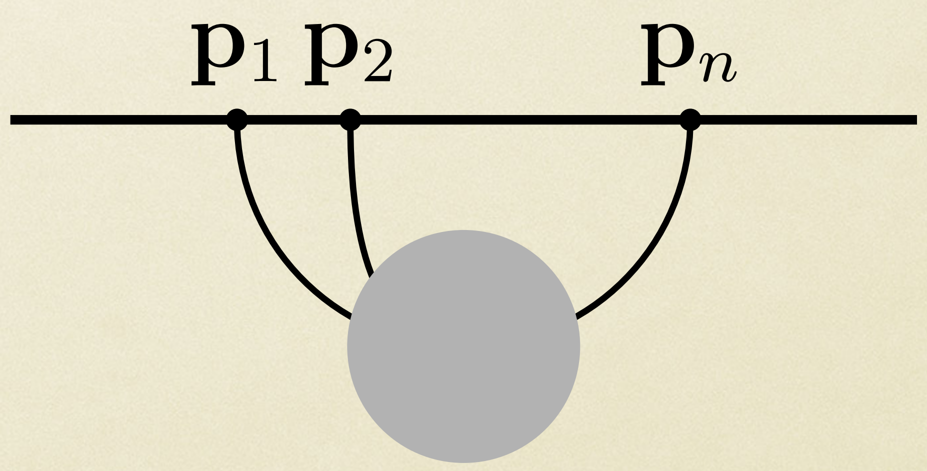
- **Goal:** correlation functions in an FRW cosmology $a(\eta) = (\eta/\eta_0)^{-(1+\varepsilon)}$

$$\Psi_{\text{FRW}}(E_v, E_I) = \int_0^\infty \prod_v d\omega_v \left(\prod_v \omega_v\right)^\varepsilon \Psi_{\text{flat}}(E_v + \omega_v, E_I)$$

rational function of E_v and E_I
("energies" associated with the vertices and the internal edges)

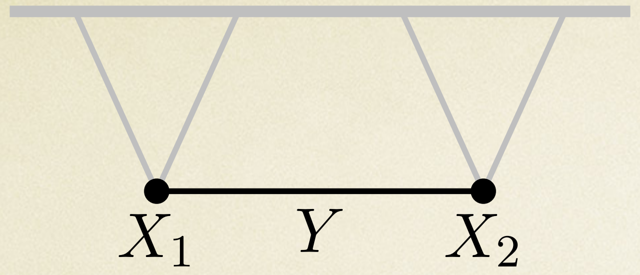
- **Twisted period integrals**

$$I(C, D; n; \varepsilon) = \int_0^\infty dx_1 \cdots dx_m\, P(x) \prod_I (C_{Ij}x_j + D_I)^{-n_I+\varepsilon_I} \quad =$$



The cosmological wavefunction satisfies a differential equation, which governs how it changes as the external kinematics are varied.

Arkani-Hamed, Baumann, Hillmann, Joyce, Lee, Pimentel



$$I = \int dz_1 \wedge dz_2 \frac{(z_1 z_2)^\epsilon}{(z_1+y_1+1)(z_2+y_2+1)(z_1+z_2+y_1+y_2)}$$

Brunello & P.M. (2023)
[unpublished]

- **Twisted Period Integrals**

$$I = \int_{\mathcal{C}} u(z_1, z_2)\varphi(z_1, z_2) \qquad u = (z_1 z_2)^\epsilon (D_1 D_2 D_3)^\gamma \qquad D_1 = (z_1+y_1+1)\,, \quad D_2 = (z_2+y_2+1)\,, \quad D_3 = (z_1+z_2+y_1+y_2)$$

γ is a regulator

$$\omega = d\log(u) = \omega_1 dz_1 + \omega_2 dz_2 \qquad \omega_1 = \frac{\gamma(2y_1+y_2+2z_1+z_2+1)}{(y_1+z_1+1)(y_1+y_2+z_1+z_2)} + \frac{\epsilon}{z_1} \qquad \omega_2 = \frac{\gamma(y_1+2y_2+z_1+2z_2+1)}{(y_2+z_2+1)(y_1+y_2+z_1+z_2)} + \frac{\epsilon}{z_2}$$

- **Number of MIs = dimH and bases choice**

$$\omega_2 = 0 \qquad \nu_2 = 2 \qquad e^{(2)} = h^{(2)} = \left\{\frac{1}{D_1}, \frac{1}{D_2}\right\}$$

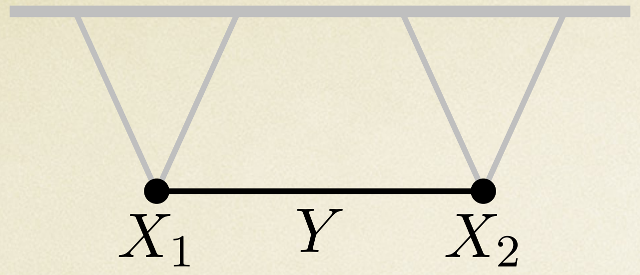
- **2 MIs in the internal layer**

$$\begin{cases}\omega_1 = 0\\ \omega_2 = 0\end{cases} \qquad \nu_{21} = 4 \qquad e^{(21)} = h^{(21)} = \left\{\frac{1}{\epsilon D_3^2}, \frac{1}{D_1 D_3}, \frac{1}{D_2 D_3}, \frac{1}{D_1 D_2 D_3}\right\}$$

- **4 MIs in the external layer**

- **Intersection Matrix**

$$C = \begin{pmatrix} \frac{(\gamma+\epsilon)^2}{\gamma(\gamma^2-1)\epsilon^2(3\gamma+2\epsilon)} & -\frac{\gamma+\epsilon}{(\gamma-1)\gamma\epsilon(3\gamma+2\epsilon)} & -\frac{\gamma+\epsilon}{(\gamma-1)\gamma\epsilon(3\gamma+2\epsilon)} & \frac{1}{\gamma\epsilon-\gamma^2\epsilon} \\ -\frac{\gamma+\epsilon}{\gamma(\gamma+1)\epsilon(3\gamma+2\epsilon)} & \frac{2(\gamma+\epsilon)^2}{\gamma^2(2\gamma+\epsilon)(3\gamma+2\epsilon)} & \frac{1}{3\gamma^2+2\gamma\epsilon} & \frac{1}{\gamma^2} \\ -\frac{\gamma+\epsilon}{\gamma(\gamma+1)\epsilon(3\gamma+2\epsilon)} & \frac{1}{3\gamma^2+2\gamma\epsilon} & \frac{2(\gamma+\epsilon)^2}{\gamma^2(2\gamma+\epsilon)(3\gamma+2\epsilon)} & \frac{1}{\gamma^2} \\ -\frac{1}{\gamma^2\epsilon+\gamma\epsilon} & \frac{1}{\gamma^2} & \frac{1}{\gamma^2} & \frac{3}{\gamma^2} \end{pmatrix}$$



$$I = \int dz_1 \wedge dz_2 \frac{(z_1 z_2)^\epsilon}{(z_1 + y_1 + 1)(z_2 + y_2 + 1)(z_1 + z_2 + y_1 + y_2)}$$

Brunello & P.M. (2023) [unpublished]

- **4 MIs**

$$e^{(21)} = \left\{ \frac{1}{\epsilon D_3^2}, \frac{1}{D_1 D_3}, \frac{1}{D_2 D_3}, \frac{1}{D_1 D_2 D_3} \right\}$$

- **System of Differential Equations**

$$\partial_x \langle e_i | = \Omega_{ij} \langle e_j |$$

- **Master Decomposition Formula**

$$\Omega_{ij} = \langle (\partial_x + \sigma_x) e_i | h_k \rangle \, (\mathbf{C}^{-1})_{kj}$$

after taking the limit $\gamma \to 0$:

- **Canonical system**

$$\Omega_{y_1} = \begin{pmatrix} \frac{2\epsilon}{y_1+y_2+1} & 0 & 0 & 0 \\ -\frac{\epsilon}{y_1+1} & \frac{\epsilon}{y_1+1} & 0 & 0 \\ \frac{\epsilon}{y_1} & 0 & \frac{\epsilon}{y_1} & 0 \\ \frac{\epsilon}{y_1(y_1+1)} & 0 & \frac{\epsilon}{y_1(y_1+1)} & \frac{\epsilon}{y_1+1} \end{pmatrix}$$

$$\Omega_{y_2} = \begin{pmatrix} \frac{2\epsilon}{y_1+y_2+1} & 0 & 0 & 0 \\ \frac{\epsilon}{y_2} & \frac{\epsilon}{y_2} & 0 & 0 \\ -\frac{\epsilon}{y_2+1} & 0 & \frac{\epsilon}{y_2+1} & 0 \\ \frac{\epsilon}{y_2(y_2+1)} & \frac{\epsilon}{y_2(y_2+1)} & 0 & \frac{\epsilon}{y_2+1} \end{pmatrix}$$

☑ **Cohomology-based methods for cosmological correlations @ tree level** Pokraka et al. (2023) Gasparotto, Mazloumi, Xu (2024)

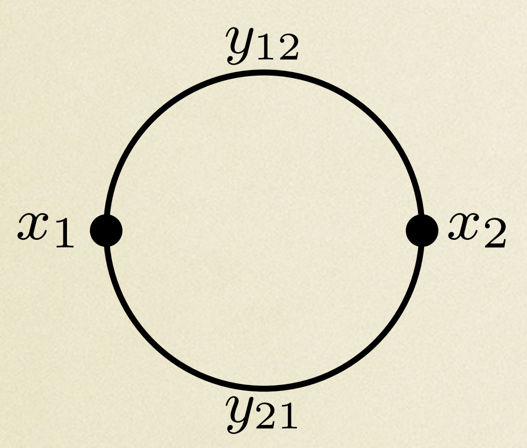
☑ **Differential Equations for cosmological correlations @ tree level** Arkani-Hamed, Baumann, Hillmann, Joyce, Lee, Pimentel (2023)

Cosmological Integrals @ 1-loop

Benincasa, Brunello, Mandal, Vazão, & PM (2024)

- Mapping cosmological integrals to **QFT-like integrals in momentum space**, with **semi-integer denominator powers**
- From momentum-space to **Baikov representation** to cast them as **twisted period integrals**

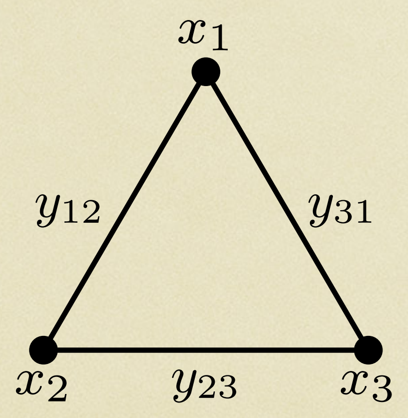
Two-site graph



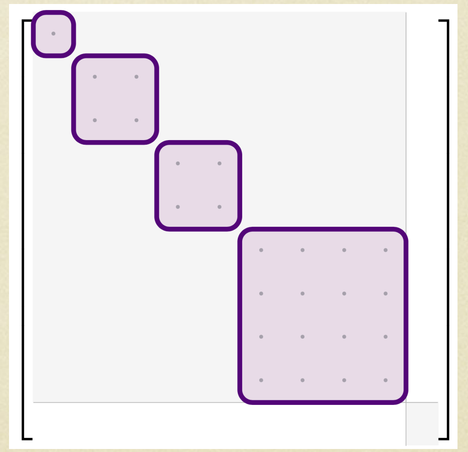
- ☑ Linear algebra from **Algebraic Geometry and Syzygy equations**
- ☑ Linear algebra from **Intersection Theory**
- ☑ (y-integration) **Canonical Differential Equations for $\nu = 6$ MIs:** polylog structure
- ☑ (y-integration) **Analytic solution**
- ☑ **Site-weight** x-integration: Mellin Transform and **Method of Brackets**
- ☑ **Analytic solution:** back of a envelope result

$$
\begin{aligned}
\mathcal{I}_{(2,\,1)} = {} & \frac{2^{-3-2\alpha}\pi^{3/2}(X_1+X_2)^{1+2\alpha}\csc(\pi\alpha)^2\Gamma\left(-\frac{1}{2}-\alpha\right)}{\Gamma[-\alpha]}\left(2-\frac{1}{\epsilon}-\log\!\left(4\pi e^{\gamma_E}P^2\right)\right) \\
& + \frac{\pi^{3/2}\csc^2(\pi\alpha)}{8(\alpha\ +1)^2 P}\Bigg[-4\sqrt{\pi}\left((P+X_1)^{\alpha+1}-2\ (X_1-P)^{\alpha+1}\right)(P+X_2)^{\alpha+1} \\
& - \frac{4^{-\alpha}\Gamma\left(-\alpha\ -\frac{1}{2}\right)(X_1+X_2)^{2\alpha+2}}{\Gamma\ (-\alpha)}\,{}_2F_1\left(1,-2(\ \alpha+1);-\alpha;\frac{P+X_1}{X_1+X_2}\right)\Bigg] \\
& + \frac{\pi^2\csc(\pi\alpha)\csc(2\pi\alpha)(P+X_1)^{\alpha}}{4\alpha+2}\Bigg[-2(P+X_1)\left((P-X_2)^{\alpha}+(-1)^{\alpha}(P+X_2)^{\alpha}\right) \\
& + (-1)^{\alpha}(X_1-X_2)(P+X_1)^{\alpha}\,{}_2F_1\left(1-\alpha,-2\alpha;1-2\alpha;\frac{X_1-X_2}{P+X_1}\right) \\
& + (X_1+X_2)(P+X_1)^{\alpha}\,{}_2F_1\left(1-\alpha,-2\alpha;1-2\alpha;\frac{X_1+X_2}{P+X_1}\right)\Bigg] \\
& - \frac{\pi^{5/2}4^{-\alpha-1}\csc(\pi\alpha)\csc(2\pi\alpha)}{\Gamma(-\alpha\ )\Gamma\left(\alpha+\frac{3}{2}\right)(P+X_1)}\Bigg[(-1)^{\alpha}(X_1-X_2)^{2\alpha+2}\,{}_3F_2\left(1,1,\alpha+2;2,2\ \alpha+3;\frac{X_1-X_2}{P+X_1}\right) \\
& + (X_1+X_2)^{2\alpha+2}\ {}_3F_2\left(1,1,\alpha+2;2,2\alpha+3;\frac{X_1+X_2}{P+X_1}\right)\Bigg] \\
& + \frac{\pi^{5/2}2^{-2\alpha-1}\csc(\pi\alpha)\csc(2\pi\ \alpha)\left((-1)^{\alpha}\ (X_1-X_2)^{2\alpha+1}+(X_1+X_2)^{2\alpha+1}\right)}{\Gamma\ (-\alpha)\Gamma\left(\alpha+\frac{3}{2}\right)}\log\left(\frac{P+X_1}{P}\right) \\
& + (X_1\leftrightarrow X_2).
\end{aligned}
$$

Three-site graph

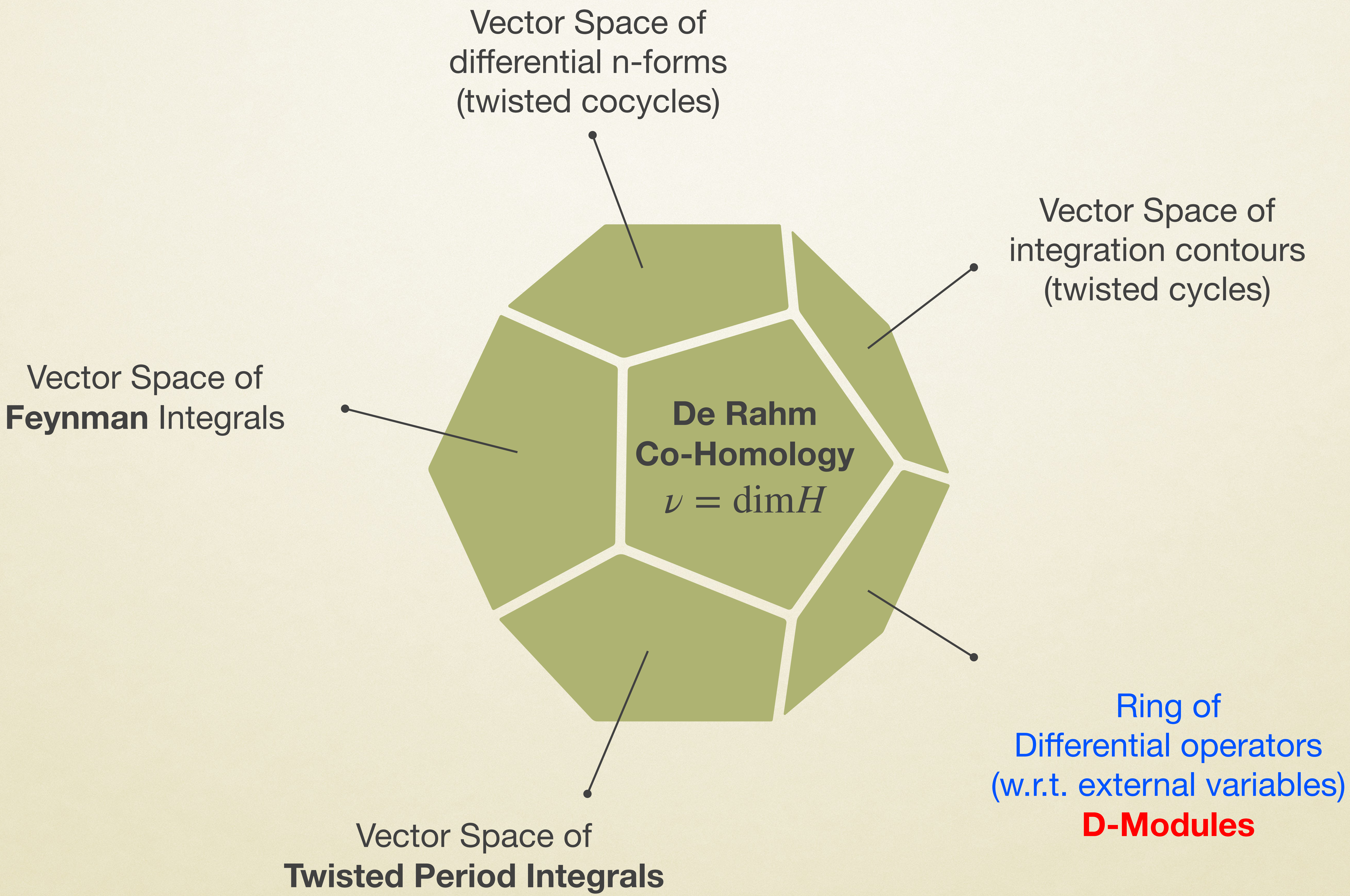


- ☑ Linear algebra from **Algebraic Geometry and Syzygy equations**
- ☑ Linear algebra from **Intersection Theory**
- ☑ (y-integration) **Differential Equations for $\nu = 41$ MIs**: polylog and **elliptic** structure



DEQ:
structure of the
elliptic sector
(4x4)-block

De Rham Thm & Vector Spaces *Isomorphism*



Differential Space of Twisted Period Integrals: Annihilators and D-modules

Chestnov, Flieger, Matsubara-Heo, Takayama, Torres Bobadilla, **& P.M.** (soon)

- **Annihilators** of Twisted Period Integral I $\quad \mathrm{Ann}(I) \ni \mathbf{A}_p : \quad \mathbf{A}_p \cdot I = 0$ $\quad$ p = order of the differential operator

Systems of Differential equations for twisted period integrals:

- to compute them

- to detect/investigate their symmetries

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- **(Twisted) Griffiths' theorem** if $\quad I =$ (twisted) period integral, $\quad \exists \; d\gamma(I) : \quad \int_\Gamma d\gamma(I) = 0$ Griffiths (Vanhove, De La Cruz)

Differential Space of Twisted Period Integrals: Annihilators and D-modules

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A New Algorithm to build Annihilators:

- **Annihilators** from **Griffiths' theorem**

$$\mathbf{A}_p \equiv \mathbf{D}_p + \mathbf{D}_{p-1} \qquad \mathbf{A}_p \cdot I = \int_\Gamma d\gamma(I) = 0$$

- Conjecture **Annihilators** as **D-module Generators**

$$\mathbf{A}_p \cdot I = 0 \quad \Rightarrow \quad \partial_{x_1}^{(i_1)} \cdots \partial_{x_n}^{(i_n)} (\mathbf{A}_p \cdot I) = 0$$

Macaulay system

- D-module basis: **Standard monomials (Std)** form Macaulay system solving
- rank = **# of Std**

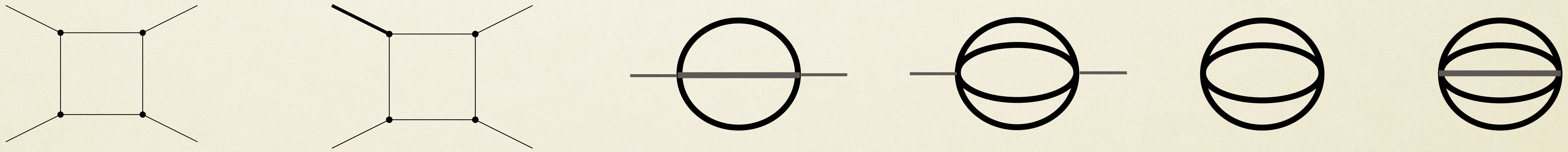
Differential Space of Twisted Period Integrals: Annihilators and D-modules

Chestnov, Flieger, Matsubara-Heo, Takayama, Torres Bobadilla, **& P.M.** (soon)

- **D-module:** ambient space to derive system of **Pfaffian equations** and **Picard-Fuchs (PF)operators**
  [**exact** dependence on dimensional/regulating parameters, and **no surface-term** involved]

==> rank = dim(de Rham Co-homology groups) = # of master integrals = order of PF operator <==

- **Applications**
  - **@ Hypergeometric Integrals** $_2F_1$, $_3F_2$
  - **@ Feynman Integrals**



  - **@ Witten Diagrams (AdS4, dim.reg.)**

Proving the conjecture <==> D-module ~ De Rham Cohomology Group

Extending Gel'Fand Kapranov Zelevinski's theorem to restricted integrals

To Conclude:

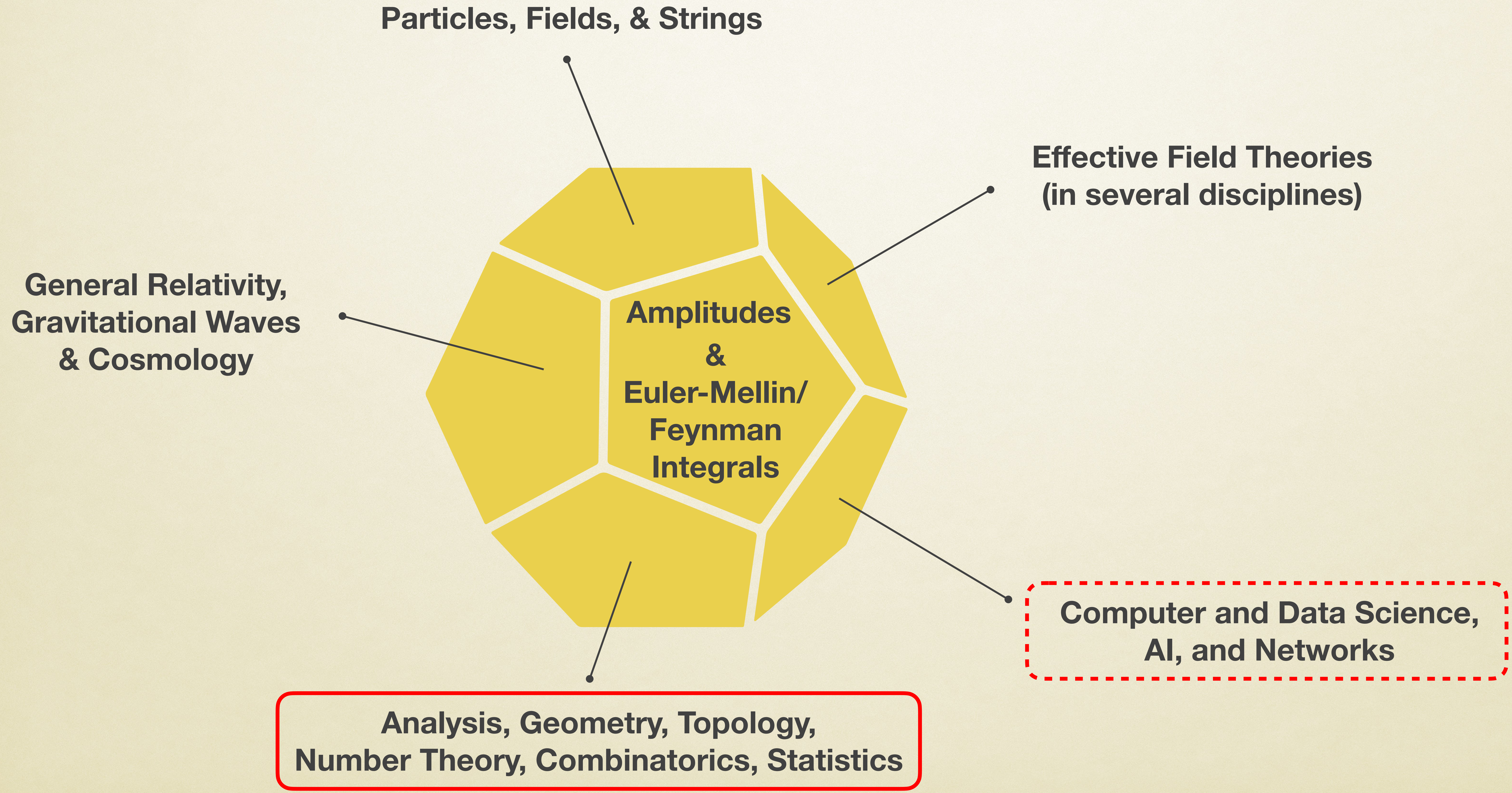
A unifying vision on Calculus in Fundamental Physics

Twisted Period Integrals				
Area	High-energy Physics and String theory	General Relativity	Cosmology	Algebraic and Differential Geometry
Target	Scattering Amplitudes	Interaction Hamiltonians	Correlator Functions	Euler-Mellin Integrals

Underpinning Correspondence: Physics vs Math

PHYSICS	Quantum Field Theory	Feynman Integrals	Integrals Vector Space	Master Integrals	Integration by Parts Identities	Differential Equations	Quadratic Integral Relations
MATH	Differential and Algebraic Geometry	Twisted Period Integrals	Co-Homology Groups	Master Forms and Contours	Contiguity Relations	Pfaffian Systems	Riemann Twisted Period Relations

Scattering Amplitudes & Multiloop Calculus: interdisciplinary toolbox



Summary

- **Feynman Integrals and Scattering Amplitudes**
  - Multi-loop Calculus from Quantum Field Theory to Effective Filed Theory of Gravity
  - Precision Gravitational Wave Physics: PM and PN corrections

- ***The ubiquitous De Rahm Theory***
  - Intersection Theory for Twisted de Rham co-homology
  - Analyticity & Unitarity vs Differential and Algebraic Geometry, Topology, Number Theory, Combinatorics, Statistics

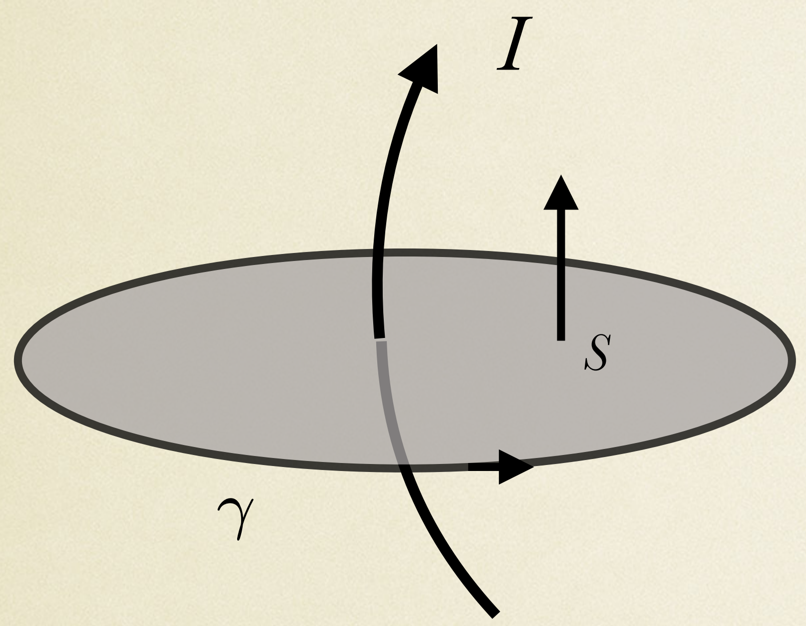
- **Vector Space Structures**
  - Vector-space dimensions = dimension of co-homology group = *counting holes* = number of independent Integrals
  - Intersection Numbers ~ **Scalar Product** for Feynman (Twisted Period) Integrals

- **Differential Space Structure**
  - Annihilators and D-Modules
  - Isomorphism of D-Module vs de Rham cohomology group

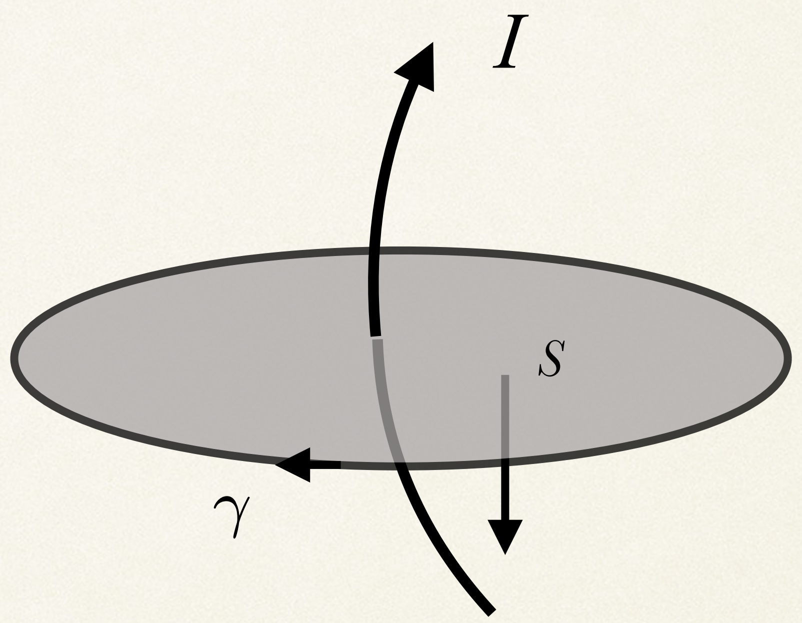
- **General algorithm for Physics and Math applications**
  - key: Co-Homology Group Isomorphisms
  - Triggering interdisciplinarity: interwinement between Fundamental Physics, Geometry and Statistics: fluxes ~ period integrals ~ statistical moments
    - Feynman Integrals
    - Euler-Mellin Integrals
    - Orthogonal polynomials
    - QM Matrix Elements
    - Fourier integrals
    - D-modules & GKZ theory
    - Gluing methods in N=4 SYM
    - Green's functions in QFT
    - Correlator functions in Cosmology
    - … … …

Ampere's Law

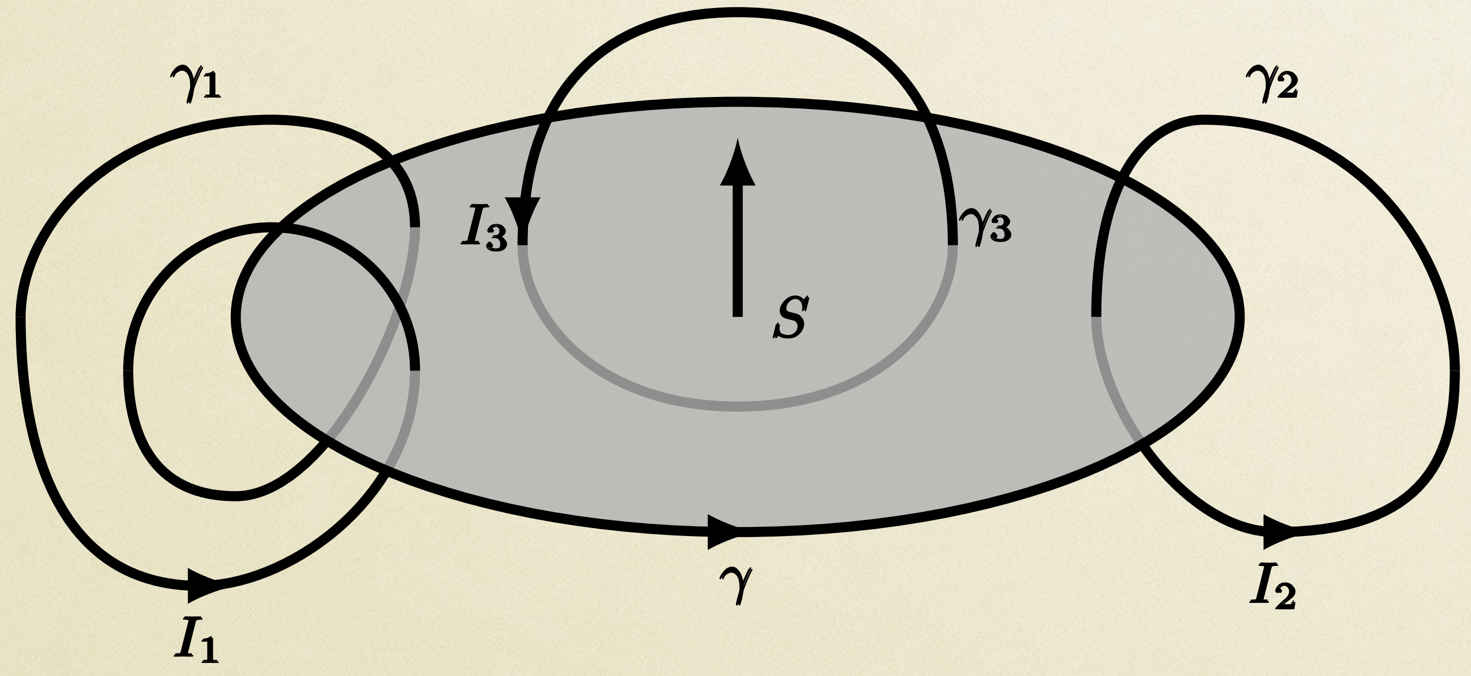
Cacciatori & P.M.



$$\oint_\gamma \mathbf{B}\cdot d\vec{\ell} = \mu_0 I$$



$$\oint_\gamma \mathbf{B}\cdot d\vec{\ell} = -\mu_0 I$$



- **Integral decomposition by geometry**

$$\oint_\gamma \mathbf{B}\cdot d\vec{\ell} = \sum_k (\pm n_k) \oint_{\gamma_k} \mathbf{B}\cdot d\vec{\ell} = \mu_0 \sum_k (\pm n_k) I_k$$

Master Contributions

Gauss' Linking Number

$$n_k = \mathrm{Link}(\gamma_k, \gamma)$$

$\mathrm{Link}(\gamma_1,\gamma) = +2$, $\mathrm{Link}(\gamma_2,\gamma) = -1$, and $\mathrm{Link}(\gamma_3,\gamma) = 0$

The unreasonable effectiveness of mathematics

E. Wigner

Wigner was referring to the mysterious phenomenon in which areas of pure mathematics, originally constructed without regard to application, are suddenly discovered to be exactly what is required to describe the structure of the physical world.

M. Berry

Based on:

- **PM**, Mizera
  Feynman Integral and Intersection Theory
  JHEP 1902 (2019) 139 [arXiv: 1810.03818]

- Frellesvig, Gasparotto, Laporta, Mandal, **PM**, Mattiazzi, Mizera
  Decomposition of Feynman Integrals in the Maximal Cut by Intersection Numbers
  JHEP 1095 (2019) 153 [arXiv: 1901.11510]

- Frellesvig, Gasparotto, Mandal, **PM**, Mattiazzi, Mizera
  Vector Space of Feynman Integrals and Multivariate Intersection Numbers
  Phys. Rev. Lett. 123 (2019) 20, 201602 [arXiv 1907.02000]

- Frellesvig, Gasparotto, Laporta, Mandal, **PM**, Mattiazzi, Mizera
  Decomposition of Feynman Integrals by Multivariate Intersection Numbers.
  JHEP 03 (2021) 027 [arXiv 2008.04823]

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  Macaulay Matrix for Feynman Integrals: linear relations and intersection numbers.
  JHEP09 (2022) 187 [arXiv: 2204.12983]

- Cacciatori & **PM,**
  Intersection Numbers in Quantum Mechanics and Field Theory.
  2211.03729 [hep-th].

- ***Brunello, Chestnov, Crisanti***, Frellesvig, Mandal & **PM**
  Intersection Numbers, Polynomial Division & Relative Cohomology
  JHEP09(2024)015 [arXiv: 2401.01897]

- ***Brunello, Crisanti***, ***Giroux***, ***Smith*** & **PM,**
  Fourier Calculus from Intersection Theory
  Phys.Rev.D 109 (2024) 9, 094047 [arXiv: 2311.14432]

- ***Brunello, Chestnov,*** & **PM,**
  Intersection Numbers from Companion Tensor Algebra
  2408.16668 [hep-th].

- ***Benincasa, Brunello, Mandal, Vazão,*** & **PM,**
  On one-loop corrections to the Bunch-Davies wavefunction of the universe
  2408.16386 [hep-th].

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  2411.07330 [hep-th], accepted by PRL

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  Gravitational spin-orbit Hamiltonian at NNNLO in the post-Newtonian framework
  JHEP 03 (2023) 130

- Mandal, **PM**, Patil, Steinhoff,
  Gravitational quadratic in spin Hamiltonian at NNNLO in the post-Newtonian framework
  JHEP 07 (2023) 128

- Mandal, **PM**, Patil, Silva, Steinhoff,
  Gravitoelctric dynamical tides at second post-Newtonian framework
  JHEP 11 (2023) 067

- Mandal, **PM**, Patil, Silva, Steinhoff,
  Renormalizing Love: tidal effects at third post-Newtonian framework
  JHEP 02 (2024) 188

- Mandal, **PM**, Patil, Silva, Steinhoff,
  Radiating Love: adiabatic tidal fluxes and modes up to NNL post-Newtonian order
  JHEP 05 (2025) 008

- Bini, Damour, Geralico, Laporta, **PM**,
  Gravitational dynamics at O(G^6): perturbative gravitational scattering meets experimental mathematics
  2008.09389 [gr-qc]

- Bini, Damour, Geralico, Laporta, **PM**,
  Gravitational scattering at seventh order in G: non-local contribution at the sixth Post-Newtonian order
  Pays Rev D 103 (2021) 4, 044038

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P. Benincasa, F. Vazao, B. Eden, M. Gottwald, T. Scherdin

Extra Slides

i) Orthogonal Polynomials

Laguerre $L_n^{(\rho)}$, Legendre P_n, Tchebyshev T_n, Gegenbauer $C_n^{(\rho)}$, and Hermite H_n polynomials:

$$I_{nm} \equiv \int_\Gamma \mu\, P_n P_m\, dz = f_n\, \delta_{nm} \; = \int_\Gamma \mu\, \varphi = c_1\, E_1$$

$$\varphi \equiv P_n\, P_m\, dz$$

Type	u	ν	$\hat{e}_i$	**C**-matrix	ρ_0	E_1	c_1
$L_n^{(\rho)}$	$z^\rho \exp(-z)$	1	1	ρ	–	$\Gamma(1+\rho)$	$(\rho+1)(\rho+2)\cdots(\rho+n)/n!$
P_n	$(z^2-1)^\rho$	1	1	$2\rho/(4\rho^2-1)$	0	2	$1/(2n+1)$
T_n	$(1-z^2)^\rho$	1	1	$2\rho/(4\rho^2-1)$	$-1/2$	π	$1/2$
$C_n^{(\rho)}$	$(1-z^2)^{\rho-1/2}$	1	1	$(1-2\rho)/(4\rho(\rho-1))$	–	$\sqrt{\pi}\Gamma(1/2+\rho)/\Gamma(1+\rho)$	$\rho(2\rho(2\rho+1)\cdots(2\rho+n-1))/((n+\rho)n!)$
H_n	$z^\rho \exp(-z^2)$	2	$1, 1/z$	$\text{diagonal}(1/2, 1/\rho)$	0	$\sqrt{\pi}$	$2^n n!$

Let us observe that, in the case of Hermite polynomials, $\nu = 2$, yielding $\varphi = c_1\, e_1 + c_2\, e_2$, but $c_2 = 0$, due to the adopted basis

ii) Matrix Elements in QM

Harmonic Oscillator. (for unitary mass and pulsation, $m = 1 = \omega$)

$$\langle z|n\rangle = \psi_n(z) = e^{-\frac{z^2}{2}}\, W_n(z)\ , \quad \text{with} \quad W_n(z) \equiv N_n\, H_n(z)\ , \qquad N_n \equiv 1/\sqrt{(2^n n! \sqrt{\pi})}$$

- **Position operator**

$$\langle m|z^k|n\rangle = \int_{-\infty}^{\infty} dz\, \psi_m(z)\, z^k\, \psi_n(z) = \int_\Gamma \mu\, \varphi = c_1\, E_1\ , \quad \text{with} \quad \mu \equiv e^{-z^2}\ , \quad \text{and} \quad \varphi \equiv W_m(z)\, z^k\, W_n(z)\, dz.$$

Type	u	ν	$\hat{e}_i$	**C**-matrix	ρ_0	E_1
W_n	$z^\rho \exp(-z^2)$	2	$1, 1/z$	diagonal$(1/2, 1/\rho)$	0	$\sqrt{\pi}$

ii) Matrix Elements in QM

Hydrogen Atom. (for unitary Bohr radius $a_0 = 1$)

$$\langle z|n,\ell\rangle = R_{n,\ell}(z) = e^{-\frac{z}{n}} W_{n,\ell}(z) \,, \quad \text{with} \quad W_{n,\ell}(z) \equiv N_{n\ell} \left(\frac{2z}{n}\right)^{\ell} L^{2\ell+1}_{(n-\ell-1)}\left(\frac{2z}{n}\right) \qquad N_{n\ell} = (2/n)^{3/2}\sqrt{(n-\ell-1)!/(2n\,(n+\ell)!)}$$

- **Position operator**

$$\langle n_1,\ell|z^k|n_2,\ell\rangle = \int_0^{\infty} dz\, z^2\, R_{n_1,\ell}(z)\, z^k\, R_{n_2,\ell}(z) = \int_{\Gamma} \mu\, \varphi = c_1\, E_1 \,, \quad \text{with} \quad \mu \equiv z^2\, e^{-z\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}, \text{ and } \varphi \equiv W_{n_1,\ell}(z)\, z^k\, W_{n_2,\ell}(z)$$

Type	u	ν	$\hat{e}_i$	**C**-matrix	ρ_0	E_1
$W_{n,\ell}$	$z^{\rho+2}\exp(-z(n_1+n_2)/(n_1 n_2))$	1	1	$(n_1 n_2/(n_1+n_2))^2(2+\rho)$	0	$2(n_1 n_2/(n_1+n_2))^3$

i) Green's Function

Single field, ϕ^4-theory

real scalar field $\phi(x)$ $\quad S_E \equiv S_0 + \varepsilon S_1$, with $S_0 = (\gamma/2)\,\phi^2(x)$, and $S_1 = \phi^4(x)$

$$\int \mathscr{D}\phi\, \phi(x_1)\cdots\phi(x_n)\, e^{-S_E} = G_n \int \mathscr{D}\phi\, e^{-S_E}$$

$$\int_\Gamma \mu\, \varphi = G_n E_1\,, \quad \text{with} \quad \mu \equiv e^{-S_E}\,, \quad \varphi \equiv \phi(x_1)\cdots\phi(x_n)\,\mathscr{D}\phi\,, \quad E_1 \equiv \int_\Gamma \mu\, e_1\,, \quad \text{and} \quad e_1 \equiv \mathscr{D}\phi$$

Free theory. The n-point Green's function $G_n^{(0)}$ $\quad \phi(x) \equiv z \quad$ $\mu \equiv e^{-S_0} \quad \varphi = z^n\, dz$

Type	u	ν	$\hat{e}_i$	**C**-matrix	ρ_0	E_1	c_1
$G_n^{(0)}$	$z^\rho \exp(-\gamma z^2/2)$	2	$1, 1/z$	diagonal$(1/\gamma, 1/\rho)$	0	not needed	$(n-1)!!/\gamma^{n/2}$

for even n

ii) Kontsevich-Witten tau-function

$$Z_{KW} \equiv \frac{\int d\Phi \exp\left[-\mathrm{Tr}\left(-\frac{i}{3!}\Phi^3 + \frac{\Lambda}{2}\Phi^2\right)\right]}{\int d\Phi \exp\left[-\mathrm{Tr}\left(\frac{\Lambda}{2}\Phi^2\right)\right]}$$

- **Univariate Model** Itzykson-Zuber (1992)

$Z_{KW} = \sum_{n=0}^{\infty} Z_{KW}^{(n)}$ $\qquad \int_\Gamma \mu\,\varphi = c_1 E_1 \qquad c_1 = Z_{KW}^{(n)}$

$\varphi \equiv N_n z^{6n}, \qquad N_n \equiv \varepsilon^{2n} \qquad \varepsilon \equiv i/(3!)(\Lambda/2)^{-3/2}$

Type	u	ν	$\hat{e}_i$	**C**-matrix	ρ_0	E_1	c_1
$Z_{KW}^{(n)}$	$z^\rho \exp(-z^2)$	2	$1, 1/z$	diagonal$(1/2, 1/\rho)$	0	not needed	$(-2/9)^n\,(\Lambda^{-3n}/(2n)!)\,\prod_{j=0}^{3n-1}(j+1/2)$

GKZ Hypergeometric Systems

- **Euler-Mellin Integral / A-Hypergeometric function** $$f_\Gamma(z) = \int_\Gamma g(z;x)^{\beta_0}\, x_1^{-\beta_1} \cdots x_n^{-\beta_n}\, \frac{dx}{x}$$

Bernstein, Saito, Sturmfels, Takayama, Matsubara-Heo, Agostini, Fevola, Sattelberger, Tellen, De La Crux,...

$$\frac{dx}{x} := \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} \qquad u(\mathbf{x}) = g(z,x)^{\beta_0}\, x_1^{-\beta_1} \cdots x_n^{-\beta_n}$$

$$g(z;x) = \sum_{i=1}^N z_i\, x^{\alpha_i} \qquad x^{\alpha_i} := x_1^{\alpha_{i,1}} \cdots x_n^{\alpha_{i,n}} \qquad A = \begin{pmatrix} a_1 & \dots & a_N \end{pmatrix} \quad (n+1)\times N \text{ matrix} \qquad a_i := (1, \alpha_i)$$

$$\mathrm{Ker}(A) = \{u = (u_1, \dots, u_N) \in \mathbb{Z}^N \,|\, \sum_{j=1}^N u_j\, a_j = \mathbf{0}\}$$

- **Gelfand-Kapranov-Zelevinsky (GKZ) system of PDEs**

$$E_j\, f_\Gamma(z) = 0\ ,$$
$$\Box_u\, f_\Gamma(z) = 0\ ,$$

- **Generators**

$$E_j = \sum_{i=1}^N a_{j,i}\, z_i\, \frac{\partial}{\partial z_i} - \beta_j\ , \qquad j = 1, \dots, n+1$$

$$\Box_u = \prod_{u_i>0} \left(\frac{\partial}{\partial z_i}\right)^{u_i} - \prod_{u_i<0} \left(\frac{\partial}{\partial z_i}\right)^{-u_i}\ , \quad \forall u \in \mathrm{Ker}(A)\,.$$

GKZ D-Module and De Rham Cohomolgy group

- **Weyl Algebra:** E_j $\Box_u$ can be regarded as elements of a Weyl algebra

$$\mathcal{D}_N = \mathbb{C}[z_1, \dots, z_N]\langle \partial_1, \dots, \partial_N \rangle \quad , \quad [\partial_i, \partial_j] = 0 \quad , \quad [\partial_i, z_j] = \delta_{ij}$$

GKZ system as the left $\mathcal{D}_N$-module $\mathcal{D}_N / H_A(\beta)$

$$H_A(\beta) = \sum_{j=1}^{n+1} \mathcal{D}_N \cdot E_j + \sum_{u \in \mathrm{Ker}(A)} \mathcal{D}_N \cdot \Box_u$$

- **Standard Monomials** $\mathrm{Std} := \{\partial^k\}$ found by Groebner basis Hibi, Nishiyama, Takayama (2017)

The holonomic rank equals the number of independent solutions to the system of PDEs

$$r = n! \cdot \mathrm{vol}(\Delta_A)$$

- **Isomorphism**

$$\mathcal{D}_N / H_A(\beta) \simeq \mathbb{H}^n$$

GKZ D-module — nth-Cohomology group

Intersection Numbers for n-forms (V) from Pfaffian D-module systems

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama **& P.M.** (2022)

Let $\{e_i\}_{i=1}^r$ be a basis for $\mathbb{H}^n$ and $\{h_i\}_{i=1}^r$ a basis for $\mathbb{H}^{n\vee}$

$\varphi \in \mathbb{H}^n$ in terms of $\{e_i\}_{i=1}^r$

- **Thm : Isomorphism**

nth-Cohomology group $\simeq$ **GKZ D-module**

Gelf'and Kapranov Zelevinsky)1990)

Euler-Mellin Integrals

Differential Operators (w.r.t. external variables)

Intersection Numbers for n-forms (V) from Pfaffian D-module systems

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$\varphi \in \mathbb{H}^n$ in terms of $\{e_i\}_{i=1}^r$

- **Thm : Isomorphism**

nth-Cohomology group $\simeq$ **GKZ D-module** — Gelf'and Kapranov Zelevinsky (1990)

nth-Cohomology group → **Euler-Mellin Integrals**

GKZ D-module → **Differential Operators (w.r.t. external variables)**

Pfaffian Systems: for Master Integrals (alias Master forms)

$$\partial_x \langle e_i| = \mathbf{\Omega}_{ij} \langle e_j|$$

$$\mathbf{\Omega} = \mathbf{\Omega}(d, x)$$

- **Pfaffian Matrix**

$\langle e_i|$, $\langle e_j|$ → **Basis of the Cohomology group**

$\mathbf{\Omega}$ → **Integral decomposition (IBP/InterX)**

Intersection Numbers for n-forms (V) from Pfaffian D-module systems

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama **& P.M.** (2022)

Let $\{e_i\}_{i=1}^{r}$ be a basis for $\mathbb{H}^n$ and $\{h_i\}_{i=1}^{r}$ a basis for $\mathbb{H}^{n\vee}$

$\varphi \in \mathbb{H}^n$ in terms of $\{e_i\}_{i=1}^{r}$

- **Thm : Isomorphism**

nth-Cohomology group $\simeq$ **GKZ D-module**

Gelf'and Kapranov Zelevinsky)1990)

Euler-Mellin Integrals — **Differential Operators (w.r.t. external variables)**

Pfaffian Systems: for Master Integrals (alias Master forms) & for D-operators (alias Std mon's)

$$\partial_x \langle e_i| = \mathbf{\Omega}_{ij} \langle e_j|$$

$$\mathbf{\Omega} = \mathbf{\Omega}(d, x)$$

- **Pfaffian Matrix**

Basis of the D-Operators

Macaulay Matrix method

Chestnov, Gasparotto, Mandal, Munch, Matsubara-Heo, Takayama **& P.M.** (2022)

Chestnov,,, Munch, Matsubara-Heo, Takayama **& P.M.** (2023)