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New Frontiers in Theoretical Physics
Cortona 22-05-2025

hadronic spectral densities from the lattice

a new route to QCD phenomenology

why we should look at hadronic spectral densities?

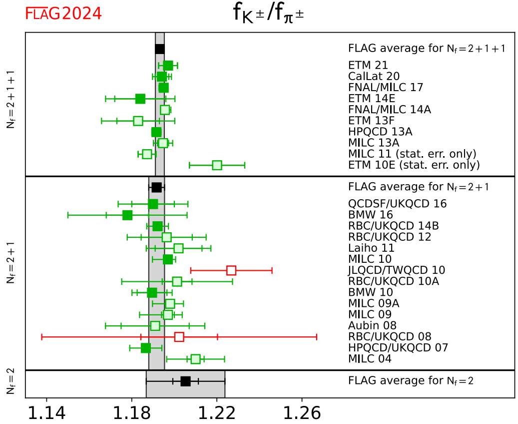
or, more explicitly,

what the hell do these obscure mathematical objects have to do with QCD phenomenology?

you might have heard many times people like me, the Lattice Guy (LG), to say

LG: LQCD entered the precision era!

FLAG arXiv:2411.04268



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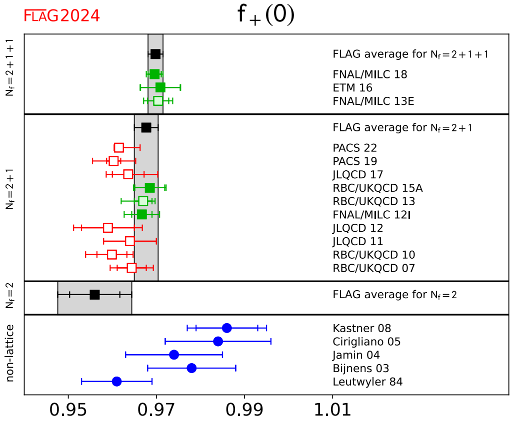

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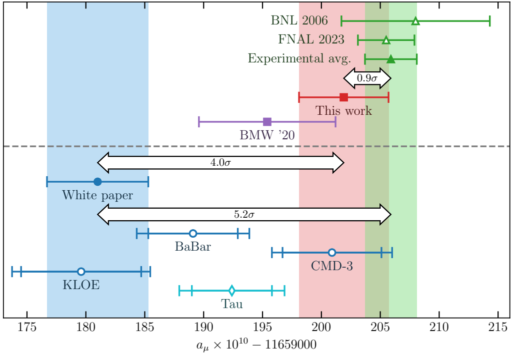
BMW arXiv:2407.10913

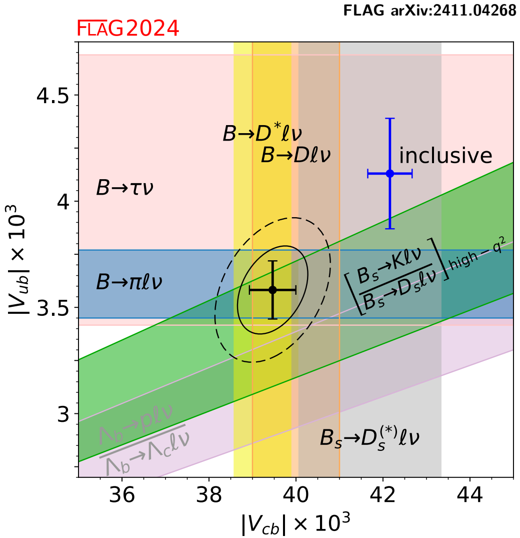


you might have heard many times people like me, the Lattice Guy (LG), to say

LG: LQCD entered the precision era!

and maybe then you, the Pheno Guy (PG), said:

PG: come on! you are only able to compute f_π and a couple of other things. what about inclusive semileptonic B decays? non-leptonic decays?



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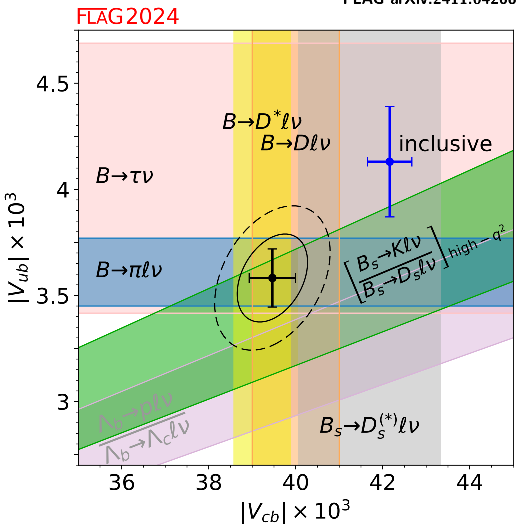
PG: come on! you are only able to compute f_π and a couple of other things. what about inclusive semileptonic B decays? non-leptonic decays?

LG: well, you know, we are working in euclidean time, there are problems of analytical continuation, . . .

PG: you can't do it, right?

LG: ok, let's see. . .

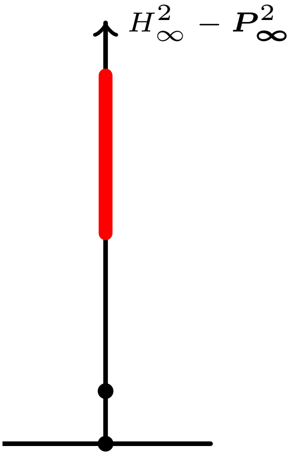
FLAG arXiv:2411.04268



in QFT, in order to get a state, we have to probe the spectrum of the hamiltonian by acting on the vacuum with interpolating operators

let's chose an operator with the quantum numbers of the B-meson and set the spatial momentum to zero,

$$|\Psi(t)\rangle = \int d^3x \, O_B(x) \, |0\rangle$$



in minkowsky-time QM we have

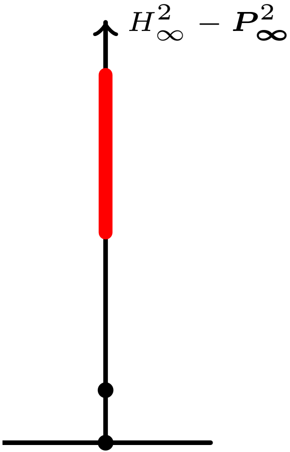
$$|\Psi(t)\rangle = e^{-itH}|\Psi\rangle = \int_{m_B}^{\infty} dE\, e^{-itE}\,\delta(H-E)\,|\Psi\rangle$$

the operatorial δ-function can be expressed in terms of states as

$$\delta(H-E) = \sum_{X(E)} |X(E)\rangle\langle X(E)|$$

and we can use Fourier's transform, a razor, to select any energy

$$\int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{itE_\star}|\Psi(t)\rangle = \sum_{X(E_\star)} |X(E_\star)\rangle\langle X(E_\star)|\Psi\rangle$$



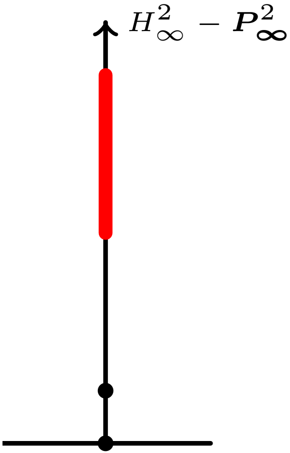
in euclidean-time QM we have

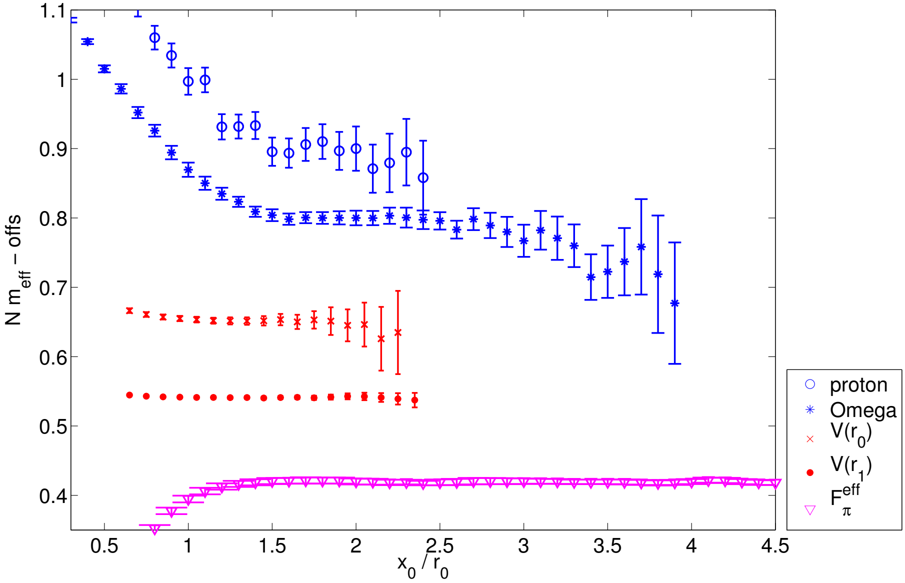
$$|\Psi(t)\rangle = e^{-tH}|\Psi\rangle = \int_{m_B}^{\infty} dE\, e^{-tE}\,\delta(H-E)\,|\Psi\rangle$$

the physics is untouched but we don't have the Fourier transform!

we have, though, a very (too) efficient way to isolate single-particle states

$$|\Psi(t)\rangle \quad \xrightarrow{t\mapsto\infty} \quad e^{-tm_B}|B\rangle\frac{\langle B|O_B(0)|0\rangle}{2m_B} \;\times\; \left\{1+O(e^{-2tm_\pi})\right\}$$

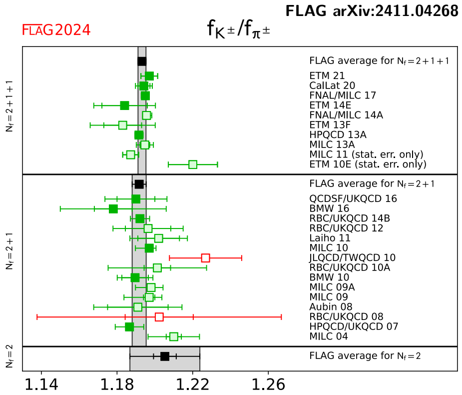




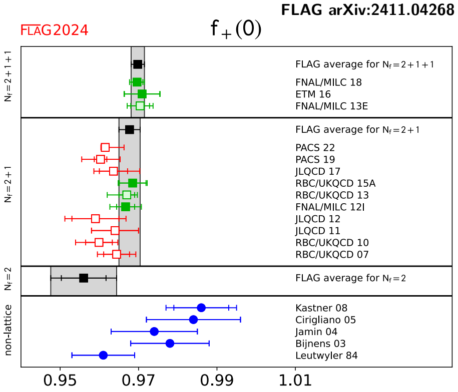
N m_{eff} − offs
x_0 / r_0
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.5
1
1.5
2
2.5
3
3.5
4
4.5
proton
Omega
$V(r_0)$
$V(r_1)$
F_π^{eff}

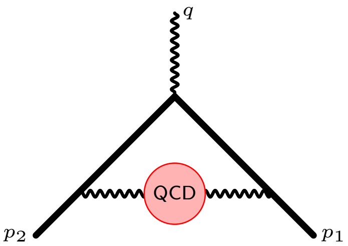
$$\langle 0|J_W^\alpha(0)|\pi\rangle$$

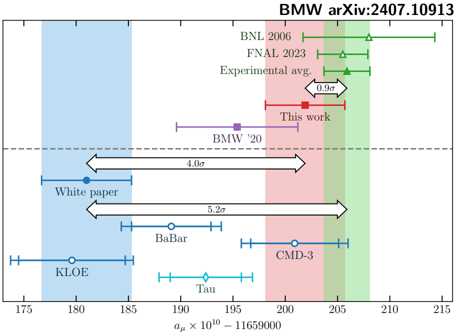
$$\langle 0|J_W^\alpha(0)|K\rangle$$



$$\langle \pi | J_W^\alpha(0) | K \rangle$$







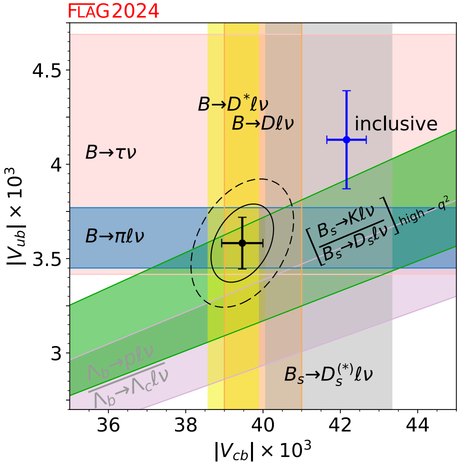
more on this later but, thinking in QCD+QED,

$$\langle \mu | J^\alpha(0) | \mu \rangle$$

exclusive:

$$\langle D|J_W^{\alpha}(0)|B\rangle$$

FLAG arXiv:2411.04268



inclusive:

$$\frac{d\Gamma^{B\mapsto X_u \ell\bar{\nu}}}{dE_\ell \, d|\boldsymbol{p}_X|} \propto \int_{E_\pi}^{\infty} dE \, K_{\alpha\beta}(E) \, H^{\alpha\beta}(E)$$

where the leptonic part can be seen as a smearing kernel

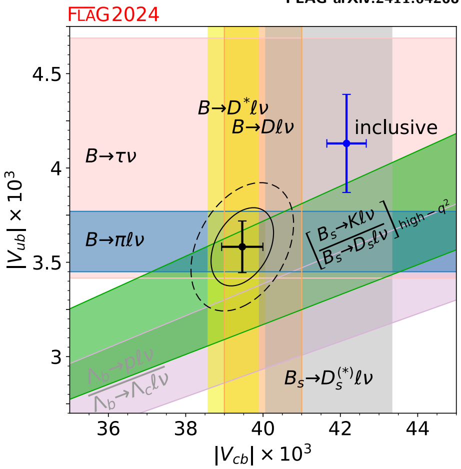
$$K_{\alpha\beta}(E) = L_{\alpha\beta}(E)\theta(E_{\max} - E)$$

for the hadronic spectral density

$$H^{\alpha\beta}(E) = \langle B | J_W^\alpha(0)\delta(H-E)\delta^3(\boldsymbol{P} - \boldsymbol{p}_X) J_W^\beta(0) | B \rangle$$

that we need for $E \gg E_\pi = \sqrt{m_\pi^2 + \boldsymbol{p}_X^2}$

FLAG arXiv:2411.04268



spectral densities are key objects in QFT

the Fourier transforms of Wightman's functions in minkowsky space

$$W(x_1, \cdots x_{n-1}) = \langle 0|\hat{O}_1 e^{-i\hat{P}\cdot x_1} \hat{O}_2 \cdots e^{-i\hat{P}\cdot x_{n-1}} \hat{O}_n|0\rangle$$

$$\rho(p_1, \cdots p_{n-1}) = \int \frac{dx_1 \cdots dx_{n-1}}{(2\pi)^{n-1}} \, e^{i\sum_i p_i\cdot x_i} \, W(x_1, \cdots x_{n-1})$$

$$= \langle 0|\hat{O}_1 (2\pi)^3 \delta^4(\hat{P} - p_1) \hat{O}_2 \cdots (2\pi)^3 \delta^4(\hat{P} - p_{n-1}) \hat{O}_n|0\rangle$$

r.haag Phys.Rev. 112 1958
d.ruelle Helv.Phys.Acta 35 1962
a.wightman Proc.Symp.Pure Math.Amer.Math.Soc. 28 1976

from their knowledge it is possible to study any hadronic process the theoretical connection between

euclidean lattice correlators

spectral densities

and generic S-matrix elements

is known since a while!!

j.barata, k.fredenhagen CMP 138 (1991)
j.barata Rev.Math.Phys. (1993)
j.bulava, m.t.hansen Phys.Rev.D 100 (2019)
a.patella, n.tantalo JHEP 01 (2025)

Commun. Math. Phys. 138, 507–519 (1991)

Communications in Mathematical Physics

Particle Scattering in Euclidean Lattice Field Theories

J. C. A. Barata [1,*] **and K. Fredenhagen** [2]

[1] Instituto de Física a Universidade de São Paulo, Cep 05508, Caixa Postal 20516, São Paulo, SP, Brazil
[2] Freie Universität Berlin, Fachbereich Physik, Institut für Theorie der Elementarteilchen, Arnimallee 14, 1000 Berlin 33, FRG

Received June 20, 1990

Abstract. A Haag-Ruelle Scattering Theory for Euclidean Lattice Field Theories is developed.

S-Matrix Elements in Euclidean Lattice Theories [1]

João C. A. Barata.[2 3 4]
Institut für Theoretische Physik der Freien Universität Berlin.
Arnimallee 14, 1000 Berlin 33.

Abstract: Expressions for S-matrix elements involving Euclidean expectations are presented.

mathematically, the problem is reduced to that of an inverse laplace-transform

to be performed numerically

by starting from a finite and noisy set of input data

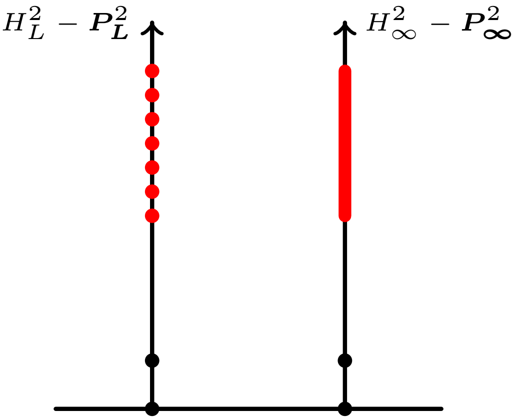
moreover. . .

Axiom W1: For each test function f, i.e. for a function with a compact support and continuous derivatives of any order, there exists a set of operators $O_1(f), \cdots, O_n(f)$ which, together with their adjoints, are defined on a dense subset of the Hilbert state space, containing the vacuum. **The fields O are operator-valued tempered distributions**. The Hilbert state space is spanned by the field polynomials acting on the vacuum (cyclicity condition).

spectral densities are distributions and **must be smeared**

this is particularly important on finite volumes where

$$\rho_L(E) = \sum_n w_n(L)\,\delta\left(E_n(L) - E\right)$$



PHYSICAL REVIEW D **99,** 094508 (2019)

Extraction of spectral densities from lattice correlators

Martin Hansen,[1] Alessandro Lupo,[2] and Nazario Tantalo[3]
[1] *INFN Roma Tor Vergata, Via della Ricerca Scientifica 1, I-00133 Rome, Italy*
[2] *University of Rome Tor Vergata, Via della Ricerca Scientifica 1, I-00133 Rome, Italy*
[3] *University of Rome Tor Vergata and INFN Roma Tor Vergata, Via della Ricerca Scientifica 1, I-00133 Rome, Italy*

having all this in mind, we developed a method (that my friend j.bulava then called HLT) that allows to extract **smeared spectral densities** from lattice correlators

$$\Gamma_\sigma(L) = \int_{E_0}^{\infty} dE\, K_\sigma(E)\rho_L(E)\ , \qquad \Gamma = \lim_{\sigma\mapsto 0}\lim_{L\mapsto\infty} \Gamma_\sigma(L)$$

other methods are available on the market

a.rothkopf EPJ Web Conf. 274, 01004 (2022)

j.bulava PoS LATTICE2022 (2023) 231

...

and whenever i have a student named alessandro i devise a new one...

Eur. Phys. J. C (2024) 84:32
https://doi.org/10.1140/epjc/s10052-024-12399-0

THE EUROPEAN PHYSICAL JOURNAL C

Check for updates

Regular Article - Theoretical Physics

Teaching to extract spectral densities from lattice correlators to a broad audience of learning-machines

Michele Buzzicotti[a], **Alessandro De Santis**[b], **Nazario Tantalo**[c]

University and INFN of Roma Tor Vergata, Via della Ricerca Scientifica 1, 00133 Rome, Italy

let's see how the HLT method works...

$$\rho(E) = \langle F | O_2 \, \delta\left(H - E\right) \delta^3\left(\boldsymbol{P} - \boldsymbol{p}\right) O_1 | I \rangle$$

$$C(t) = \int_{E_0}^{\infty} dE \, e^{-tE} \, \rho(E)$$

$$t = a\tau \; , \qquad \tau = 1, \cdots, T$$

$$K(\infty) = 0 \; , \qquad K(\infty) \in \mathcal{L}^2[E_0, \infty]$$

$$K(E) = \sum_{\tau=1}^{\infty} g_\tau \, e^{-a\tau E}$$

$$\int_{E_0}^{\infty} dE \, K(E) \, \rho(E) = \lim_{T \mapsto \infty} \sum_{\tau=1}^{T} g_\tau \, C(a\tau)$$

we have seen that the hadronic quantities that have been computed very precisely on the lattice are matrix elements of local operators between single-particle states and/or the vacuum . . .

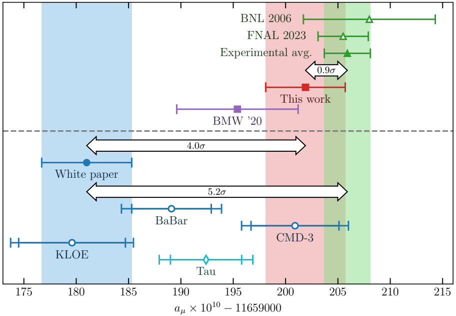
$$\int_{2m_\pi}^{\infty} dE\, K(E)\, R(E) = \lim_{T\mapsto\infty} \sum_{\tau=1}^{T} g_\tau\, C(a\tau)$$

$$C(t) = \int d^3x\, \langle 0|J^i_{\rm em}(x) J^i_{\rm em}(0)|0\rangle$$

$$C(t) \sim e^{-2m_\pi it} \mapsto e^{-2m_\pi t}$$

$$g_t \sim e^{m_\mu it} \mapsto e^{m_\mu t}$$

BMW arXiv:2407.10913

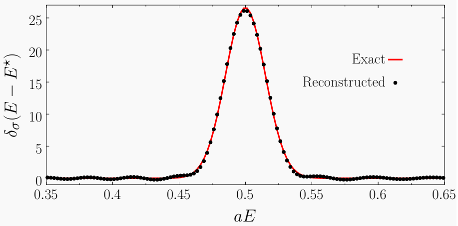


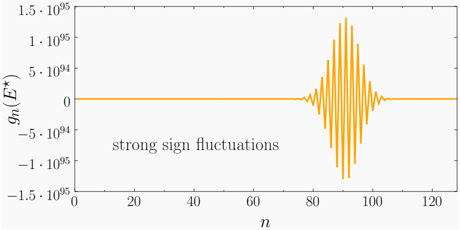
but, actually, also integrals of euclidean correlators times coefficients that, when there are no problems of analytical continuation, are smooth and well behaving

$$A_{\mathrm{n}}[\boldsymbol{g}] = \int_{E_0}^{\infty} dE\, w_{\mathrm{n}}(E) \left| K(E) - \sum_{\tau=1}^{T} g_\tau\, e^{-\tau a E} \right|^2$$

$$d_{\mathrm{n}}[\boldsymbol{g}] = \sqrt{\frac{A_{\mathrm{n}}[\boldsymbol{g}]}{A_{\mathrm{n}}[\boldsymbol{0}]}}$$

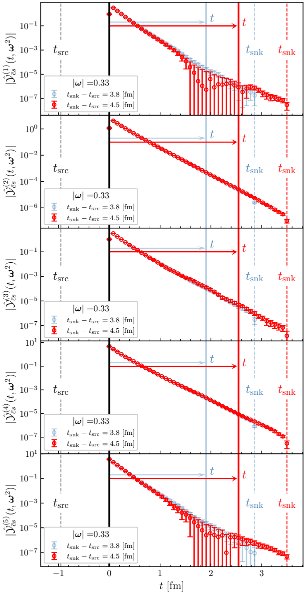
$$\int_{E_0}^{\infty} dE\, K(E)\, \rho(E) = \lim_{T \mapsto \infty} \sum_{\tau=1}^{T} g_\tau\, C(a\tau)$$





in general, the coefficients are all but smooth...

ETMC, $D_s \mapsto X\ell\bar{\nu}$



$$A_{\rm n}[\boldsymbol{g}] = \int_{E_0}^{\infty} dE\, w_{\rm n}(E) \left| K(E) - \sum_{\tau=1}^{T} g_\tau\, e^{-\tau a E} \right|^2$$

$$d_{\rm n}[\boldsymbol{g}] = \sqrt{\frac{A_{\rm n}[\boldsymbol{g}]}{A_{\rm n}[\boldsymbol{0}]}}$$

$$\int_{E_0}^{\infty} dE\, K(E)\, \rho(E) = \lim_{T \mapsto \infty} \sum_{\tau=1}^{T} g_\tau\, C(a\tau)$$

in general, the coefficients are all but smooth...

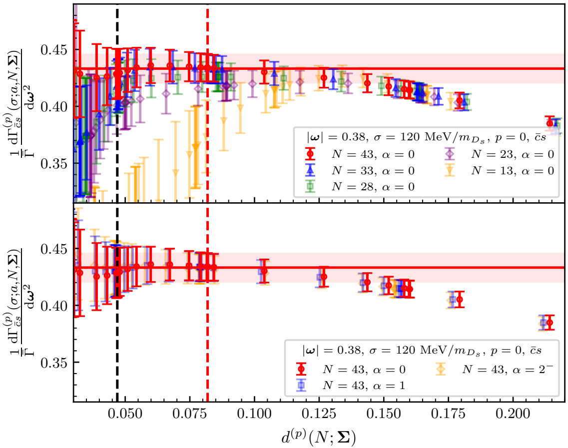
and the correlators are noisy!!

$$\int_{E_0}^{\infty} dE\, K(E)\, \rho(E) = \lim_{T \mapsto \infty} \sum_{\tau=1}^{T} g_\tau\, C(a\tau)$$

$$A_{\rm n}[\boldsymbol{g}] = \int_{E_0}^{\infty} dE\, w_{\rm n}(E) \left| K(E) - \sum_{\tau=1}^{T} g_\tau\, e^{-\tau a E} \right|^2$$

$$B[\boldsymbol{g}] = \sum_{\tau_{1,2}=1}^{T} g_{\tau_1} g_{\tau_2}\, {\rm Cov}_{\tau_1 \tau_2}$$

$$W_{\rm n}[\boldsymbol{g}] = A_{\rm n}[\boldsymbol{g}] + \lambda B[\boldsymbol{g}]$$



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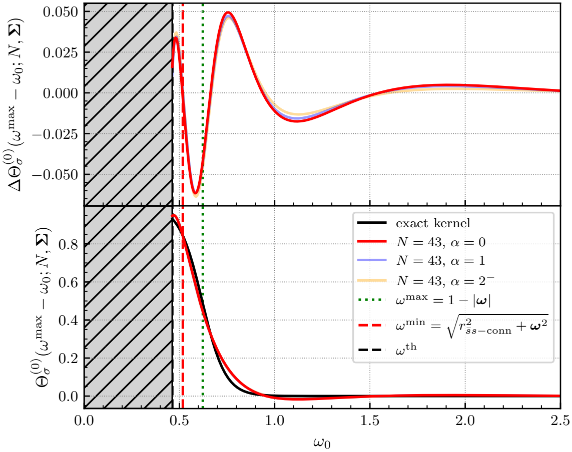
$$A_{\mathrm{n}}[\boldsymbol{g}] = \int_{E_0}^{\infty} dE\, w_{\mathrm{n}}(E) \left| K(E) - \sum_{\tau=1}^{T} g_\tau\, e^{-\tau a E} \right|^2$$

$$B[\boldsymbol{g}] = \sum_{\tau_{1,2}=1}^{T} g_{\tau_1} g_{\tau_2}\, \mathrm{Cov}_{\tau_1 \tau_2}$$

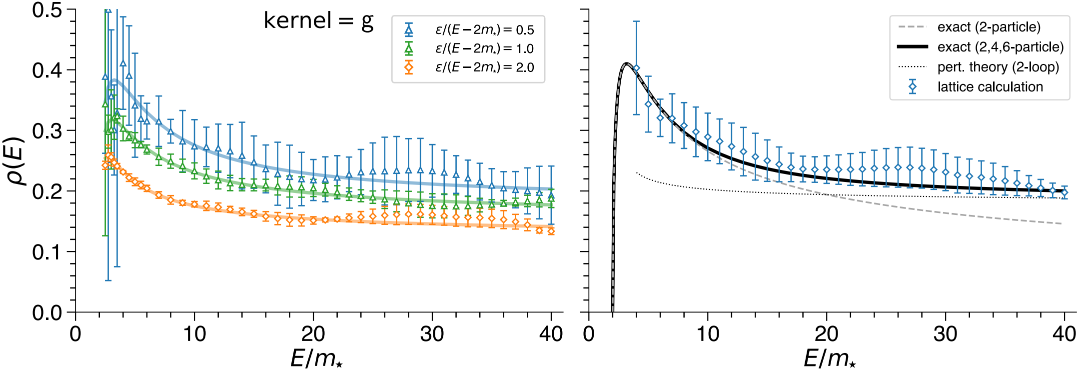
$$W_{\mathrm{n}}[\boldsymbol{g}] = A_{\mathrm{n}}[\boldsymbol{g}] + \lambda B[\boldsymbol{g}]$$



PG: look, i'm exhausted!

what the hell do these obscure mathematical objects have to do with QCD phenomenology??

LG: look, after barata and fredenhagen, a robust numerical method was needed...



the R-ratio in the non-linear $O(3)$ sigma-model in two dimensions

PG: come on, a model in two dimensions. . .

what the hell do these obscure mathematical objects have to do with QCD phenomenology????

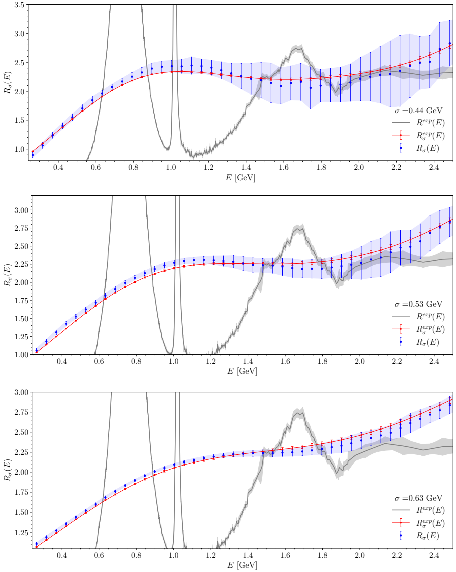
LG: ok, let's try again. . .

PHYSICAL REVIEW LETTERS **130,** 241901 (2023)

Probing the Energy-Smeared *R* Ratio Using Lattice QCD

Constantia Alexandrou,[1,2] Simone Bacchio,[2] Alessandro De Santis,[3] Petros Dimopoulos,[4]
Jacob Finkenrath,[2] Roberto Frezzotti,[3] Giuseppe Gagliardi,[5] Marco Garofalo,[6] Kyriakos Hadjiyiannakou,[1,2]
Bartosz Kostrzewa,[7] Karl Jansen,[8] Vittorio Lubicz,[9] Marcus Petschlies,[6] Francesco Sanfilippo,[5] Silvano Simula,[5]
Nazario Tantalo,[3,*] Carsten Urbach,[6] and Urs Wenger[10]

(Extended Twisted Mass Collaboration (ETMC))



$$\sigma \simeq 500 \text{ MeV}\,, \qquad \frac{\Delta R_\sigma}{R_\sigma} \simeq 2\%$$

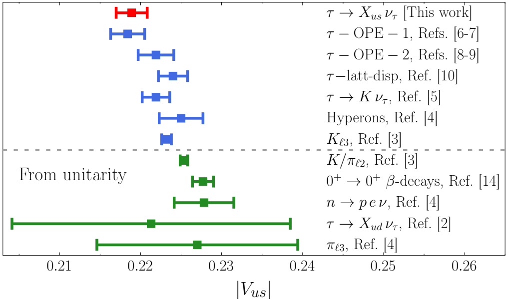
f.sanfilippo talk

PHYSICAL REVIEW LETTERS **132,** 261901 (2024)

Inclusive Hadronic Decay Rate of the τ Lepton from Lattice QCD: The $\bar{u}s$ Flavor Channel and the Cabibbo Angle

Constantia Alexandrou,[1,2] Simone Bacchio,[2] Alessandro De Santis,[3] Antonio Evangelista,[3] Jacob Finkenrath,[4] Roberto Frezzotti,[3] Giuseppe Gagliardi,[5] Marco Garofalo,[6] Bartosz Kostrzewa,[7] Vittorio Lubicz,[8] Simone Romiti,[6] Francesco Sanfilippo,[5] Silvano Simula,[5] Nazario Tantalo,[3] Carsten Urbach,[6] and Urs Wenger[9]

(Extended Twisted Mass Collaboration)



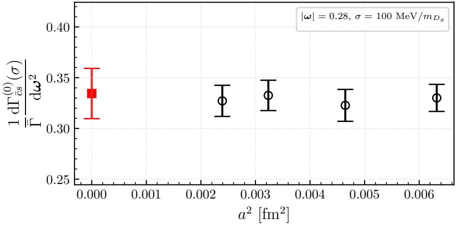
$$\frac{\Delta V_{us}}{V_{us}} \simeq 0.9\%$$

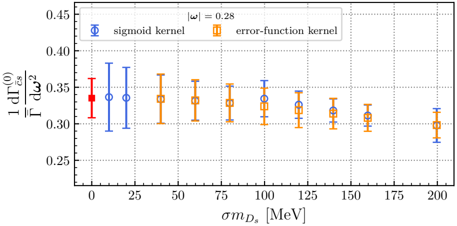
exp. dominated

arXiv:2504.06064, arXiv:2504.06063

Inclusive semileptonic decays of the D_s meson:
Lattice QCD confronts experiments

Alessandro De Santis [1,2] Antonio Evangelista [3] Roberto Frezzotti [3] Giuseppe Gagliardi [4,5] Paolo Gambino [6] Marco Garofalo [7] Christiane Franziska Groß [7] Bartosz Kostrzewa [7] Vittorio Lubicz [4,5] Francesca Margari [3] Marco Panero [6,8] Francesco Sanfilippo [5] Silvano Simula [5] Antonio Smecca [9] Nazario Tantalo [3] and Carsten Urbach [7]





arXiv:2504.06064, arXiv:2504.06063

Inclusive semileptonic decays of the D_s meson:
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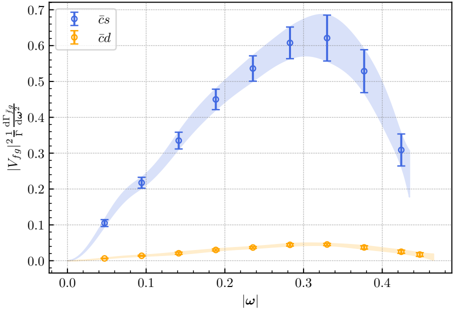
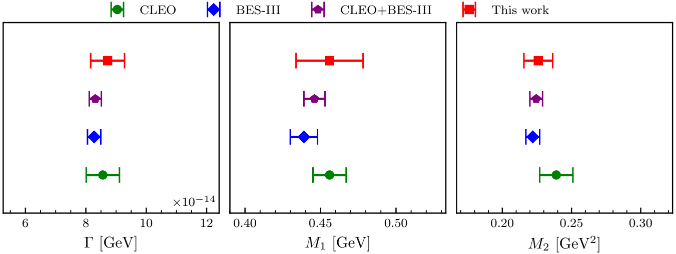
Alessandro De Santis [1,2] Antonio Evangelista [3] Roberto Frezzotti [3] Giuseppe Gagliardi [4,5]
Paolo Gambino [6] Marco Garofalo [7] Christiane Franziska Groß [7] Bartosz Kostrzewa [7]
Vittorio Lubicz [4,5] Francesca Margari [3] Marco Panero [6,8] Francesco Sanfilippo [5]
Silvano Simula [5] Antonio Smecca [9] Nazario Tantalo [3] and Carsten Urbach [7]

Extended Twisted Mass Collaboration





$$\frac{\Delta\Gamma}{\Gamma} \simeq 6\%$$ stat. dominated

r.frezzotti et al. Phys.Rev.D 108 (2023)
a.patella, n.tantalo JHEP 01 (2025)

not yet $B \mapsto X\ell\bar{\nu}$ but we are working on it...

as well as on non-local weak matrix elements...

and exclusive non-leptonic processes...

j.barata, k.fredenhagen CMP 138 (1991)
...
m.t.hansen, h.meyer, d.robaina Phys.Rev.D 96 (2017)
s.hashimoto, PTEP 2017 (2017)
m.hansen, a.lupo, n.tantalo, Phys.Rev.D 99 (2019)
j.bulava, m.t.hansen Phys.Rev.D 100 (2019)
p.gambino, s.hashimoto Phys.Rev.Lett. 125 (2020)
...
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to summarize

the bring-home message from the LG to the PG is:

we have realized that axiomatic field theory opened a new route to QCD phenomenology!

ANY AMPLITUDE can be written in terms of euclidean correlators as

$$\mathcal{A}_{n\mapsto m} = \sum_{t_1,\dots t_{n+m}} g_{t_1,\dots t_{n+m}}\, C(t_1,\cdots,t_m)$$

and is therefore computable and the spectral-density approach WORKS!

$$\mathcal{A}_{n\mapsto m} = \sum_{t_1,\ldots t_{n+m}} g_{t_1,\ldots t_{n+m}} C(t_1,\cdots,t_m)$$

a new route has been opened...

$$\mathcal{A}_{n\mapsto m} = \sum_{t_1,\ldots t_{n+m}} g_{t_1,\ldots t_{n+m}} \, C(t_1, \cdots, t_m)$$

a new route has been opened...

but this doesn't mean that it will be a comfortable journey...

$$\mathcal{A}_{n\mapsto m} = \sum_{t_1,\ldots t_{n+m}} g_{t_1,\ldots t_{n+m}} \, C(t_1,\cdots,t_m)$$

a new route has been opened. . .

but this doesn't mean that it will be a comfortable journey. . .

therefore, it is even more important than before to

invest on as much precise as possible lattice calculations in order to

exploit the full discovery potential of past and future experiments!