Spin correlations for di-boson systems: a look from every angle

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mostly based on M. Grossi, GP, A. Vicini JHEP 12 (2024) 120 [2409.16731 [hep-ph]]

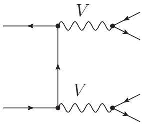
Introduction

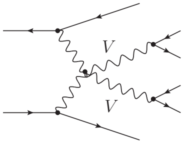
Motivations

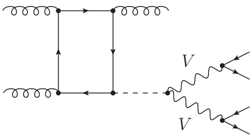
Precise measurements of multi-boson processes with Run-3 & High-Lumi LHC data.

Polarisation & spin correlations of EW bosons

- non trivial to extract
- crucial probes of gauge and scalar sector in SM
- discriminate between SM and new physics
- inputs to contruct quantum observables

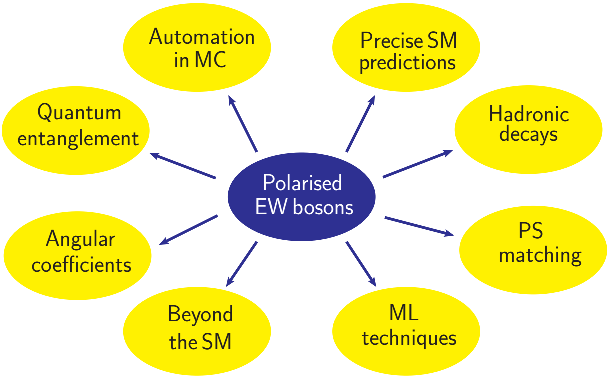






Di-boson: simplest, non-trivial spin correlations with EW bosons

What is needed from the theory side?



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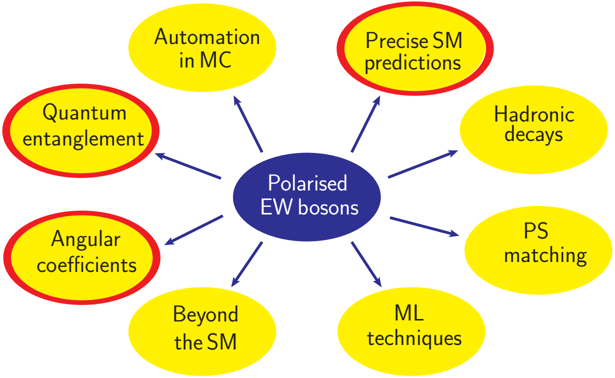
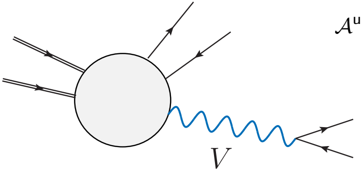


General framework

Amplitude structure

Single resonant boson (in pole/narrow-width approximation):



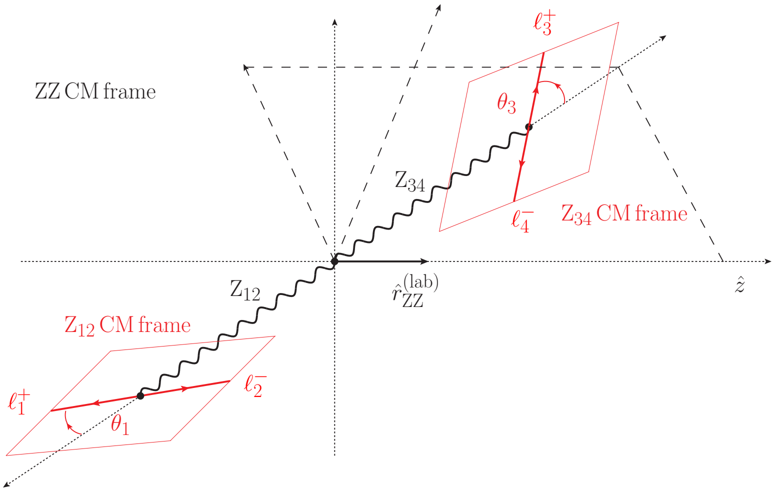
$$\begin{aligned}\mathcal{A}^{\text{unpol}} &= \mathcal{P}_\mu \, \frac{-g^{\mu\nu}}{k^2 - M_V^2 + iM_V\Gamma_V} \, \mathcal{D}_\nu \\ &= \mathcal{P}_\mu \, \frac{\sum_{\lambda'} \varepsilon^{\mu}_{\lambda'} \varepsilon^{*\nu}_{\lambda'}}{k^2 - M_V^2 + iM_V\Gamma_V} \, \mathcal{D}_\nu \\ &= \sum_{\lambda'} \mathcal{P}_\mu \, \frac{\varepsilon^{\mu}_{\lambda'} \varepsilon^{*\nu}_{\lambda'}}{k^2 - M_V^2 + iM_V\Gamma_V} \, \mathcal{D}_\nu = \sum_{\lambda'} \mathcal{A}_{\lambda'}\end{aligned}$$

At the cross section level:

$$|\mathcal{A}^{\text{unpol}}|^2 = \underbrace{\sum_{\lambda} |\mathcal{A}_\lambda|^2}_{\text{incoherent sum}} + \underbrace{\sum_{\lambda \neq \lambda'} \mathcal{A}^*_\lambda \mathcal{A}_{\lambda'}}_{\text{interference terms}}$$

Polarisation vectors are defined in a specific Lorenzt frame.

Two bosons: geometric visualisation



Angular dependence for boson pairs

Two resonant bosons $(\theta_1, \phi_1, \theta_3, \phi_3)$ are ℓ_1^+, ℓ_3^+ angles in each V rest frame, w.r.t. V direction in some Lorentz frame (e.g. VV-CM):

$$\frac{d\sigma}{\mathrm{d}\cos\theta_1\,\mathrm{d}\phi_1\,\mathrm{d}\cos\theta_3\,\mathrm{d}\phi_3\,\mathrm{d}X} = \frac{d\sigma}{\mathrm{d}X}\Bigg[\frac{1}{(4\pi)^2} + \\ +\frac{1}{4\pi}\sum_{\ell=1}^{2}\sum_{m=-\ell}^{\ell}\alpha^{(1)}_{\ell m}(X)\,Y_{\ell m}(\theta_1,\phi_1) + \frac{1}{4\pi}\sum_{\ell=1}^{2}\sum_{m=-\ell}^{\ell}\alpha^{(3)}_{\ell m}(X)\,Y_{\ell m}(\theta_3,\phi_3) \\ +\sum_{\ell_1=1}^{2}\sum_{\ell_3=1}^{2}\sum_{m_1=-\ell_1}^{\ell_1}\sum_{m_3=-\ell_3}^{\ell_3}\gamma_{\ell_1 m_1 \ell_3 m_3}(X)\,Y_{\ell_1 m_1}(\theta_1,\phi_1)\,Y_{\ell_3 m_3}(\theta_3,\phi_3)\Bigg]$$

80 independent coefficients [Rahaman Singh 2109.09345, Aguilar-Saavedra et al. 2209.13441] extracted with projections or asymmetries

$$\int_{-1}^{1} d\cos\theta^* \int_0^{2\pi} d\phi^*\; Y_{\ell m}(\theta^*,\phi^*)\,\frac{d\sigma}{d\cos\theta^*\,d\phi^*\,dX} = \alpha_{\ell m} = \alpha_{\ell m}(X)$$

Remark: bosons correlated if $\alpha^{(1)}_{\ell_1,m_1}\cdot\alpha^{(3)}_{\ell_3,m_3} \neq \gamma_{\ell_1 m_1 \ell_3 m_3}$.

Issues in angular-coefficient extraction

In principle, coefficients extracted directly from data through projections up to $\ell_{1,3} = 2$ of the angular distributions $d\sigma/d\Omega_1 d\Omega_3$.

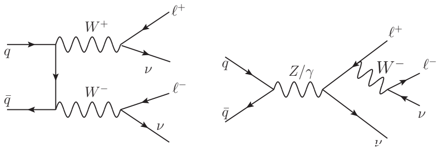
In practice, the analytic structure of $d\sigma/d\Omega_1 d\Omega_3$:

1. assumes two spin-1 resonances, and subsequent two-body decays,
2. its coefficient values not invariant under Lorentz boosts,
3. is not described by $\ell_{1,3} \leq 2$ if selection cuts/ν-reconstruction applied,
4. is not fully model independent (if NP in both production and decay)

Off-shell effects and two-body decays

Selecting resonant diagrams

Factorised amplitude not possible for all contributions



(1) Non-resonant diagrams regarded as non-resonant background.

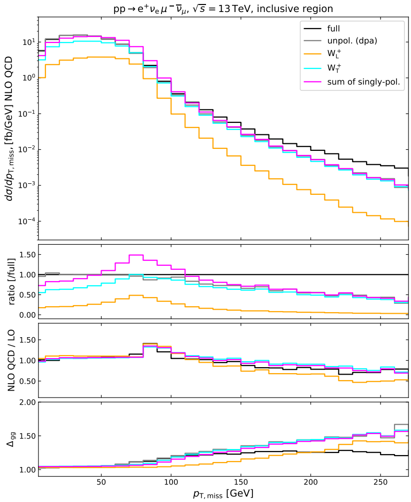
(2) Resonant diagrams treated with

DPA: double-pole approximation [Denner et al. 0006307]

NWA: spin-correlated narrow-width approximation [Artoisenet et al. 1212.3460]

(3) Compute polarisation and spin correlations

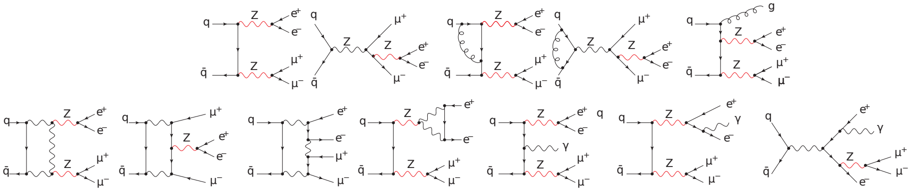
Off-shell effects in EW production: $\mathrm{W^+ W^-}$



$\mathrm{pp} \to \mathrm{e}^+\nu_\mathrm{e}\mu^-\bar{\nu}_\mu + X$ @ NLO QCD
fully inclusive [Denner GP 2006.14867]

- typically small in single-boson production with jets [Pellen et al. 2109.14336, 2204.12394]
- $\lesssim 1\%$ for Z bosons, owing to physical invariant-mass cuts [Denner GP 2107.06579, Le Baglio 2203.01470]
- larger for W bosons: up to 50% in $\mathrm{W^+ W^-}$ for $p_{\mathrm{T,\,mis}} \gtrsim 200\mathrm{GeV}$ [Denner GP 2006.14867, Poncelet Popescu 2102.13583]

Off-shell effects in EW production: ZZ



DPA vs full off-shell in $e^+e^-\mu^+\mu^-$ @ NLO EW+QCD [Grossi GP Vicini 2409.16731]

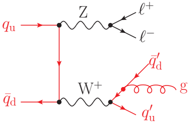
	σ			$\alpha^{(1)}_{2,0}$		
	full off-shell	DPA	δ	full off-shell	DPA	δ
LO	21.483(8)	21.209(7)	−1.28%	0.02977(2)	0.02982(3)	+0.16%
NLO QCD	28.63(1)	28.22(1)	−1.43%	0.02807(5)	0.02806(5)	−0.04%
NLO EW	19.15(1)	18.94(1)	−1.10%	0.02932(5)	0.02944(5)	+0.41%

Two-body decays

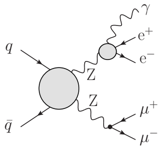
Angular analytic structure ($\ell \leq 2$) only valid for 2-body decays of EW bosons.

$\ell > 2$ spherical harmonics appear in 3-body decays: QED/QCD radiation off decays

QCD corrections to hadronic decays
[Denner Haitz GP 2211.09040]



EW corrections to leptonic decays
[Denner Haitz GP 2107.06579, Le Baglio 2203.01470,
Denner Haitz GP 2311.16031, Dao Le 2311.17027]



Angles defined after jet clustering/ lepton dressing

Small effects in QED, a bit larger in QCD: (≥ 3)-prong topologies suppressed

Radiative corrections and frame dependence

QCD + EW corrections in ZZ production

Computed NLO QCD+EW [Grossi GP Vicini 2409.16731] with PowHeg [Chiesa et al. 2005.12146, GP Zanderighi 2311.05220] and MoCaNLO [Denner GP 2107.06579] for $|M_{\ell\ell} - \mathrm{M_Z}| <$10GeV.

	LO	NLO QCD	NLO EW
$\alpha^{(1)}_{2,0}$	0.03009(11)	0.02794(13)	0.02960(11)
$\alpha^{(3)}_{2,0}$	0.03006(7)	0.02796(6)	0.02964(7)
$\gamma_{2,0,2,0}$	0.00188(2)	0.00168(2)	0.00187(2)
$\alpha^{(1)}_{1,0}$	−0.00001(9)	−0.00097(10)	−0.00004(9)
$\alpha^{(3)}_{1,0}$	0.00012(13)	−0.00086(15)	0.00018(14)
$\gamma_{1,0,1,0}$	−0.00173(3)	−0.00148(3)	−0.00043(3)
$\alpha^{(1)}_{2,-2}/\sqrt{3}$	−0.00967(7)	−0.00993(9)	−0.00991(7)
$\alpha^{(3)}_{2,-2}/\sqrt{3}$	−0.00973(4)	−0.01003(4)	−0.00996(4)

- large QCD effects on all coefficients
- QCD radiation changes longitudinal components
- large EW effect on $\gamma_{1,0,1,0}$: change in left-right balance
- higher orders needed: NNLO + gg missing.

Level of correlation between two (longitudinal) bosons, R_c [ATLAS 2211.09435]:

$$R_c = \frac{f_{\mathrm{LL}}}{f^{(1)}_{\mathrm{L}}\, f^{(3)}_{\mathrm{L}}} = \frac{1 - 4\sqrt{5\pi}\,\alpha^{(1)}_{2,0} - 4\sqrt{5\pi}\,\alpha^{(3)}_{2,0} + 80\pi\,\gamma_{2,0,2,0}}{1 - 4\sqrt{5\pi}\,\alpha^{(1)}_{2,0} - 4\sqrt{5\pi}\,\alpha^{(3)}_{2,0} + 80\pi\,\alpha^{(1)}_{2,0}\,\alpha^{(3)}_{2,0}} = \left\{ \begin{array}{lll} \mathrm{ZZ} & \mathrm{WZ} & \\ 1.89(3) & 2.39(3) & \mathrm{LO} \\ 1.73(2) & 1.31(2) & \mathrm{NLO\ QCD{+}EW} \end{array} \right.$$

$\ell = 1$ coefficients at NLO EW

Large NLO EW corrections to $\ell = 1$ coeff.: where do they come from?

Not from non-factorisable corr.
DPA vs off-shell for $\gamma_{1,0,1,0} = (3\,\eta_\ell{}^2/16\pi)\,(f_{--} + f_{++} - f_{-+} - f_{+-})$

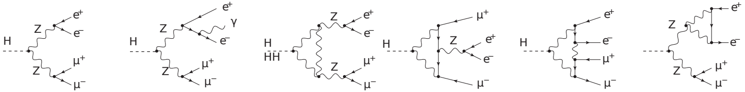
	DPA	off-shell
LO	$-0.001776(5)$	$-0.001708(2)$
NLO EW	$-0.000412(6)$	$-0.000411(9)$

Spin-analysing power η_ℓ ($Z \to \ell^+\ell^-$) $\longrightarrow$ -40% loop corr. ("LEP physics")
G_μ scheme (`MoCaNLO + Recola`):

$$\eta_\ell^{\mathrm{LO}} = \frac{1 - 4s_w^2}{1 - 4s_w^2 + 8s_w^4} = 0.21356\ldots \quad \longrightarrow \quad \eta_\ell^{\mathrm{NLO\,EW}} = 0.1313(4)$$

Further correction from NLO EW effects on $f_{\pm\pm}$, $f_{\pm\mp}$

Frame dependence and real-photon corrections in $h \to 4\ell$



Massless charged leptons dressed with real photons: IR-safe observables

Quantum tomography in Higgs rest frame (H-rest) or dressed 4ℓ CM frame (4ℓ-CM):

$M_{\ell^+\ell^-} > 10$ GeV	LO	NLO EW (4ℓ-CM)	δ_{EW}	NLO EW (H-rest)	δ_{EW}
$\alpha^{(1)}_{2,0}$	$-0.04792(4)$	$-0.04498(6)$	-6.1%	$-0.04215(6)$	-12.0%
$\alpha^{(3)}_{2,0}$	$-0.04792(4)$	$-0.04506(6)$	-6.0%	$-0.04224(6)$	-11.9%
$\gamma_{1,0,1,0}$	$0.00117(1)$	$0.00012(2)$	-90%	$0.00011(2)$	-91%
$\gamma_{2,0,2,0}$	$0.01097(1)$	$0.01079(2)$	-1.6%	$0.01019(2)$	-7.1%
$\gamma_{1,1,1,-1}$	$-0.00185(2)$	$-0.00048(2)$	-74%	$-0.00047(2)$	-75%
$\gamma_{2,1,2,-1}$	$-0.00776(1)$	$-0.00779(2)$	$+0.4\%$	$-0.00715(2)$	-7.9%
$\gamma_{2,2,2,-2}$	$0.00493(2)$	$0.00489(2)$	-0.8%	$0.00481(2)$	-2.4%

NLO EW corrections to $\ell = 2$ coeff.'s smaller in the 4ℓ-CM definition.

Smaller effects from lepton-dressing details.

Quantum observables in h $\rightarrow 4\ell$

Entanglement

PH criterion [Peres 9604005, Horodecki 9703004]: two spin-1 from decay of a spin-0 are entangled if and only if $\gamma_{2,1,2,-1} \neq 0$ or $\gamma_{2,2,2,-2} \neq 0$ at LO [Aguilar-Saavedra et al. 2209.13441]. What about NLO EW?

$M_{\ell^+\ell^-} > 10\,\text{GeV}$	LO	NLO EW (4ℓ-CM)	δ_{EW}	NLO EW (H-rest)	δ_{EW}
$\gamma_{2,1,2,-1}$	$-0.00776(1)$	$-0.00779(2)$	$+0.4\%$	$-0.00715(2)$	-7.9%
$\gamma_{2,2,2,-2}$	$0.00493(2)$	$0.00489(2)$	-0.8%	$0.00481(2)$	-2.4%

NLO EW corr. small, most zeros in ρ_{LO} still zeros at NLO [Grossi GP Vicini 2409.16731]

$$\alpha^{(1)}_{2,0} = \alpha^{(3)}_{2,0}, \gamma_{\ell,m,\ell,-m} = \gamma_{\ell,-m,\ell,m}\,, \qquad 20\pi\left(\gamma_{2,2,2,-2} + \gamma_{2,0,2,0}\right) = 1$$

$$40\pi\gamma_{2,2,2,-2} = \sqrt{20\pi}\alpha^{(1)}_{2,0} + 1\,, \qquad 5\eta_\ell^2\,\gamma_{2,1,2,-1} = \gamma_{1,1,1,-1}$$

$$\eta_\ell^2\,\left(1 - 20\pi\,\gamma_{2,0,2,0}\right) = 4\pi\gamma_{1,0,1,0}$$

A priori, no guarantee $\gamma_{2,1,2,-1} \neq 0$ or $\gamma_{2,2,2,-2} \neq 0$ still sufficient and necessary conditions: full NLO spin-density matrix needed, then apply PH

Bell-inequality violation

Quantum state close to maximal entanglement: Bell non-locality can be probed.

CGLMP inequality [Collins et al. 0106024] suitable for two qutrits: violated if

$$\mathcal{I}_3 = \langle \mathcal{O}_{\text{Bell}} \rangle = \text{Tr}\left(\rho\, \mathcal{O}_{\text{Bell}}\right) > 2\,.$$

For a scalar decaying into two spin-1 [Barr 2106.01377, Aguilar-Saavedra et al. 2209.13441],

$$\mathcal{I}_3 = \frac{1}{2} + \frac{4\sqrt{3}}{9} - \sqrt{5\pi}\left(1 - \frac{8\sqrt{3}}{9}\right)\alpha_{2,0} - 40\pi\left(\frac{2}{3} + \frac{4\sqrt{3}}{9}\right)\gamma_{2,1,2,-1} + \frac{20\pi}{3}\,\gamma_{2,2,2,-2}\,.$$

For $M_{\ell^+\ell^-} > 10\text{GeV}$ [Grossi GP Vicini 2409.16731] $\mathcal{I}_3$ robust under EW corr. (not guaranteed a priori)

$$\mathcal{I}_3^{(\text{LO})} = 2.671(2)\,, \qquad \mathcal{I}_3^{(\text{NLO EW},4\ell)} = 2.682(4)\,, \qquad \mathcal{I}_3^{(\text{NLO EW,H})} = 2.571(4)\,.$$

Importance of reference-frame choice for the spin quantisation.

$\rightarrow$ next talk by Priyanka

Fiducial cuts

Selection-cut effects

Angular expansion ($\ell \leq 2$ spherical harmonics) valid fully differentially,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta^*\,\mathrm{d}\phi^*\,\mathrm{d}X}\,.$$

Still valid after integration over independent variables X, if no cuts are applied on decay products,

$$\int_{\text{inc}} \frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta^*\,\mathrm{d}\phi^*\,\mathrm{d}X}\mathrm{d}X = \frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta^*\,\mathrm{d}\phi^*}\,.$$

Fiducial cuts on decay products introduce angular modulation f_{cut} which is not a combination of $\ell \leq 2$ spherical harmonics,

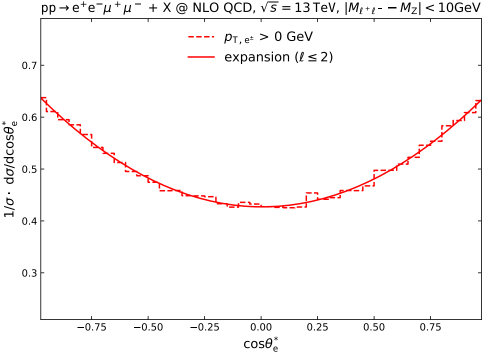
$$\int_{\text{cut}} \frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta^*\,\mathrm{d}\phi^*\,\mathrm{d}X}\mathrm{d}X = \frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta^*\,\mathrm{d}\phi^*} f_{\text{cut}}(\theta^*,\phi^*)\,.$$

Therefore

$$\int_{-1}^{1} d\cos\theta^* \int_0^{2\pi} d\phi^*\; Y_{\ell m}(\theta^*,\phi^*)\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta^*\,\mathrm{d}\phi^*} f_{\text{cut}}(\theta^*,\phi^*) \neq \sigma_{\text{cut}}\alpha_{\ell m}\,.$$

Expansion up to $\ell \leq 2$: ZZ production

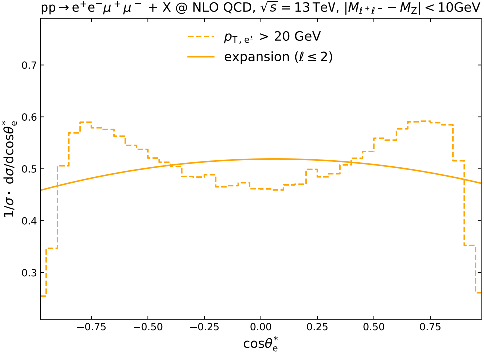
$\mathrm{pp} \to \mathrm{e}^+\mathrm{e}^-\mu^+\mu^- + X$: polar decay angle of first Z boson (defined in ZZ CM frame)



fully inclusive

Expansion up to $\ell \leq 2$: ZZ production

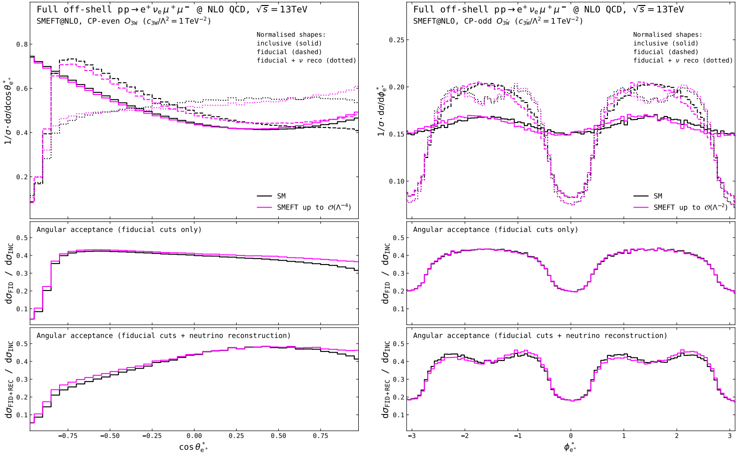
$\mathrm{pp} \to \mathrm{e}^+\mathrm{e}^-\mu^+\mu^- + X$: polar decay angle of first Z boson (defined in ZZ CM frame)



cut setup

Extrapolation and model (in)dependence

Fiducial cuts disrupt $\ell \leq 2$ expansion: extrapolation needed.
Compared WZ($3\ell\nu$) angular acceptances [Grossi GP Vicini 2409.16731] in SM and in presence of dim-6 operators O_{3W} (CP-even), $O_{3\tilde{W}}$ (CP-odd) [El Faham GP Vryonidou 2405.19083].



Failure of Standard-Model extrapolation

Standard SM extrapolation [Grossi GP Vicini 2409.16731]:

- take cut SMEFT distributions (mock data)
- undo cuts according to SM angular acceptance (with/without ν-reco.)
- apply usual projection to extract coefficients

$c_{3\,\mathrm{W}}$, $c_{3\,\tilde{\mathrm{W}}}$ =1TeV^{-2}	CP-even $O_{3\,\mathrm{W}}$		CP-odd $O_{3\,\tilde{\mathrm{W}}}$	
	$\alpha^{(1)}_{1,0}$	$\alpha^{(1)}_{2,0}$	$\alpha^{(1)}_{2,-2}$	$\alpha^{(1)}_{2,2}$
SM, truth	−0.0464(1)	0.0283(1)	−0.0122(2)	0.0001(2)
SMEFT, truth	**−0.0411(1)**	**0.0313(1)**	**−0.0120(2)**	**−0.0066(2)**
Extrapol., no ν-reco.	−0.038(2)	0.039(2)	−0.0116(3)	−0.0082(3)
Extrapol., with ν-reco.	−0.049(2)	0.041(2)	−0.0119(3)	0.0035(3)

Fails in presence of (small) NP effects.

Model-independent extrapolation: polarised-template for decay-like observables?

Conclusions

Conclusions

Rich phenomenology from decay angles in di-boson (or two-qutrit) systems at LHC:

- ★ off-shell and radiative-decay effects: small for EW prod., larger for $h \to 4\ell$
- ★ QCD and EW corr. sizeably change angular coeff.
- ★ choose suitable Lorentz frame, especially with radiative corr.
- ★ LO picture may be enough for some quantum obs.
- ★ scrutinise NLO effects: tools available
- ★ fiducial cuts/ν-reco. disrupt $\ell \leq 2$ expansion: extrapolation needed

Backup

Lorentz-frame dependence in WZ production

Helicity states defined in specific Lorentz frame: same structure, coeff. values change.

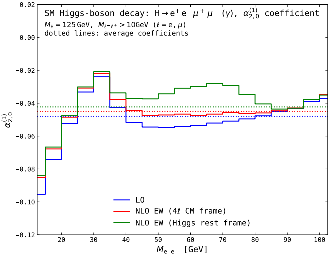
- modified helicity coord. sys (VV CM) [ATLAS 1902.05759] more natural than lab., reference axis only affects azimuthal coefficients

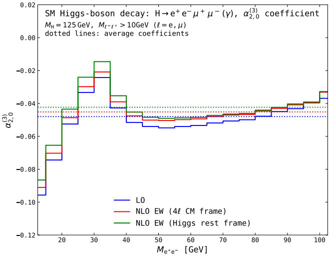
DPA fractions	LO	NLO QCD	K-factor
VV CM frame			
$W^+_L Z_L$	7.9%	5.7%	1.31
$W^+_L Z_T$	10.6%	15.5%	2.65
$W^+_T Z_L$	9.9%	14.7%	2.68
$W^+_T Z_T$	70.9%	63.5%	1.62
LAB frame			
$W^+_L Z_L$	5.7%	6.0%	1.91
$W^+_L Z_T$	16.5%	17.6%	1.93
$W^+_T Z_L$	22.9%	21.4%	1.69
$W^+_T Z_T$	55.0%	54.8%	1.80

- Lorentz-frame choice changes size of higher-order corrections, even QCD ISR in fully leptonic channel. WZ in fiducial setup subject to giant QCD K-factors [Rubin Salam Sapeta 1006.2144] that differ for various polarisation modes [Denner GP 2010.07149].

More on off-shell effects $h \to 4\ell$

Dependence of coefficients on $M_{\ell^+\ell^-}$ [Grossi GP Vicini 2409.16731]. $\alpha_{(2,0)}$ in figure.
Setup: $M_{e^+e^-}, M_{\mu^+\mu^-} > 10\text{GeV}$.





Strong dependence of coefficients on off-shell-ness already at LO. 4ℓ-CM definition better behaved at NLO EW.

$\alpha^{(1)}_{2,0} = \alpha^{(3)}_{2,0}$ relation broken at NLO EW if unequal cuts applied on $M_{\ell\ell}$.

Spin-correlations change value with tighter $M_{\ell\ell}$ cuts but preserve symmetric charachter ($\gamma_{2,+2,2,-2} = \gamma_{2,-2,2,+2}$).

Experimental results

We cannot directly measure the spin state of EW bosons but we can:

⋆ Extract spin-sensitive coefficients from angular distributions

Run-1 analyses mostly relied on angular-coefficient extraction:

- W/Z+jets [CMS 1104.3829, 1504.03512, 2008.04174, ATLAS 1203.2165, 1606.00689]
- $t\bar{t}$ [CMS 1605.09047, ATLAS 1612.02577, 2005.03799]

⋆ Perform fits of LHC data with polarised templates

Run-2 analyses mostly rely fits with polarised templates:

- WZ/ZZ [ATLAS 1902.05759, CMS 2110.11231, ATLAS 2211.09435, 2402.16365, 2310.04350]
- $W^\pm W^\pm$ scattering [CMS 2009.09429]

Two-qutrit formalism

Generic spin-density matrix:

$$\rho = \frac{1}{9}\left[(\mathbb{I}_3 \otimes \mathbb{I}_3) + A^{(1)}_{l_1,m_1}\left(T^{l_1}_{m_1} \otimes \mathbb{I}_3\right) + A^{(3)}_{l_3,m_3}\left(\mathbb{I}_3 \otimes T^{l_3}_{m_3}\right) + C_{l_1,m_1,l_3,m_3}\left(T^{l_1}_{m_1} \otimes T^{l_3}_{m_3}\right)\right]$$

T^l_m: irreducible tensor representations of the boson spin.

Our original, fully decayed formula is nothing but:

$$\frac{1}{\sigma}\frac{d\sigma}{d\Omega_1 d\Omega_3} = \left(\frac{3}{4\pi}\right)^2 \mathrm{Tr}\left\{\rho\left(\Gamma^{(1)} \otimes \Gamma^{(3)}\right)^{\mathsf{T}}\right\}.$$

$\Gamma^{(1,3)}$: decay matrices.

$$\sqrt{40\pi}\,\alpha^{(i)}_{2,m_i} = A^{(i)}_{2,m_i}\,, \qquad 40\pi\gamma_{2,m_1,2,m_3} = C_{2,m_1,2,m_3}\,, \qquad 8\pi\gamma_{1,m_1,1,m_3}/\eta_\ell^2 = C_{1,m_1,1,m_3}\,.$$

$$\mathrm{Tr}\left(\mathbb{I}_3\Gamma^{\mathsf{T}}\right) \propto Y_{00}(\theta,\phi)\,, \qquad \mathrm{Tr}\left(T^l_m\Gamma^{\mathsf{T}}\right) \propto Y_{lm}(\theta,\phi)$$

$$\begin{aligned}\mathcal{O}_{\mathrm{Bell}} \quad = \quad & \frac{2}{3\sqrt{3}}(T^1_1 \otimes T^1_1 - T^1_0 \otimes T^1_0 + T^1_1 \otimes T^1_{-1}) + \frac{1}{12}(T^2_2 \otimes T^2_2 + T^2_2 \otimes T^2_{-2}) \\ + \quad & \frac{1}{2\sqrt{6}}(T^2_2 \otimes T^2_0 + T^2_0 \otimes T^2_2) - \frac{1}{3}(T^2_1 \otimes T^2_1 + T^2_1 \otimes T^2_{-1}) + \frac{1}{4}T^2_0 \otimes T^2_0\,, +\mathrm{h.c.}\end{aligned}$$

Polar-angle correlations

$$
\begin{aligned}
\alpha^{(i)}_{1,0} &= \frac{1}{4}\sqrt{\frac{3}{\pi}}\,\eta_i\,\left(f^{(i)}_{+} - f^{(i)}_{-}\right)\,, \\
\alpha^{(i)}_{2,0} &= \frac{1-3\,f^{(i)}_{\mathrm{L}}}{4\sqrt{5\pi}}\,, \\
\gamma_{1,0,1,0} &= 3\,\eta_1\eta_3\,\frac{f_{--}+f_{++}-f_{-+}-f_{+-}}{16\pi}\,, \\
\gamma_{2,0,2,0} &= \frac{1-3f^{(1)}_{\mathrm{L}}-3f^{(3)}_{\mathrm{L}}+9f_{\mathrm{L\,L}}}{80\pi}\,, \\
\gamma_{1,0,2,0} &= \sqrt{\frac{3}{5}}\,\eta_1\,\frac{3\left(f_{-\,\mathrm{L}}-f_{+\,\mathrm{L}}\right)-\left(f^{(1)}_{-}-f^{(1)}_{+}\right)}{16\pi}\,, \\
\gamma_{2,0,1,0} &= \sqrt{\frac{3}{5}}\,\eta_3\,\frac{3\left(f_{\mathrm{L}-}-f_{\mathrm{L}+}\right)-\left(f^{(3)}_{-}-f^{(3)}_{+}\right)}{16\pi}\,.
\end{aligned}
$$