Maximum Magic in Two-Qubit States

Based on 2502.17550 with Qiaofeng Liu and Ian Low

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Magic

What separates “quantum” from “classical”?

- Entanglement
- Supremacy in quantum computing

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- Supremacy in quantum computing

Entanglement is not enough to guarantee advantage of quantum computing

- Gottesman-Knill theorem: Classical algorithms can efficiently simulate stabilizer states going through quantum circuits consisting of Clifford gates
- Clifford gates: single qubit Hadamard gate H and phase gate S; 2-qubit CNOT gate

Clifford gates

- Consider a qubit: $|\psi\rangle = c_1|0\rangle + c_2|1\rangle \to \{c_1, c_2\}$

$$H = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

- Consider 2 qubits:
$|\psi\rangle = c_1|00\rangle + c_2|01\rangle + c_3|10\rangle + c_4|11\rangle \to \{c_1, c_2, c_3, c_4\}$

$$\mathsf{CNOT}_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- Clifford gates + T-gate → Universal quantum computation

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$

Stabilizer states

Stabilizer states: The computational basis going through Clifford gates

- 1-qubit examples: $\{1,0\} = |0\rangle$, $\{1,1\}/\sqrt{2} \propto (|0\rangle + |1\rangle)$, $\{1,i\}/\sqrt{2}, \cdots$
- 2-qubit examples: $\{1,0,0,0\} = |00\rangle$, $\{1,0,0,1\}/\sqrt{2} \propto |00\rangle + |11\rangle, \cdots$

The amount of "magic", i.e. non-stabilizerness: How much a state enables an advantage of quantum computing over classical algorithms

- Quantum resource theory: Clifford operations are "free", non-Clifford operations cost "currency"
- The amount of magic for a state corresponds to how much "currency" is needed to generate it from stabilizer states

Stabilizer Rényi entropy

A measure of magic: Stabilizer Rényi entropy (SRE)

Leone, Oliviero, Hamma, *Phys.Rev.Lett.* 128 (2022) 5, 050402

$$M_\alpha(|\psi\rangle) = \frac{1}{1-\alpha} \log \sum_{\mathcal{O} \in \mathcal{W}(d)} \frac{1}{d} |\langle \psi | \mathcal{O} | \psi \rangle|^{2\alpha},$$

where $\mathcal{W}(d)$ is the Weyl-Heisenberg (WH) group

- For a single qubit: $\mathcal{W}(2) = \{I, X, Y, Z\}$
- For n-qubit systems: $\mathcal{W}(2)^{\otimes n}$

Bound on SRE: WH SIC-POVM

It is quite straightforward to see that states corresponding to the WH symmetric informationally complete positive operator-valued measure (SIC-POVM) saturates SRE

Cuffaro, Fuchs, 2412.21083

- SIC-POVM: A set of projection operators $\Pi_i = |\psi_i\rangle\langle\psi_i|$, $i = 1, 2, \cdots, d^2$, such that

$$\mathrm{Tr}\,(\Pi_i \Pi_j) = |\langle\psi_i|\psi_j\rangle|^2 = \begin{cases} 1, & i = j \\ \frac{1}{d+1} & i \neq j \end{cases} .$$

- WH SIC: $\{|\psi_i\rangle\}$ consists of an orbit of the WH group
-

$$M_\alpha(|\psi\rangle) \leq \frac{1}{1-\alpha} \log \frac{1 + (d-1)(d+1)^{1-\alpha}}{d}, M_2(|\psi\rangle) \leq \log \frac{d+1}{2}.$$

Examples of WH SIC

For a single qubit,

$$\left\{\sqrt{6}+\sqrt{2},(1-i)\sqrt{2}\right\},\ \left\{\sqrt{3}+2-i,-1+\sqrt{3}i\right\},$$
$$\left\{\sqrt{3}-i,\sqrt{3}+2+i\right\},\ \left\{-1+\sqrt{3}i,\sqrt{3}+2-i\right\}$$

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$M_2(|\psi\rangle) = \log(3/2)$.
One can perform a search for maximal SRE by parameterizing $|\psi\rangle = c_1|0\rangle + c_2|1\rangle$ using

$$c_1 = \cos(\theta/2), \quad c_2 = \sin(\theta/2)e^{i\phi},$$

Then

$$M_2(|\psi\rangle) = -\log \frac{8\sin^4\theta\cos 4\phi + 4\cos 2\theta + 7\cos 4\theta + 53}{64}$$

Maximal magic for 2-qubit

It is known that for n-qubit systems, WH SICs only exist for $n = 1$ or 3

Godsil, Roy, *Eur.J.Combinatorics* 30 (2009) 246

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For the 2-qubit system, we can parameterize
$|\psi\rangle = c_1 |00\rangle + c_2 |01\rangle + c_3 |10\rangle + c_4 |11\rangle$ using

$$c_1 = \sin\theta_1 \sin\theta_2 e^{i\phi_1}, \quad c_2 = \sin\theta_1 \cos\theta_2 e^{i\phi_2},$$
$$c_3 = \cos\theta_1 \sin\theta_3 e^{i\phi_3}, \quad c_4 = \cos\theta_1 \cos\theta_3.$$

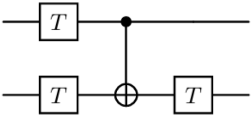
Then $M_2(|\psi\rangle) = -\log \Xi_2(|\psi\rangle)$,

$$\begin{aligned}
\Xi_2(|\psi\rangle) \;=\; & \frac{3}{32}\sin^4(2\theta_1)\sin^2(2\theta_2)\sin^2(2\theta_3)\left[2\sin^2\phi_3\sin^2(\phi_2-\phi_1)+2\sin^2\phi_2\sin^2(\phi_3-\phi_1)+2\sin^2\phi_1\sin^2(\phi_2-\phi_3)\right.\\
& \left.+1\right] + 2\sin^4\theta_1\cos^4\theta_1\left[24\sin^2\theta_2\cos^2\theta_2\sin^2\theta_3\cos^2\theta_3\left(\cos^2\phi_3\cos^2(\phi_2-\phi_1)+\cos^2\phi_2\cos^2(\phi_3-\phi_1)\right.\right.\\
& \left.+\cos^2\phi_1\cos^2(\phi_2-\phi_3)\right)+\cos^4\theta_2\left(\sin^4\theta_3(\cos(4\phi_2-4\phi_3)+6)+\cos^4\theta_3(\cos(4\phi_2)+6)\right)\\
& \left.+\sin^4\theta_2\left(\sin^4\theta_3(\cos(4\phi_3-4\phi_1)+6)+\cos^4\theta_3(\cos(4\phi_1)+6)\right)\right]\\
& +2\sin^8\theta_1\sin^4\theta_2\cos^4\theta_2(\cos(4\phi_2-4\phi_1)+6)+\sin^8\theta_1\sin^8\theta_2+\sin^8\theta_1\cos^8\theta_2\\
& +\cos^8\theta_1\left[2\sin^4\theta_3\cos^4\theta_3(\cos(4\phi_3)+6)+\sin^8\theta_3+\cos^8\theta_3\right].
\end{aligned}$$

WH MUBs saturates magic

480 states saturate SRE with $\max M_2(|\psi\rangle) = \log(16/7) < \log(5/2)$

- Example: $\{1, i, i, i\}/2$, which can be generated by the stabilizer state $\{1, 1, 1, 1\}/2$ going through the following circuit:



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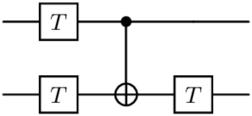

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The maximal magic states as well as stabilizer states form mutually unbiased bases (MUBs)

- Two orthonormal bases $\{|\psi_i\rangle\}$ and $\{|\chi_i\rangle\}$ are MU if $|\langle\psi_i|\chi_j\rangle|^2 = 1/d$ for any i, j
- The number of MUBs for dimension d is bounded by $d + 1$, which can be saturated when d is a prime power
- A set of WH MUBs consist of d MUBs that form a single orbit of the WH group

MUBs for 2 qubits

60 stabilizer states can be grouped into 15 orthonormal bases

- Each basis corresponds to a single orbit of the WH group
- 9 of the bases not entangled ($\Delta = 0$), the other 6 maximally entangled ($\Delta = 1$)

480 states can be grouped into 30 sets of WH MUBs

- Each one of the basis in the stabilizer states can form 5 MUBs with 2 sets of the WH MUBs
- $\Delta = 0$ stabilizer states are associated with $\Delta = 1/\sqrt{2}$ maximal magic states, $\Delta = 1$ stabilizer states with $\Delta = 1/2$ maximal magic states

MUB example

Example:

- Stabilizer states that form an orthonormal basis, which is also an orbit of the WH group, with $\Delta = 0$

$$\{1,0,0,0\},\{0,0,1,0\},\{0,1,0,0\},\{0,0,0,1\}$$

- 4 MUBs formed by maximal magic states, corresponding to an orbit of $\mathcal{W}(2)^{\otimes 2}$, with $\Delta = 1/\sqrt{2}$

$$\begin{aligned}&\{1,-1,-1,i\},\{1,-1,1,-i\},\{1,1,-1,-i\},\{1,1,1,i\};\\&\{-1,i,1,-1\},\{i,1,i,-i\},\{-1,-i,1,1\},\{i,-1,i,i\};\\&\{-1,1,i,-1\},\{-1,1,-i,1\},\{i,i,1,-i\},\{i,i,-1,i\};\\&\{i,-1,-1,1\},\{1,i,-i,i\},\{1,-i,i,i\},\{i,1,1,1\}.\end{aligned}$$

Conclusions

- Conjecture: WH MUBs saturates SRE when WH SICs do not exist, when $\alpha \geq 2$
- Do these states that saturate SRE also saturates other measures of magic?
- What kind of theory can saturate magic for 2-qubits in its final state from particle scattering?

Liu, Low, **ZY**, 2503.03098