Physics and Geometry of Line Bundles over Calabi-Yau Manifolds

Andrei Constantin

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Work done in collaboration with Steve Abel, Lara Anderson, Callum Brodie, Cristofero Fraser-Taliente, James Gray, Thomas Harvey, Luca Nutricati, Lucas Leung, Andre Lukas, Burt Ovrut, Eran Palti, Fabian Ruehle and Elijah Sheridan.

2410.17704, 2402.01615, 2401.14463, 2306.03147, 2112.12107, 1307.4787

THE ROYAL SOCIETY

Pollica Workshop on Calabi-Yau Manifolds, 6 June 2025

VACUUM CONFIGURATIONS FOR SUPERSTRINGS

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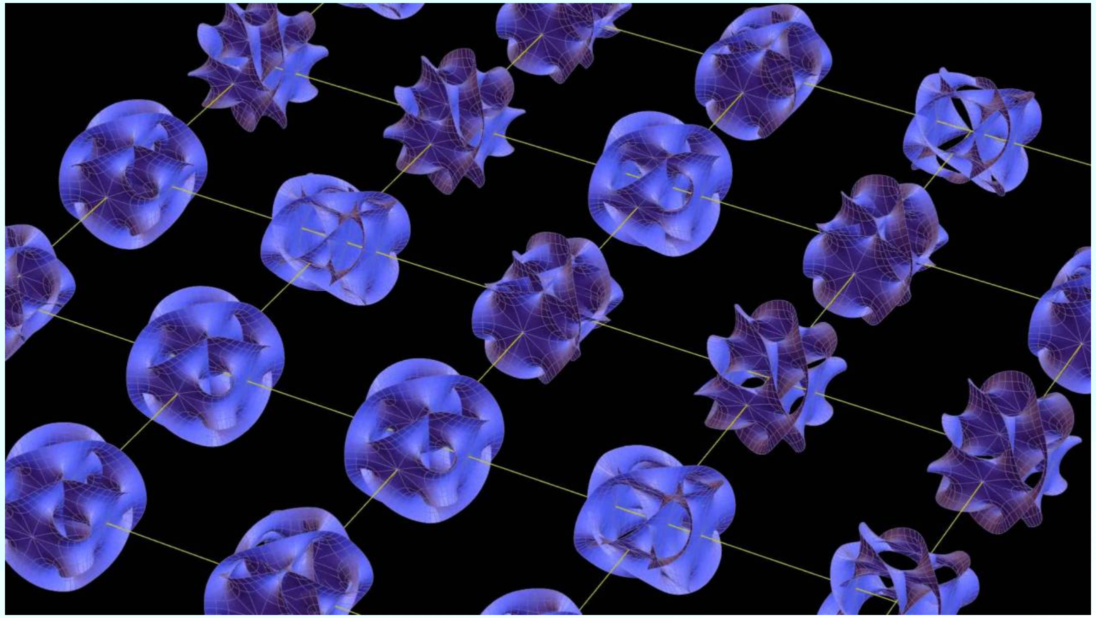
Received 2 January 1985

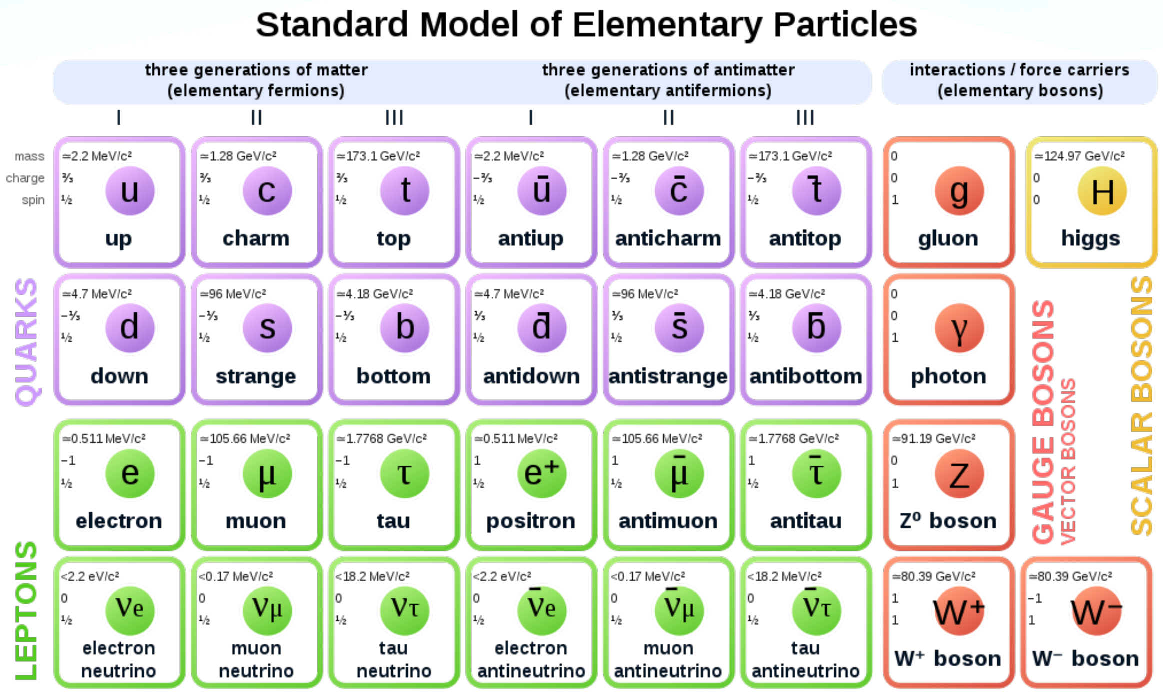
We study candidate vacuum configurations in ten-dimensional O(32) and $E_8 \times E_8$ supergravity and superstring theory that have unbroken $N = 1$ supersymmetry in four dimensions. This condition permits only a few possibilities, all of which have vanishing cosmological constant. In the $E_8 \times E_8$ case, one of these possibilities leads to a model that in four dimensions has an E_6 gauge group with four standard generations of fermions.

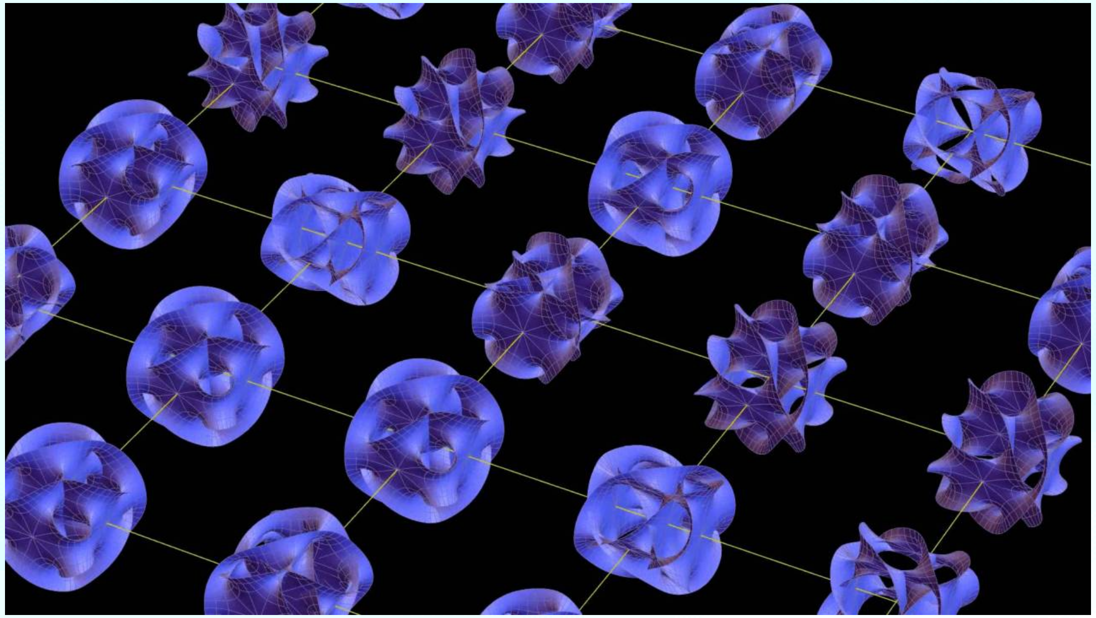
mathematical abstraction meets physical necessity

Nucl. Phys. B 258 (1985)



String theory becomes predictive only after specifying a vacuum solution.





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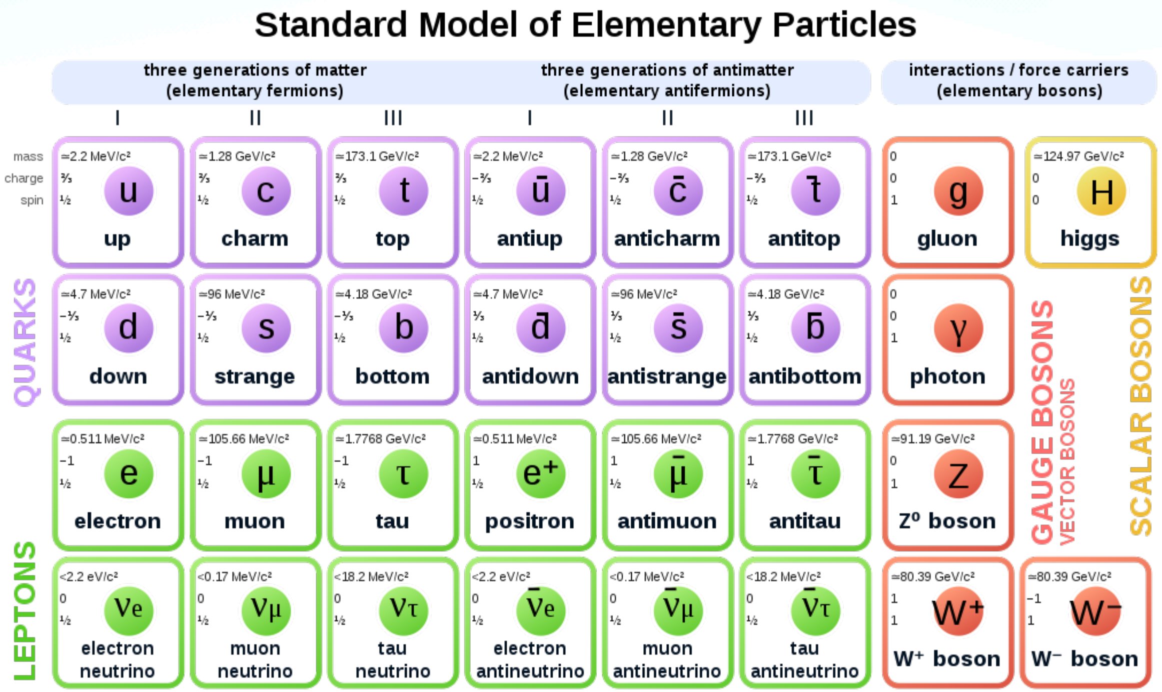
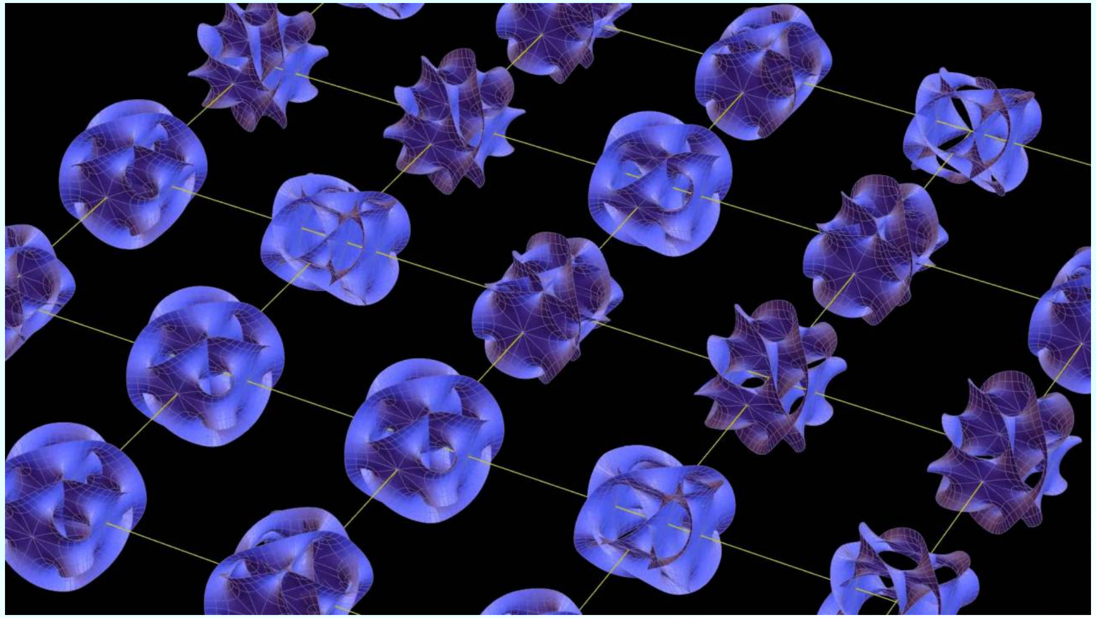
Two difficulties: 1) too many options;
2) the map from geometry to physics is too complicated.





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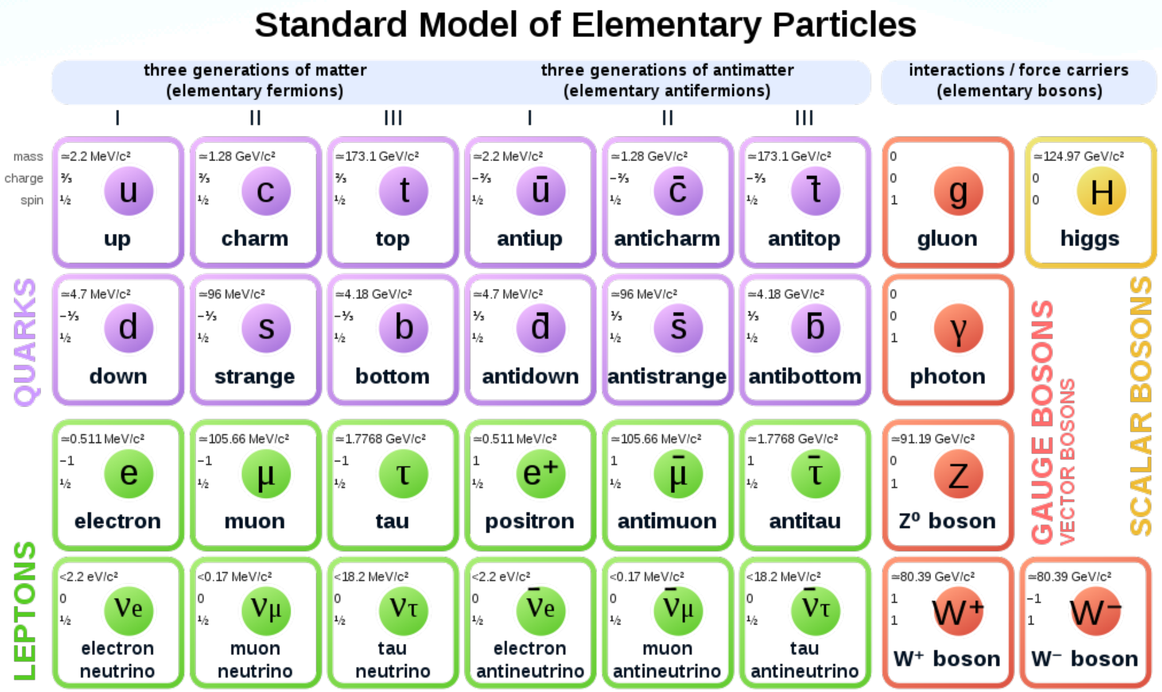
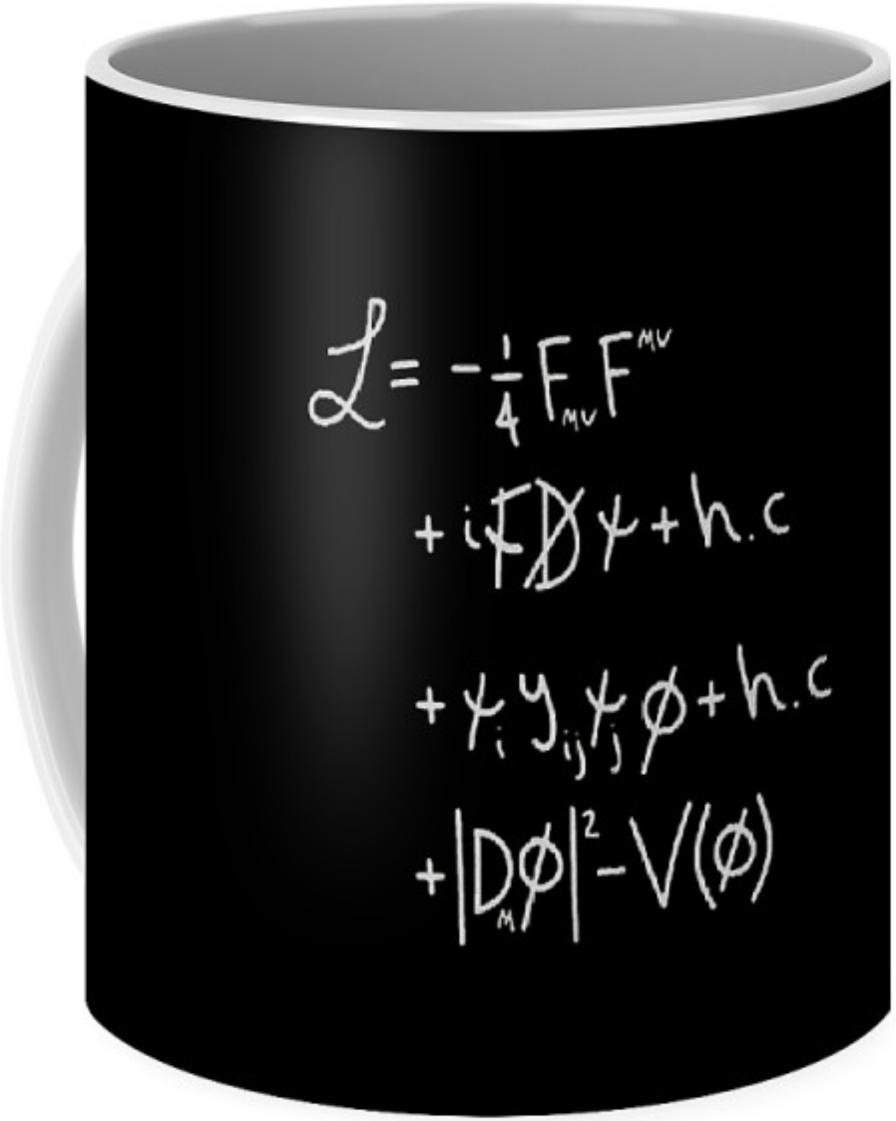
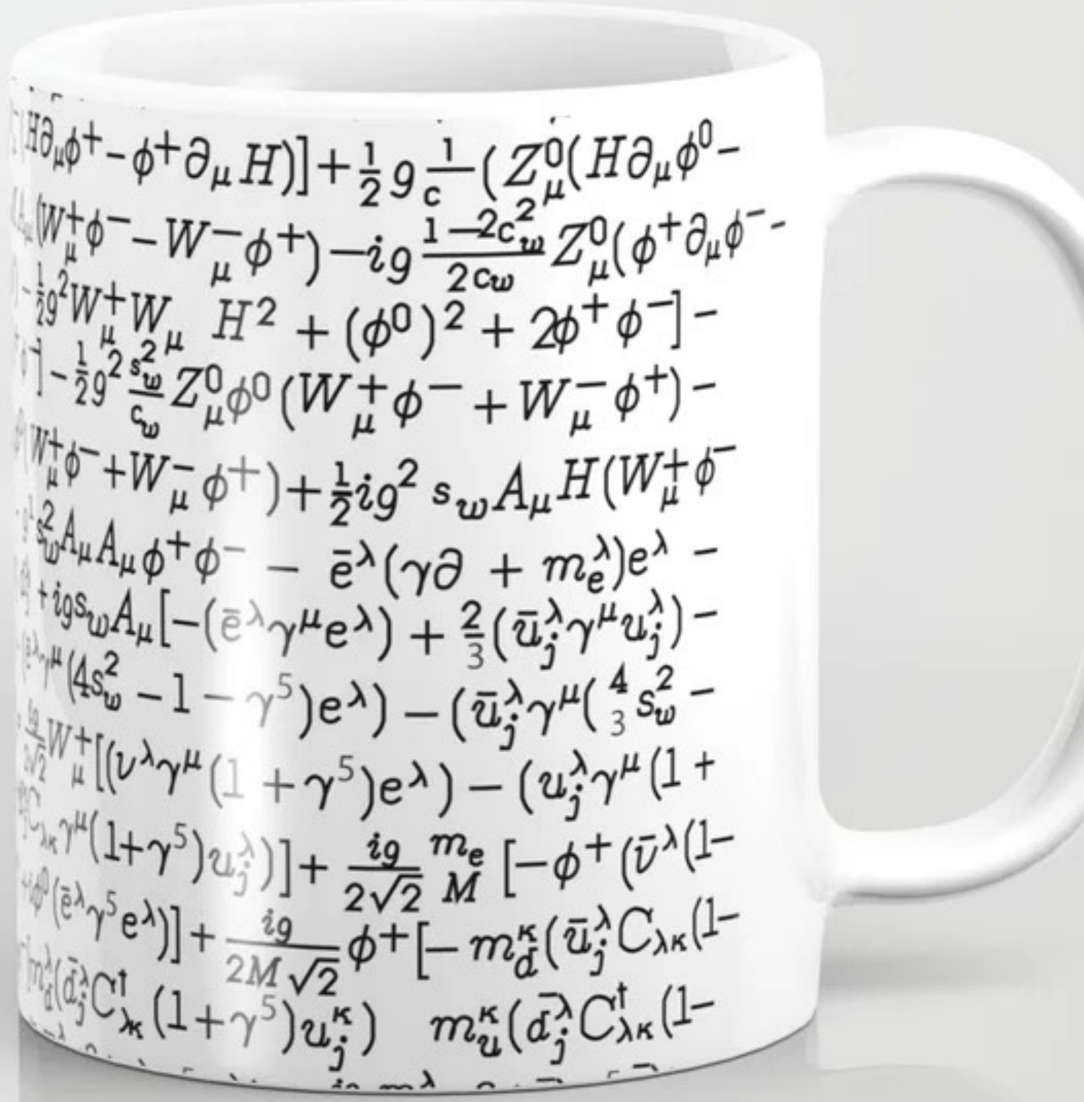


No realistic bottom-up approach.

Some hand-crafted models:
Greene, Kirklin, Miron, Ross, 1986
Braun, Candelas, Davies, Ronagi 2009+2011
Bouchard, Ronagi, 2005
Braun, He, Ovrut, Pantev, 2005


$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$
$+ i\bar{\psi} \not{D} \psi + h.c$
$+ \psi_i y_{ij} \psi_j \phi + h.c$
$+ |D_\mu \phi|^2 - V(\phi)$



Three steps

- identify models that have the correct gauge group and particle spectrum
- compute Yukawa couplings (quark and lepton masses and mixing parameters) as functions of the moduli
- stabilise all moduli; understanding non-perturbative physics is typically required at this step

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In this paper, we will discuss some considerations, which, if valid, come very close to determining K uniquely. We require

(i) The geometry to be of the form $\mathcal{M}_4 \times K$, where $\mathcal{M}_4$ is a maximally symmetric spacetime.

(ii) There should be an unbroken $N = 1$ supersymmetry in four dimensions. General arguments [10] and explicit demonstrations [11] have shown that supersymmetry may play an essential role in resolving the gauge hierarchy or Dirac large numbers problem. These arguments require that supersymmetry is unbroken at the Planck (or compactification) scale.

(iii) The gauge group and fermion spectrum should be realistic.

[Candelas, Horowitz, Strominger, Witten, 1985]

Enumerating Calabi-Yau Manifolds: Placing Bounds on the Number of Diffeomorphism Classes in the Kreuzer-Skarke List

Aditi Chandra, Andrei Constantin, Cristofero S. Fraser-Taliente, Thomas R. Harvey, and Andre Lukas*

The diffeomorphism class of simply connected smooth Calabi-Yau threefolds with torsion-free cohomology is determined via certain basic topological invariants: the Hodge numbers, the triple intersection form, and the second Chern class. In the present paper, we shed some light on this classification by placing bounds on the number of diffeomorphism classes present in the set of smooth Calabi-Yau threefolds constructed from the Kreuzer-Skarke (KS) list of reflexive polytopes up to Picard number six. The main difficulty arises from the comparison of triple intersection numbers and divisor integrals of the second Chern class up to basis transformations. By using certain basis-independent invariants, some of which appear here for the first time, we are able to place lower bounds on the number of classes. Upper bounds are obtained by explicitly identifying basis transformations, using constraints related to the index of line bundles. Extrapolating our results, we conjecture that the favorable entries of the KS list of reflexive polytopes lead to some 10^{400} diffeomorphically distinct Calabi-Yau threefolds.

Fortsch. Phys. 72 (2024)

$$\sim 10^{400}\ \mathrm{CY}_3$$

See also: *Counting Calabi-Yau Threefolds*, Gendler, MacFadden, McAllister, Moritz, 2310.06820

dim. 1: all genus-one curves are diffeomorphic
dim. 2: all K3 are diffeomorphic to each other
dim. 3: diffeomorphism classes classified by the "Wall data": Hodge numbers, triple intersection numbers $d_{ijk} = D_i \cdot D_j \cdot D_k$ and second Chern class $c_{2,i} = c_2(X) \cdot D_i$

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Difficulty: hard to tell when two Wall data are equivalent up to an integral transformation on $H^2(X, \mathbb{Z})$.
Solution: use $GL(n, \mathbb{Z})$-invariants and line bundle data.

$$\begin{pmatrix} 0 & 0 & 8 & 60 & 112 & 164 & 216 & 268 \\ 0 & 0 & 2 & 36 & 70 & 104 & 138 & 172 \\ 0 & 0 & 0 & 20 & 40 & 60 & 80 & 100 \\ 0 & 0 & 0 & 10 & 20 & 30 & 40 & 50 \\ 0 & 0 & 0 & 4 & 8 & 12 & 16 & 20 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 36 \\ 0 & 0 & 0 & 0 & 0 & 0 & 20 & 40 \\ 0 & 0 & 0 & 0 & 0 & 10 & 20 & 30 \\ 0 & 0 & 0 & 0 & 4 & 8 & 12 & 16 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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Counting string theory standard models

Andrei Constantin [a,b], Yang-Hui He [c,d,e,*], Andre Lukas [f]

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[b] *Mansfield College, Oxford University, OX1 3TF, Oxford, UK*
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[d] *Department of Mathematics, City, University of London, EC1V0HB, UK*
[e] *School of Physics, NanKai University, Tianjin, 300071, China*
[f] *Rudolf Peierls Centre for Theoretical Physics, Oxford University, 1 Keble Road, Oxford, OX1 3NP, UK*

A B S T R A C T

We derive an approximate analytic relation between the number of consistent heterotic Calabi-Yau compactifications of string theory with the exact charged matter content of the standard model of particle physics and the topological data of the internal manifold: the former scaling exponentially with the number of Kähler parameters. This is done by an estimate of the number of solutions to a set of Diophantine equations representing constraints satisfied by any consistent heterotic string vacuum with three chiral massless families, and has been computationally checked to hold for complete intersection Calabi-Yau threefolds (CICYs) with up to seven Kähler parameters. When extrapolated to the entire CICY list, the relation gives $\sim 10^{23}$ string theory standard models; for the class of Calabi-Yau hypersurfaces in toric varieties, it gives $\sim 10^{723}$ standard models.

Phys. Lett. B 792 (2019)

$\sim 10^{723}$ three-family models per CY_3

Cf. with the famous 10^{500} IIB flux compactifications [Douglas, 2003] and $10^{272,000}$ F-theory flux compactifications on a single 4-fold [Taylor & Wang, 2015]

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Physics of Line Bundles on Calabi-Yau Threefolds

Heterotic string compactifications on CY 3-folds with line bundles

$E_8 \times E_8$ Heterotic string - from N=1 supersymmetric theory in 10d to the N=1 in 4d:

- $X_{10} = X_6 \times M_4$
- $E_8 \to G_{\text{bundle}} \times G_{\text{GUT}}, \quad G_{\text{GUT}} \to G_{\text{finite}} \times G_{\text{SM}}$
- matter fields: $\mathbf{248} \to (\mathbf{1}, \text{Ad}_{G_{\text{GUT}}}) \oplus (\bigoplus_i (R_i, r_i)) \qquad n_{r_i} = h^1(X, V_{R_i})$

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To preserve N=1 susy in 4d:

- X_6 must be Calabi-Yau, $R_{a\bar{b}} = 0$
- V must be holomorphic and poly-stable, $F_{ab} = F_{\bar{a}\bar{b}} = g^{a\bar{b}} F_{a\bar{b}} = 0$
- anomaly cancellation: $c_2(V) \leq c_2(TX)$
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Simplest setting: $\quad V = \oplus_{a=1}^{5} L_a, \quad G_{\text{bundle}} = S(U(1)^5)$ and $G_{\text{GUT}} = SU(5) \times S(U(1)^5)$.

SM multiplets in $\bar{\mathbf{5}}_{\mathbf{e}_a+\mathbf{e}_b}, \mathbf{10}_{\mathbf{e}_a}$; Higgs pair: $(\mathbf{5}_{-\mathbf{e}_a-\mathbf{e}_b}, \bar{\mathbf{5}}_{\mathbf{e}_a+\mathbf{e}_b})$; bundle moduli: $\mathbf{1}_{\mathbf{e}_a-\mathbf{e}_b}$

PROBLEM: What is the number $N = N(h, c_{2,i}, d_{ijk})$ of rank five line bundle sums $V = \oplus_{a=1}^{5} L_a$, where $L_a = \mathcal{O}_X(\mathbf{k}_a)$ such that the following constraints are satisfied:

E_8 embedding: $c_1(V) = \sum_{a=1}^{5} k_a^i \stackrel{!}{=} 0$ for all $i = 1, \ldots, h$;

Anomaly cancellation:

$$c_{2,i}(V) = -\frac{1}{2} d_{ijk} \sum_a k_a^j k_a^k \stackrel{!}{\leq} c_{2,i} \text{ for all } i = \ldots, h;$$

Supersymmetry/Zero Slope: there is a common solution t^i to the vanishing slopes

$$\mu(L_a) = d_{ijk} k^i t^j t^k \stackrel{!}{=} 0 \text{ for } a = 1, \ldots, 5$$

such that $J = t^i J_i \in$ interior of the Kahler cone;

Particle generations: the chiral asymmetry is six, i.e.

$$\text{ind}(V) = \frac{1}{6} d_{ijk} \sum_a k_a^i k_a^j k_a^k \stackrel{!}{=} -3.$$

Heterotic line bundle models: searches

- Situation about 12 years ago: only a handful of models that recovered the SM spectrum were known
- Systematic searches: in 2013 we undertook a massive search, scanning essentially over some

10^{40} (X, V)-pairs; this resulted in several million heterotic line bundle models with three families

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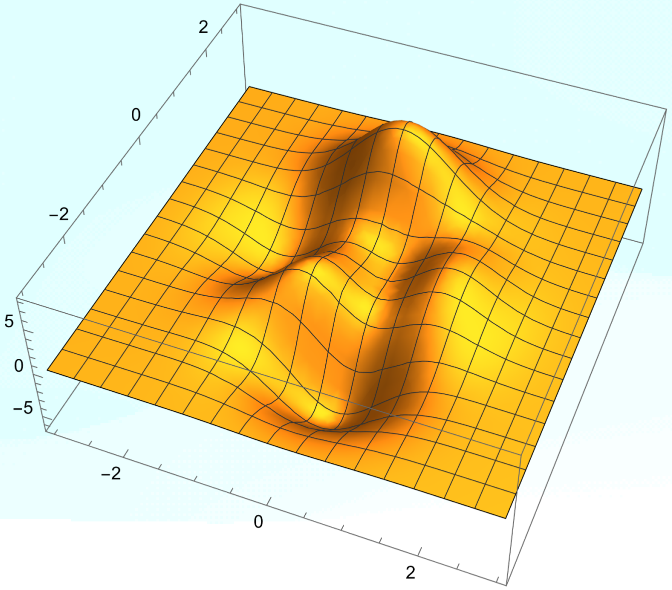
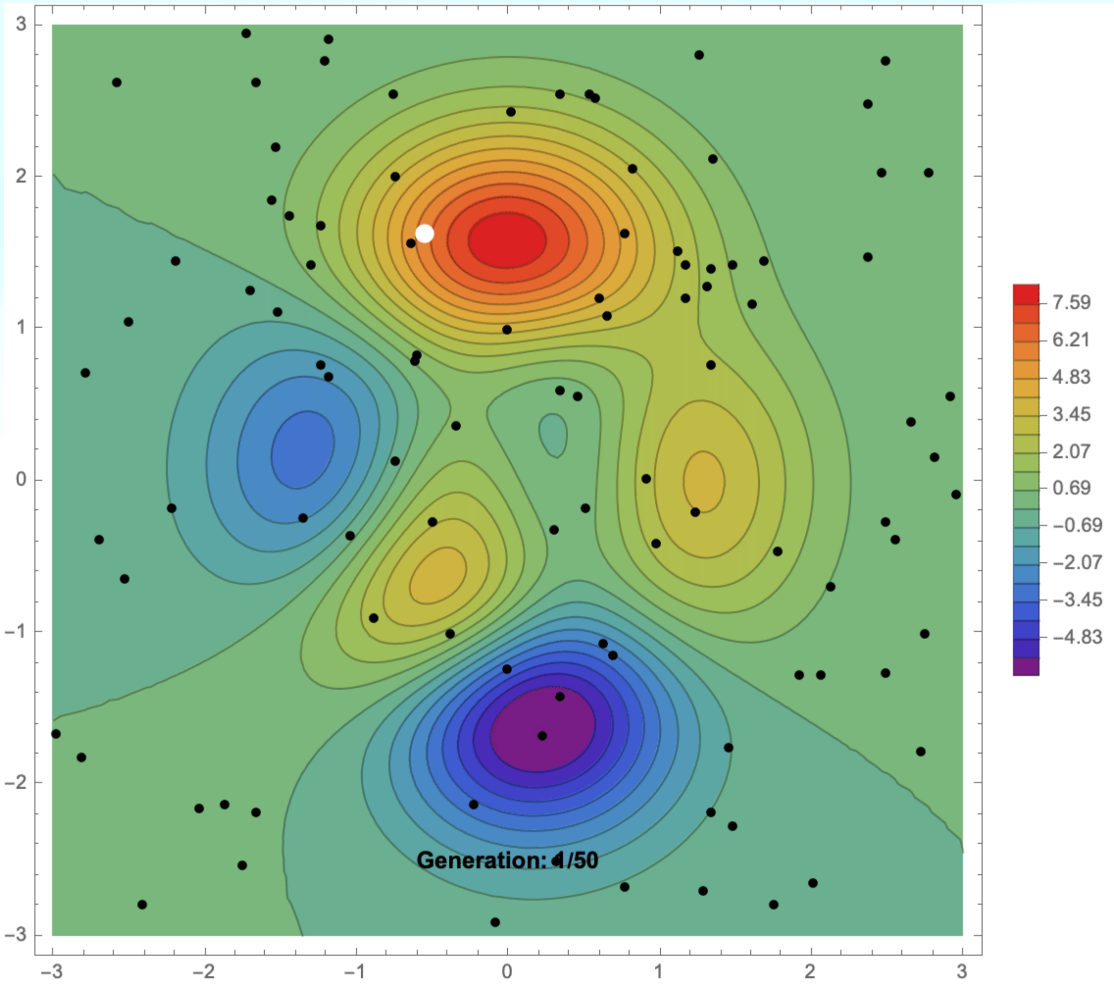
- Heuristic searches: more recently, we used Genetic Algorithms and Reinforcement Learning to search even larger regions of the string landscape. We also used Quantum Annealing 'intrinsic' mutation to enhanced the GAs performance.
- New three-family models can now be identified on demand (thousands per day) or generated using AI.

[Larfors, Schneider '20]

[Abel, AC, Harvey, Lukas, Nutricati '23]

Genetic Algorithms in Pictures

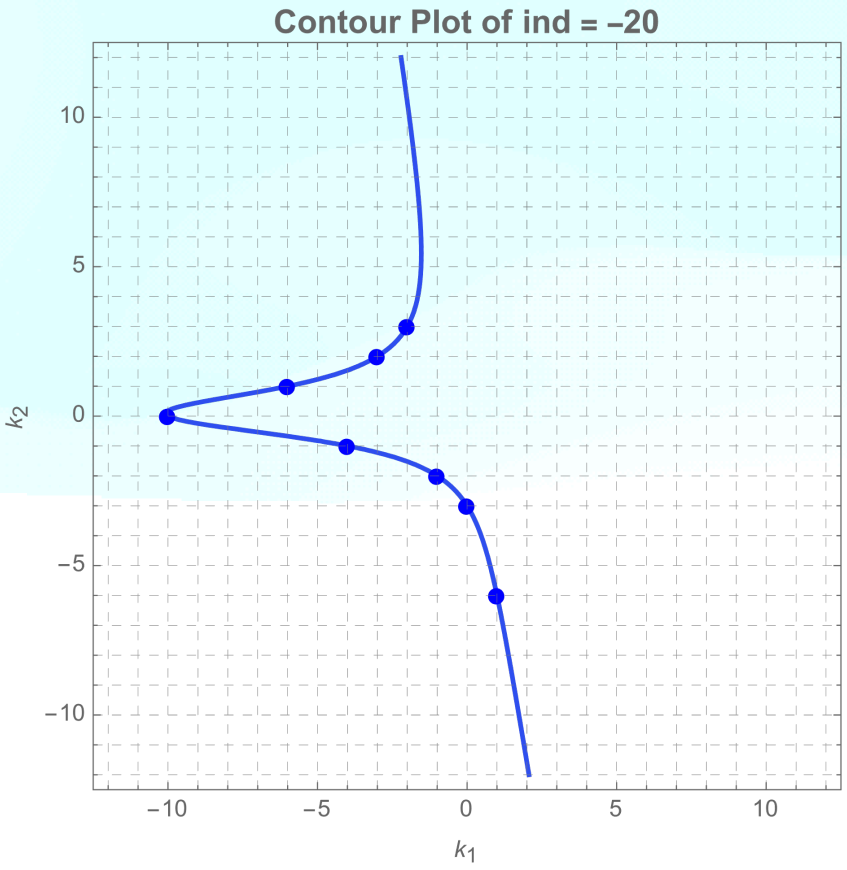




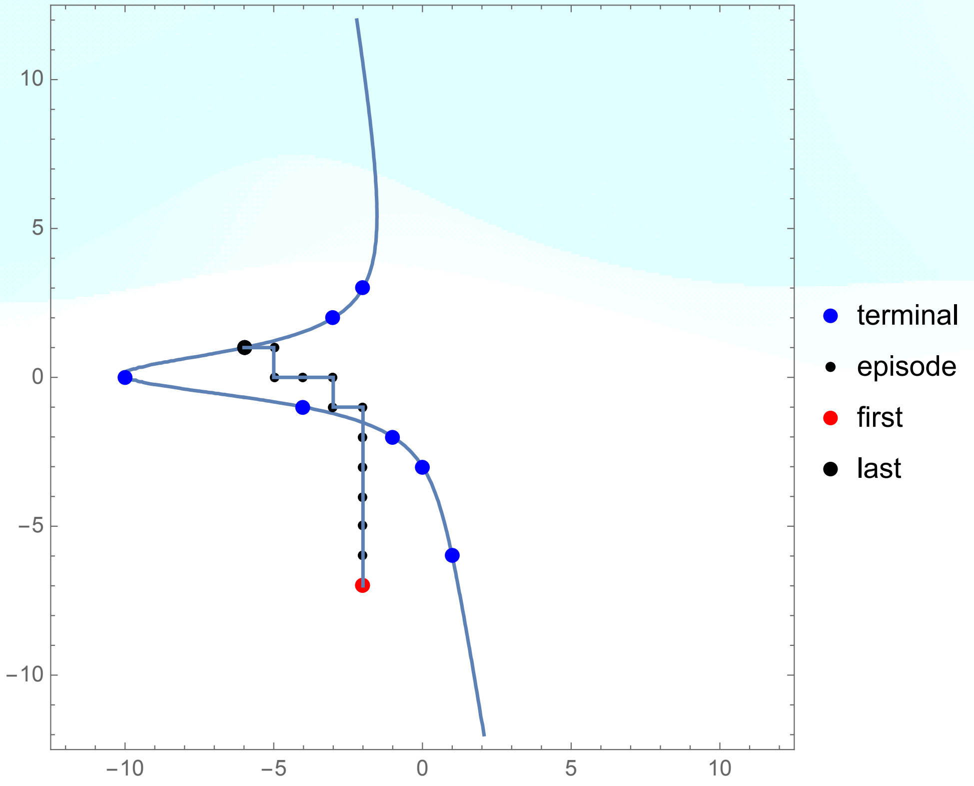
In this example:

- Population size of 100 individuals
- Binary encoding/decoding for the chromosomes
- 16-bit chromosomes (8 bits for x-coordinate, 8 bits for y-coordinate)
- Tournament selection for parent selection
- Single-point crossover with crossover rate of 80%
- Bit-flip mutation rate of 3%
- Evolution over 50 generations
- Elitism to preserve the best solution

Reinforcement Learning in Pictures



$$X = \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^3 \end{matrix} \left[\begin{matrix} 2 \\ 4 \end{matrix} \right]^{2,86}$$

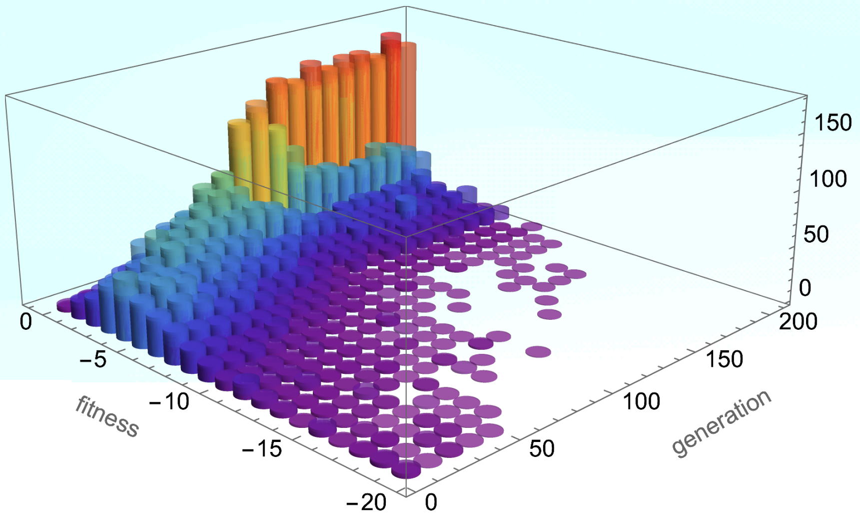


$$\chi(X, \mathcal{O}_X(D)) = \frac{1}{6}D^3 + \frac{1}{12}c_2(X) \cdot D$$

$$= \frac{1}{6}\left(4k_1k_2^2 + \left(4k_1k_2 + \left(4k_1 + 2k_2\right)k_2\right)k_2\right) + \frac{1}{12}\left(24k_1 + 44k_2\right)$$

Mathematical structure of RL: Markov Decision Processes.
Simplest version: policy-based RL.
The policy is controlled by a NN and learnt without any prior knowledge of the environment.

Search results



Manifold	h	$\|\Gamma\|$	Range	GA	Scan	Found	Explored
7862	4	2	[-7,8]	5	5	100%	10^{-10}
7862	4	4	[-7,8]	30	31	97%	10^{-10}
7447	5	2	[-7,8]	38	38	100%	10^{-14}
7447	5	4	[-7,8]	139	154	90%	10^{-14}
5302	6	2	[-7,8]	403	442	93%	10^{-19}
5302	6	4	[-7,8]	722	897	80%	10^{-19}
4071	7	2	[-3,4]	11,937	N/A	N/A	10^{-14}

[Abel, AC, Harvey, Lukas, Nutricati ’23]

Comparison with systematic scans: virtually the same results while scanning only a fraction of $\sim 10^{-20}$!!
Comparison between GA and RL: very different philosophies, similar results.

Beyond 3 generations

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- fast cohomology computations to decide the presence of Higgs pairs, exotic matter, bundle moduli
- use the additional (effectively global) $U(1)$-symmetries to constrain the superpotential

The Yukawa couplings take the form:

$$\begin{aligned} \text{up sector:} \quad & (\text{singlet insertions}) \times H^u_{-\mathbf{e}_a-\mathbf{e}_b} \mathbf{10}^i_{\mathbf{e}_c} \mathbf{10}^j_{\mathbf{e}_d} \\ \text{down sector:} \quad & (\text{singlet insertions}) \times H^d_{\mathbf{e}_a+\mathbf{e}_b} \bar{\mathbf{5}}^i_{\mathbf{e}_c+\mathbf{e}_d} \mathbf{10}^j_{\mathbf{e}_e} \end{aligned}$$

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- out of the millions of models line bundle models that have the correct MSSM spectrum, we were able to identify a few dozen that can accommodate — somewhere in the moduli space — the empirical flavour parameters in the SM. In these models the $\mu H\bar{H}$ term is also under control thanks to the $U(1)$-symmetries. [AC, Leung, Lukas, Nutricati ’25]

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- full knowledge of the compactification geometry required to compute the O(1)-coefficients: the Ricci-flat metric on X, hermitian Yang-Mills connection on V, harmonic bundle-valued forms

Beyond 3 generations

Going beyond the basic check of having 3 chiral families of fermions involves:

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w.i.p.
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An example

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Up Yukawa ->
$$\begin{pmatrix} \phi_{4,2} + \Phi_3\, \phi_{5,2}^2 & \Phi_3\, \phi_{5,2} & \Phi_3\, \phi_{5,2} \\ \Phi_3\, \phi_{5,2} & \Phi_3 & \Phi_3 \\ \Phi_3\, \phi_{5,2} & \Phi_3 & \Phi_3 \end{pmatrix}$$

Down Yukawa ->
$$\begin{pmatrix} \phi_{1,4}^2\, \phi_{3,1}\, \phi_{4,2} + \phi_{1,4}\, \phi_{1,5}\, \phi_{3,1}\, \phi_{5,2} & \phi_{1,4}^2\, \phi_{3,1}\, \phi_{4,2} + \phi_{1,4}\, \phi_{1,5}\, \phi_{3,1}\, \phi_{5,2} & \phi_{1,4}^2\, \phi_{3,1}\, \phi_{4,2} + \phi_{1,4}\, \phi_{1,5}\, \phi_{3,1}\, \phi_{5,2} \\ \phi_{1,4}\, \phi_{1,5}\, \phi_{3,1} + \Phi_1\, \phi_{5,2} & \phi_{1,4}\, \phi_{1,5}\, \phi_{3,1} + \Phi_1\, \phi_{5,2} & \phi_{1,4}\, \phi_{1,5}\, \phi_{3,1} + \Phi_1\, \phi_{5,2} \\ \phi_{1,4}\, \phi_{1,5}\, \phi_{3,1} + \Phi_1\, \phi_{5,2} & \phi_{1,4}\, \phi_{1,5}\, \phi_{3,1} + \Phi_1\, \phi_{5,2} & \phi_{1,4}\, \phi_{1,5}\, \phi_{3,1} + \Phi_1\, \phi_{5,2} \end{pmatrix}$$

VEVs --> $\langle| \Phi_1 \to 0.011186,\ \Phi_2 \to 0,\ \Phi_3 \to 0.0208645,\ \Phi_4 \to 0,\ \Phi_5 \to 0 |\rangle\ \langle| \phi_{2,1} \to 0,\ \phi_{3,1} \to 0.220866,\ \phi_{1,4} \to 0.157553,$
$\phi_{1,5} \to 0.116906,\ \phi_{5,1} \to 0,\ \phi_{2,3} \to 0,\ \phi_{2,4} \to 0,\ \phi_{4,2} \to 0.589303,\ \phi_{5,2} \to 0.334451,\ \phi_{4,3} \to 0,\ \phi_{5,3} \to 0,\ \phi_{5,4} \to 0 |\rangle$

o1 Up coeffs -->
$$\begin{pmatrix} \{-3.48814, 6.81476\} & \{-0.0970636\} & \{0.405479\} \\ \{-6.88697\} & \{-0.00205929\} & \{0.00507336\} \\ \{0.336461\} & \{0.663477\} & \{-0.280755\} \end{pmatrix}$$

o1 Down coeffs -->
$$\begin{pmatrix} \begin{pmatrix} 1.07321 \\ -3.93794 \end{pmatrix} & \begin{pmatrix} 5.31902 \\ 3.78552 \end{pmatrix} & \begin{pmatrix} -6.95226 \\ -1.50959 \end{pmatrix} \\ \begin{pmatrix} 0.582884 \\ -0.670534 \end{pmatrix} & \begin{pmatrix} 1.25548 \\ -1.27911 \end{pmatrix} & \begin{pmatrix} -0.76408 \\ 0.773568 \end{pmatrix} \\ \begin{pmatrix} -1.31787 \\ 0.638412 \end{pmatrix} & \begin{pmatrix} 0.411061 \\ 0.729356 \end{pmatrix} & \begin{pmatrix} -1.12268 \\ 0.156689 \end{pmatrix} \end{pmatrix}$$

o1 Leptons coeffs -->
$$\begin{pmatrix} \begin{pmatrix} 0.913761 \\ -0.377272 \end{pmatrix} & \begin{pmatrix} 3.91872 \\ -1.6784 \end{pmatrix} & \begin{pmatrix} 6.64024 \\ -2.99491 \end{pmatrix} \\ \begin{pmatrix} -0.756866 \\ 0.818604 \end{pmatrix} & \begin{pmatrix} -5.31757 \\ 2.85963 \end{pmatrix} & \begin{pmatrix} 0.114206 \\ -0.0983683 \end{pmatrix} \\ \begin{pmatrix} -2.10909 \\ 2.28786 \end{pmatrix} & \begin{pmatrix} -1.30161 \\ 0.317915 \end{pmatrix} & \begin{pmatrix} 3.54416 \\ -3.76934 \end{pmatrix} \end{pmatrix}$$

{Up Higgs, Down Higgs} --> {84.5001, 152.104}

$\{m_t,\ m_c,\ m_u\}$ --> {172.4, 1.27, 0.00216}

$\{m_b,\ m_s,\ m_d\}$ --> {4.18, 0.093, 0.00467}

$\{m_\tau,\ m_\mu,\ m_e\}$ --> {1.77682, 0.1057, 0.000511}

CKM matrix -->
$$\begin{pmatrix} 0.970323 & 0.241786 & 0.0035785 \\ 0.2417 & 0.969315 & 0.0448244 \\ 0.00736923 & 0.0443591 & 0.998988 \end{pmatrix}$$

Geometry of Line Bundles on Calabi-Yau Threefolds

Line bundle cohomology for low energy string spectra

repr.	cohomology	total number	required for MSSM
$\mathbf{1}_{a,b}$	$H^1(X, L_a \otimes L_b^*)$	$\sum_{a,b} h^1(X, L_a \otimes L_b^*) = h^1(X, V \otimes V^*)$	-
$\mathbf{5}_{a,b}$	$H^1(X, L_a^* \otimes L_b^*)$	$\sum_{a<b} h^1(X, L_a^* \otimes L_b^*) = h^1(X, \wedge^2 V^*)$	n_h
$\overline{\mathbf{5}}_{a,b}$	$H^1(X, L_a \otimes L_b)$	$\sum_{a<b} h^1(X, L_a \otimes L_b) = h^1(X, \wedge^2 V)$	$3\|\Gamma\| + n_h$
$\mathbf{10}_a$	$H^1(X, L_a)$	$\sum_a h^1(X, L_a) = h^1(X, V)$	$3\|\Gamma\|$
$\overline{\mathbf{10}}_a$	$H^1(X, L_a^*)$	$\sum_a h^1(X, L_a^*) = h^1(X, V^*)$	0

Line bundle cohomology with spectral sequences

$X \subset \mathcal{A} = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \ldots \times \mathbb{P}^{n_m}$

Let $L \to X$ be a line bundle over X and $\mathcal{L}_{\mathcal{A}}$ the corresponding line bundle.

Write the Koszul complex associated with L:

$$0 \;\to\; \mathcal{L}_{\mathcal{A}} \otimes \wedge^{K} \mathcal{N}^* \;\to\; \mathcal{L}_{\mathcal{A}} \otimes \wedge^{K-1} \mathcal{N}^* \;\to\; \ldots \;\to\; \mathcal{L}_{\mathcal{A}} \;\to\; L \;\to\; 0$$

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Line bundles on $\mathbb{P}^n$. Cohomology dimensions given by the Bott formula:

$$h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = \binom{k+n}{n} = \frac{1}{n!}(1+k)\ldots(n+k)\ , \text{ if } k \geq 0, \text{ and 0 otherwise.}$$

$$h^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = 0\ , \text{ if } 0 < i < n\ .$$

$$h^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = \binom{-k-1}{-n-k-1} = \frac{1}{n!}(-n-k)\ldots(-1-k)\ , \text{ if } k \leq -n-1,$$

and 0 otherwise.

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and 0 otherwise.

For spectral sequence technology, see, e.g., Hübsch' *CY Bestiary.*

The Leray spectral sequence machinery can be automatised.

[CIPro package, Anderson, AC, Gray, He, Lee, Lukas, to appear]

[pyCICY by Larfors & Schneider '19]

Computational cost of cohomology calculations with spectral sequences: $\sim O\left(\left(\rho(X)^{\dim(X)}\deg(L)^{\dim(X)}\right)^3\right)$

Example: for a line bundle of (multi)-degree 10 on a Calabi-Yau threefold

with $h^{1,1}(X) = \rho(X) = 4$ Kähler parameters, the estimate is

$\sim 10^{14}$ elementary operations

which reaches the limits of a standard machine

Train a neural network?

Here is some data for $h^0(S, L = \mathcal{O}_S(k_1, k_2))$, where S is the Hirzebruch surface F_1 for $-8 \le k_i \le 8$.

0	0	0	0	0	0	0	0	9	17	24	30	35	39	42	44	45
0	0	0	0	0	0	0	0	8	15	21	26	30	33	35	36	36
0	0	0	0	0	0	0	0	7	13	18	22	25	27	28	28	28
0	0	0	0	0	0	0	0	6	11	15	18	20	21	21	21	21
0	0	0	0	0	0	0	0	5	9	12	14	15	15	15	15	15
0	0	0	0	0	0	0	0	4	7	9	10	10	10	10	10	10
0	0	0	0	0	0	0	0	3	5	6	6	6	6	6	6	6
0	0	0	0	0	0	0	0	2	3	3	3	3	3	3	3	3
0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

One can blindly train a NN to predict these numbers. Most of the time they come out right, with the occasional error of ± 1.

[Ruehle '17]

```
{0, 0}
{162, 162}
{0, 0}
{62, 62}
{1, 0}
{0, 0}
{36, 36}
{0, 0}
{0, 0}
{0, 0}
{0, 0}
{0, 0}
{0, 0}
{0, 0}
{0, 0}
{0, 0}
{0, 0}
{0, 0}
{0, 0}
{28, 28}
{0, 0}
{251, 252}
{0, 0}
{63, 63}
{95, 95}
{45, 45}
```

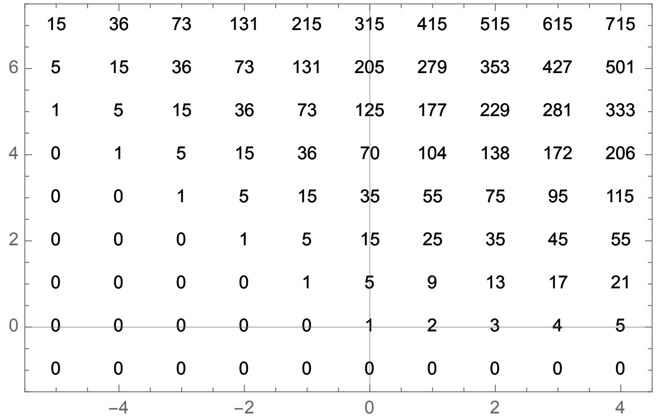
```
{0, 0}
{0, 0}
{0, 0}
{0, 0}
{0, 0}
{0, 0}
{37, 36}
{0, 0}
{0, 0}
{0, 0}
{135, 135}
{1, 0}
{0, 0}
{0, 0}
{21, 21}
{119, 120}
{0, 0}
{0, 0}
{0, 0}
```

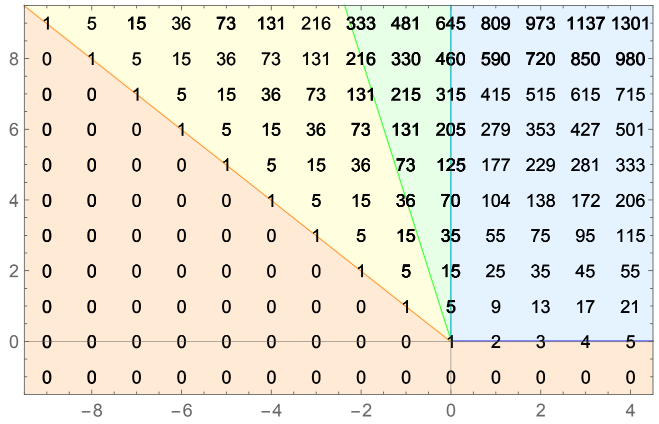
An exercise in pattern recognition

$$X = \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^4 \end{matrix} \left[\begin{matrix} 1 & 1 \\ 4 & 1 \end{matrix} \right]^{2,86}$$

look at patterns in the data for

$h^0(X, L),\ L \in \mathrm{Pic}(X)$





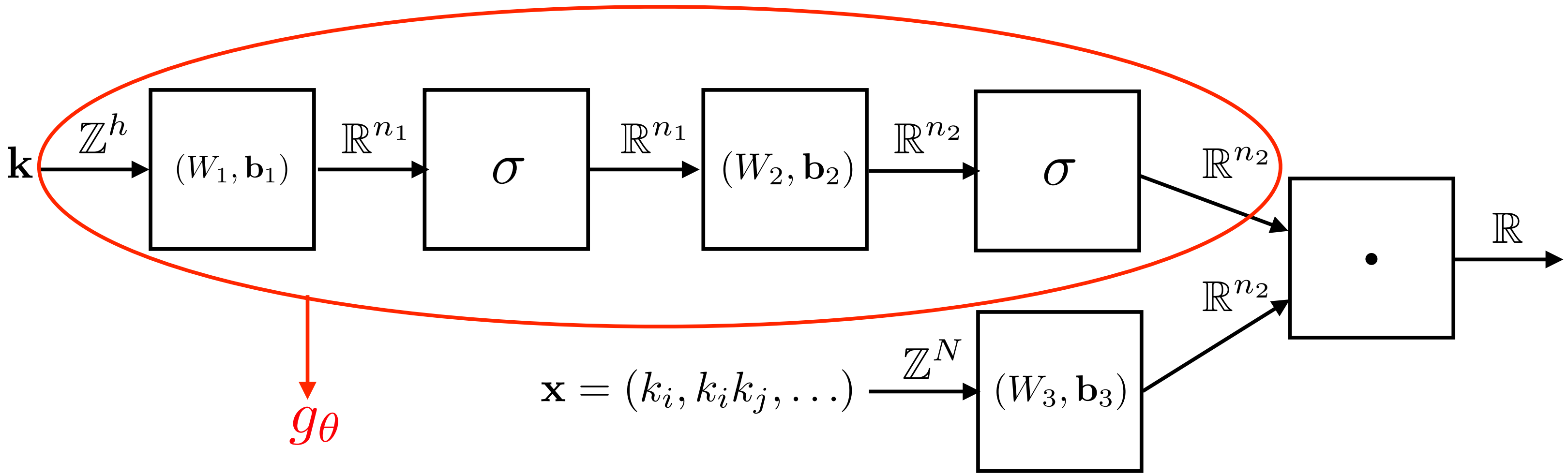
region in eff. cone	$h^0(X, L = \mathcal{O}_X(D = k_1 D_1 + k_2 D_2))$
blue	$2k_1(1+k_2^2) + \frac{5}{6}k_2(5+k_2^2)$
green	$2k_1(1+k_2^2) + \frac{5}{6}k_2(5+k_2^2) + \frac{8}{3}k_1(1-k_1^2)$
yellow	$2k_1(1+k_2^2) + \frac{5}{6}k_2(5+k_2^2) + \frac{8}{3}k_1(1-k_1^2) + \frac{1}{2}(1-(4k_1+k_2)^2)\left\lceil \frac{4k_1+k_2}{-3} \right\rceil$
$k_1 > 0,\ k_2 = 0$	$k_1 + 1$
$-k_1 = k_2 \geq 0$	1

[AC, Lukas ’18]

[Larfors, Schneider ’19]

[Brodie, AC, Lukas ‘21]

It is possible to train a neural network (supervised learning) to identify the different regions and the formulae that hold within each.

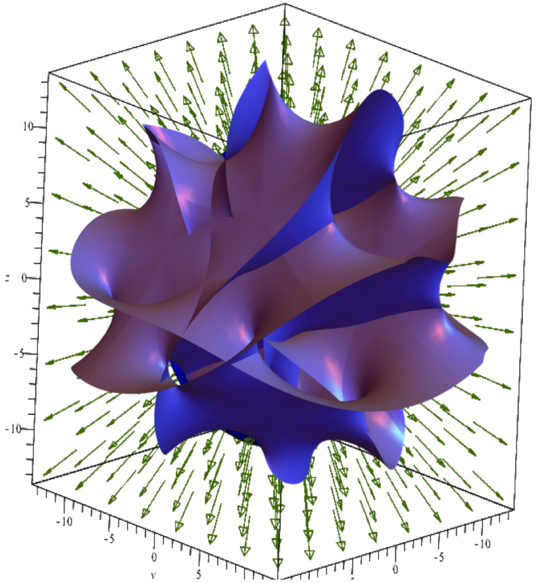


[Brodie, AC, Deen, Lukas, 1906.08730]
see also: [Klaewer, Schlechter, 1809.02547]

The training data consists of pairs $(\mathbf{k}, h^i(X, \mathcal{O}_X(\mathbf{k})))$.

Drawback: the amount of training data is limited by the slow algorithmic computation. For larger Picard number manifolds it is not feasible to generate enough training data. Nevertheless, this ML exercise was useful to generate conjectures.

topological data of (X, V)

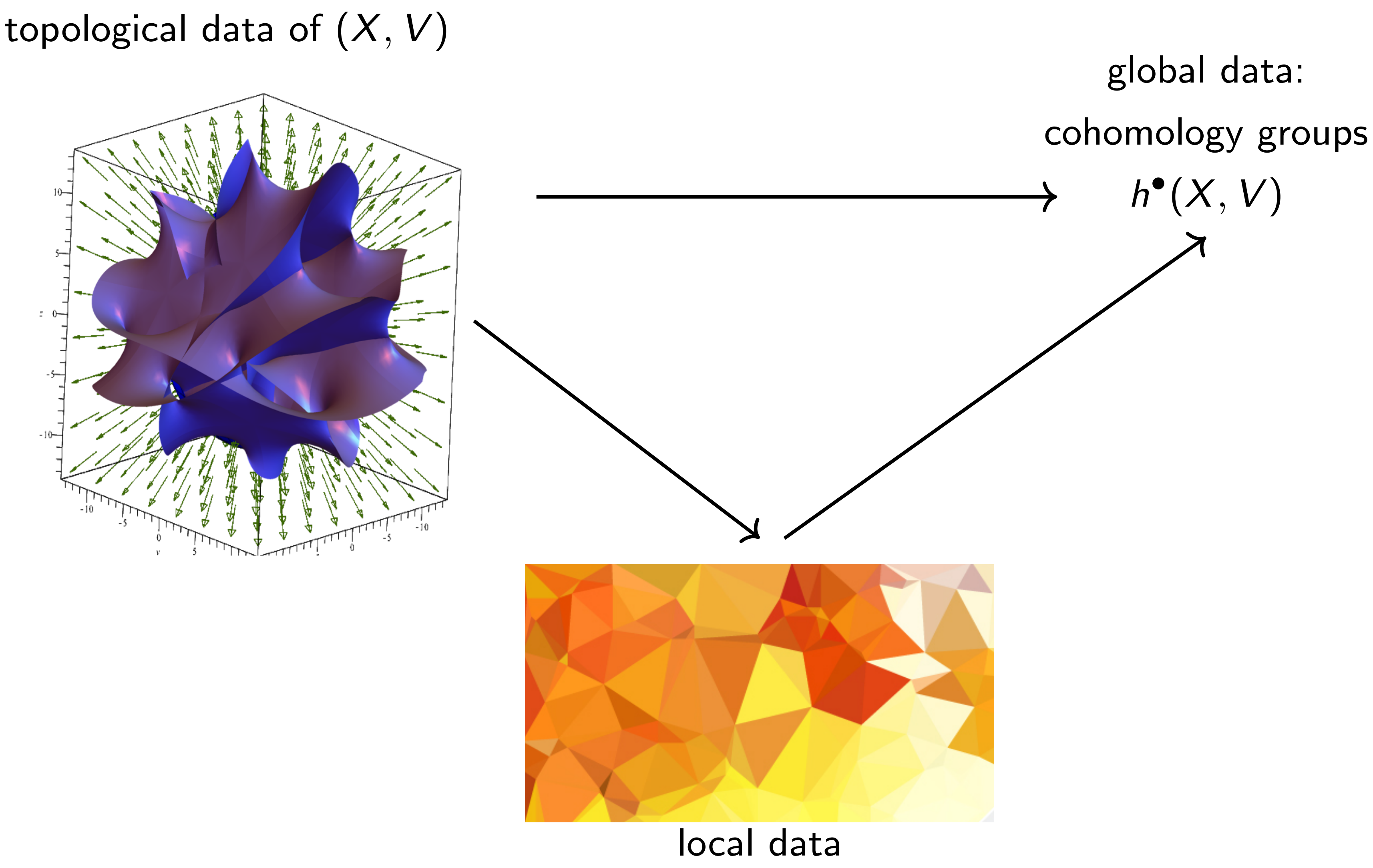


global data:

cohomology groups

$h^\bullet(X, V)$

local data


topological data of (X, V)
global data:
cohomology groups
$h^{\bullet}(X, V)$
local data

Quasi-topological formula for individual cohomologies on surfaces

Hirzebruch-Riemann-Roch theorem (X cplx, V holom.):

$$\chi(X,V) = \sum_{i=0}^{\dim(X)} (-1)^i h^i(X,V) = \int_X \mathrm{ch}(V)\cdot \mathrm{td}(X)$$

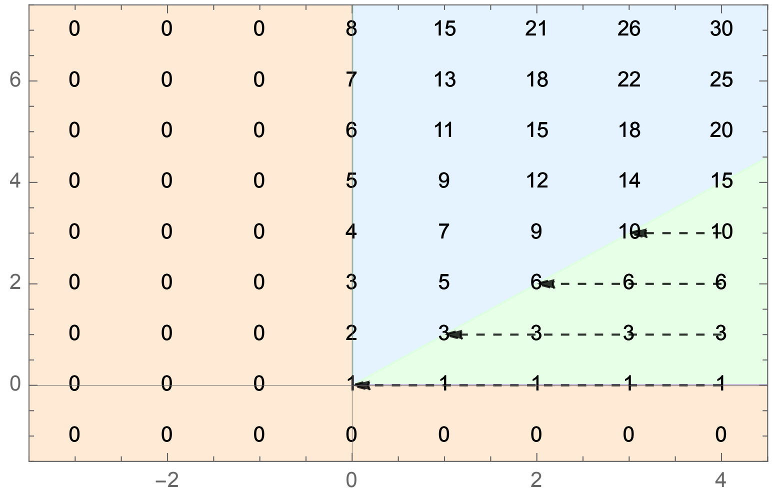
Theorem: line bundle cohomology formula for toric surfaces

Let S be a smooth projective toric surface, and D an effective integral divisor with Zariski decomposition $D = P + N$. Then

$$h^0(S,\mathcal{O}_S(D)) = \chi(S,\mathcal{O}_S(\lfloor P \rfloor))\,.$$

Explicitly, if D lies in the Zariski chamber $\Sigma_{i_1,\ldots i_n}$, obtained by translating a codimension n face F of the nef cone along the set of dual Mori cone generators $\{\mathcal{M}_{i_1},\mathcal{M}_{i_2},\ldots\mathcal{M}_{i_n}\}$ orthogonal (with respect to the intersection form) to the face F, then

$$h^0(S,\mathcal{O}_S(D)) = \chi\left(S,\mathcal{O}_S\left(D - \sum_{k=1}^{n}\left\lceil -D\cdot\mathcal{M}^\vee_{i_k,\{i_1,\ldots,i_n\}}\right\rceil \mathcal{M}_{i_k}\right)\right)\,.$$



h^0 data for del Pezzo of degree 8

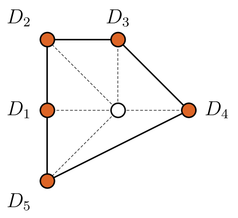
Similar theorems for K3 surfaces and (generalised) del Pezzo surfaces.

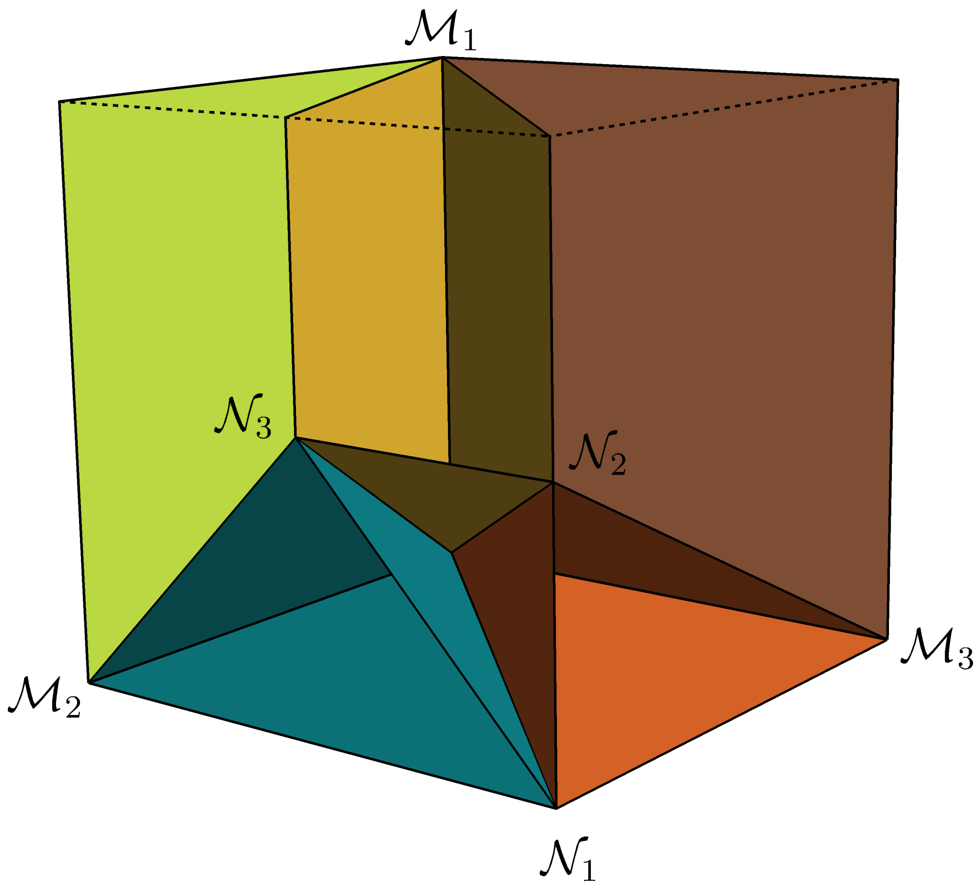
Information needed to write a formula: the intersection form and the generators of the Mori cone

[Brodie, AC '20]

A Picard number 3 toric surface

D_1	D_2	D_3	D_4	D_5
1	0	0	1	0
0	1	0	2	1
0	0	1	1	1





Line bundle cohomology on Calabi-Yau threefolds

Features of line bundle cohomology on Calabi-Yau threefolds

- We studied: CICY three-folds, smooth quotients thereof by freely acting discrete symmetries, (hypersurfaces) in toric varieties.
- We know empirically that analytic formulae exist for all cohomology groups. By Serre duality, it is enough to understand the zeroth and the first cohomologies.
- The Picard group splits into various cones, in each of which the zeroth cohomology can be computed as an index.

Two types of cones

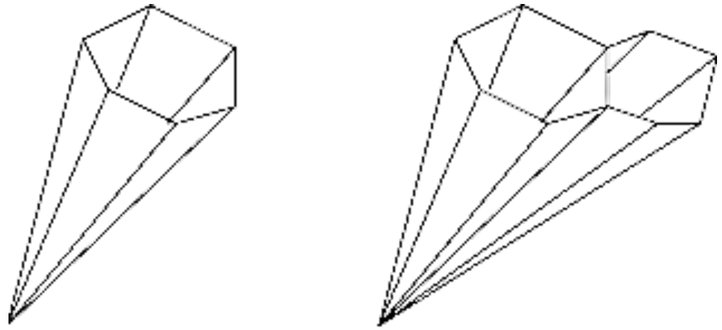
- In the Kähler cone $\mathcal{K}(X)$, due to Kodaira's vanishing theorem

$$h^0(X, L) = \chi(X, L)$$

where the Euler characteristic of $L = \mathcal{O}_X(D)$, on a Calabi-Yau 3-fold is

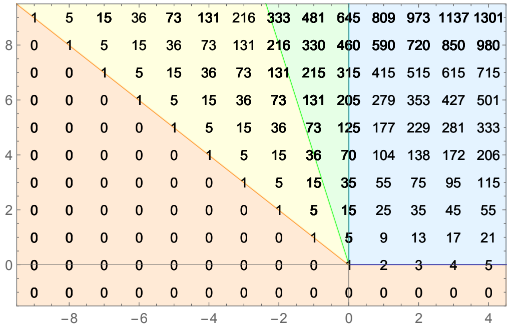
$$\chi(X, \mathcal{O}_X(D)) = \frac{1}{6}D^3 + \frac{1}{12}c_2(X) \cdot D$$

- Some CY3s have 'other Kähler cones': these are really the Kähler cones of the threefolds related to X by a sequence of flops



- There can also be Zariski chambers, similar to the case of complex surfaces

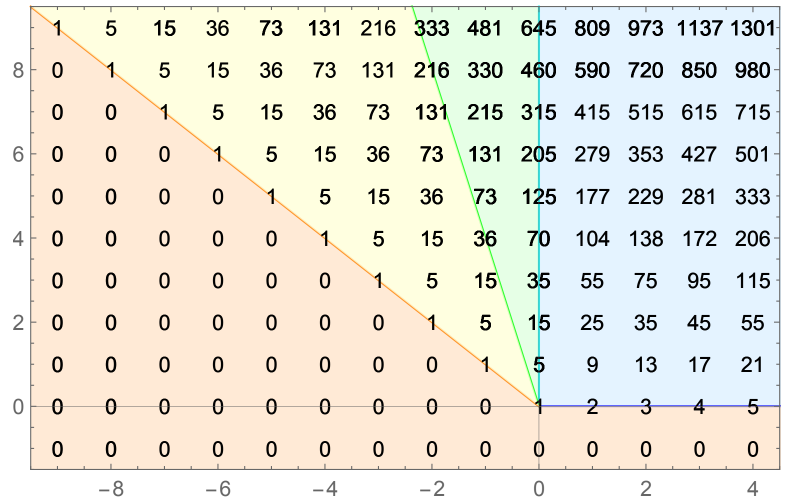
Zariski chambers. The other type of zeroth cohomology chambers that arise are Zariski chambers. Here is an example.



$$X = \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^4 \end{matrix} \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}^{2,86}$$

In each chamber, the zeroth cohomology can be written as an index.

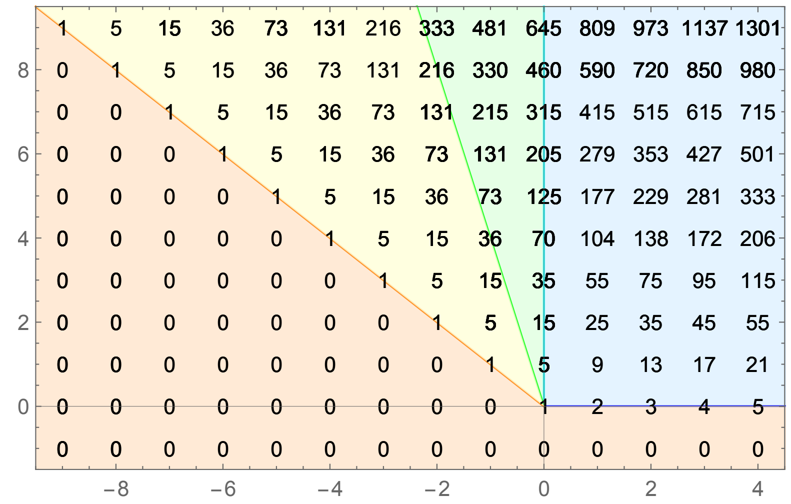
region in eff. cone	$h^0(X, L = \mathcal{O}_X(D = k_1 D_1 + k_2 D_2))$
$\mathcal{K}(X)$	$\chi(X, \mathcal{O}_X(D))$
$\overline{\mathcal{K}}(X') \setminus \{\mathcal{O}_X\}$	$\chi(X', \mathcal{O}_{X'}(D'))$
$\overline{\Sigma}$	$\chi\left(X', \mathcal{O}_{X'}\left(D' - \left\lceil \frac{D' \cdot \tilde{C}_2'}{\Gamma' \cdot \tilde{C}_2'} \right\rceil \Gamma'\right)\right)$
$k_1 > 0,\ k_2 = 0$	$\chi(\mathbb{P}^1, (D \cdot C_1) H_{\mathbb{P}^1})$
$-k_1 = k_2 \geq 0$	1



Cohomology series

The Hilbert-Poincaré series associated with the coordinate ring of X is

$$HS(X, t_1, t_2) = \frac{(1 - t_1 t_2)\left(1 - t_1 t_2^4\right)}{(1 - t_1)^2 (1 - t_2)^5} \ .$$



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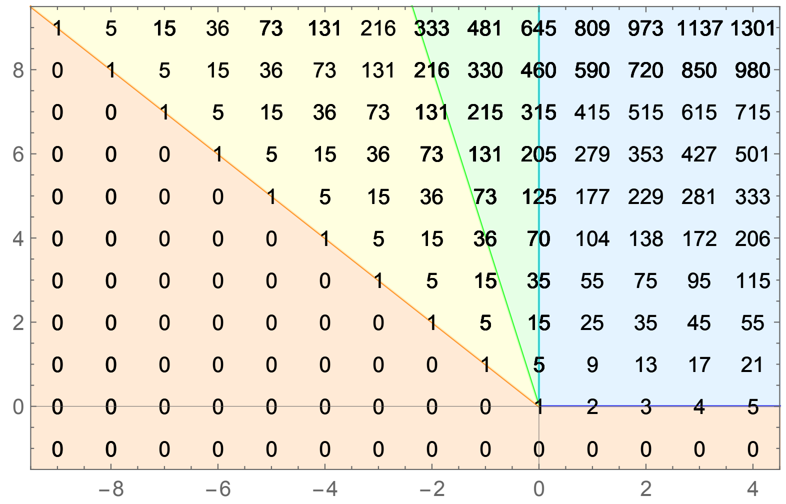
To construct the zeroth cohomology series, note that X can be flopped to a complete intersection X' in a toric variety [25] with a weight system and weights for the defining equations given by

$$X' \sim \begin{array}{ccccc|cc} z_1 & z_2 & y_1 & \ldots & y_5 & P_1' & P_2' \\ \hline 1 & 1 & 0 & \ldots & 0 & 1 & 1 \\ -1 & -4 & 1 & \ldots & 1 & 0 & 0 \end{array} \sim \begin{array}{ccccc|cc} z_1 & z_2 & y_1 & \ldots & y_5 & P_1' & P_2' \\ \hline -1 & -1 & 0 & \ldots & 0 & -1 & -1 \\ 4 & 1 & 1 & \ldots & 1 & 5 & 5 \end{array} \tag{3.52}$$

which corresponds to the Hilbert-Poincaré series

$$HS(X', t_1, t_2) = \frac{(1 - t_1^{-1} t_2^5)^2}{(1 - t_1^{-1} t_2)^2 (1 - t_2)^5 (1 - t_1^{-1} t_2^4)} \; . \tag{3.53}$$



Cohomology series

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$$HS(X,t_1,t_2) = \frac{(1-t_1t_2)\left(1-t_1t_2^4\right)}{(1-t_1)^2\,(1-t_2)^5}\ .$$

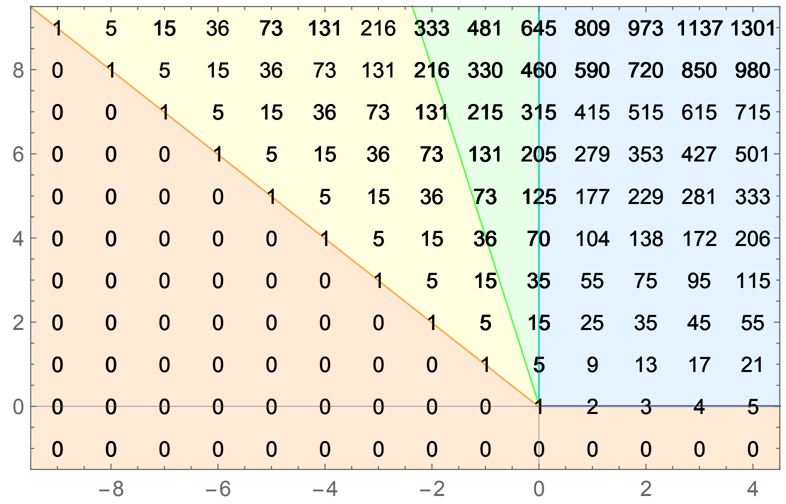
$$HS(X',t_1,t_2) = \frac{(1-t_1^{-1}t_2^5)^2}{(1-t_1^{-1}t_2)^2(1-t_2)^5(1-t_1^{-1}t_2^4)}\ . \tag{3.53}$$

Both X and the flopped threefold X' are resolutions of the same singular manifold X_{sing} which belongs to the deformation family $\mathbb{P}^4[5]$ as discussed in [25]. As such, we construct the generating function for the zeroth line bundle cohomology on X (and also on X') from the following contributions

$$CS^0(X,t_1,t_2) = \left(\frac{(1-t_1t_2)(1-t_1t_2^4)}{(1-t_1)^2(1-t_2)^5}\ ,\ \begin{matrix} t_2 & t_1 \\ 0 & 0 \end{matrix}\right) + \left(\frac{(1-t_1^{-1}t_2^5)^2}{(1-t_1^{-1}t_2)^2(1-t_2)^5(1-t_1^{-1}t_2^4)}\ ,\ \begin{matrix} t_2 & t_1 \\ 0 & 0 \end{matrix}\right) - \left(\frac{1+t_2^5}{(1-t_2)^5}\ ,\ \begin{matrix} t_2 \\ 0 \end{matrix}\right)\ ,$$

where the correction term is such that:

$$\left.\frac{(1-t_1t_2)(1-t_1t_2^4)}{(1-t_1)^2(1-t_2)^5}\right|_{t_1=0} + \left.\frac{(1-t_1^{-1}t_2^5)^2}{(1-t_1^{-1}t_2)^2(1-t_2)^5(1-t_1^{-1}t_2^4)}\right|_{t_1=\infty} - \frac{1+t_2^5}{(1-t_2)^5} = HS(\mathbb{P}^4[5],t_2) \tag{3.54}$$



Cohomology series

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$$HS(X,t_1,t_2) = \frac{(1-t_1t_2)\left(1-t_1t_2^4\right)}{(1-t_1)^2\,(1-t_2)^5} \,.$$

$$HS(X',t_1,t_2) = \frac{(1-t_1^{-1}t_2^5)^2}{(1-t_1^{-1}t_2)^2(1-t_2)^5(1-t_1^{-1}t_2^4)} \,. \tag{3.53}$$

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$$CS^0(X,t_1,t_2) = \left(\frac{(1-t_1t_2)(1-t_1t_2^4)}{(1-t_1)^2(1-t_2)^5} \,,\ \begin{matrix} t_2 & t_1 \\ 0 & 0 \end{matrix} \right) + \left(\frac{(1-t_1^{-1}t_2^5)^2}{(1-t_1^{-1}t_2)^2(1-t_2)^5(1-t_1^{-1}t_2^4)} \,,\ \begin{matrix} t_2 & t_1 \\ 0 & 0 \end{matrix} \right) - \left(\frac{1+t_2^5}{(1-t_2)^5} \,,\ \begin{matrix} t_2 \\ 0 \end{matrix} \right) ,$$

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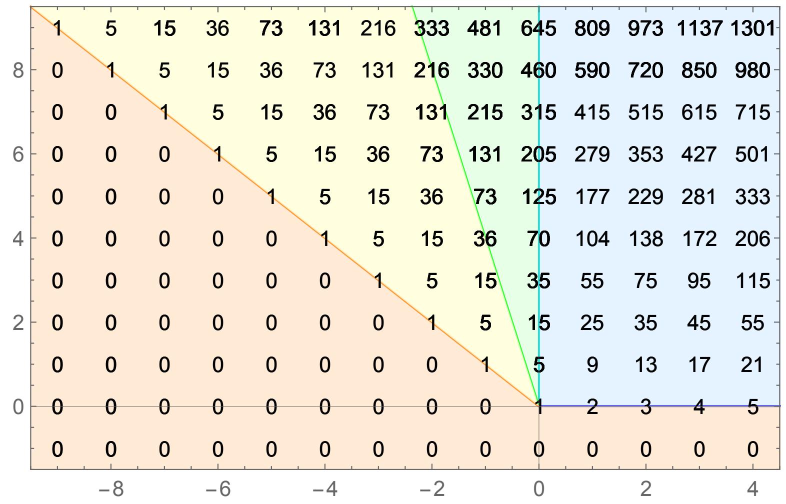


Cohomology series

The Hilbert-Poincaré series associated with the coordinate ring of X is

$$HS(X, t_1, t_2) = \frac{(1 - t_1 t_2)\left(1 - t_1 t_2^4\right)}{(1 - t_1)^2 (1 - t_2)^5} \ .$$

So the zeroth line bundle cohomology data encodes the information about **the two birational models** X and X' (their triple intersection numbers and second Chern classes), related by a flop, as well as about **the singular threefold that lies in the 'middle' of the flop**. In particular, it encodes the **GV invariant associated with the collapsing curve class** involved in the flop.

It also know about the way in which X' **degenerates as the Kähler form approaches the Zariski wall**. This is encoded by the data around the wall, which corresponds to

$$HS(\mathbb{P}_{111113}[44], t) = \frac{(1 - t^4)^2}{(1 - t)^5 (1 - t^3)} = 1 + 5t + 15t^2 + 36t^3 + 73t^4 + 131t^5 + \ldots$$

It also knows that X **degenerates as a K3 fibration** over $\mathbb{P}^1$ as the Kähler form approaches the boundary of the movable cone that is also a boundary of the effective cone.

Conjecture 5. *Let X be a general complete intersection of two hypersurfaces of bi-degrees $(1,1)$ and $(1,4)$ in $\mathbb{P}^1 \times \mathbb{P}^4$, belonging to the deformation family with configuration matrix*

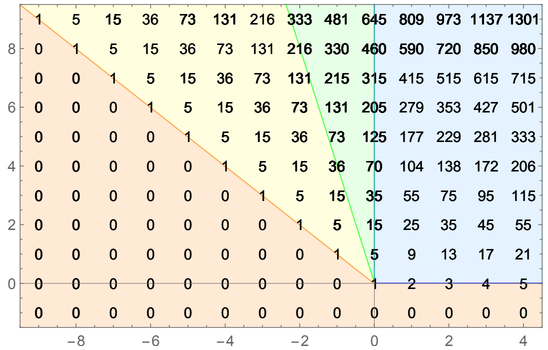
$$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^4 \end{matrix} \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} . \tag{1.16}$$

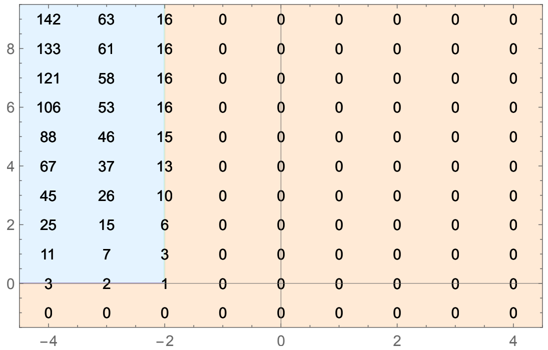
The effective, movable and nef cones of X are given by

$$\begin{gathered} \mathrm{Eff}(X) = \mathbb{R}_{\geq 0} H_1 + \mathbb{R}_{\geq 0}(H_2 - H_1), \ \mathrm{Mov}(X) = \mathbb{R}_{\geq 0} H_1 + \mathbb{R}_{\geq 0}(4H_2 - H_1) \\ \mathrm{Nef}(X) = \mathbb{R}_{\geq 0} H_1 + \mathbb{R}_{\geq 0} H_2 \ , \end{gathered} \tag{1.17}$$

where $H_1 = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^4}(1,0)|_X$ and $H_2 = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^4}(0,1)|_X$. We propose the following generating functions for all line bundle cohomology dimensions in the entire Picard group of X:

$$\begin{aligned} CS^0(X, \mathcal{O}_X) &= \left(\frac{(1-t_2)^2 \left(1-t_2^4\right)^2}{(1-t_1)^2 (1-t_2)^5 \left(1-t_1^{-1} t_2\right) \left(1-t_1^{-1} t_2^4\right)} , \begin{matrix} t_2 & t_1 \\ 0 & 0 \end{matrix} \right) \\ CS^1(X, \mathcal{O}_X) &= \left(\frac{(1-t_2)^2 \left(1-t_2^4\right)^2}{(1-t_1)^2 (1-t_2)^5 \left(1-t_1^{-1} t_2\right) \left(1-t_1^{-1} t_2^4\right)} , \begin{matrix} t_2 & t_1 \\ \infty & 0 \end{matrix} \right) \\ CS^2(X, \mathcal{O}_X) &= \left(\frac{(1-t_2)^2 \left(1-t_2^4\right)^2}{(1-t_1)^2 (1-t_2)^5 \left(1-t_1^{-1} t_2\right) \left(1-t_1^{-1} t_2^4\right)} , \begin{matrix} t_2 & t_1 \\ 0 & \infty \end{matrix} \right) \\ CS^3(X, \mathcal{O}_X) &= \left(\frac{(1-t_2)^2 \left(1-t_2^4\right)^2}{(1-t_1)^2 (1-t_2)^5 \left(1-t_1^{-1} t_2\right) \left(1-t_1^{-1} t_2^4\right)} , \begin{matrix} t_2 & t_1 \\ \infty & \infty \end{matrix} \right) \end{aligned} \tag{1.18}$$





[AC ’24]

A Picard number 3 example

z_0	z_1	z_2	z_3	z_4	z_5	z_6
1	0	0	1	−1	0	1
4	1	1	1	1	0	0
2	0	0	0	1	1	0

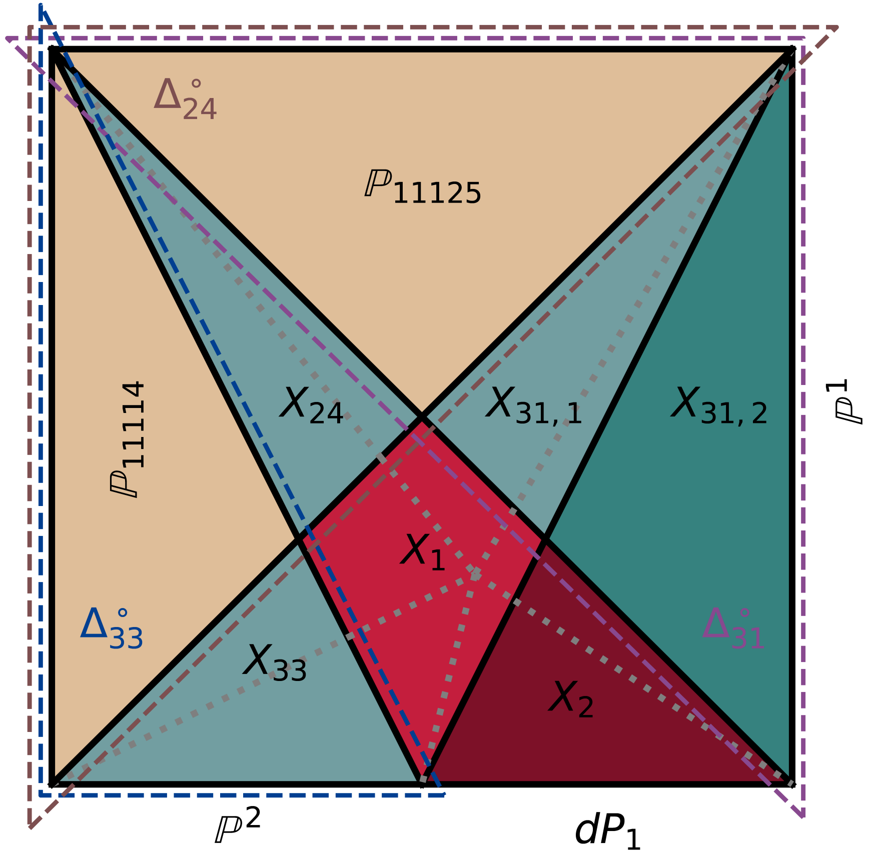


Figure 14: A slice of the effective cone of the example CY hypersurface discussed in §4.6. In red are the two Kähler cones of the two Picard number 3 CYs X_1 and X_2. In cyan are Zariski chambers corresponding to the Picard number 2 CYs associated to the reflexive polytopes $\Delta^\circ_{24}, \Delta^\circ_{31}, \Delta^\circ_{33}$ (the full secondary fans of these polytopes, each embedded in this secondary fan, are outlined with dashed lines). In beige are Zariski chambers associated to weighted projective spaces. Black lines delineate chamber boundaries in the CY effective cone; gray dashed line are flips of the toric variety which do not affect this chamber structure. Walls of the effective cone are labeled when they correspond to non-trivial toric varieties of lower dimension.

[AC, Lukas, Sheridan '25]

Cohomology series: examples in arbitrary dimension, Fano, CY and general type included

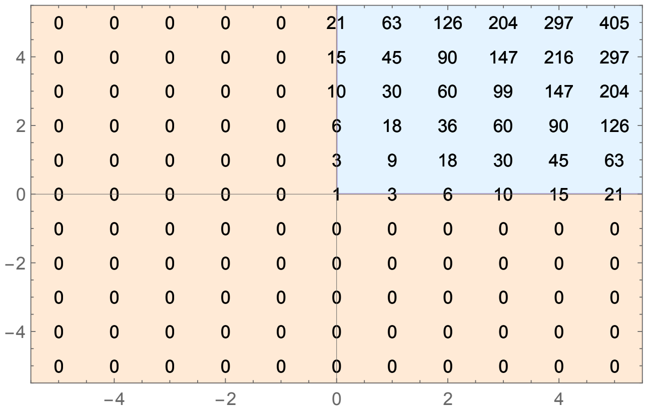
Hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^n$. Moving up in dimension, we propose the following.

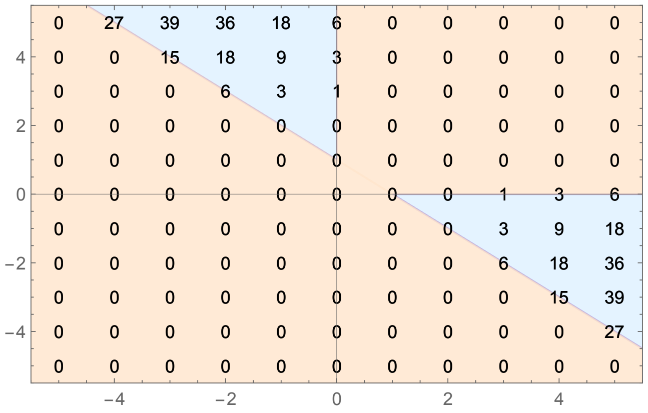
Conjecture 3. *Let X be a general hypersurface of bi-degree (d,e) in $\mathbb{P}^1 \times \mathbb{P}^{n\geq 3}$ with $d \leq n$ and e arbitrary or d arbitrary and $e = 1$. Denote $H_1 = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^n}(1,0)|_X$ and $H_2 = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^n}(0,1)|_X$. Then in the basis $\{H_1, H_2\}$,*

$$\begin{aligned}
CS^0(X,\mathcal{O}_X) &= \left(\frac{(1-t_2^e)^{d+1}}{(1-t_1)^2(1-t_2)^{n+1}(1-t_1^{-1}t_2^e)^d}, \begin{matrix} t_2 & t_1 \\ 0 & 0 \end{matrix} \right) = \sum_{m_1,m_2\in\mathbb{Z}} h^0(X,\mathcal{O}_X(m_1H_1+m_2H_2))t_1^{m_1}t_2^{m_2} \\
CS^1(X,\mathcal{O}_X) &= \left(\frac{(1-t_2^e)^{d+1}}{(1-t_1)^2(1-t_2)^{n+1}(1-t_1^{-1}t_2^e)^d}, \begin{matrix} t_2 & t_1 \\ 0 & \infty \end{matrix} \right) = \sum_{m_1,m_2\in\mathbb{Z}} h^1(X,\mathcal{O}_X(m_1H_1+m_2H_2))t_1^{m_1}t_2^{m_2} \\
(-1)^n CS^{n-1}(X,\mathcal{O}_X) &= \left(\frac{(1-t_2^e)^{d+1}}{(1-t_1)^2(1-t_2)^{n+1}(1-t_1^{-1}t_2^e)^d}, \begin{matrix} t_2 & t_1 \\ \infty & 0 \end{matrix} \right) = \sum_{m_1,m_2\in\mathbb{Z}} h^{n-1}(X,\mathcal{O}_X(m_1H_1+m_2H_2))t_1^{m_1}t_2^{m_2} \\
(-1)^n CS^{n}(X,\mathcal{O}_X) &= \left(\frac{(1-t_2^e)^{d+1}}{(1-t_1)^2(1-t_2)^{n+1}(1-t_1^{-1}t_2^e)^d}, \begin{matrix} t_2 & t_1 \\ \infty & \infty \end{matrix} \right) = \sum_{m_1,m_2\in\mathbb{Z}} h^{n}(X,\mathcal{O}_X(m_1H_1+m_2H_2))t_1^{m_1}t_2^{m_2}
\end{aligned} \tag{1.10}$$

and all intermediate line bundle cohomologies vanish.

[AC ’24]
[Pollock, Szendroi ’25]





The bicubic CY3

Conjecture 3.25. (Conjecture 6) *Let X be a general hypersurface of bidegree $(3,3)$ in $\mathbb{P}^2 \times \mathbb{P}^2$. Let $H_1 = \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,0)|_X$ and $H_2 = \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(0,1)|_X$. Defining*

$$G(x,y) = \frac{(x^{-1}y)^3((1+x-y)^3 - 1 + 3x(1-y))}{(1-x^{-1}y)^3(1-y)^3}\ , \tag{3.90}$$

all line bundle cohomology dimensions on X are encoded in the following generating functions, written in the basis $\{H_1, H_2\}$:

$$\begin{aligned}
CS^0(X,\mathcal{O}_X) &= 1 + \left(G(t_1,t_2)\,,\ \begin{matrix} t_1 & t_2 \\ 0 & 0 \end{matrix}\right) + \left(G(t_2,t_1)\,,\ \begin{matrix} t_1 & t_2 \\ 0 & 0 \end{matrix}\right) \\
CS^1(X,\mathcal{O}_X) &= 0 + \left(G(t_1,t_2)\,,\ \begin{matrix} t_1 & t_2 \\ \infty & 0 \end{matrix}\right) + \left(G(t_2,t_1)\,,\ \begin{matrix} t_1 & t_2 \\ 0 & \infty \end{matrix}\right) \\
-CS^2(X,\mathcal{O}_X) &= 2 + \left(G(t_1,t_2)\,,\ \begin{matrix} t_1 & t_2 \\ 0 & \infty \end{matrix}\right) + \left(G(t_2,t_1)\,,\ \begin{matrix} t_1 & t_2 \\ \infty & 0 \end{matrix}\right) \\
-CS^3(X,\mathcal{O}_X) &= 1 + \left(G(t_1,t_2)\,,\ \begin{matrix} t_1 & t_2 \\ \infty & \infty \end{matrix}\right) + \left(G(t_2,t_1)\,,\ \begin{matrix} t_1 & t_2 \\ \infty & \infty \end{matrix}\right)
\end{aligned} \tag{3.91}$$

[AC ’24]

Conjecture 7. *Let X be a general complete intersection Calabi-Yau threefold in the deformation family given by the configuration matrix*

$$\begin{matrix}\mathbb{P}^4 \\ \mathbb{P}^4\end{matrix}\begin{bmatrix} 2 & 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 \end{bmatrix}$$

and let $H_1 = \mathcal{O}_{\mathbb{P}^4\times\mathbb{P}^4}(1,0)|_X$ and $H_2 = \mathcal{O}_{\mathbb{P}^4\times\mathbb{P}^4}(0,1)|_X$. The effective cone decomposes into a doubly infinite sequence of Mori chambers corresponding to the nef cones of isomorphic Calabi-Yau threefolds connected to X through a sequence of flops, of the form

$$K^{(n)} = \mathbb{R}_{\geq 0}(a_{n+1}H_1 - a_n H_2) + \mathbb{R}_{\geq 0}(a_n H_1 - a_{n-1}H_2) \tag{1.24}$$

where a_n is given by

$$a_n = \frac{(3+2\sqrt{2})^n - (3-2\sqrt{2})^n}{4\sqrt{2}}\ , \quad (a_n) = \ldots -204, -35, -6, -1, 0, 1, 6, 35, 204, \ldots \tag{1.25}$$

such that $K^{(0)} = \mathrm{Nef}(X)$. A generating function for all line bundle cohomology dimensions can be written in the basis $\{H_1, H_2\}$ in terms of the functions

$$\begin{aligned} G_n(t_1,t_2) &= \frac{(1-(t_1^{a_{n+1}}t_2^{-a_n})^2)(1-(t_1^{a_n}t_2^{-a_{n-1}})^2)(1-t_1^{a_n+a_{n+1}}t_2^{-a_{n-1}-a_n})^3}{(1-t_1^{a_{n+1}}t_2^{-a_n})^5(1-t_1^{a_n}t_2^{-a_{n-1}})^5} \\ C_n(t_1,t_2) &= \frac{(1-(t_1^{a_n}t_2^{-a_{n-1}})^2)(1-(t_1^{a_n}t_2^{-a_{n-1}})^3)}{(1-t_1^{a_n}t_2^{-a_{n-1}})^5}\ , \end{aligned} \tag{1.26}$$

as follows:

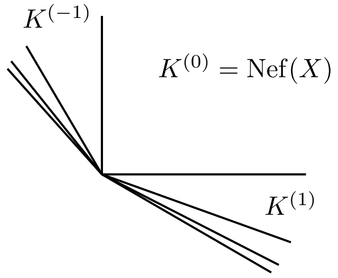
$$CS^0(X,\mathcal{O}_X) = \left(\sum_{n=-\infty}^{0} G_n(t_1,t_2)+C_n(t_1,t_2)\,,\ \begin{matrix} t_2 & t_1 \\ 0 & 0\end{matrix}\right) + \left(\sum_{n=1}^{\infty} G_n(t_1,t_2)+C_n(t_1,t_2)\,,\ \begin{matrix} t_1 & t_2 \\ 0 & 0\end{matrix}\right)$$

$$\begin{aligned} CS^1(X,\mathcal{O}_X) = &\left(\sum_{n=-\infty}^{0} G_n(t_1,t_2)+C_n(t_1,t_2)\,,\ \begin{matrix} t_2 & t_1 \\ 0 & \infty\end{matrix}\right) /.\, \{\textit{remove terms } t_1^{\alpha}t_2^{\beta} \textit{ with } \alpha+\beta<0\} + \\ &\left(\sum_{n=0}^{\infty} G_n(t_1,t_2)+C_n(t_1,t_2)\,,\ \begin{matrix} t_1 & t_2 \\ 0 & \infty\end{matrix}\right) /.\, \{\textit{remove terms } t_1^{\alpha}t_2^{\beta} \textit{ with } \alpha+\beta<0\} \end{aligned}$$

Non-Mori dream spaces

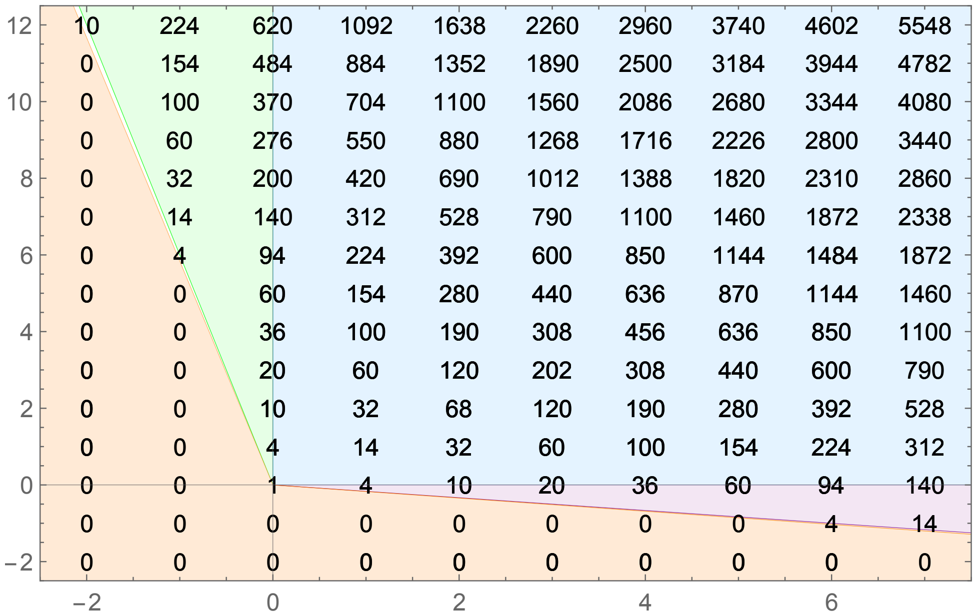
Mori dream space X: $\mathrm{Cox}(X)$ is finitely generated.

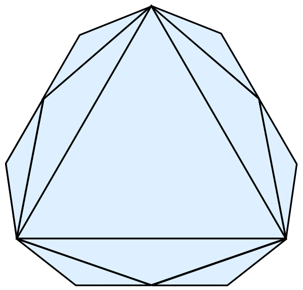
$$\mathrm{Cox}(X) = \bigoplus_{\mathcal{L}\in \mathrm{Pic}(X)} H^0(X,\mathcal{L})$$



Side remark: infinite sequences of flops. Many CICY 3-folds and hypersurfaces in toric varieties admit infinite sequences of flops. Here is an example.

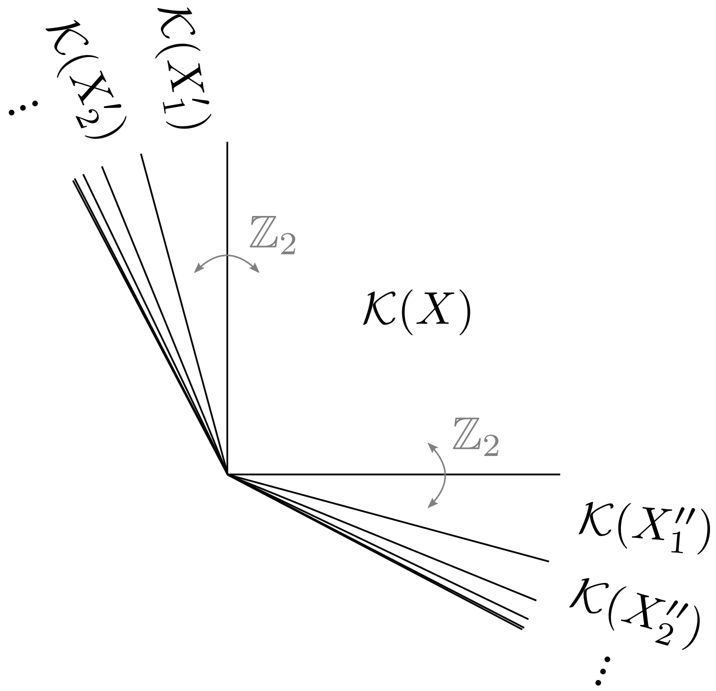
$$X = \begin{matrix} \mathbb{P}^3 \\ \mathbb{P}^3 \end{matrix} \left[\begin{matrix} 2 & 1 & 1 \\ 2 & 1 & 1 \end{matrix} \right]^{2,66} \qquad L = \mathcal{O}_X(k_1 D_1 + k_2 D_2)$$

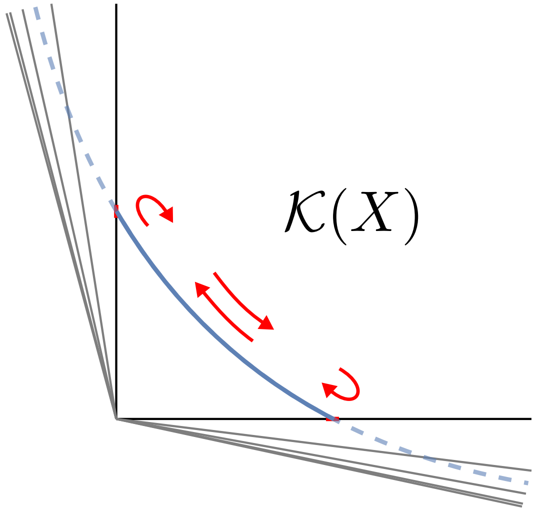




New features: infinitely many Kähler cones. The effective cone (in this case the extended Kähler cone) turns out to be irrational.

Infinite Flop Chains, the Distance Conjecture and the Kawamata-Morrison Conjecture





[Brodie, AC, Lukas, Ruehle ’21]

Why should one care?

The existence of line bundle cohomology formulae / generating functions greatly simplifies the analysis of heterotic line bundle models. Calculations that would otherwise take minutes or hours, are now virtually instantaneous.

Moreover, these expressions are of mathematical interest in themselves. We have examples in arbitrary dimension ≥ 2 including varieties of Fano, semi-Fano, CY and general type, including also non-Mori dream spaces and complex structure dependence.
Aim: convert geometry into algebraic data.

Two surprises:

1. evidence that such generating functions exist
2. the same generating function, expanded around different points, encodes the zeroth and higher cohomology of all line bundles.

Generating functions carry a lot of numerical information about the variety.
Do they uniquely determine the variety? A similar question has been asked for the regularised quantum period of Fano varieties, which is a generating function for certain Gromov-Witten invariants.

[Coates, Kasprzyk, Pitton, Tveiten '21]

Thank you for listening!

Summary

Connecting String Theory and particle Physics: a hard, but worthwhile problem.
AI tools likely to bring the solution within reach.

The size of the string landscape: the spectacular success of heuristic search methods seems to indicate that this is no longer a problem.

Fast line bundle cohomology computations: an essential tool for model building.

Computation of physical parameters (quark and lepton masses): now feasible in realistic string models.

ML Tutorial

ML and Neural Network basics

- One should think of Machine Learning in terms of fitting functions with a large number of parameters. AlexNet: millions of parameters. GPT-4: (estimated to) trillions of parameters. Us: ~80 billion neurons.

- Neural networks provide a versatile and structured recipe for constructing such functions by composing linear (affine) and non-linear functions:

$$y = f(x) = \underbrace{f_n \circ \underbrace{f_{n-1} \circ \ldots \circ \underbrace{f_2 \circ \underbrace{f_1(\overbrace{\mathbf{x}}^{=\mathbf{z}^{(0)}})}_{=\mathbf{z}^{(1)}}}_{=\mathbf{z}^{(2)}}}_{=\mathbf{z}^{(n-1)}}}_{=\mathbf{z}^{(n)}}$$

- The free parameters are placed in the linear (affine) parts. Parameter optimisation is often carried out using first order algorithms such as gradient descent.

Linear Regression with Linear Model

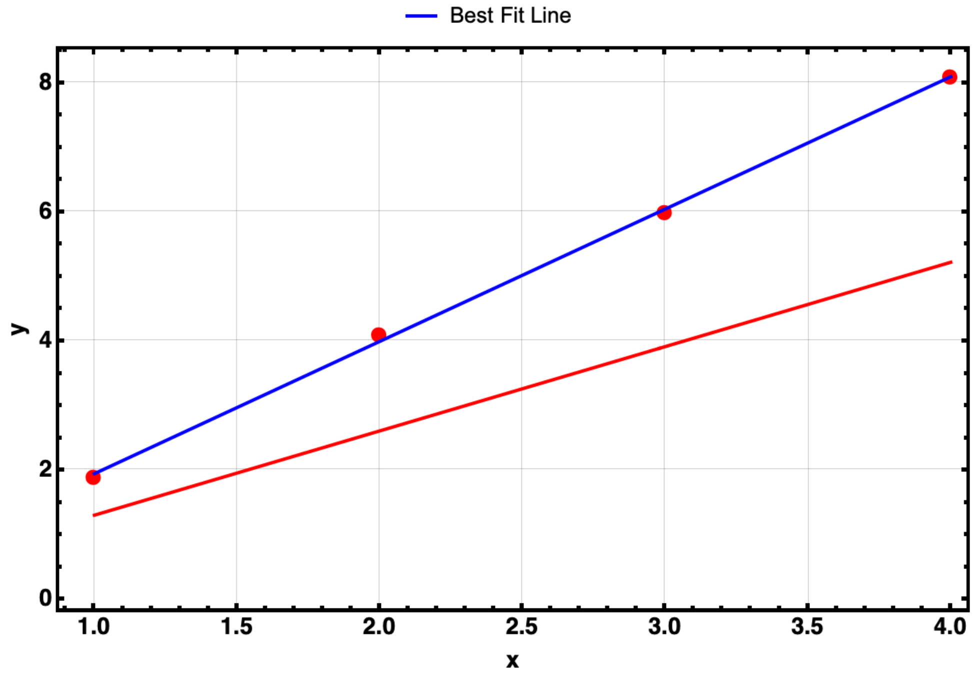
Data: $\mathcal{D} = \{(1,1.9),(2,4.1),(3,6.0),(4,8.1)\}; \quad N=4.$

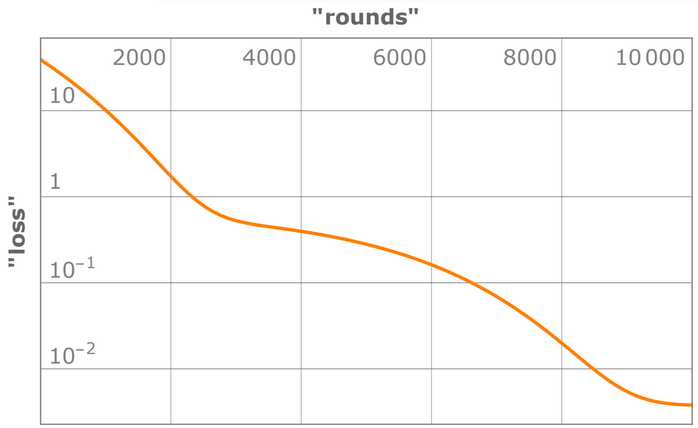
Linear model $f_\theta(x) = ax+b$. **Parameters**: $\theta = \{a,b\}$.

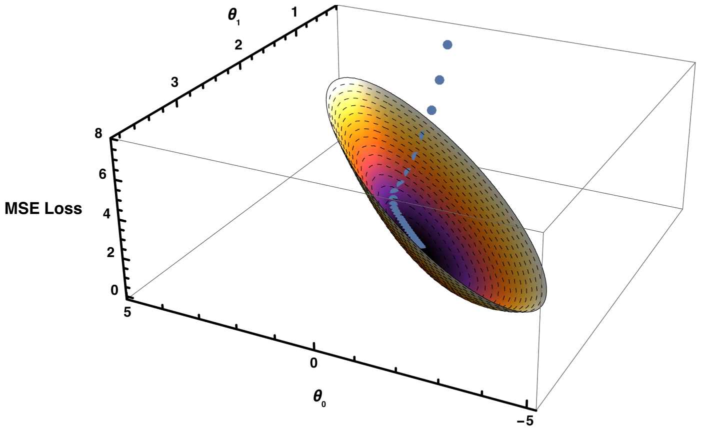
Mean square **loss function**:

$$\mathcal{L}(\theta,\mathcal{D}) = \langle (y-(ax+b))^2 \rangle_{\mathcal{D}} = \frac{1}{N}\sum_{\alpha=1}^{N}(y_\alpha - (ax_\alpha + b))^2$$

```
net = NetChain[LinearLayer[1],
        "Input" -> "Scalar", "Output" -> "Scalar"];
trainedNet = NetTrain[net, data, MaxTrainingRounds->10000]
```

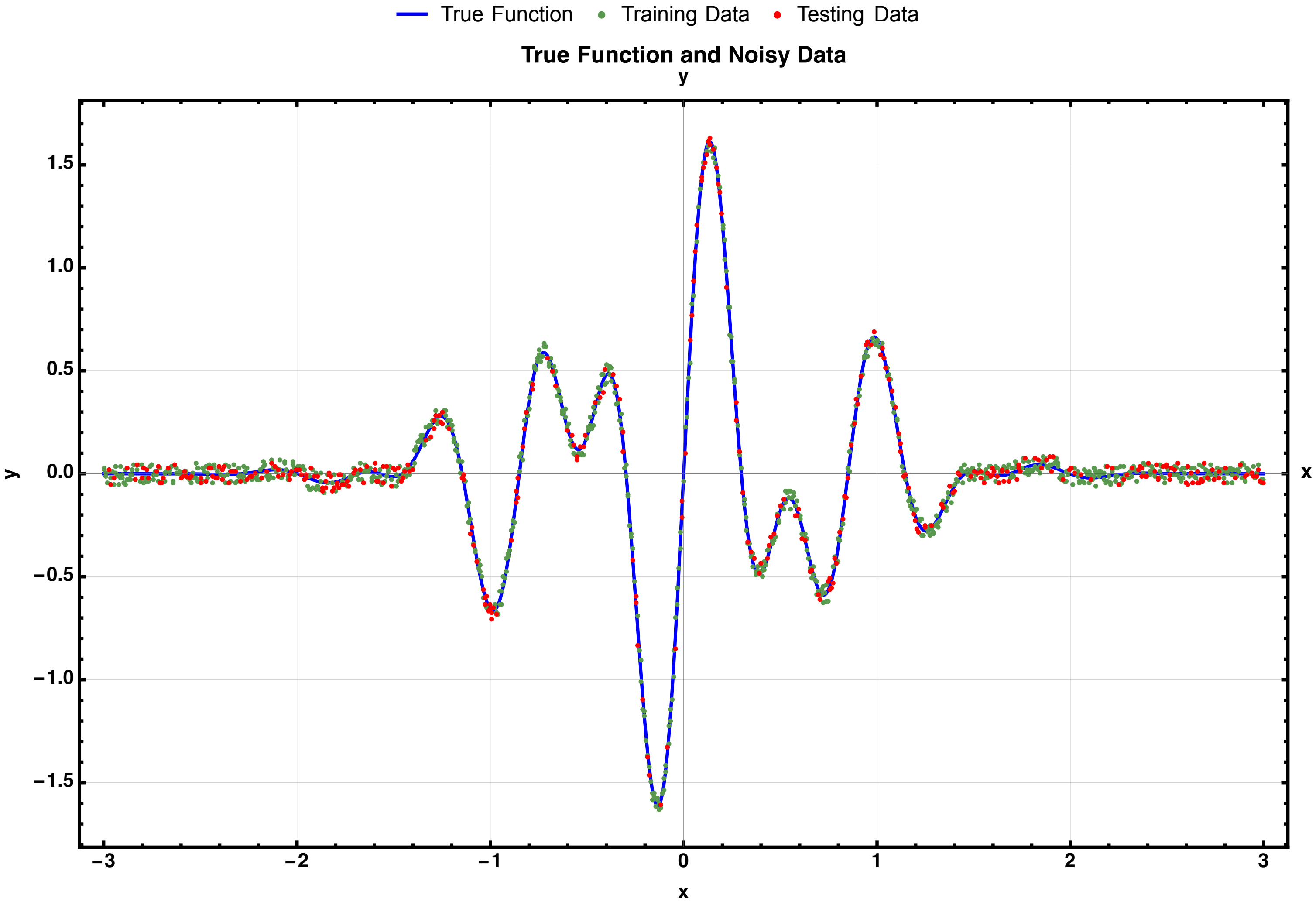




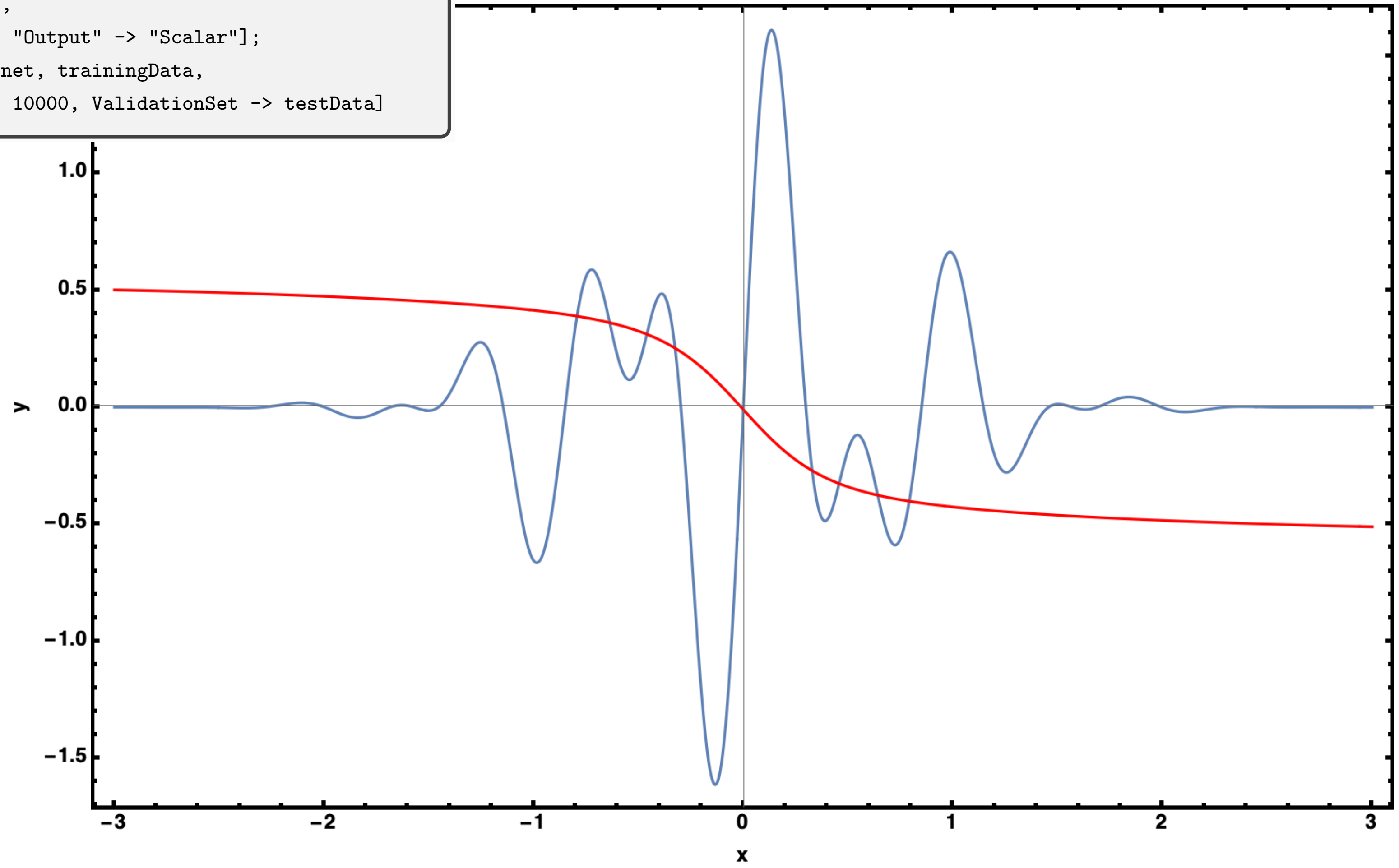


$$\theta \;\rightarrow\; \theta - \eta \nabla_\theta \mathcal{L}$$

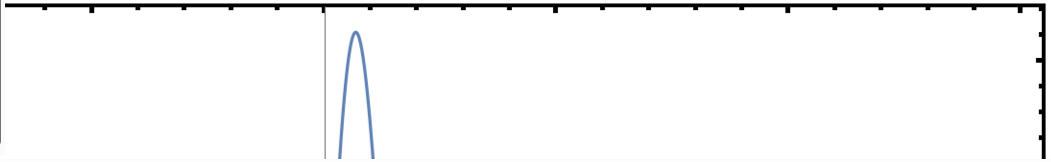
Non-Linear Regression

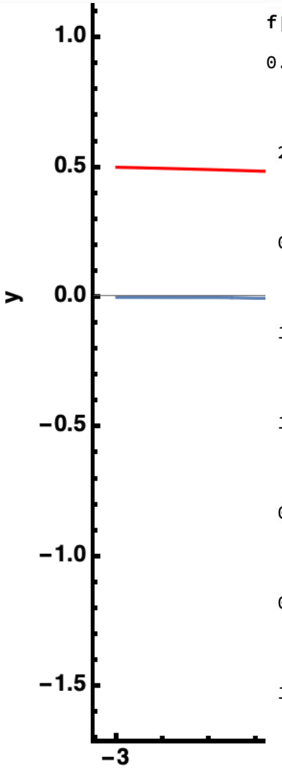


```
net = NetChain[LinearLayer[10], Tanh, LinearLayer[10],
  Tanh, LinearLayer[1],
  "Input" -> "Scalar", "Output" -> "Scalar"];
trainedNet = NetTrain[net, trainingData,
  MaxTrainingRounds -> 10000, ValidationSet -> testData]
```



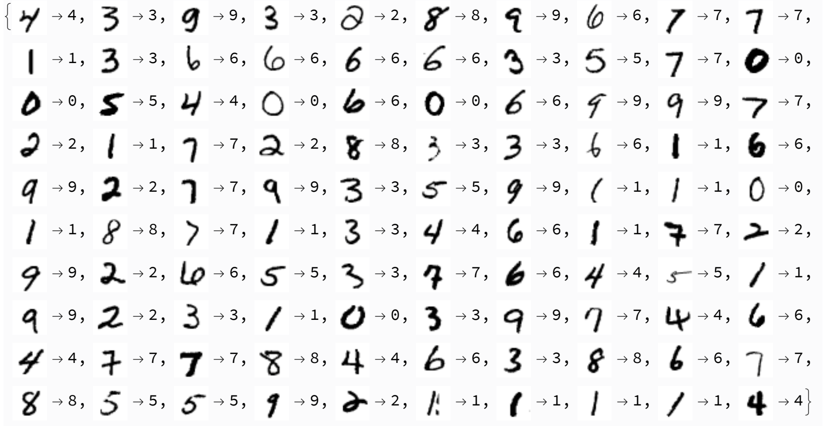
```
net = NetChain[LinearLayer[10], Tanh, LinearLayer[10],
  Tanh, LinearLayer[1],
  "Input" -> "Scalar", "Output" -> "Scalar"];
trainedNet = NetTrain[net, trainingData,
  MaxTrainingRounds -> 10000, ValidationSet -> testData]
```





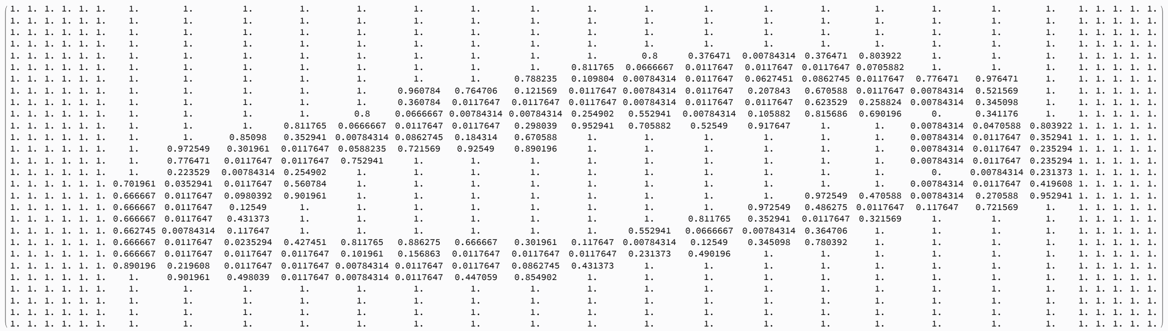

```
f[x]
0.17661 + 0.703751 Tanh[0.204803 + 0.00520699 Tanh[2.15607 - 7.44641 x] + 1.14988 Tanh[0.610416 - 4.64819 x] +
    0.596274 Tanh[1.60028 - 2.82762 x] - 0.139781 Tanh[2.0049 - 2.05248 x] + 1.41275 Tanh[1.33975 - 1.19419 x] - 1.83663 Tanh[0.488284 + 1.95824 x] -
    1.40558 Tanh[1.2765 + 2.80008 x] - 0.781195 Tanh[5.46282 + 3.94157 x] - 1.76049 Tanh[4.21767 + 5.05674 x] - 1.29827 Tanh[0.914676 + 5.71618 x]] -
 2.75689 Tanh[0.163854 + 1.52014 Tanh[2.15607 - 7.44641 x] + 0.934268 Tanh[0.610416 - 4.64819 x] + 0.147245 Tanh[1.60028 - 2.82762 x] -
    0.037561 Tanh[2.0049 - 2.05248 x] - 0.36906 Tanh[1.33975 - 1.19419 x] - 0.946579 Tanh[0.488284 + 1.95824 x] - 0.157913 Tanh[1.2765 + 2.80008 x] -
    0.437372 Tanh[5.46282 + 3.94157 x] + 0.00617781 Tanh[4.21767 + 5.05674 x] - 1.21075 Tanh[0.914676 + 5.71618 x]] +
 0.779261 Tanh[0.216784 + 0.129262 Tanh[2.15607 - 7.44641 x] + 0.246823 Tanh[0.610416 - 4.64819 x] + 1.01958 Tanh[1.60028 - 2.82762 x] -
    0.110342 Tanh[2.0049 - 2.05248 x] + 2.35819 Tanh[1.33975 - 1.19419 x] - 2.86023 Tanh[0.488284 + 1.95824 x] - 1.41244 Tanh[1.2765 + 2.80008 x] -
    2.18076 Tanh[5.46282 + 3.94157 x] - 2.11799 Tanh[4.21767 + 5.05674 x] - 1.09177 Tanh[0.914676 + 5.71618 x]] +
 1.47308 Tanh[0.0986917 + 0.0714758 Tanh[2.15607 - 7.44641 x] + 0.313793 Tanh[0.610416 - 4.64819 x] - 1.24099 Tanh[1.60028 - 2.82762 x] -
    0.256933 Tanh[2.0049 - 2.05248 x] - 1.13638 Tanh[1.33975 - 1.19419 x] + 0.954097 Tanh[0.488284 + 1.95824 x] + 1.00679 Tanh[1.2765 + 2.80008 x] +
    1.11952 Tanh[5.46282 + 3.94157 x] + 0.112738 Tanh[4.21767 + 5.05674 x] - 0.85862 Tanh[0.914676 + 5.71618 x]] +
 1.3983 Tanh[0.163903 - 0.0163045 Tanh[2.15607 - 7.44641 x] + 0.0811762 Tanh[0.610416 - 4.64819 x] - 0.311803 Tanh[1.60028 - 2.82762 x] -
    0.325621 Tanh[2.0049 - 2.05248 x] + 0.629629 Tanh[1.33975 - 1.19419 x] - 0.75579 Tanh[0.488284 + 1.95824 x] + 0.322376 Tanh[1.2765 + 2.80008 x] -
    3.47232 Tanh[5.46282 + 3.94157 x] - 2.23701 Tanh[4.21767 + 5.05674 x] - 0.284133 Tanh[0.914676 + 5.71618 x]] +
 0.837281 Tanh[0.172128 - 0.00604862 Tanh[2.15607 - 7.44641 x] + 0.266826 Tanh[0.610416 - 4.64819 x] + 2.40392 Tanh[1.60028 - 2.82762 x] +
    2.3114 Tanh[2.0049 - 2.05248 x] + 1.50169 Tanh[1.33975 - 1.19419 x] - 0.538229 Tanh[0.488284 + 1.95824 x] - 0.297384 Tanh[1.2765 + 2.80008 x] -
    0.582774 Tanh[5.46282 + 3.94157 x] - 0.216548 Tanh[4.21767 + 5.05674 x] + 0.0199791 Tanh[0.914676 + 5.71618 x]] +
 0.298102 Tanh[0.138128 + 1.59492 Tanh[2.15607 - 7.44641 x] + 0.813672 Tanh[0.610416 - 4.64819 x] + 2.00805 Tanh[1.60028 - 2.82762 x] +
    0.0844229 Tanh[2.0049 - 2.05248 x] + 0.856589 Tanh[1.33975 - 1.19419 x] - 0.392794 Tanh[0.488284 + 1.95824 x] - 0.338992 Tanh[1.2765 + 2.80008 x] +
    0.209547 Tanh[5.46282 + 3.94157 x] - 0.539457 Tanh[4.21767 + 5.05674 x] + 0.0503122 Tanh[0.914676 + 5.71618 x]] +
 1.4728 Tanh[0.33742 + 0.216924 Tanh[2.15607 - 7.44641 x] + 0.500515 Tanh[0.610416 - 4.64819 x] - 0.873752 Tanh[1.60028 - 2.82762 x] +
    2.35402 Tanh[2.0049 - 2.05248 x] + 3.72717 Tanh[1.33975 - 1.19419 x] - 0.233875 Tanh[0.488284 + 1.95824 x] + 0.264878 Tanh[1.2765 + 2.80008 x] +
    0.0393616 Tanh[5.46282 + 3.94157 x] - 0.313547 Tanh[4.21767 + 5.05674 x] + 0.116385 Tanh[0.914676 + 5.71618 x]] -
 2.4524 Tanh[1.0637 - 1.0223 Tanh[2.15607 - 7.44641 x] - 0.599124 Tanh[0.610416 - 4.64819 x] + 0.663922 Tanh[1.60028 - 2.82762 x] + 1.65099 Tanh[2.0049 - 2.05248 x] -
    0.556632 Tanh[1.33975 - 1.19419 x] - 0.0284082 Tanh[0.488284 + 1.95824 x] - 0.156302 Tanh[1.2765 + 2.80008 x] -
    1.45683 Tanh[5.46282 + 3.94157 x] - 0.636407 Tanh[4.21767 + 5.05674 x] + 0.12467 Tanh[0.914676 + 5.71618 x]] -
 1.00163 Tanh[0.0569403 - 0.0191611 Tanh[2.15607 - 7.44641 x] - 0.318248 Tanh[0.610416 - 4.64819 x] - 0.18598 Tanh[1.60028 - 2.82762 x] -
    0.248466 Tanh[2.0049 - 2.05248 x] - 0.916941 Tanh[1.33975 - 1.19419 x] + 3.15879 Tanh[0.488284 + 1.95824 x] + 1.50443 Tanh[1.2765 + 2.80008 x] +
    4.48518 Tanh[5.46282 + 3.94157 x] + 0.675371 Tanh[4.21767 + 5.05674 x] + 0.534663 Tanh[0.914676 + 5.71618 x]]
```

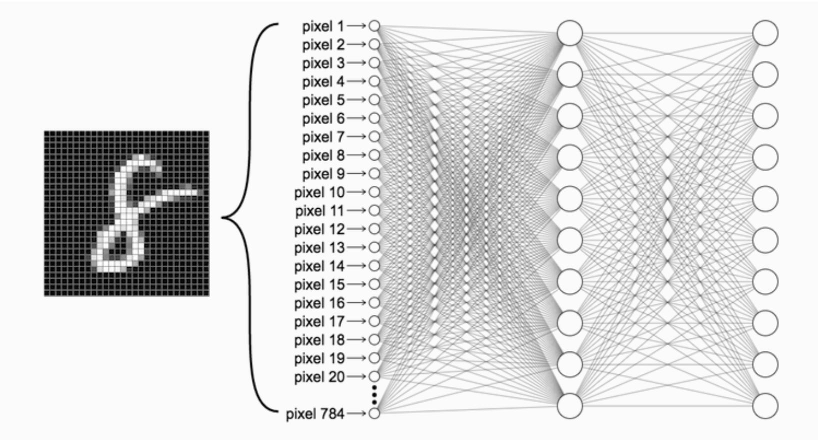
Classification



```
net = NetChain[FlattenLayer[], LinearLayer[10], Ramp,
   LinearLayer[10], SoftmaxLayer[],
   "Input" -> NetEncoder["Image", 28, 28, "Grayscale"]],
   "Output" -> NetDecoder["Class", Range[0, 9]];
trainedNet = NetTrain[net, trainingData,
   ValidationSet -> testData, MaxTrainingRounds -> 200,
   Method -> "ADAM", BatchSize -> 100]
```

```
NetEncoder["Image", 28, 28, "Grayscale"][imageName]
```





Differential Equations

Solving differential equation (LHS = RHS, BCs = 0) with neural networks:

- no training data is available
- instead, train on LHS-RHS = 0 (evaluated on a sample of points) and BCs = 0
- the neural network is simply an ansatz for the solution
- as usual, optimise the parameters with gradient descent
- avoid finite differences: the NN can be automatically differentiated w.r.t. the inputs

The simple NN from the previous examples can be successfully used to solve most undergraduate level DEs.