Machine learning CY Metrics and Yukawa couplings

Damian K. Mayorga Peña

IST, University of Lisbon

Workshop on CYs, Pollica 04.06.2025

DM
DEPARTMENT OF MATHEMATICS
TÉCNICO LISBOA

This talk is based on **arXiv:2012.15821**, **arXiv:2211.09801**, **arXiv:2401.15078**, work done in collaboration with:

Per Berglund

Giorgi Butbaia

Tristan Hübsch

Vishnu Jejjala

Challenger Mishra

Justin Tan

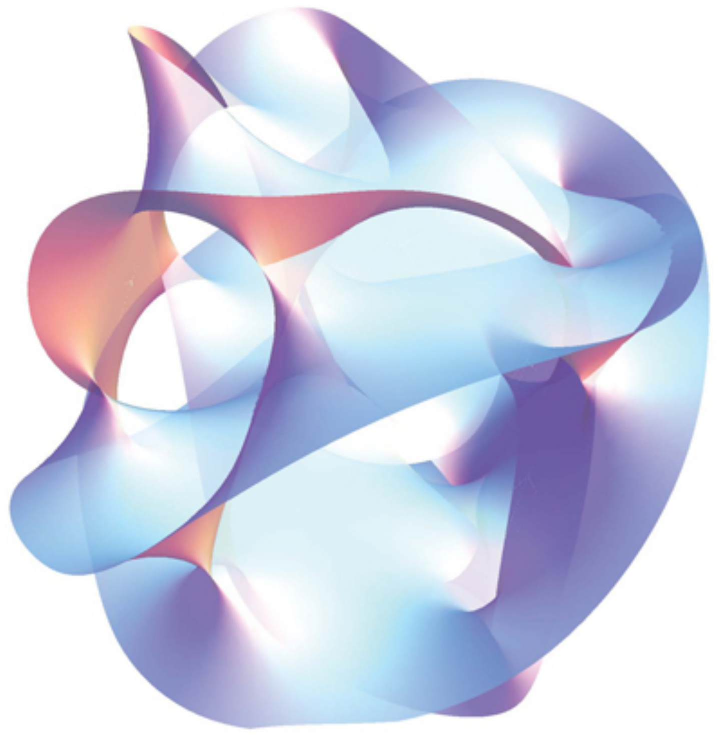
Content

$D = 10 \quad E_8 \times E_8$ Heterotic string

compactified on a Calabi-Yau threefold

Candelas, Horowitz, Strominger, Witten'85

...



Gauge structure encoded in a vector bundle V

The Ricci-flat Calabi-Yau metric

$g_{a\bar{b}}$

$D = 4 \quad \mathcal{N} = 1$ Effective field theory

with known matter content accessible via topological data

Fully fledged model requires also differential geometry data

In this talk

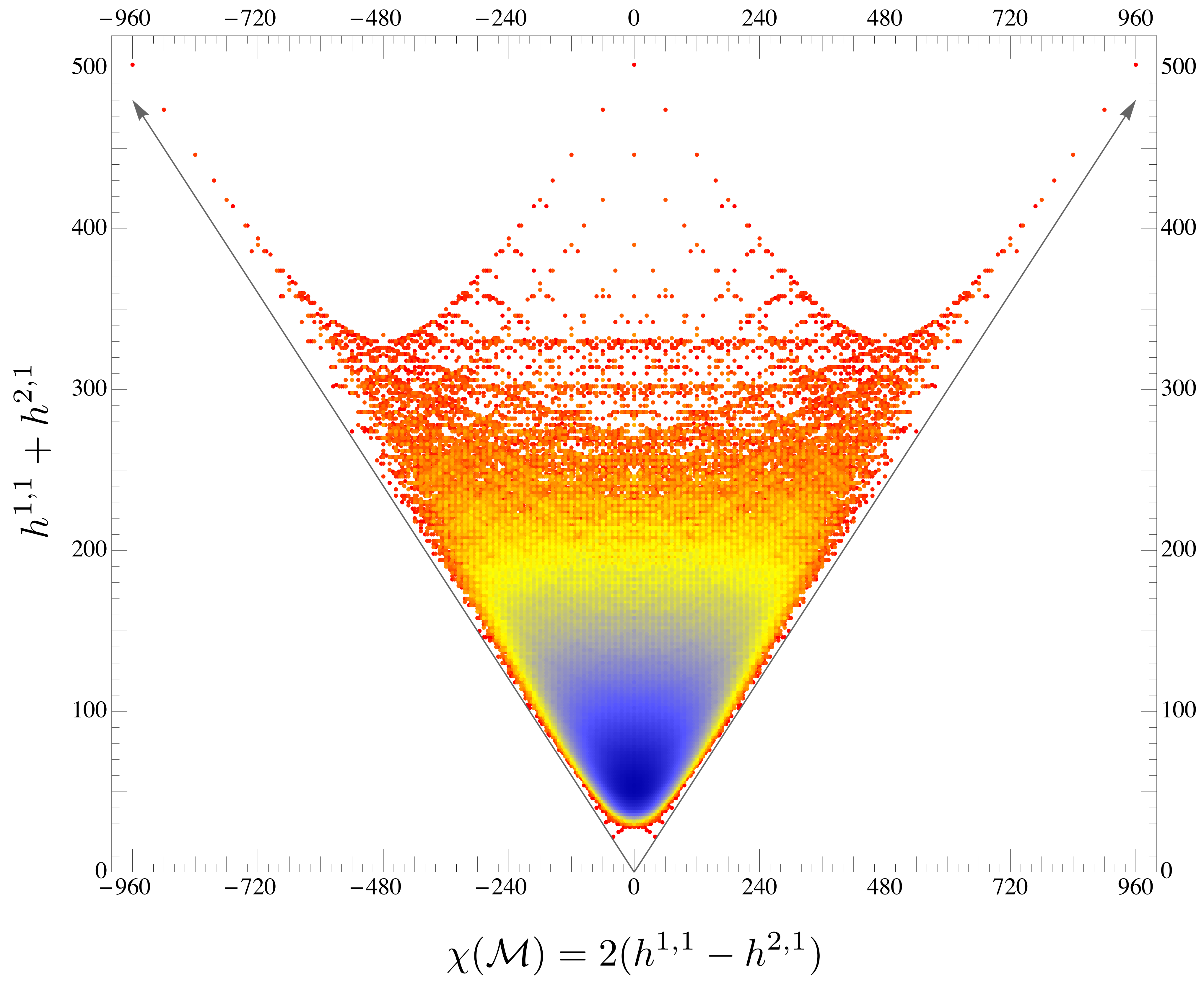
1. Calabi-Yau manifolds
2. Machine learning CY metrics
3. Yukawa couplings

1. Calabi-Yau manifolds

A (compact) Calabi-Yau manifold $\mathcal{M}$ of complex dimension n is a Kähler $(\mathcal{M}, g, J)$ manifold satisfying any (= *all*) of the following properties:

Calabi'57, Yau'77

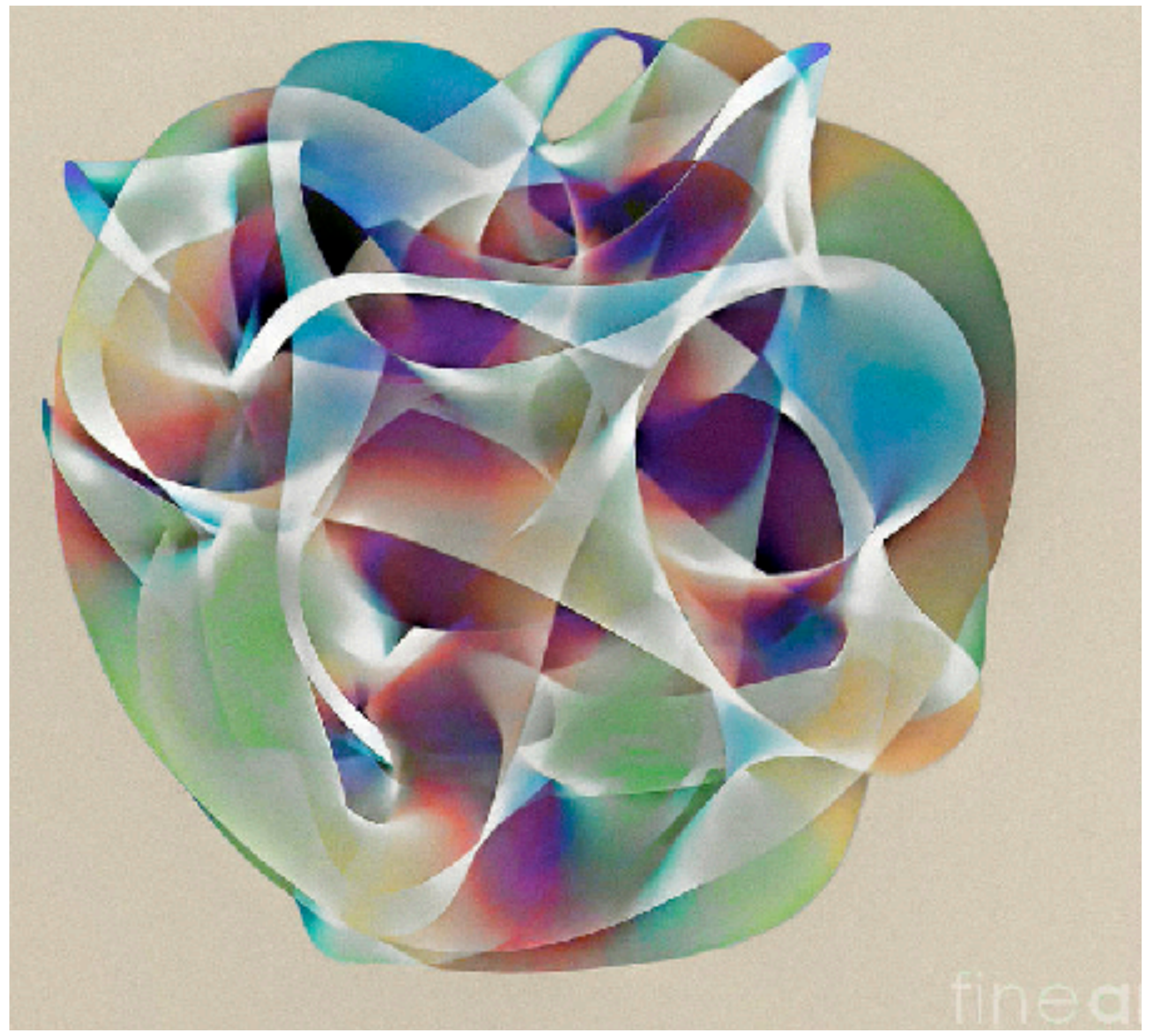
- $\mathcal{M}$ has vanishing first Chern class.
- $\mathcal{M}$ has a nowhere vanishing holomorphic top form
- $\mathcal{M}$ has a Ricci-flat Kähler metric in each Kahler class
- $\mathcal{M}$ has a Kahler metric with local holonomy $SU(3)$



cf Kreuzer, Starke'02

The parameter space of CY threefold has dimensionality $h^{1,1}(\mathcal{M}) + h^{2,1}(\mathcal{M})$. They come in mirror pairs.

2. Machine Learning the Metric (refs)



To date we do not have an analytic expression for a Ricci flat Calabi Yau metric.
With the exception of K3. Kachru, Tripathy, Zimmet'20'21

-The metric can be accessed numerically.

Headrick, Wiseman'05 Anderson, Braun Karp, Ovrut'10
Headrick, Nassar'13, Cui, Gray'19

-More recently, Machine Learning techniques have been used for this endeavour. Ashmore, Ovrut, He'19 Anderson, Gerdes, Gray, Krippendorf, Raghuram, Rühle'20
Douglas, Lakshminarasimhan, Qui'20, Jejjala, DM, Mishra'20, Ashmore, Rühle'21
Ashmore, Deen, He, Ovrut'21, Larfors, Lukas, Rühle, Schneider'21
Ashmore, He, Heyes, Ovrut'23, Gerdes, Krippendorf'23,…

Being a Kähler manifold, the Hermitian metric g can be derived from a Kähler potential

$$g_{a\bar{b}} = \partial_a \partial_{\bar{b}} K(z^a, \bar{z}^{\bar{b}})$$

The Kähler form is given by:

$$J = \frac{\mathrm{i}}{2} g_{a\bar{b}}\, dz^a \wedge \bar{z}^{\bar{b}}$$

The Ricci tensor is obtained as

$$R_{a\bar{b}} = \partial_a \partial_{\bar{b}} \log \det g$$

Simplest case: The Fubini-Study metric in the ambient space

$$K_{FS} = \frac{1}{\pi} \log(z \cdot \bar{z})$$

restricted to the hypersurface (CICY).

2. Machine Learning the Metric

Donaldson's Algortithm

1) Start with a basis of holomorphic polynomials of $\{s_\alpha\}_\alpha^{N_k}$ degree k over $\mathcal{M}$.

2) Build Kähler potential with an initial seed matrix $h^{\alpha\bar{\beta}}$, $K^{(k)} = \frac{1}{k\pi}\log(h^{\alpha\bar{\beta}} s_\alpha \bar{s}_{\bar{\beta}})$

3) Compute the quantity $H_{\alpha\bar{\beta}} = \frac{N_k}{\mathrm{Vol}_\Omega}\int_{\mathcal{M}} d\mathrm{Vol}_\Omega \left(\frac{s_\alpha \bar{s}_{\bar{\beta}}}{h^{\alpha\bar{\beta}} s_\alpha \bar{s}_{\bar{\beta}}}\right)$

4) Obtain $h^{\alpha\bar{\beta}} = (H_{\alpha\bar{\beta}})^{-1}$

5) Plug back in

6) Repeat until $h^{\alpha\bar{\beta}}$ stabilises. This is the best approximation for the flat metric at degree .

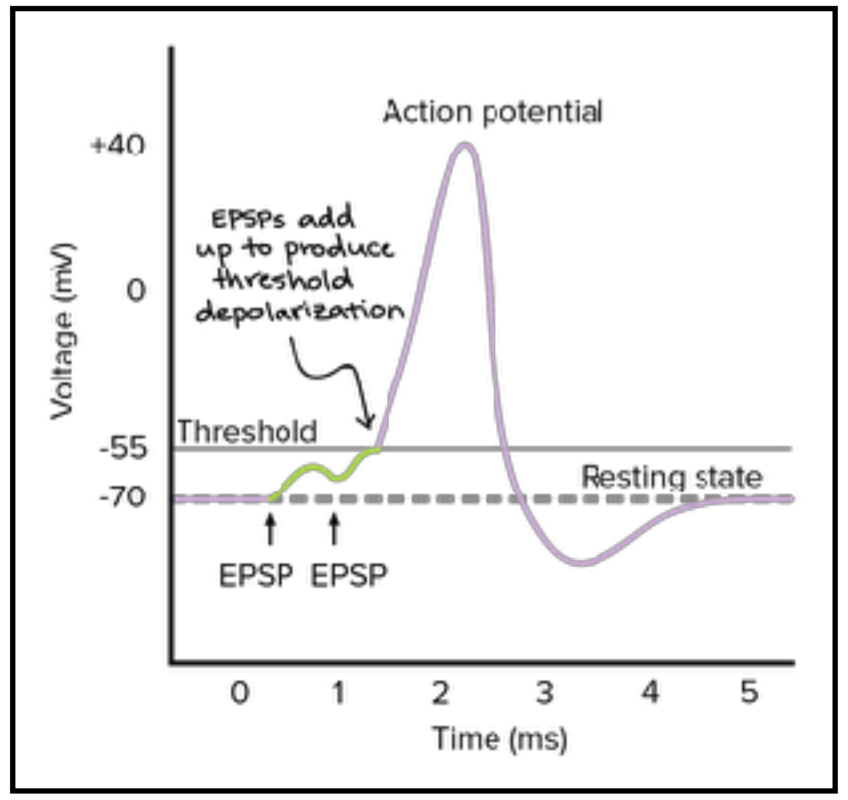
7) In the limit $k \to \infty$, the metric converges to the Ricci flat metric.

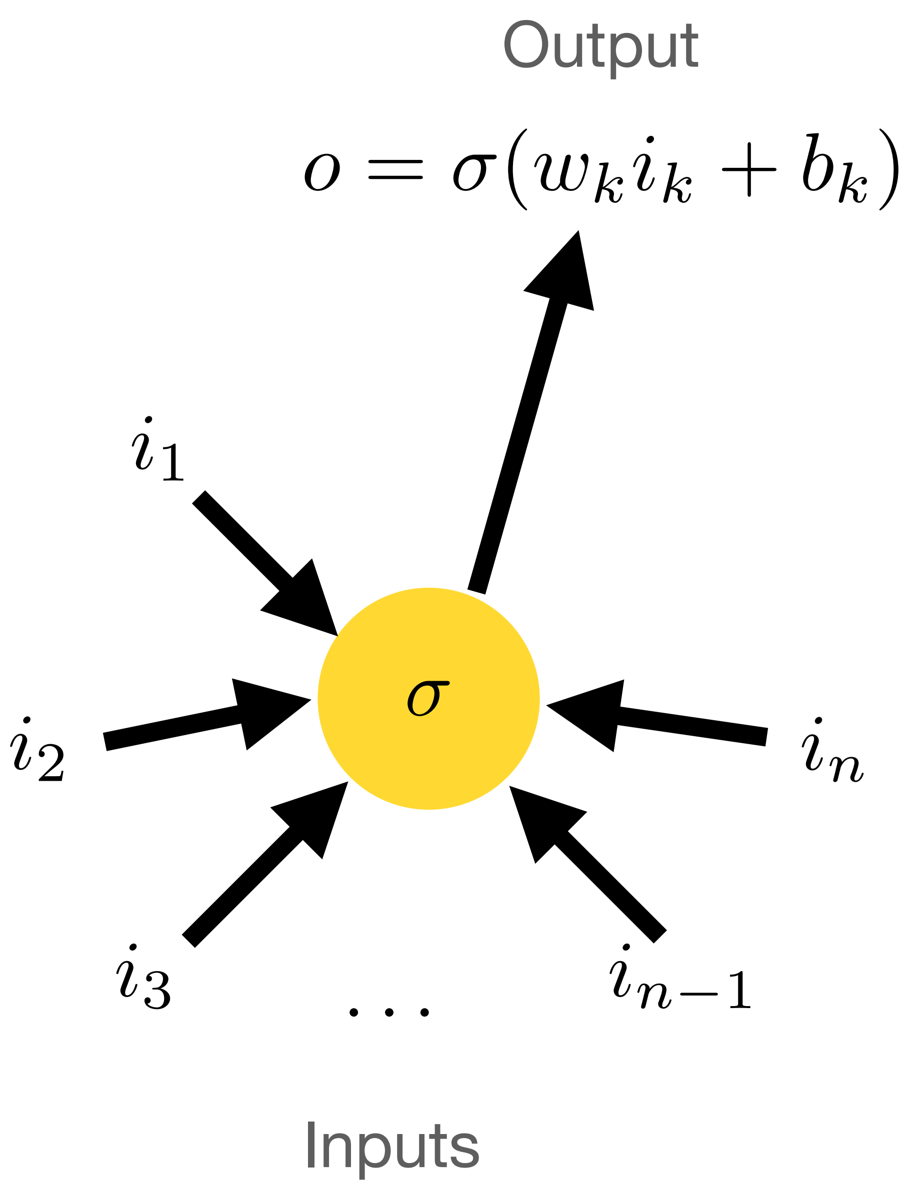
Tian'90
Donaldson'05

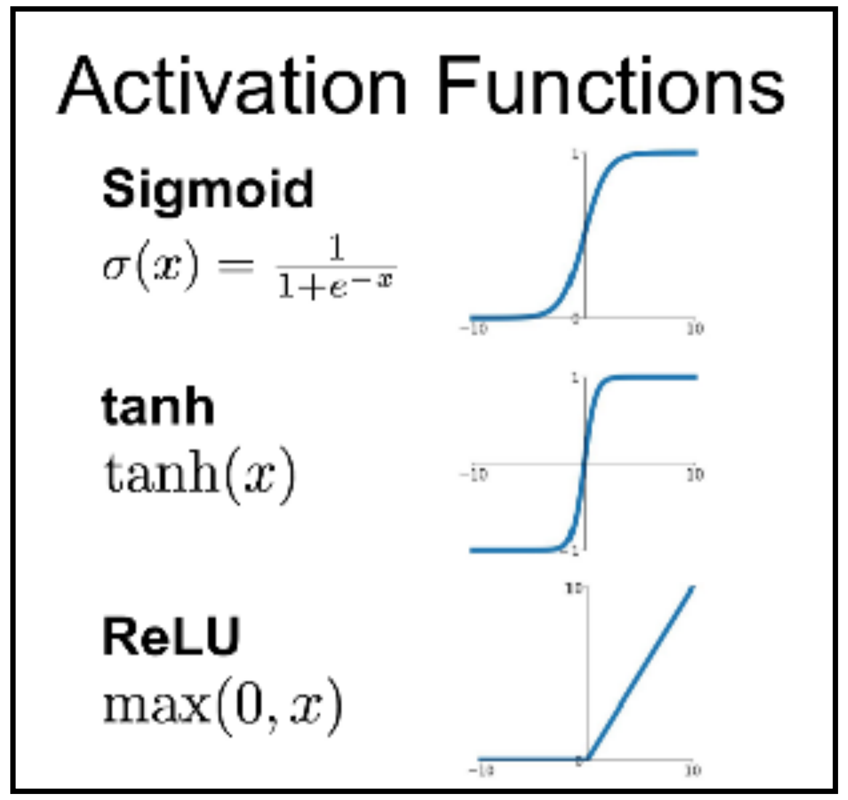
Headrick, Wiseman'05

Douglas, Karp, Lukic, Reinbacher'06
Braun, Brelidze, Douglas, Ovrut'07
Ashmore, He, Ovrut'19

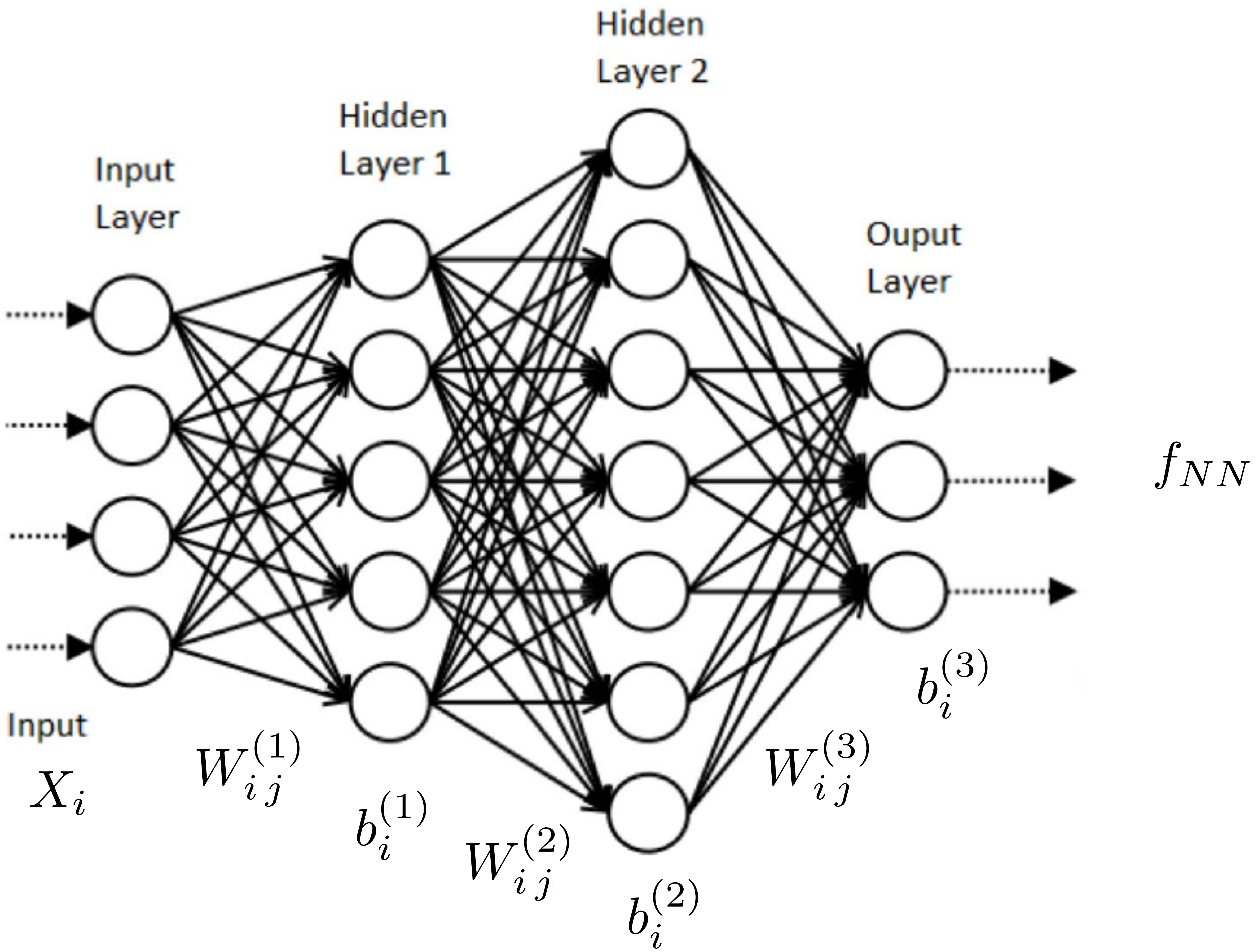
2. Machine Learning the Metric







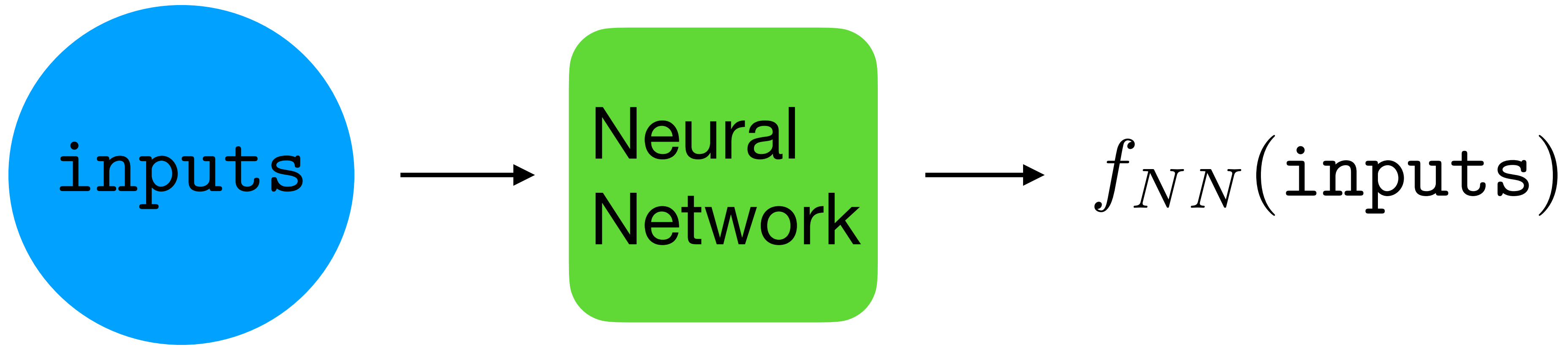
A paradigm in ML are **Artificial Neural Networks**: arrays of artificial neurons that emulate the human brain.



$$f_{NN}(X) = b_k^{(3)} + W_{kl}^{(3)}\sigma(b_l^{(2)} + W_{lm}^{(2)}\sigma(b_m^{(1)} + W_{mn}^{(1)}X_n))$$

2. Machine Learning the Metric

In this work we focus on a semi-supervised ML paradigm



Raissi, Perdikaris, Karniadakis, George’17

More flexible (than Mathematica and Matlab) packages to develop your models include:

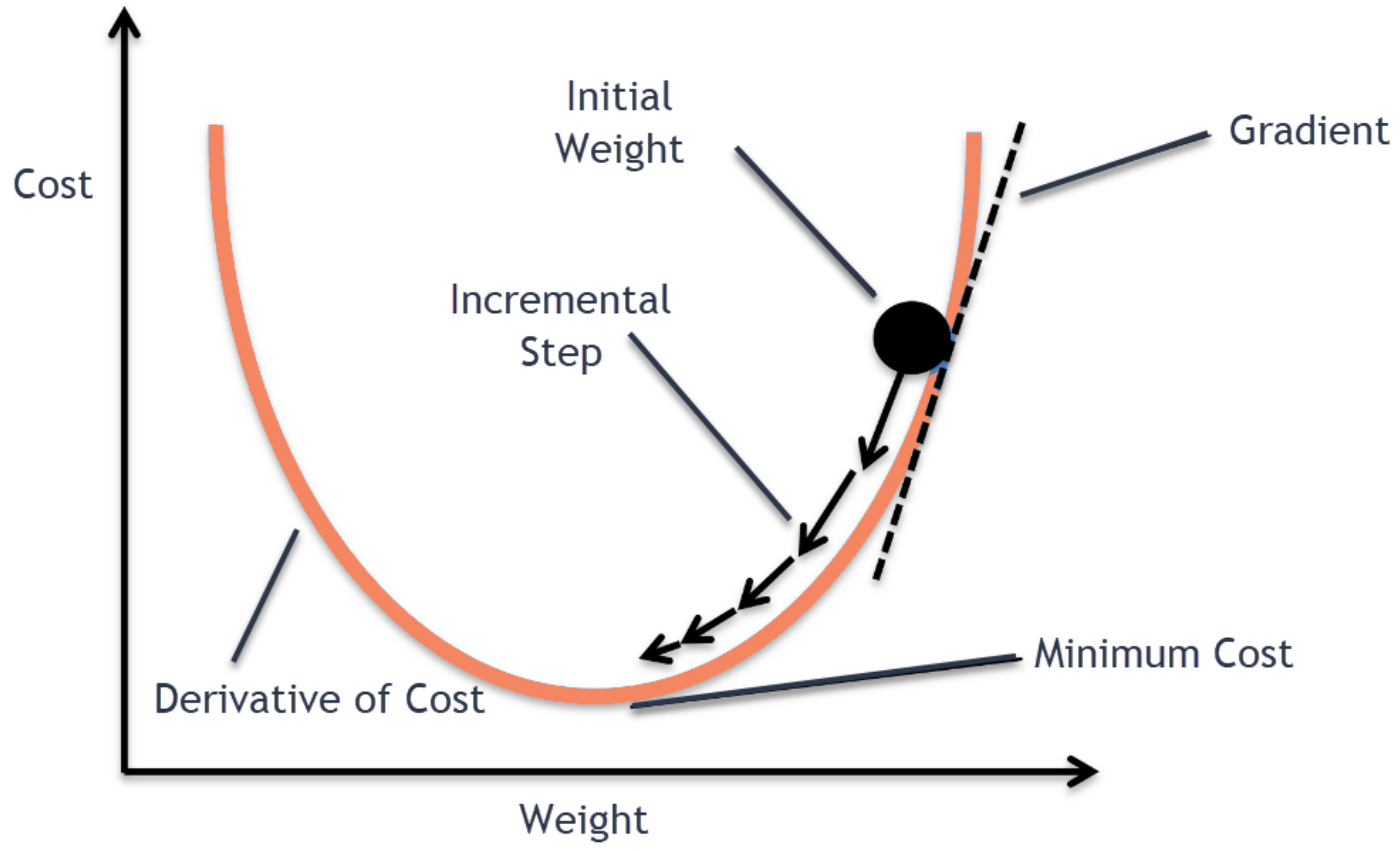
TensorFlow PyTorch JAX

The training is subject to a minimisation of a loss function

$$\mathcal{L}(f_{NN}(\texttt{inputs}))$$

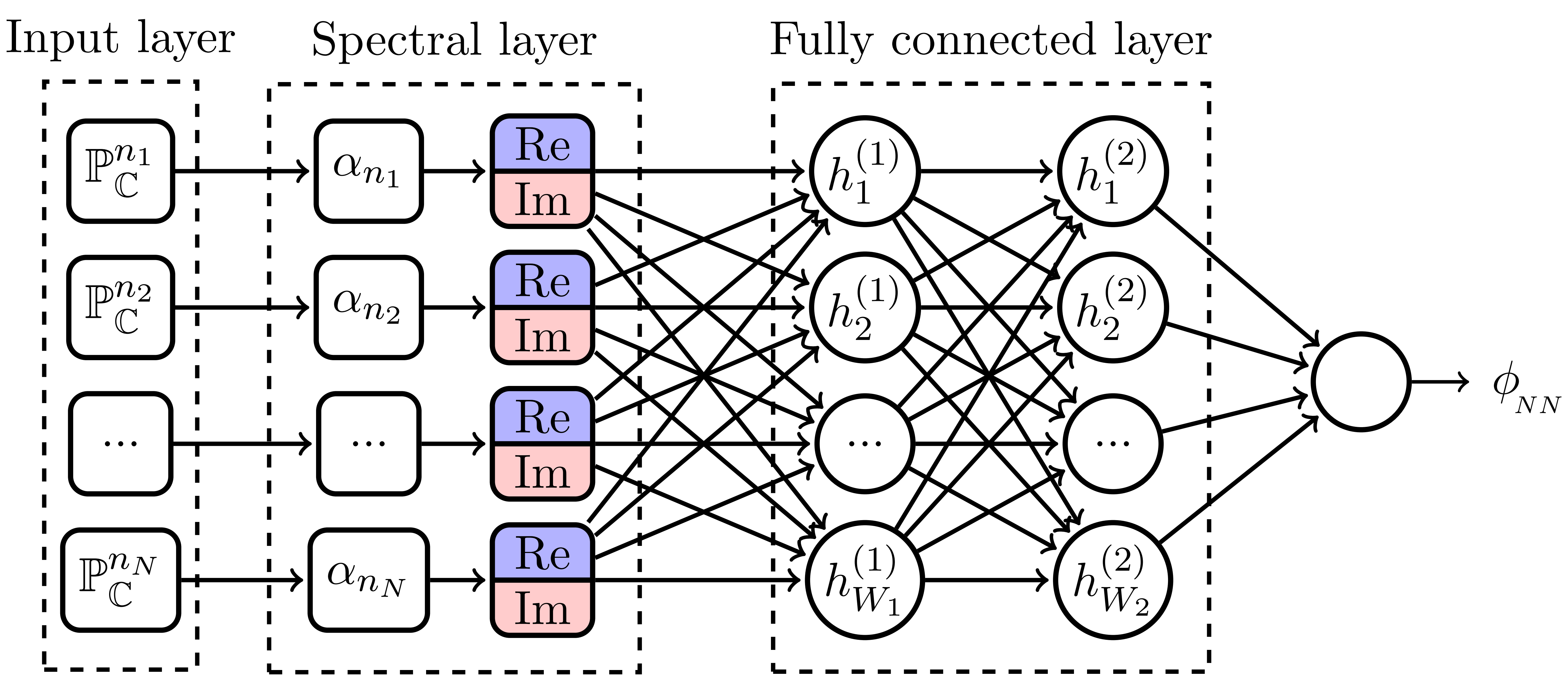
via a gradient descent procedure

$$W(t+1) = W(t) - \ell \nabla_W \mathcal{L}$$



2. Machine Learning the Metric

Spectral Neural Networks



Berglund, Butbaia, Hübsch, Jejjala, MP, Mishra, Tan'22

For every projective factor we construct the embedding

$$\alpha_{n_i} : \mathbb{P}^{n_i} \to \mathbb{C}^{n_i+1, n_i+1}$$
$$z^l \mapsto \alpha^{m,n}_{n_i} = z^m \bar{z}^n / |z|^2$$

NN inputs are real and imaginary parts.

The neural network approximation metric is

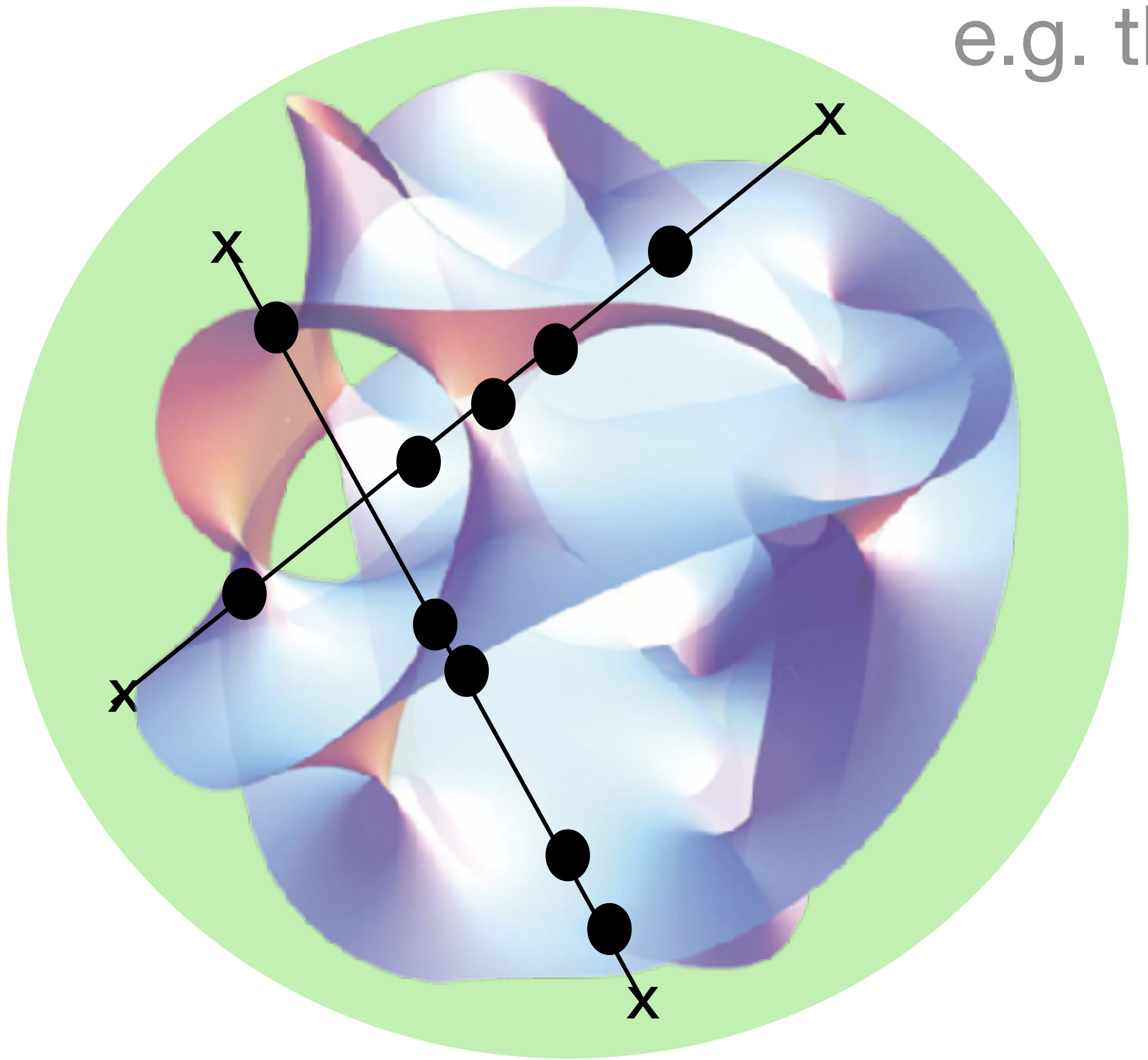
$$g_{NN} = g_s + \partial\bar{\partial}\phi_{NN}$$

with g_s a seed Kahler metric. Metrics constructed in this fashion are Kahler.

If g_{NN} is the flat metric, then ϕ_{NN} satisfies the (complex) Monge - Ampère equation $(g_s + \partial\bar{\partial}\phi_{NN})^n = \Omega \wedge \bar{\Omega}$, this enables us to construct the following loss function:

$$\sigma = \frac{1}{\mathrm{Vol}_\Omega} \int_{\mathcal{M}} d\mathrm{Vol}_\Omega \left| 1 - \frac{\mathrm{Vol}_\Omega}{\mathrm{Vol}_{g_{NN}}} \frac{\wedge^n g_{NN}}{\Omega \wedge \bar{\Omega}} \right|$$

2.1. Generating the Data



e.g. the Dwork quintic

$$\sum_i z_i^5 - 5\psi z_1 z_2 z_3 z_4 z_5 = 0 \quad \subset \mathbb{P}^4$$

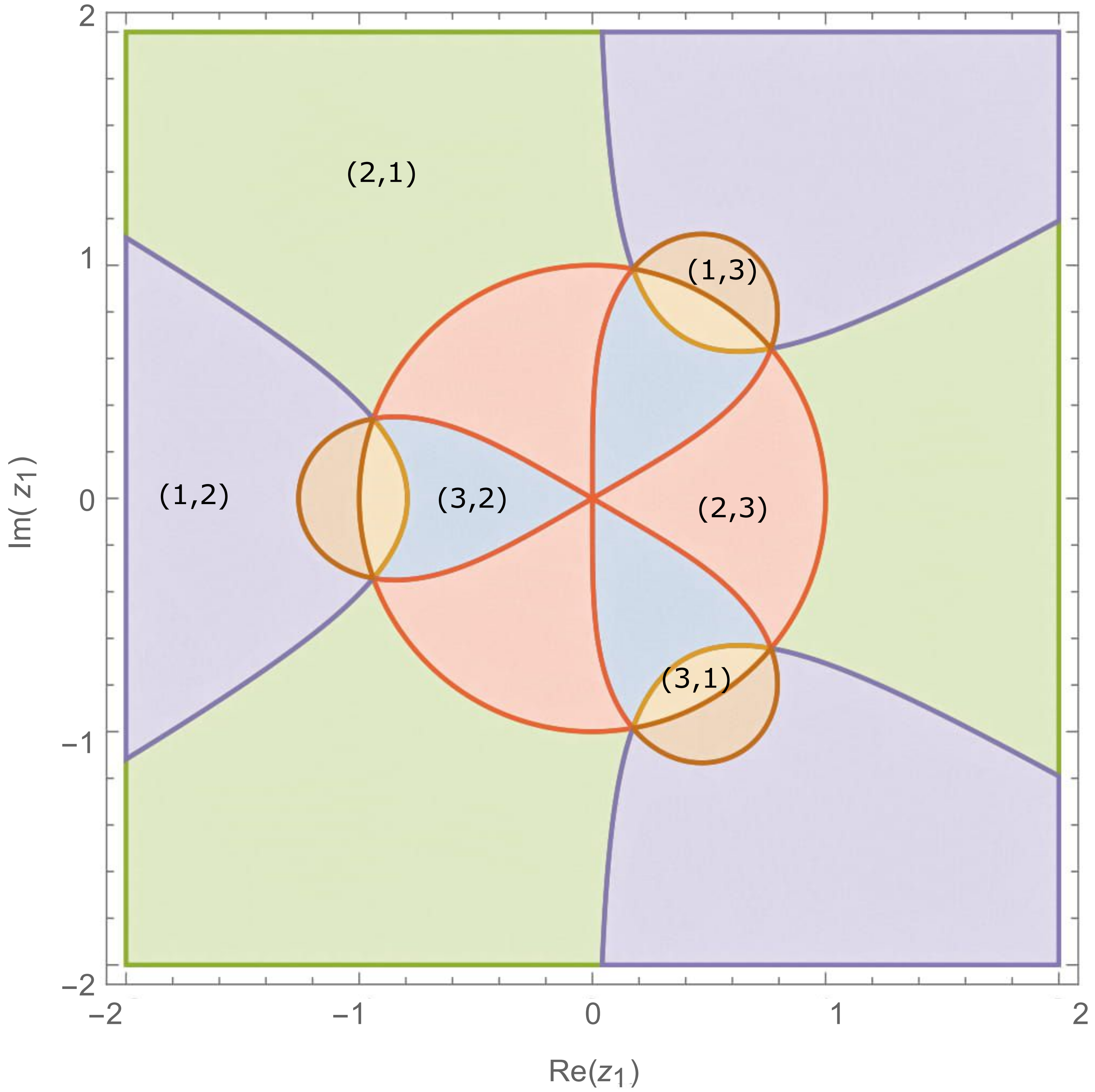
Shiffman, Zelditch'98
Braun, Brelidze, Douglas, Ovrut'07
Anderson, Braun, Karp, Ovrut'10
Ashmore, He, Ovrut'19

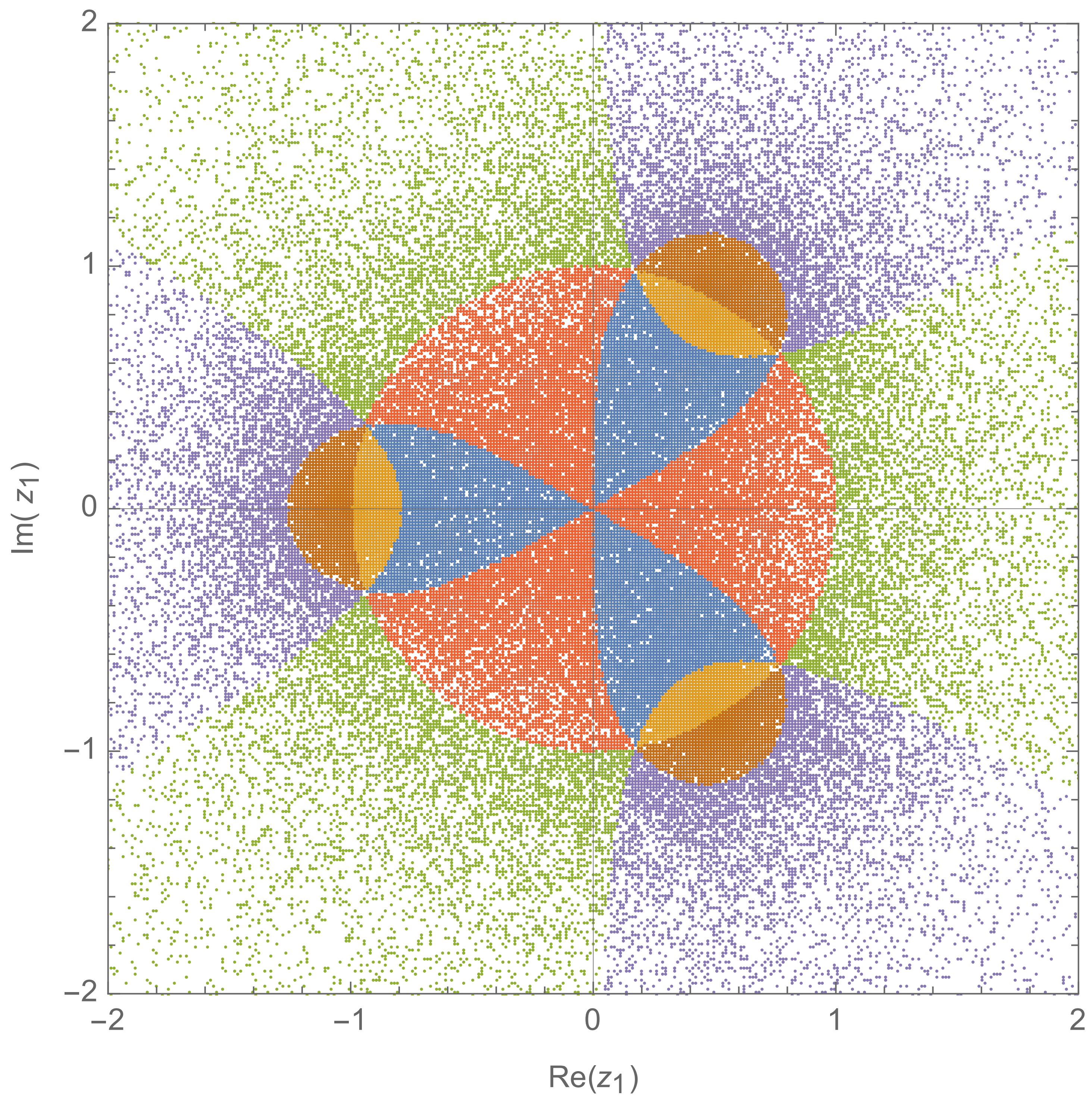
i) Start with random lines in the ambient space.
ii) Intersect each line with the hypersurface.
iii) The distribution of points is uniform with respect to the FS metric.
iv) There is a preferred patch for each point.

Sampling points in the torus

$$z_1^3 + z_2^3 + z_3^3 = 0 \subset \mathbb{P}^2$$

For the torus we have six patches fixed upon choice of the affine coordinate and the dependent coordinate. For each point in the torus there is a preferred patch.





2.1. Generating the Data

Numerical Integration

- Patches intersect over zero measure sets, hence numerical integration would be a sum over points in different patches if these would be uniformly distributed with respect to the Calabi-Yau metric.

- The sampling method provides points uniformly distributed with respect to the Fubini-Study metric restricted to the Calabi-Yau.

- Numerical integration requires to weight the sample points in order to obtain meaningful quantities

$$\int_{\mathcal{M}} d\text{Vol}_{\Omega}\, f(z,\bar{z}) = \int_{\mathcal{M}} d\text{Vol}_{FS} \left(\frac{d\text{Vol}_{\Omega}}{d\text{Vol}_{FS}}\right) f(z,\bar{z})$$

$$\int_{\mathcal{M}} d\text{Vol}_{\Omega}\, f(z,\bar{z}) = \frac{1}{N}\sum_{l=1}^{M} w(p_l) f(p_l) \qquad w(p_l) = \frac{d\text{Vol}_{\Omega}(p_l)}{d\text{Vol}_{FS}(p_l)}$$

2.2. Examples of Metrics

Probing the Cefalu Pencil

$$\mathbb{P}^3 \supset X_\lambda := \{p_\lambda(z) = 0\} : \quad p_\lambda(z) := \sum_{i=0}^{3} z_i^4 - \frac{\lambda}{3}\left(\sum_{i=0}^{3} z_i^2\right)^2$$

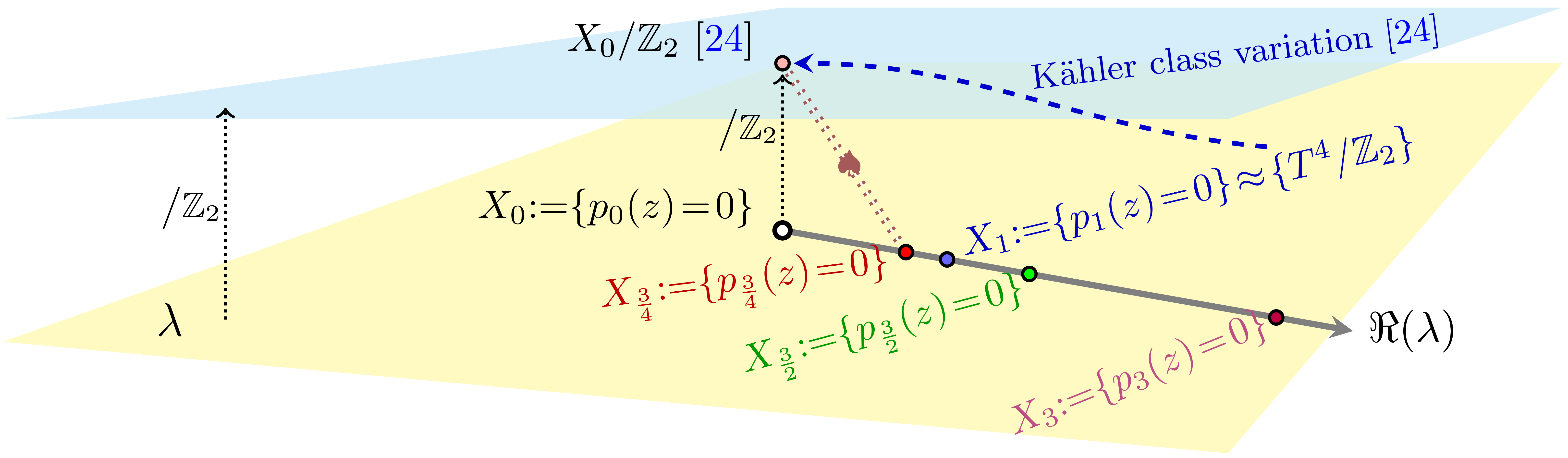
Transversality fails at:

$\lambda = 3/4$ 8 singular points

$\lambda = 1$ 16 singular points

$\lambda = 3/2$ 12 singular points

$\lambda = 3$ 4 singular points



Chern classes:

$$c_0 = 1 \ ,$$

$$c_1 = \frac{\mathrm{i}}{2\pi} \mathrm{Tr}\, \mathcal{R} \ ,$$

$$c_2 = \frac{1}{2(2\pi)^2}\left(\mathrm{Tr}\, \mathcal{R}^2 - (\mathrm{Tr}\, \mathcal{R})^2\right) \ ,$$

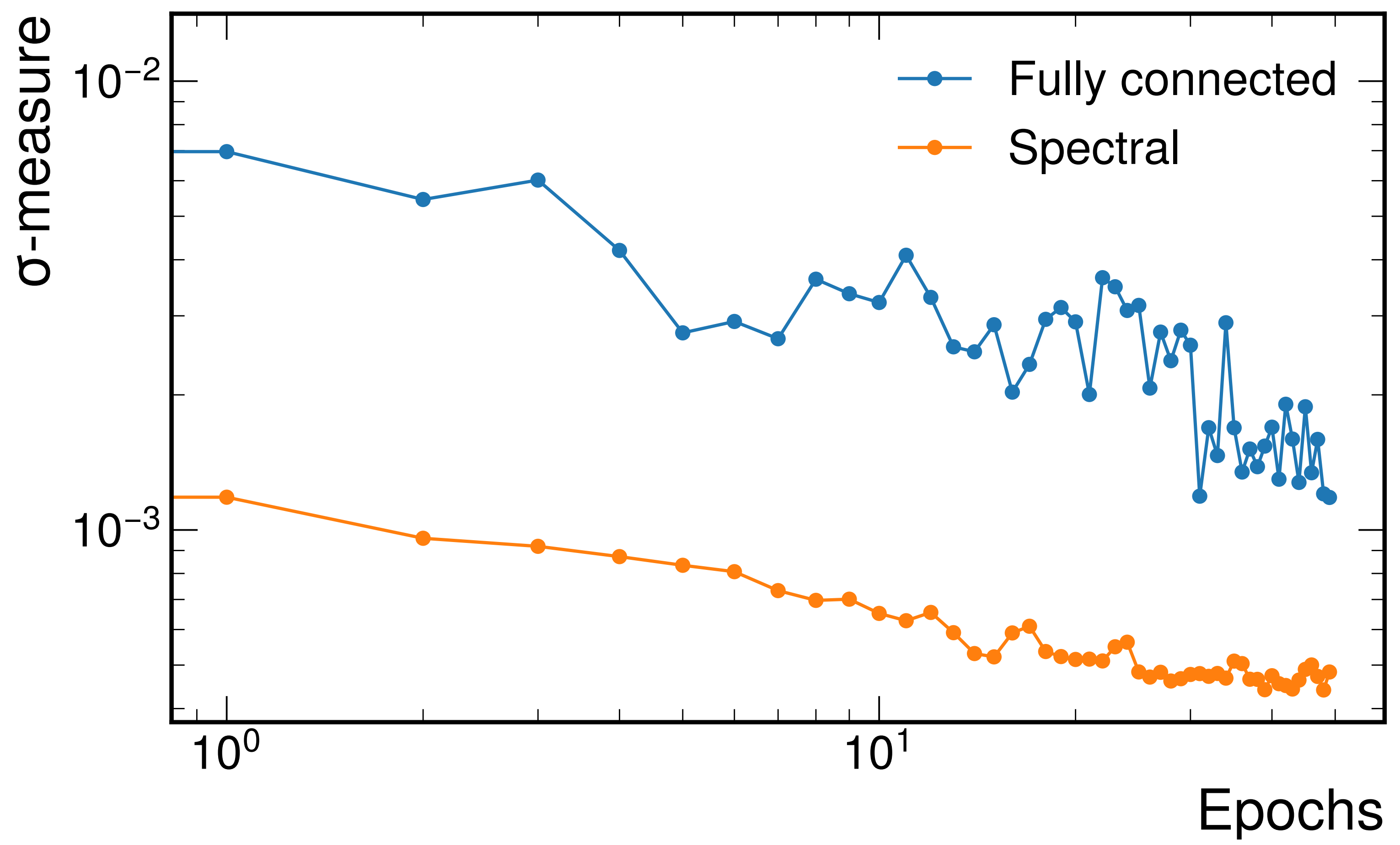
Euler Number:

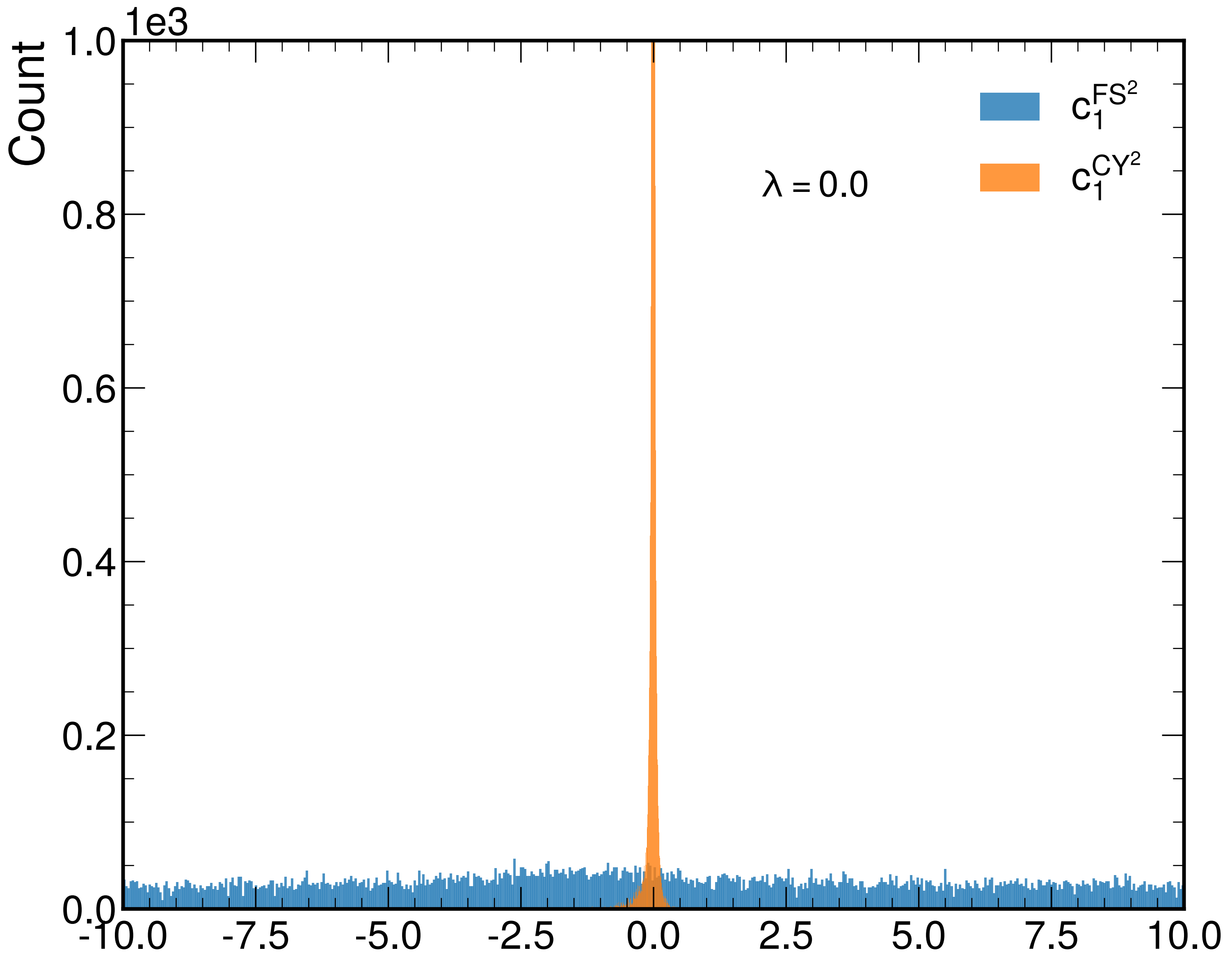
$$\chi_{\mathrm{E}} = \int_{\mathrm{K3}} c_2 = \frac{1}{2(2\pi)^2} \int_{\mathrm{K3}} \left(\mathrm{Tr}\, \mathcal{R}^2 - (\mathrm{Tr}\, \mathcal{R})^2\right)$$

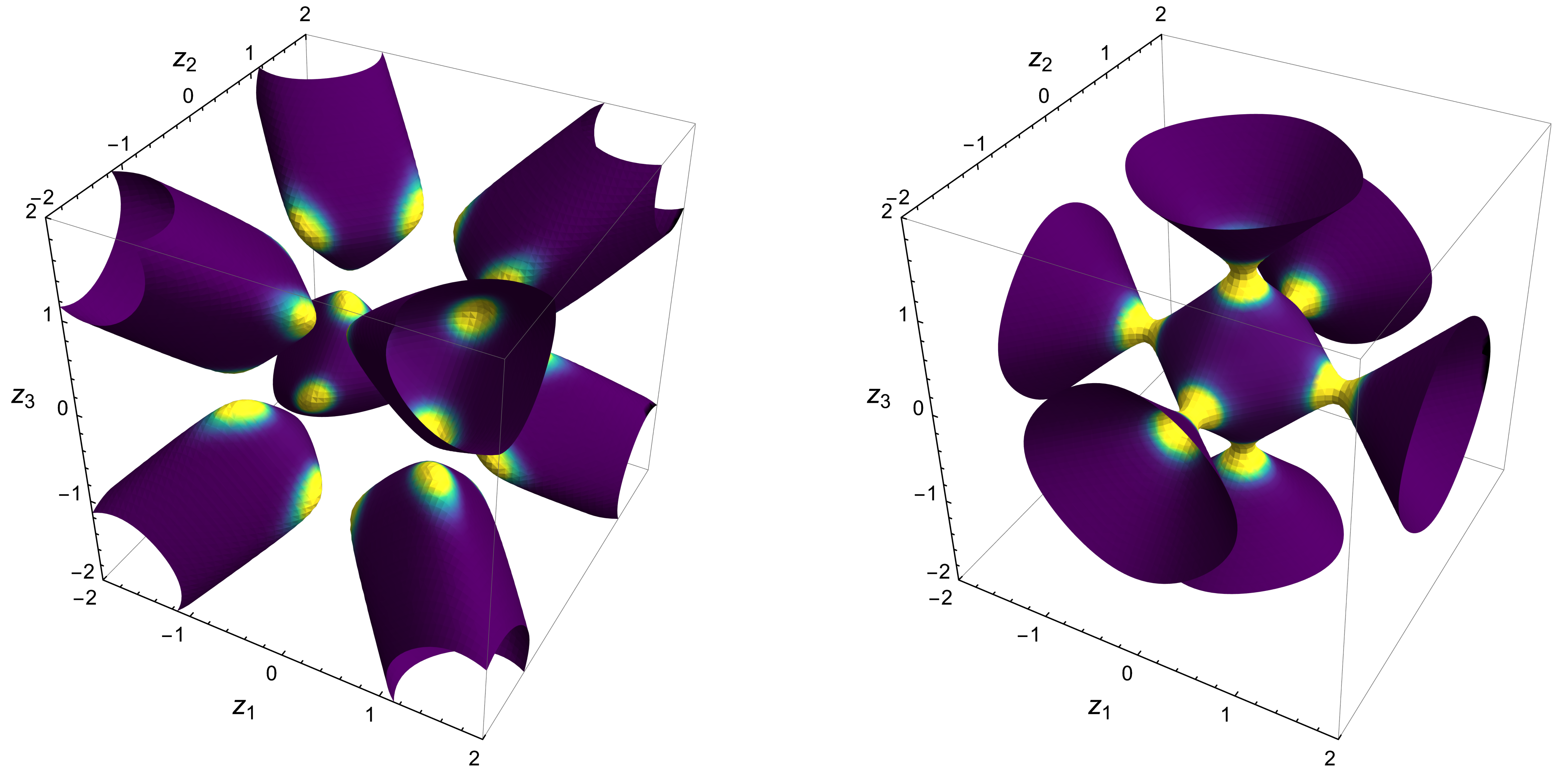
(= 24 for a smooth K3 surface)

2.2. Examples of metrics

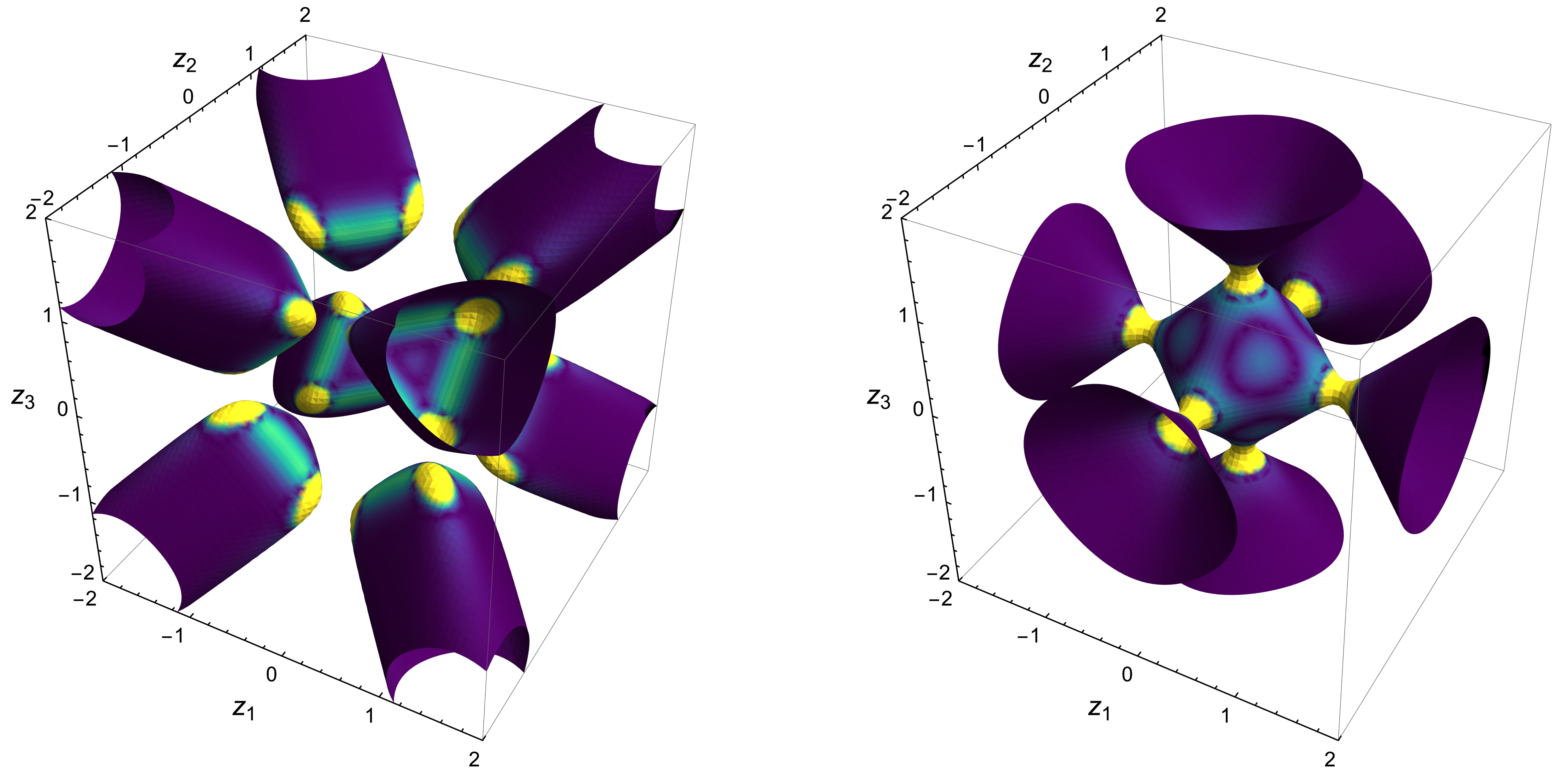
$\lambda = 0$







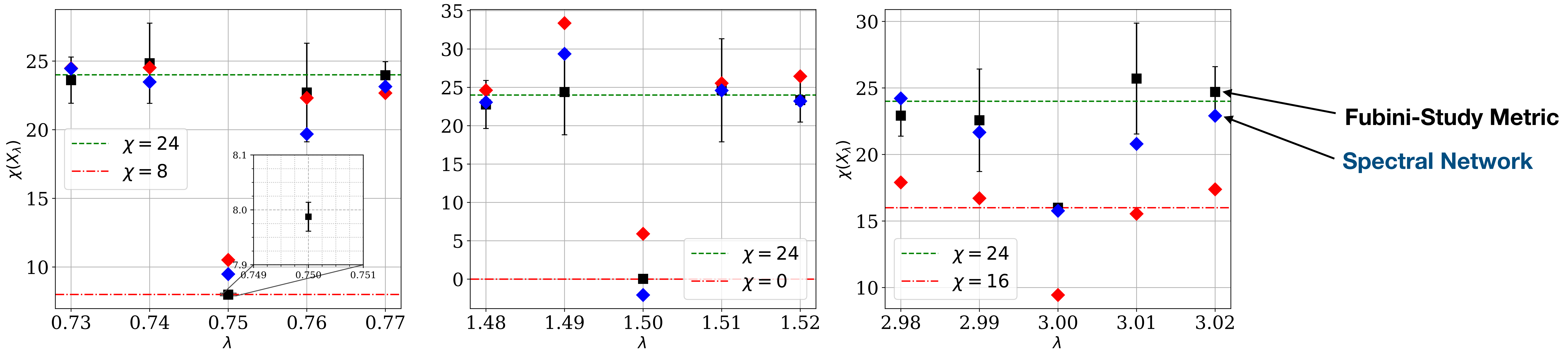
(a) $\lambda = 0.98,\ J_{\rm CY}$ (b) $\lambda = 1.48,\ J_{\rm CY}$

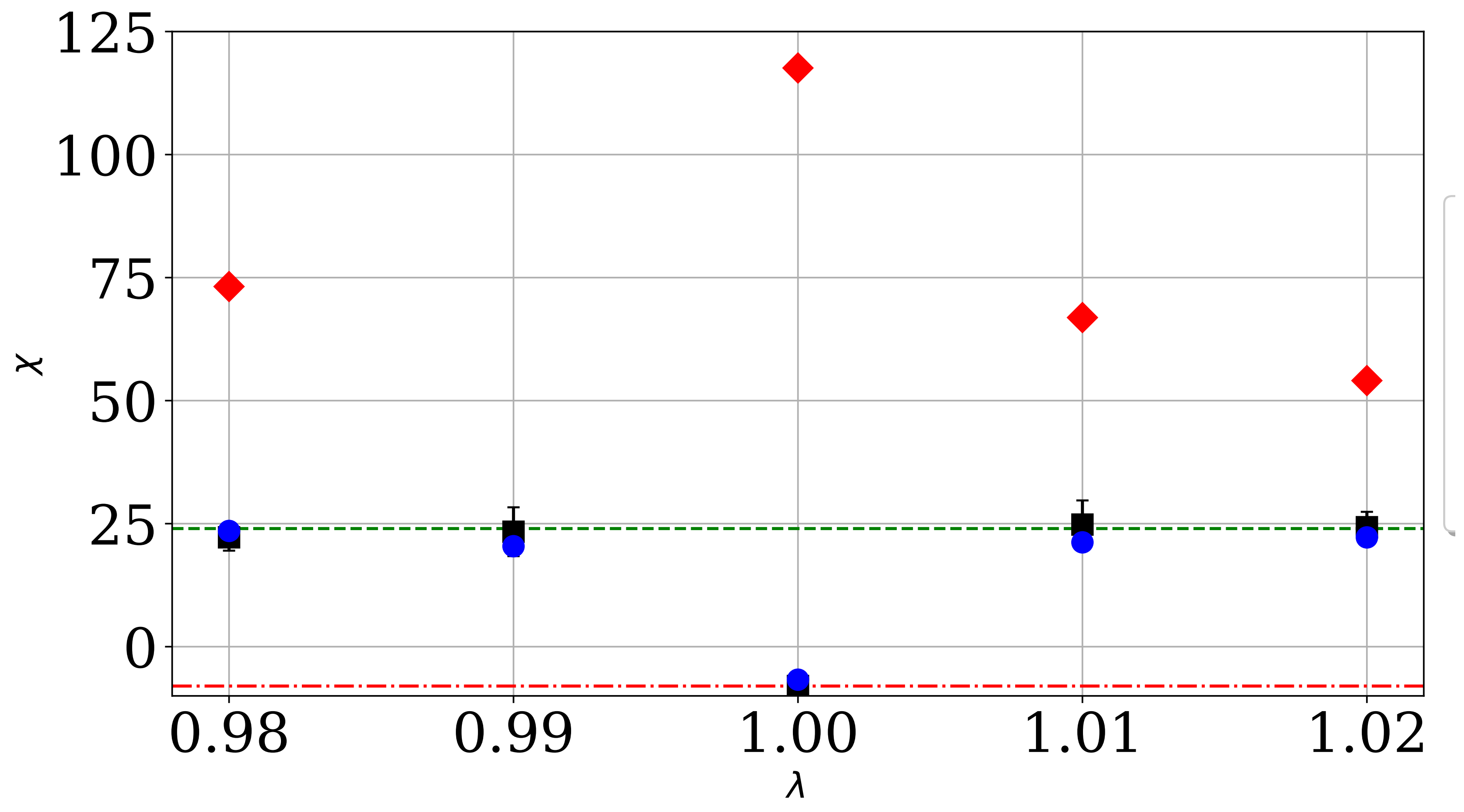


(c) $\lambda = 0.98,\ J_{\rm FS}$ (d) $\lambda = 1.48,\ J_{\rm FS}$

2.2. Examples of metrics

The Euler Number:

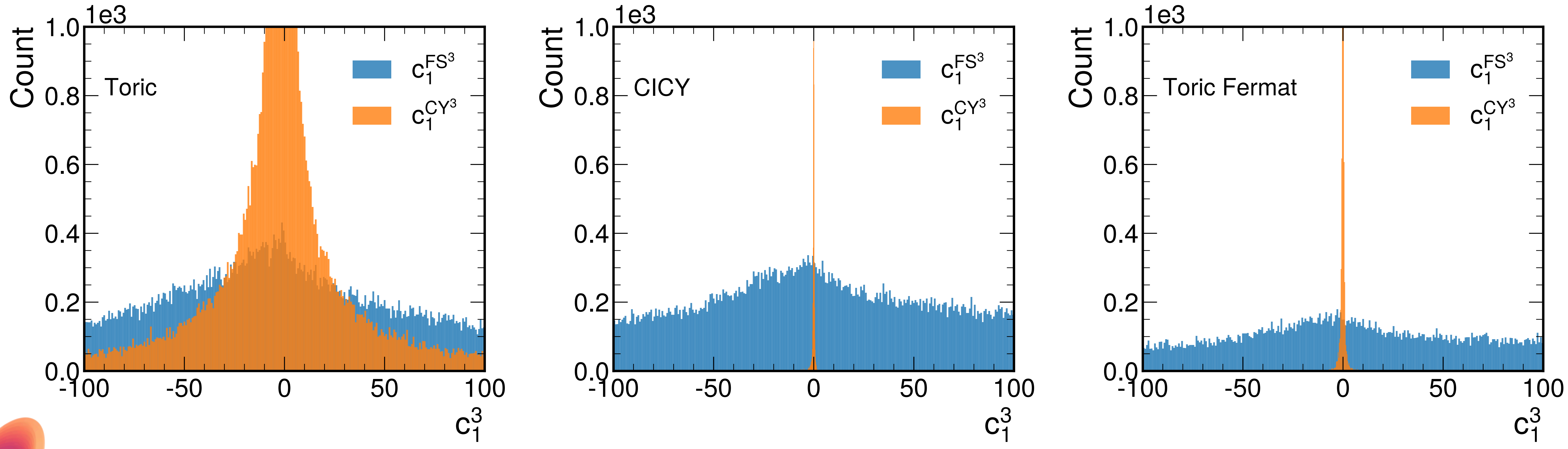


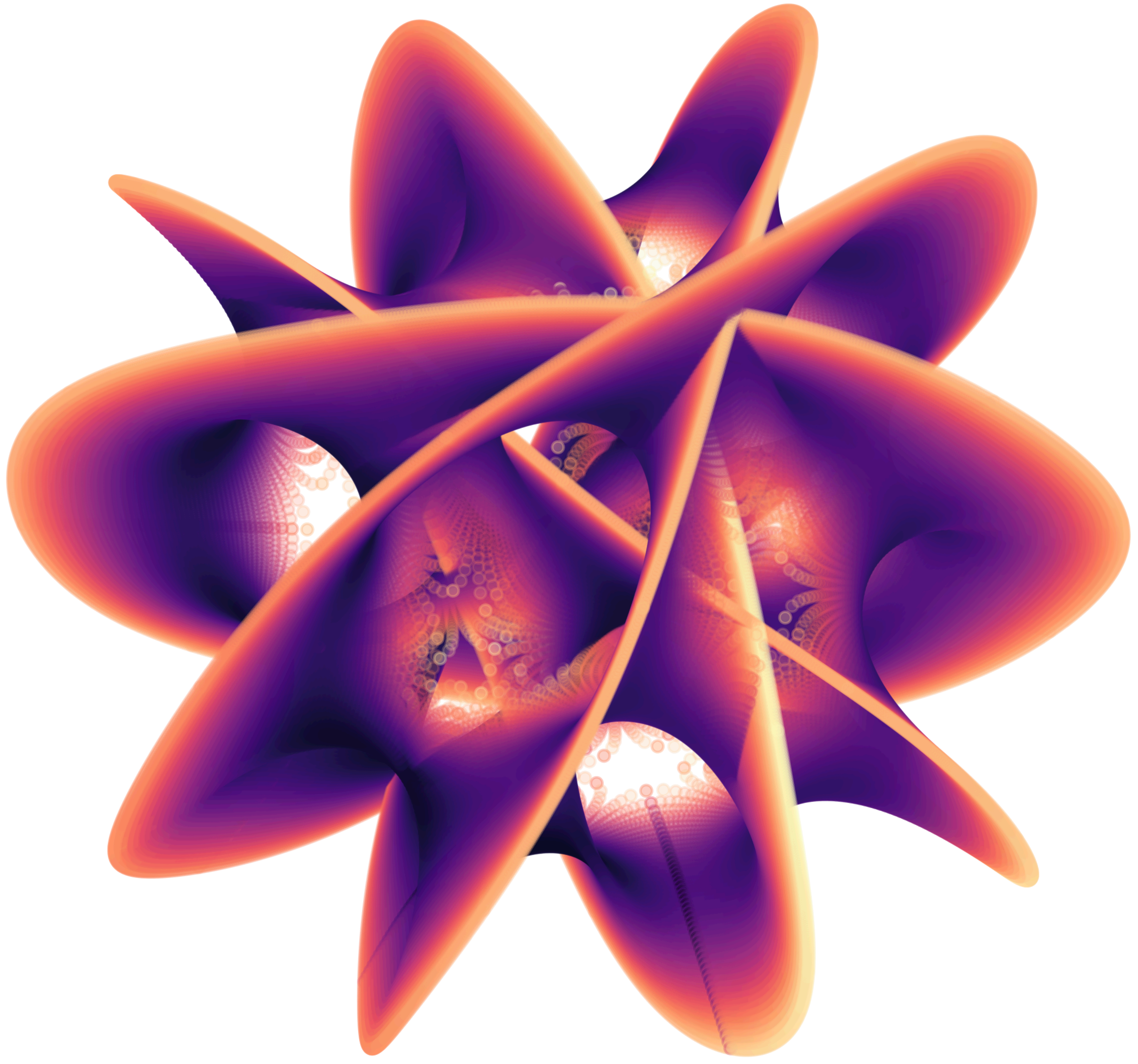


At the singular geometries one observes the following relations

$$\int_{X_s} e(J) = \deg c_F(X) - 2(-1)^{\dim X} |\mathrm{Sing}\ X|$$

2.2. Examples of metrics





We can train on various quintics, one with generic parameters, the Fermat quintic as well as its toric version.

3. Yukawa couplings

Upon compactification on a Calabi-Yau threefold X the 4D spectrum of left chiral superfields is in one-to-one correspondence with $H^1(X,V)$, where V is a holomorphic vector bundle with a given structure group $G \subset E_8$. The corresponding unnormalised trilinear couplings enter the superpotential weighted by a factor

$$\tilde{\kappa}_{ijk} = \int_X \Omega \wedge \tilde{\Omega}(a_i, a_j, a_k)$$

where $\{a_i\}_{i=1}^{h^1(V)}$ denotes a basis for $H^1(X,V)$. This expression is accessible as it is quasi-topological, since it only depends on the cohomology classes of the respective matter fields. The relevant coupling, however, is the kinetically normalised coupling

$$\kappa_{ijk} = \frac{\tilde{\kappa}_{ijk}}{\sqrt{\lambda_i \lambda_j \lambda_k}\,\mathcal{V}}$$

Given in terms of a new orthogonal basis with respect to the metric

$$G_{a\bar{b}} \sim \int_X a \wedge \bar{\star}_V b$$

3. Yukawa couplings

The simplest case is when the vector bundle is taken to be the holomorphic tangent bundle T_X. In this case the structure group $SU(3)$ breaks the symmetry to its commutant E_6. Additionally one has the isomorphism $H^1(X, T_X) = H^{(2,1)}(X)$. The number of chiral generators in this case is given by

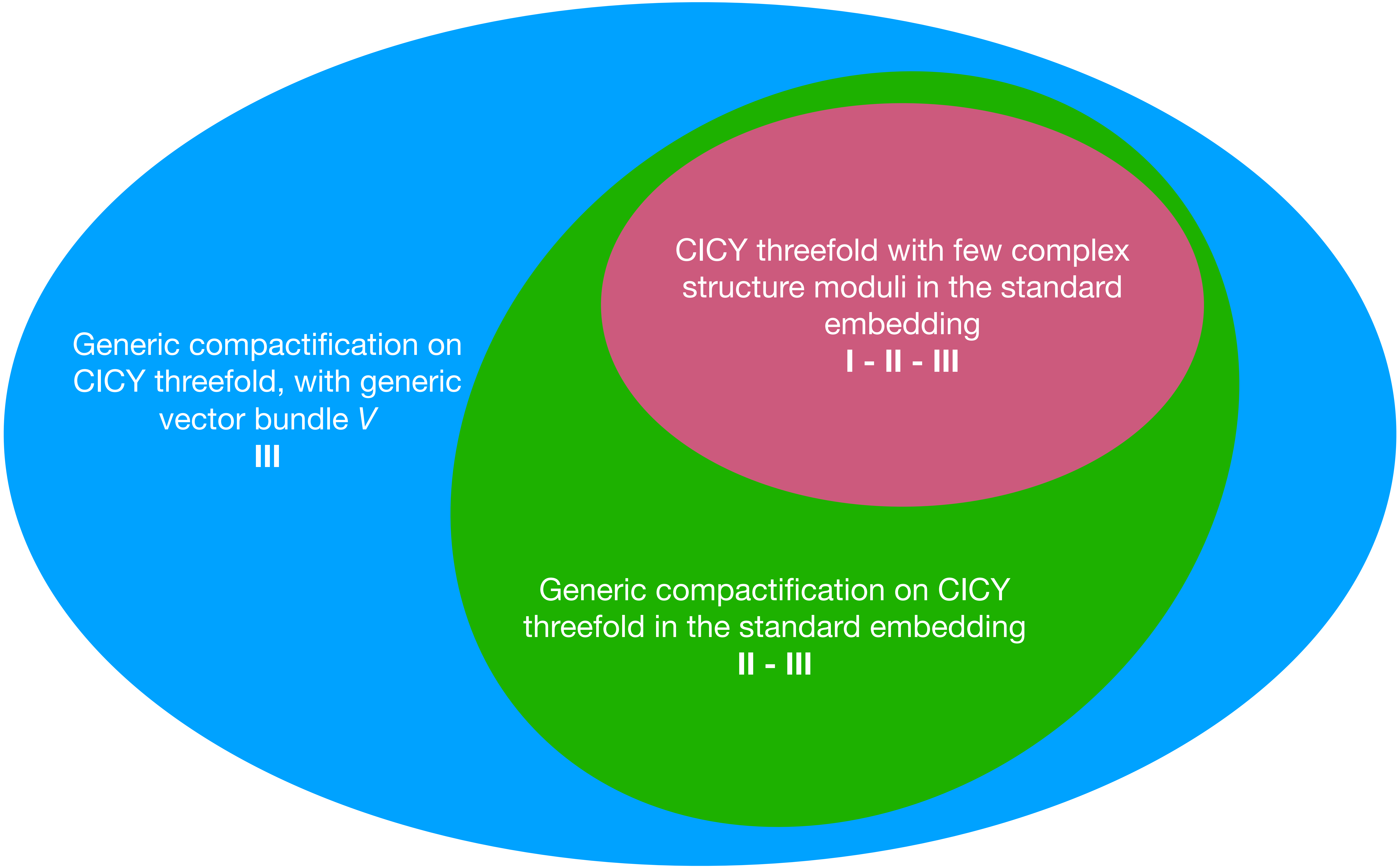
$$N_{\text{gen}} := \frac{1}{2}|\chi| = |h^{1,1} - h^{2,1}|$$

In the standard embedding the metric $G_{a\bar{b}}$ becomes:

$$G_{a\bar{b}} := \int_X a \wedge \bar{\star}_g b$$

And matches the Weil-Petersson metric (up to a conformal factor). For the standard embedding we can distinguish three different avenues to derive the kinetic normalisations.

3. Yukawa couplings



3. Yukawa couplings

I. Period computations (only available in a few examples)

Thorough knowledge of periods enables computation of Weil-Petersson metric and Yukawa couplings. This technology is only available for Calabi-Yau manifolds with few complex structure parameters.

II. Special geometry computations

Thorough knowledge of periods enables computation of Weil-Petersson metric and Yukawa couplings. This technology is only available for Calabi-Yau manifolds with one complex structure parameters. To account for complex structure deformations we write $t = (t^1, \ldots, t^m), \quad m = h^{2,1}(X)$. Locally, we interpret this construction as a fibration of X over a base B in the vicinity of a reference point t_0 .

The Weil--Petersson metric on the complex structure moduli space can be directly obtained as

$$G_{a\bar{b}} = \left(\frac{\partial \Omega_t}{\partial t^a}, \frac{\partial \Omega_t}{\partial t^b} \right) \Bigg|_{t_0} - \frac{1}{(\Omega, \Omega)} \left(\Omega, \frac{\partial \Omega_t}{\partial t^a} \right) \Bigg|_{t_0} \cdot \overline{\left(\Omega, \frac{\partial \Omega_t}{\partial t^b} \right) \Bigg|_{t_0}} .$$

The problem is then reduced to the calculation of various integrals over X, numerically computed via Monte Carlo integration in local coordinates

3. Yukawa couplings

III. Explicit harmonic representative computations

Recall that X is covered by an atlas $\{U_i\}$. On overlaps $U_i \cap U_j$, we can use the transition functions

$$f_{ij} : z_j \mapsto z_i$$

and write $f_{ij}(z_j; t)$

The Kodaira-Spencer map $\rho : T_{t_0}B \to H^1(X; T_X)$ is given by

$$\rho\left(\frac{\partial}{\partial t}\right) = \left[\left\{\frac{\partial f_{ij}(z_j, t)}{\partial t}\frac{\partial}{\partial z_i}\right\}\right]$$

Having the harmonic representatives and a harmonic projection $\mathcal{H} : H^p(X; T_X) \to H^p(X; T_X)$, the Weil-Petersson metric can be computed as follows

$$\mathcal{G}_{a\bar{b}} = \int_{X_t} \Omega(\mathcal{H}\rho(\partial/\partial t^a)) \wedge \overline{\Omega(\mathcal{H}\rho(\partial/\partial t^b))}$$

Here, $\Omega\,(\cdot)$ denotes the interior product with Ω

3.1. Machine learning harmonic one forms

The "man with a hammer" approach

Construct $H^1(X, V)$ representatives via the Kodaira-Spencer map $\Phi^a = \rho\left(\frac{\partial}{\partial t^a}\right)$

And then use a spectral network to work out the harmonic completion

with the Ansatz

$$s^a_{\mathrm{NN}} = \sum_{ijkl} \psi^{a,\mathrm{NN}}_{ijkl} \frac{\overline{\alpha^{ij}_\nu} z^k z^l}{|z|^4} g^{\mu\bar{\nu}} \cdot \frac{\partial}{\partial z^\mu} \in \Gamma(T_X)\,.$$

where $\alpha^{ij}_\nu \, dz^\nu = \imath^*(z^i dz^j - z^j dz^i)$

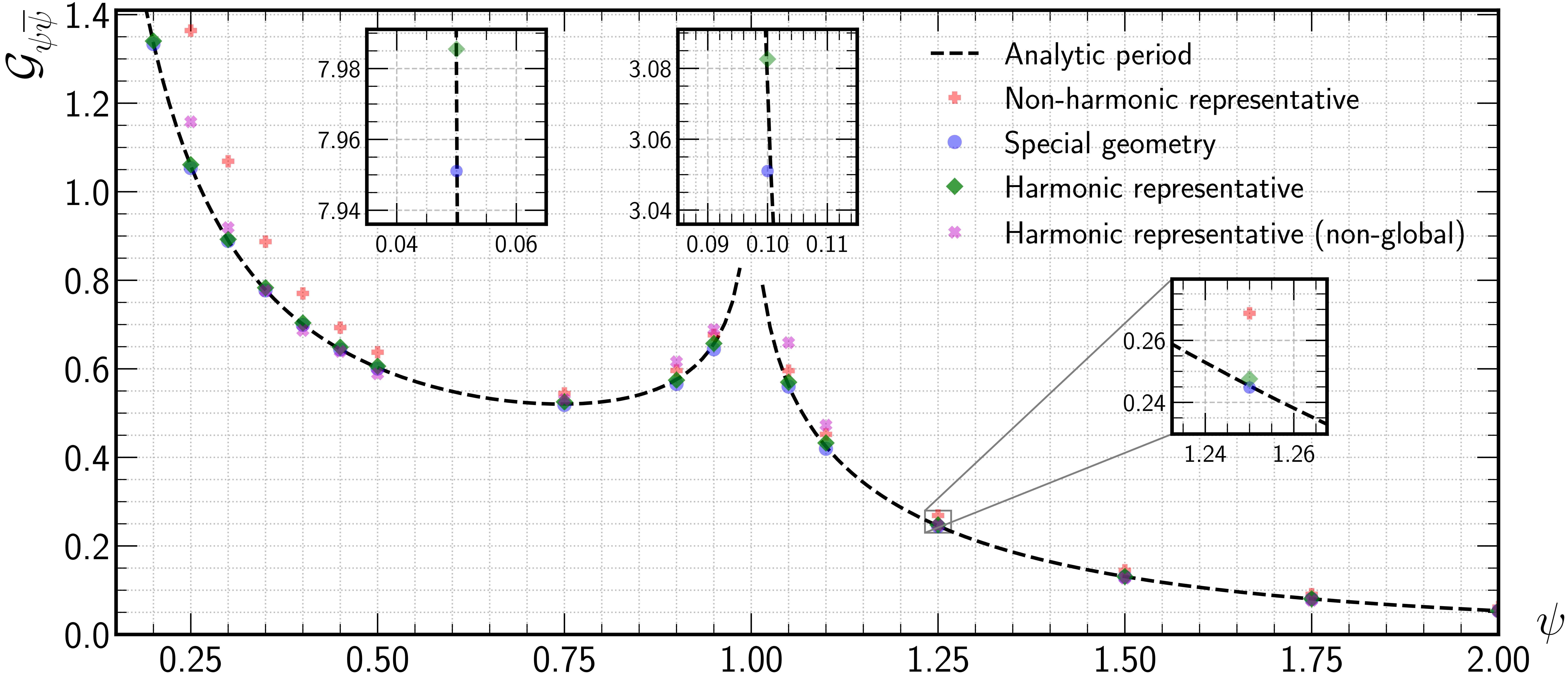
One neural network per form representative, outputs of the network are the coefficients $\psi^{a,\mathrm{NN}}_{ijkl}$

The idea is to minimise the loss

$$\mathcal{L} = (\eta^a, \Delta\eta^a) = |\partial_{T_X} \eta^a|^2$$

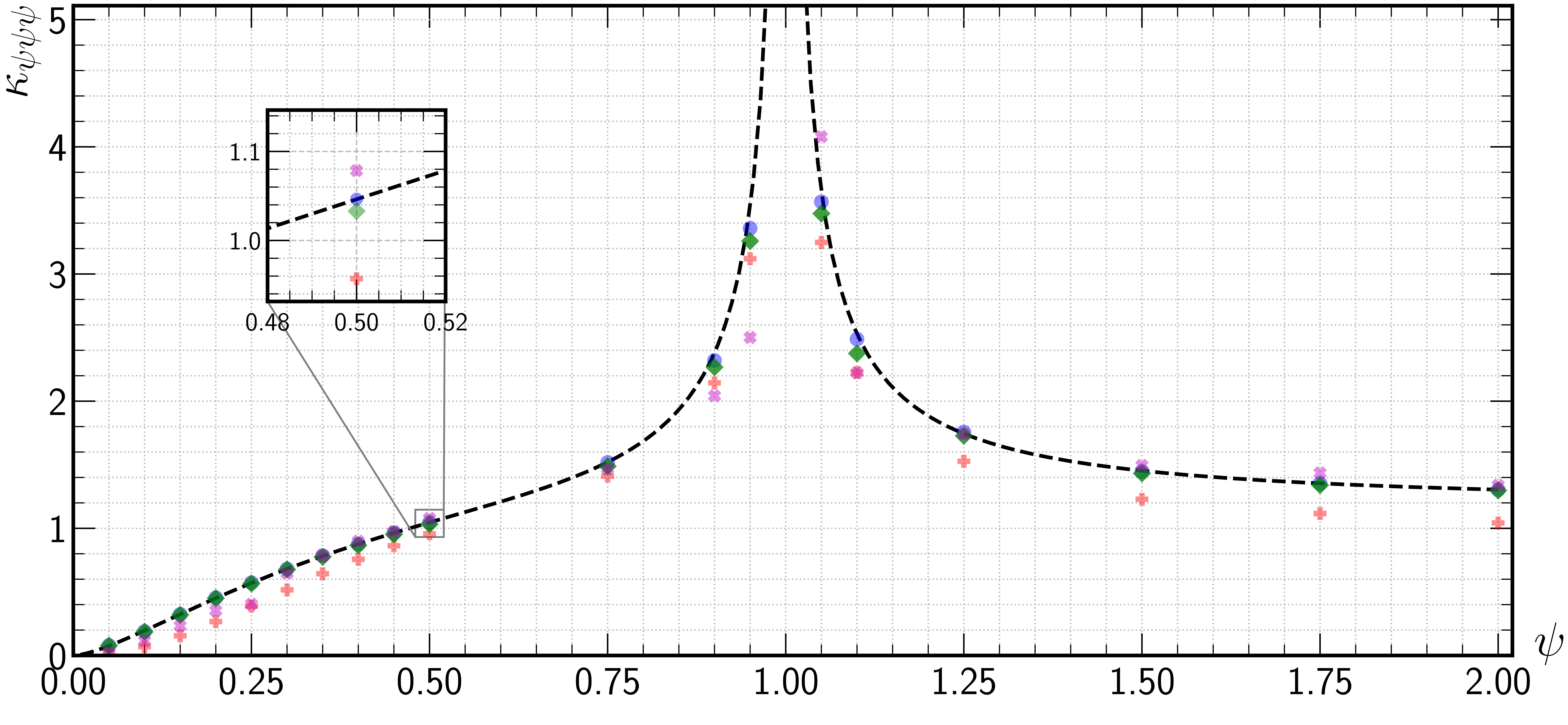
3.1. More examples

The mirror of $\mathbb{P}^5[3,3]$



3.1. More examples

The mirror of $\mathbb{P}^5[3,3]$



3.1. More Examples

The Tian - Yau manifold

$$\left[\begin{array}{c||ccc} \mathbb{P}^3 & 3 & 0 & 1 \\ \mathbb{P}^3 & 0 & 3 & 1 \end{array}\right]^{14,\ 23}_{\chi=-18} \qquad \frac{1}{3}\sum_{a=0}^{3} x_a^3 = \frac{1}{3}\sum_{a=0}^{3} y_a^3 = \sum_{a=0}^{3} x_a y_a + \epsilon \sum_{a=2}^{3} x_a y_a = 0$$

There is a freely acting $\mathbb{Z}_3$ involution

$$(x_0, x_1, x_2, x_3) \mapsto (x_0,\ \omega_3^{-1} x_1,\ \omega_3 x_2,\ \omega_3 x_3)$$
$$(y_0, y_1, y_2, y_3) \mapsto (y_0,\ \omega_3 y_1,\ \omega_3^{-1} y_2,\ \omega_3^{-1} y_3)$$

with $\omega_3 = e^{2\pi \mathrm{i}/3}$

Modding out this symmetry breaks the E_6 further down to $SU(3)^3$ (suitable choice of discrete Wilson line)

Parent E_6 model

23 $\mathbf{27}$-plets

Quotient trinification model

9 states $(\mathbf{1}, \overline{\mathbf{3}}, \mathbf{3})$ denoted by λ_i

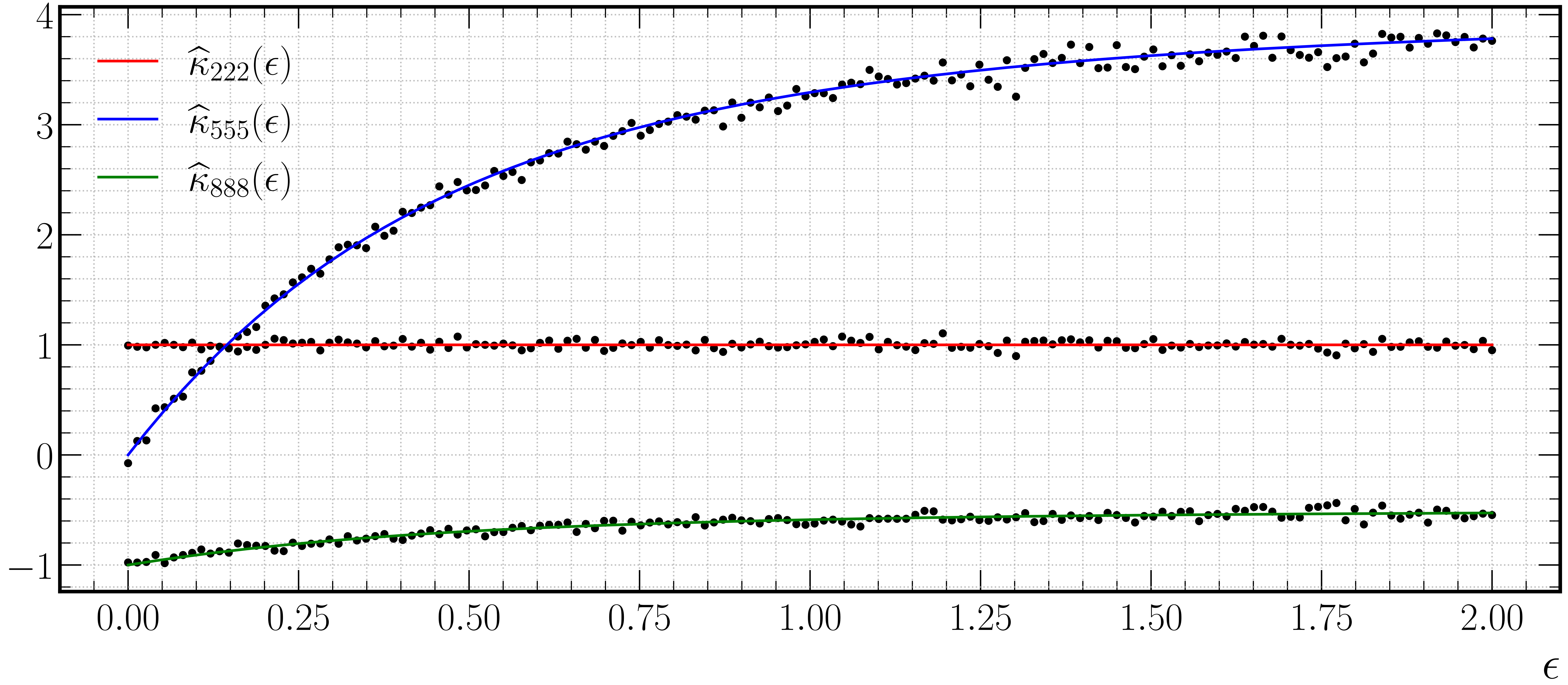
7 states $(\mathbf{3}, \mathbf{3}, \mathbf{1})$ denoted by Q_i

7 states $(\overline{\mathbf{3}}, \mathbf{1}, \overline{\mathbf{3}})$ denoted by $\overline{Q}_i$

3.1. More examples

Tian-Yau pencil becomes singular at four points in the real ϵ axis

$$\epsilon \in \{-1,\ -1-2^{-1/3},\ -2,\ -1-2^{1/3}\}$$

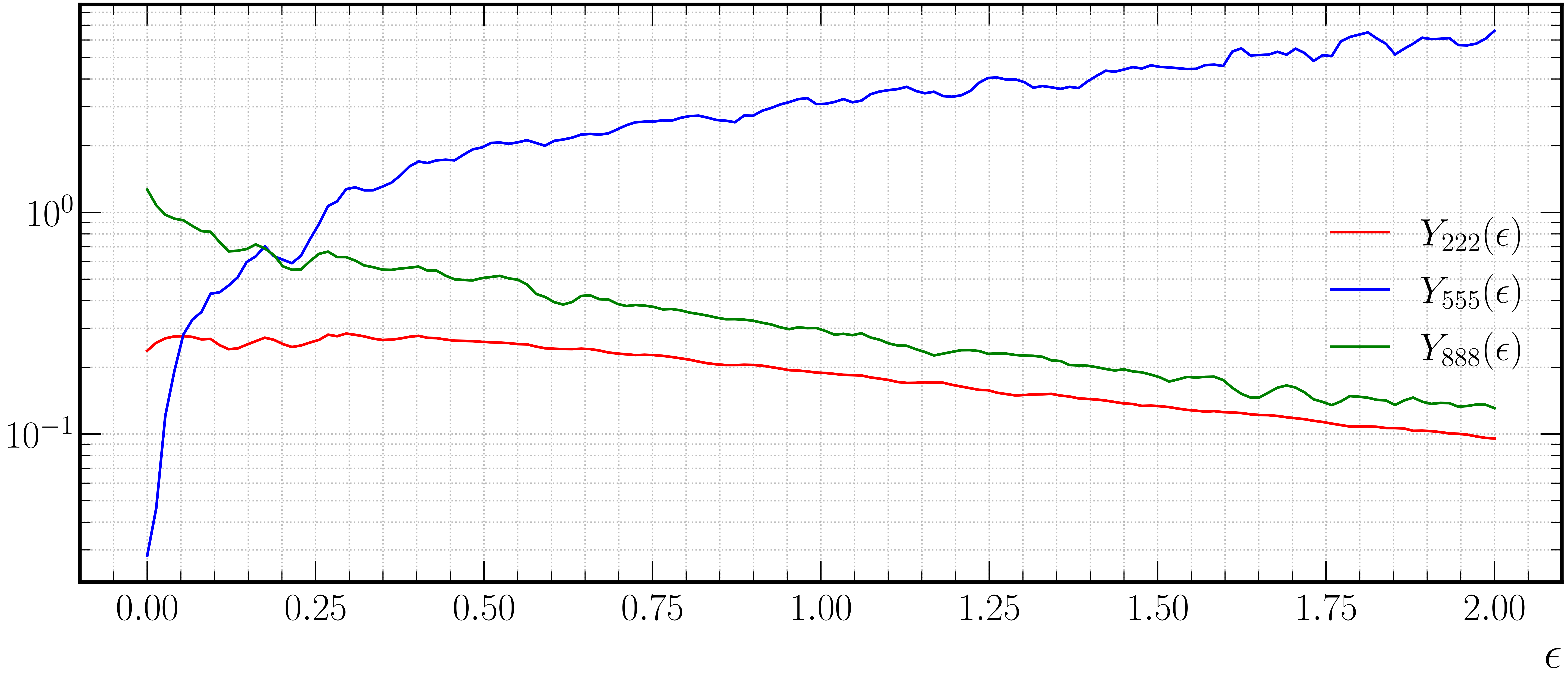


Berglund, Butbaia, Hübsch, Jejjala, MP, Mishra, Tan'24

3.1. More examples

Tian-Yau pencil becomes singular at four points in the real ϵ axis

$$\epsilon \in \{-1,\ -1-2^{-1/3},\ -2,\ -1-2^{1/3}\}$$

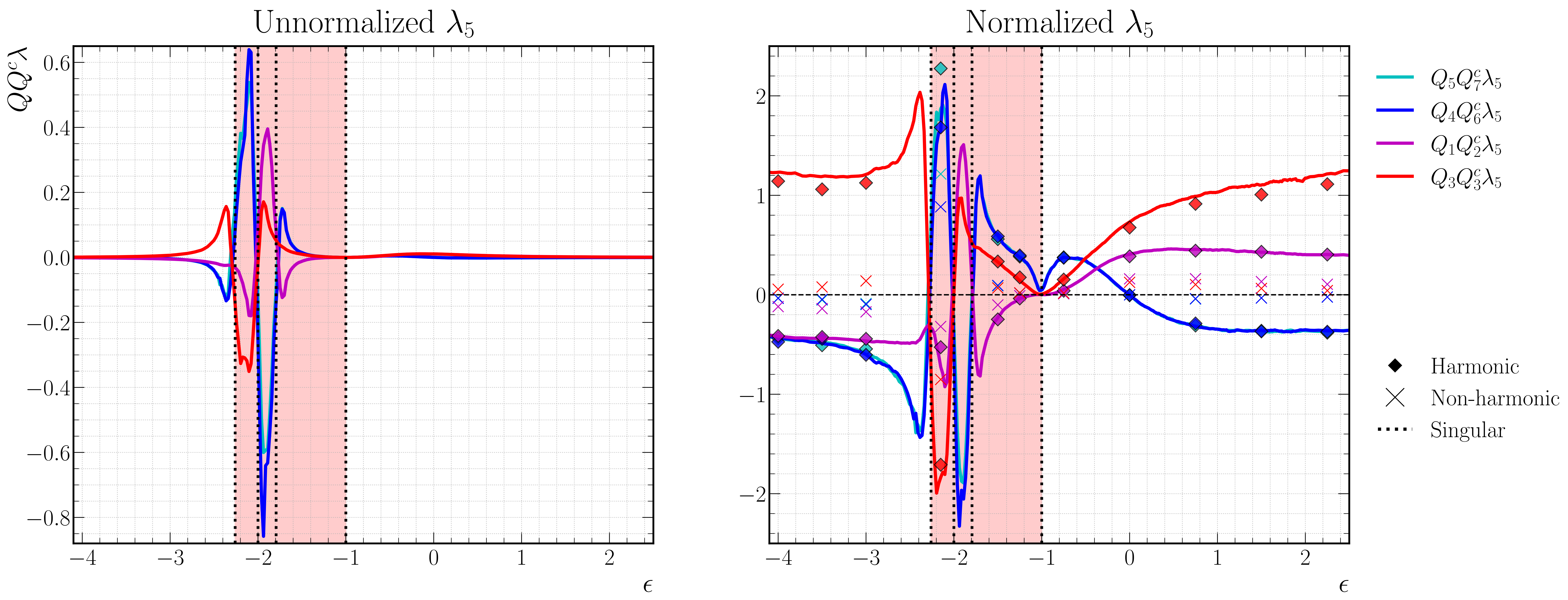


Berglund, Butbaia, Hübsch, Jejjala, MP, Mishra, Tan'24

3.1. More examples

Tian-Yau pencil becomes singular at four points in the real ϵ axis

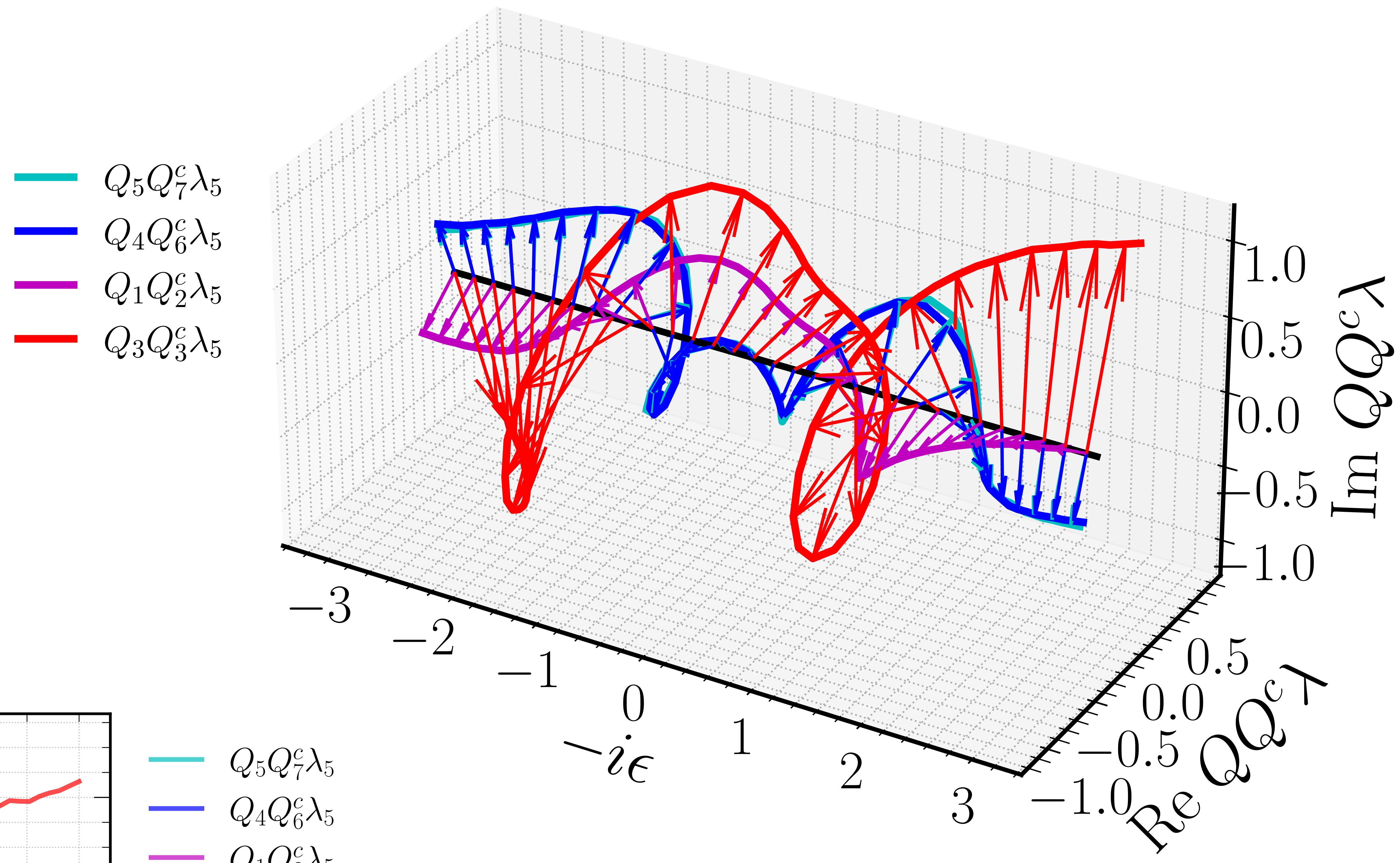
$$\epsilon \in \{-1,\ -1-2^{-1/3},\ -2,\ -1-2^{1/3}\}$$

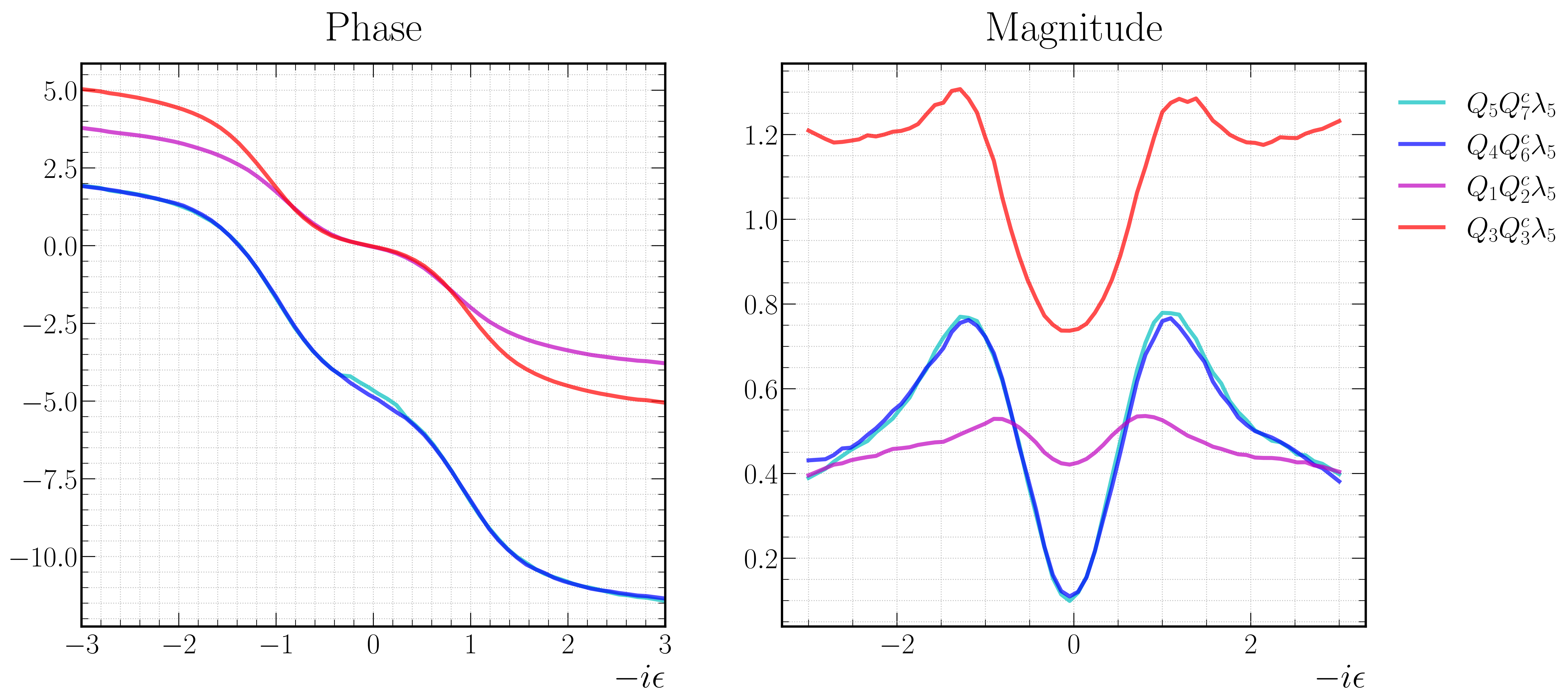


Berglund, Butbaia, Hübsch, Jejjala, MP, Mishra, Tan’24

3.1. More examples

We can also access potential CP violating phases





4. The code

Calabi–Yau Metrics, Yukawas, and Curvature

https://github.com/Justin-Tan/cymyc

arXiv:2410.19728

MLgeometry	Douglas, Lakshminarasimhan, Qi'20 Douglas'21 Douglas, Platt, Qi'24
cymetric	Larfors, Lukas, Ruehle, Schneider'21,'22 Larfors'24
cyjax	Gerdes, Krippendorf'22

README License

CYMYC

`cymyc` is a library for numerical differential geometry on Calabi-Yau manifolds written in JAX, enabling performant:

- Approximations of useful tensor fields;
- Computations of curvature-related quantities;
- Investigations of the complex structure moduli space;

in addition to many other features.

Installation

First, clone the project:

```
git clone git@github.com:Justin-Tan/cymyc.git
cd cymyc
```

Next, with a working Python installation, create a new virtual environment and run an editable install, which permits local development.

```
pip install --upgrade pip
python -m venv /path/to/venv
source /path/to/venv/bin/activate

python -m pip install -e .
```

5. Final Remarks

- Neural networks are good approximations for flat Calabi-Yau metrics, specially when Kählericity can be built in (spectral networks).
- So far applicable to CICYs only but extendable to Toric Calabi-Yaus, as well as non-standard embeddings.
- Ideally we would like an analytic expressions for the metric, is that possible?
- Go beyond the standard embedding Constantin et. al.'24
- Extensions to Spin7 or G2 manifolds? Heyes et. al. ongoing work
- Moduli dependence of Yukawas, are there generic patterns? Connections to the Swampland program? Casas, Ibañez, Marchesano'24
- Fully quantum corrected metrics? Frasier-Taliente, Harvey, Kim'24

Grazie!

5. Final Remarks

```python
import jax
from cymyc.calabi_yau import *

key = jax.random.key(0)
keys = jax.random.split(key, 3)

cy = DworkQuintic(0.0)
pts = cy.sample_points(
    key = keys[1],
    max_pts = int(1e5))

metric = RicciFlatMetric(
    key = keys[2],
    cy = cy,
    tx = optax.adamw(1e-4))

metric.fit(pts, epochs=10)



e_X = metric.chern3(
        pts.generator(
            batch_size = 100,
            include_pullbacks = True),
        pts.kappa)
print(cy.integrate(pts = pts, func_vals = e_X))
~
~
~
~
~
~
~
```

```
(cymyc) gpu@gpu-P55A-UD3 ~ $ python3 metric.py
```

```
TERMINAL /bin/bash
[2] 0:vim*                                   "gpu-P55A-UD3" 20:25 10-Dec-24
```