Relativistic spin operator

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Introduction

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How the states $|k, \sigma\rangle$ are defined?

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The standard Poincaré algebra

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\begin{aligned}
&[J_i, J_j] = i\varepsilon_{ijk} J_k, && [K_i, K_j] = -i\varepsilon_{ijk} J_k, \\
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$$U(\Lambda)|k,\sigma\rangle = \mathcal{D}^s_{\lambda\sigma}(R(\Lambda,k))|\Lambda k,\lambda\rangle,$$

$R(\Lambda,k) = L^{-1}_{\Lambda k}\Lambda L_k$ is a rotation, so called Wigner rotation,
$\mathcal{D}^s$ – spin s irreducible, unitary representation of the rotation group $SO(3)$.

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To obtain this formula one generates the vectors $|k,\sigma\rangle$ from the standard vector $|\tilde{k},\sigma\rangle$:

$$|k,\sigma\rangle = U(L_k)|\tilde{k},\sigma\rangle,$$

$\tilde{k} = m(1,0,0,0)$ – four-momentum of a particle at rest,
L_k – standard boost defined by conditions $k = L_k\tilde{k}$, $L_{\tilde{k}} = I$.

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σ – spin of a particle at rest.

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s – spin of a particle,
$\hat{W}^\mu = \frac{1}{2}\epsilon^{\nu\gamma\delta\mu}\hat{P}_\nu \hat{J}_{\gamma\delta}$ – the Pauli-Lubanski four-vector, ($\epsilon^{0123} = 1$),
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Different choices of $\hat{\mathbf{Q}}$ lead to different spin operators.

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I will discuss the following ones:

- The Newton–Wigner position operator

$$\hat{\mathbf{Q}}_{NW} = -\frac{1}{2}\left[\frac{1}{\hat{P}^0}\hat{\mathbf{K}} + \hat{\mathbf{K}}\frac{1}{\hat{P}^0}\right] - \frac{\hat{\mathbf{P}} \times \hat{\mathbf{W}}}{m\hat{P}^0(m + \hat{P}^0)}.$$

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$\hat{\mathbf{S}}_{CM}$ is not a proper spin observable.

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Explicitly, Czachor used the following spin projection operator

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The above operator has a proper spectrum but is a nonlinear function of $\mathbf{a}$.

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This operator in abstract form was for the first time discussed by Pryce in 1935.

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For spin $s = 1$:

$$S^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, S^2 = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, S^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

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Such an approach has been used in many papers, a lot of them have been devoted to a spin operator for a Dirac particle.

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Definitions of various relativistic spin operators for a Dirac particle which appear in the literature:

1. $\hat{\mathbf{S}}_D = -\frac{1}{2}\hat{\boldsymbol{\Sigma}}$
2. $\hat{\mathbf{S}}_{NW} = -\frac{|\hat{P}^0|}{2m}\hat{\boldsymbol{\Sigma}} + \frac{\hat{\mathbf{P}}(\hat{\mathbf{P}}\cdot\hat{\boldsymbol{\Sigma}})}{2m(m+|\hat{P}^0|)} + \frac{i\hat{P}^0}{2m|\hat{P}^0|}\hat{\mathbf{P}} \times \boldsymbol{\alpha}$
3. $\hat{\mathbf{S}}_{FW} = -\frac{1}{2}\hat{\boldsymbol{\Sigma}} - \frac{i\beta}{2|\hat{P}^0|}\hat{\mathbf{P}} \times \boldsymbol{\alpha} + \frac{\hat{\mathbf{P}}\times(\hat{\boldsymbol{\Sigma}}\times\hat{\mathbf{P}})}{2|\hat{P}^0|(m+|\hat{P}^0|)}$
4. $\hat{\mathbf{S}}_C = -\frac{m^2}{2\hat{P}^{02}}\hat{\boldsymbol{\Sigma}} - \frac{im\beta}{2\hat{P}^{02}}\hat{\mathbf{P}} \times \boldsymbol{\alpha} - \frac{\hat{\mathbf{P}}\cdot\hat{\boldsymbol{\Sigma}}}{2\hat{P}^{02}}\hat{\mathbf{P}}$
5. $\hat{\mathbf{S}}_F = -\frac{1}{2}\hat{\boldsymbol{\Sigma}} - \frac{i\beta}{2m}\hat{\mathbf{P}} \times \boldsymbol{\alpha}$
6. $\hat{\mathbf{S}}_{Ch} = -\frac{1}{2}\hat{\boldsymbol{\Sigma}} + \frac{i}{2m}\hat{\mathbf{P}} \times \boldsymbol{\alpha} - \frac{\hat{\mathbf{P}}\times(\hat{\boldsymbol{\Sigma}}\times\hat{\mathbf{P}})}{2m(m+|\hat{P}^0|)}$
7. $\hat{\mathbf{S}}_P = -\frac{1}{2}\beta\hat{\boldsymbol{\Sigma}} - \frac{i\alpha_3\alpha_2\alpha_1(\beta+1)\boldsymbol{\alpha}\cdot\hat{\mathbf{P}}}{2\hat{\mathbf{P}}^2}\hat{\mathbf{P}}$

$\boldsymbol{\Sigma} = \frac{1}{2i}\boldsymbol{\alpha} \times \boldsymbol{\alpha}$, $\beta = \gamma^0$, $\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$, γ^μ – Dirac matrices.

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D – Dirac, *NW* – Newton-Wigner, *FW* – Foldy-Wouthuysen (mean-spin operator), *C* – Czachor, *F* – Frenkel, *Ch* – Chakrabarti, *P* –Pryce.

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For a Dirac particle it is equivalent to the so called Foldy-Wouthuysen mean-spin operator.

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Let us consider two spin-1/2 particles with four-momenta

$$k^\mu = m(\sqrt{4x+1}, \sqrt{x}, 0, -\sqrt{3x}),$$
$$p^\mu = m(\sqrt{4x+1}, -\sqrt{x}, 0, -\sqrt{3x}),$$

where

$$x = \frac{\mathrm{W}^2}{4m^2} - 1.$$

W – invariant total energy of the two-particle system in the center of mass frame.

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We plot the left-hand-side of the CHSH inequality of two such particles in a pure scalar state for the following configuration: $\mathbf{a} = (0,0,1)$, $\mathbf{b} = (0,0,1)$, $\mathbf{c} = (\frac{\sqrt{3}}{2}, 0, -\frac{1}{2})$, $\mathbf{d} = (\frac{\sqrt{3}}{2}, 0, \frac{1}{2})$.

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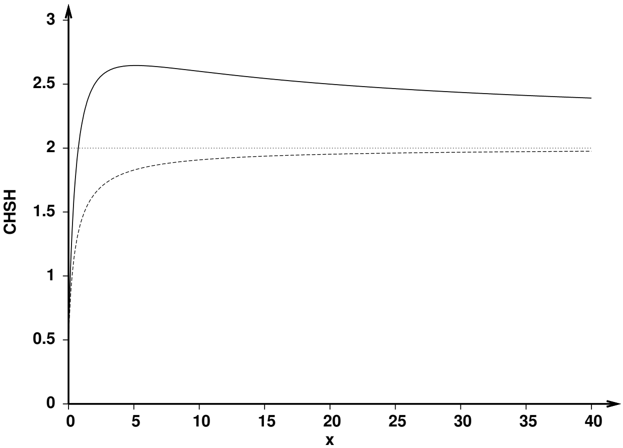

Figure: Violation of CHSH inequality by correlation functions calculated with the help of Czachor (dashed line) and Newton-Wigner (solid line) spin operators. Figure from [PRA 79 (2009) 014102]

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  is the most appropriate one.
- Which spin operator is measured at colliders??