Mirror Symmetry of Abelian Fibered Calabi-Yau Manifolds

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at Pollica Summer Workshop
"Calabi-Yau Manifolds"

based on a collaboration with
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§1. Fiber structure of CY 3 folds

X : compact Kähler manifold

$D \in H^{1,1}(X, \mathbb{Z})$

if $\left\{ \begin{array}{c} H^0(X, \mathcal{O}_X(D)) \neq \phi \\ + \\ \text{additional condition} \end{array} \right\}$, we have $\Phi_{|D|} : X \to \mathbb{P}^N$

$\cup \qquad \cup$

$x \mapsto [\varphi_0(x), \cdots, \varphi_N(x)]$

$\varphi_k \in H^0(X, \mathcal{O}_X(D))$

Theorem (Oguiso, '93)

1. Let X : a minimal CY 3-fold, D : a nef divisor on X, then $\exists m \geq 1$ s.t. we have a map

$$\Phi_{|mD|} : X \to W := \mathrm{Im}\Phi_{|mD|} \subset \mathbb{P}^N$$

2. $\Phi_{|mD|} : X \to W$ defines an algebraic fiber space classified as follows:

Classification (Oguiso):

Type	ν_D	$c_2(X).D$	
0	0	0	$X \to \{\text{pt}\}$: not interesting
I_+	1	24	$X \to W = \mathbb{P}^1$: K3 fibration
I_0	1	0	$X \to W = \mathbb{P}^1$ **abelian fibration**
II_+	2	> 0	$X \to W = S$ **elliptic fibration**
II_0	2	0	
III	3	≥ 0	$X \to W$ birational

$$\nu_D := \max\left\{k \mid D^k \not\equiv 0\right\} = \begin{cases} 0 & D = 0 \\ 1 & D \neq 0,\ D^2 D' = 0\ (\forall D') \\ 2 & D^2 D' \neq 0\ (\exists D'),\ D^3 = 0 \\ 3 & D^3 \neq 0 \end{cases}$$

§2. Abelian surfaces

Def. A: abelian surface of polarization type (d_1, d_2):

$\Leftrightarrow$ $A = \mathbb{C}^2/\Lambda$ with $\Lambda := \Omega\mathbb{Z}^4$ where

$$\Omega = \begin{pmatrix} d_1 & 0 & z_{11} & z_{12} \\ 0 & d_2 & z_{21} & z_{22} \end{pmatrix} \quad (z_{ij} = z_{ji},\ \mathrm{Im}Z > 0)$$

Some known results:

For A, we have a naural line bundle $\mathcal{L} \to A$ s.t.

$$c_1(\mathcal{L}) = \sum_i d_i dx_i \wedge dy_i = \frac{i}{2} \sum_{i,j} (\mathrm{Im}Z)^{-1}_{ij} dz_i \wedge d\bar{z}_j$$

$\in H^2(A, \mathbb{Z})$ positive class

then we have $\int_A c_1(\mathcal{L})^2 = 2d_1d_2$ and $\dim H^0(A, \mathcal{L}) = d_1d_2$.

In particular, $\Phi_{|\mathcal{L}|} : A \to \mathbb{P}^{d_1d_2-1}$ is an embedding.

($(A, \mathcal{L})$ is called an abelian surface.)

Consider $(A, \mathcal{L})$: abelian surface of type $(1, d)$ and

$$\Phi_{|\mathcal{L}|} : A \to \mathbb{P}^{d-1}.$$

Gross and Popescu (2000) studied the following problem:

Problem:
Detemine the ideal I_A which describes $\mathrm{Im}\Phi_{|\mathcal{L}|} \subset \mathbb{P}^{d-1}$.

Results:
cases $(1, d)$ $(5 \leq d \leq 10)$ are interesting from may respects.

for us, interesting because of its relations to CY3.

We study CY3s that comes from the $(1, 8)$ case.

(Studied first by Pavanelli (2003), but in more detail.)

§3. CY3-folds fibered by $(1,8)$-polarized abelian surfaces

$(A, \mathcal{L})$ abelian surface of polarization type $(1,8)$

$$\begin{array}{ccc} \Phi_{|\mathcal{L}|} : A & \to & \mathrm{Im}\Phi_{|\mathcal{L}|} \subset \mathbb{P}^7. \\ \cup & & \cup \\ p & \mapsto & [x_0(p), \cdots, x_7(p)] \end{array}$$

by **Theta functions**

Results by Gross-Popescu ('00):

The ideal with $Z(I_A) = \mathrm{Im}\Phi_{|\mathcal{L}|}$ is given by

$I_A =$ (**four quadrics**)

$+$(cubics given by 3×3 minors of a 4×4 matrix M)

Remark (Heisenberg group $\mathcal{H}_8$):

$$0 \to \mathbb{Z}_8 \times \mathbb{Z}_8 \to A \to \widehat{A} \to 0$$

$$p \mapsto t_p^* \mathcal{L} \otimes \mathcal{L}^{-1}$$

$\mathbb{Z}_8 \times \mathbb{Z}_8$ acts on the theta functions (with a level structure).

$$\left(\mathcal{H}_8 \begin{cases} \sigma : x_i \mapsto x_{i-1} \\ \tau : x_i \mapsto \xi^{-i} x_i \quad (\xi = e^{\frac{2\pi\sqrt{-1}}{8}}) \end{cases} \qquad (1 \to \mu_8 \to \mathcal{H}_8 \to \mathbb{Z}_8 \times \mathbb{Z}_8 \to 1) \right)$$

The four quadrics f_1, f_2, f_3, f_4 take the following form

$$\begin{cases} f_1 = \frac{w_0}{2}(x_0^2 + x_4^2) + w_1(x_1x_7 + x_3x_5) + w_2x_2x_6 \\ f_2 = \frac{w_0}{2}(x_1^2 + x_5^2) + w_1(x_2x_0 + x_4x_6) + w_2x_3x_7 \\ f_3 = \frac{w_0}{2}(x_2^2 + x_6^2) + w_1(x_3x_1 + x_5x_7) + w_2x_4x_0 \\ f_4 = \frac{w_0}{2}(x_3^2 + x_7^2) + w_1(x_4x_2 + x_6x_0) + w_2x_5x_1 \end{cases}$$

where $w = w(A)$ and the corresponsence $A \mapsto [w_0(A), w_1(A), w_2(A)] \in \mathbb{P}^2$ defines a **surjective** map with the fiber a pencil of abelian surfaces.

Definition: a special family of $(2,2,2,2) \subset \mathbb{P}^7$ over $\mathbb{P}^2$

$$\begin{array}{ccc} \mathcal{V} & \supset & V_w \quad \textbf{a pencil of abelian surfaces} \\ \downarrow & & \downarrow \\ \mathbb{P}^2_w & \ni & w \end{array}$$

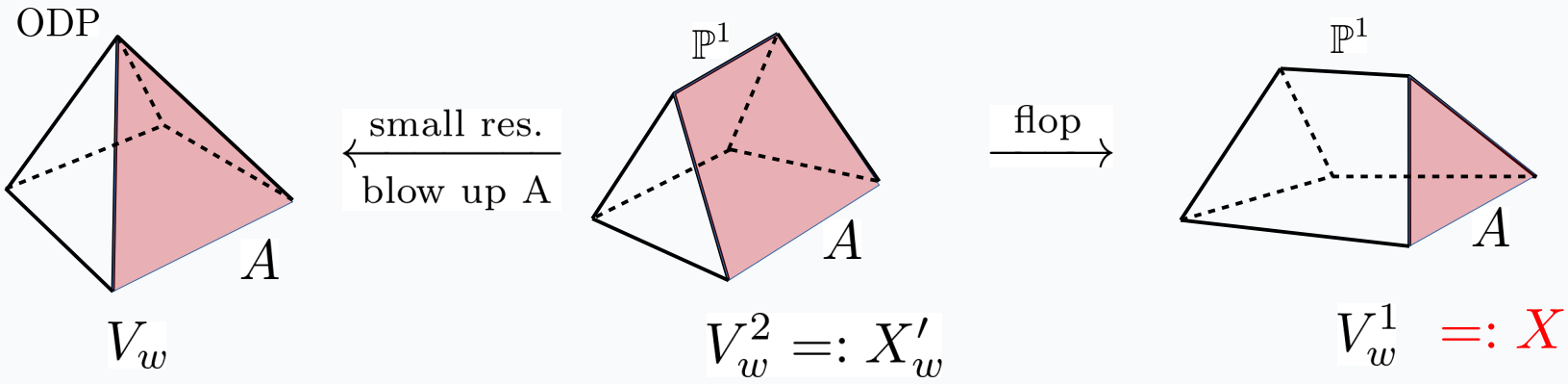
$$V_w := \{f_1(w,x) = f_2(w,x) = f_3(w,x) = f_4(w,x) = 0\} \subset \mathbb{P}^7$$

Proposition (Gross-Popescu '01) V_w has the following properties:

(1) $\mathcal{H}_8$ acts freely on V_w.

(2) V_w is a pencil of $(1,8)$-polarized abelian surfaces with 64 base points.

(3) There are small resolutions as follows:



V_w $\qquad$ $V_w^2 =: X'_w$ $\qquad$ $V_w^1 =: X$

abelian fibration over $\mathbb{P}^1$

(4) V_w^1 and V_w^2 admit free $\mathbb{Z}_8 \times \mathbb{Z}_8 (= \mathcal{H}_8)$ actions.

Thus we obtain $\begin{array}{ccc} \mathcal{V} & \supset & V_w \\ \downarrow & & \downarrow \\ \mathbb{P}^2_w & \ni & w \end{array} \rightsquigarrow \begin{array}{ccc} \mathfrak{X}' & \supset & X'_w \\ \downarrow & & \downarrow \\ \mathbb{P}^2_w & \ni & w \end{array}$ and $\begin{array}{ccc} \mathfrak{X} & \supset & X_w \\ \downarrow & & \downarrow \\ \mathbb{P}^2_w & \ni & w \end{array}$

- Hodge numbers of $X(:= V_w^1)$:

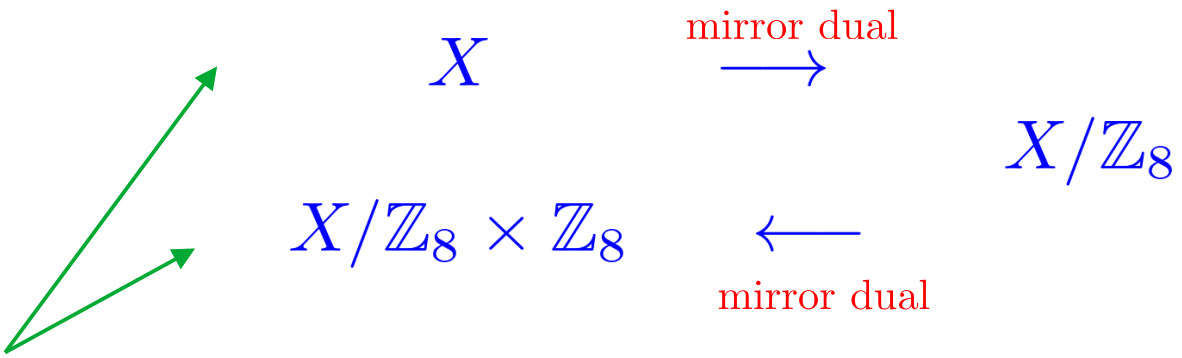
$$\begin{array}{ccccccc} & & h^{0,0} & & & & 1 \\ & & h^{1,1} & & & & 2 \\ h^{3,0} & h^{2,1} & h^{1,2} & h^{0,3} & = 1 & 2 \quad 2 & 1 \\ & & h^{2,2} & & & & 2 \\ & & h^{3,3} & & & & 1 \end{array}$$

$\rho = 2$ **and** $\chi(X) = 0$

self-mirror?

$X(= V_w^1)$ was studied together with free $\mathbb{Z}_8 \times \mathbb{Z}_8$ actions.

Conjecture (Gross,Pavanelli '08)



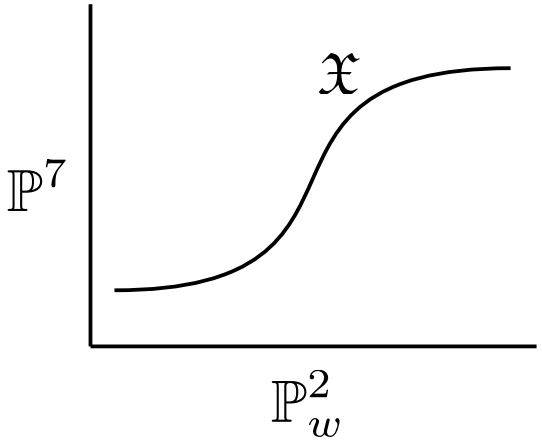
Theorem (Bak '09, Schnell '12)

X and $Y := X/\mathbb{Z}_8 \times \mathbb{Z}_8$ are derived equivalent by fiberwise FM transform.

§4. Families of $X/\mathbb{Z}_8$ and $X/\mathbb{Z}_8 \times \mathbb{Z}_8$

Summary (so far)

$$\begin{array}{ccc} \mathfrak{X} & \subset & \mathbb{P}^7 \\ \downarrow & & \\ \mathbb{P}^2_w & & \end{array}$$



$\curvearrowleft \mathcal{NH}_8$

We find a larger group $\mathcal{NH}_8$ acts on the total space $\mathfrak{X}$ as the symmetry of the ideal I_w.

$$\mathcal{H}_8 \subset \mathcal{NH}_8 \subset GL(8,\mathbb{C})$$

Problem:
Taking this into accout, find a "good" family.

Proposition. (H+Takagi, '21)

Consider two families $\mathfrak{X}_{\mathbb{Z}_8} := \mathfrak{X}/\langle\tau\rangle$ and $\mathfrak{X}_{\mathbb{Z}_8\times\mathbb{Z}_8} := \mathfrak{X}/\langle\tau,\sigma\rangle$

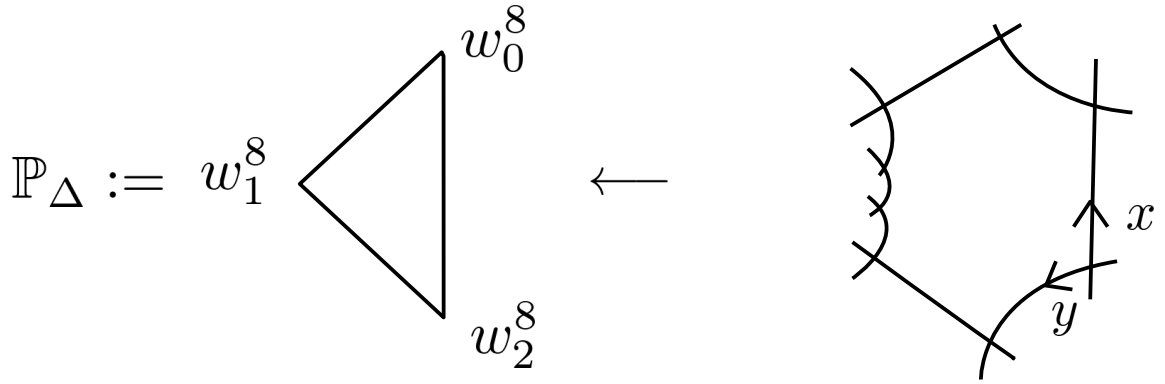
$$\begin{array}{ccc} \mathfrak{X}_{\mathbb{Z}_8} & & \mathfrak{X}_{\mathbb{Z}_8\times\mathbb{Z}_8} \\ \downarrow & \text{and} & \downarrow \\ \mathbb{P}^2_w & & \mathbb{P}^2_w \end{array}$$

Take $\mathcal{H}_8 \subset \widetilde{G}_0 \subset \mathcal{NH}_8$, then we obtain **good** families

$$\begin{array}{ccc} \mathfrak{X}_{\mathbb{Z}_8}/\widetilde{G}_0 & & \mathfrak{X}_{\mathbb{Z}_8\times\mathbb{Z}_8}/\widetilde{G}_0 \\ \downarrow & \text{and} & \downarrow \\ \mathbb{P}^2_w/\mathbb{Z}_8 =: \mathbb{P}_\Delta & & \mathbb{P}^2_w/\mathbb{Z}_8 =: \mathbb{P}_\Delta \end{array}$$

§5. Picard-Fuchs equations over $\mathbb{P}_\Delta$

The quotient $\mathbb{P}_\Delta = \mathbb{P}^2_w/\mathbb{Z}_8$ is a toric variety given by $\mathrm{Proj}\mathbb{C}[w_0^8, w_1^8, w_2^8, \cdots]$;



Proposition (Pavanelli, '03). Consider period integrals of the family

$$w(x,y) = \int_\gamma Res\Big(\frac{\sum(-1)^i x_i dx_0 \cdots d\hat{x}_i \cdots dx_7}{f_1(w)f_2(w)f_3(w)f_4(w)}\Big)$$

These satisfy the following Picard-Fuchs equations:

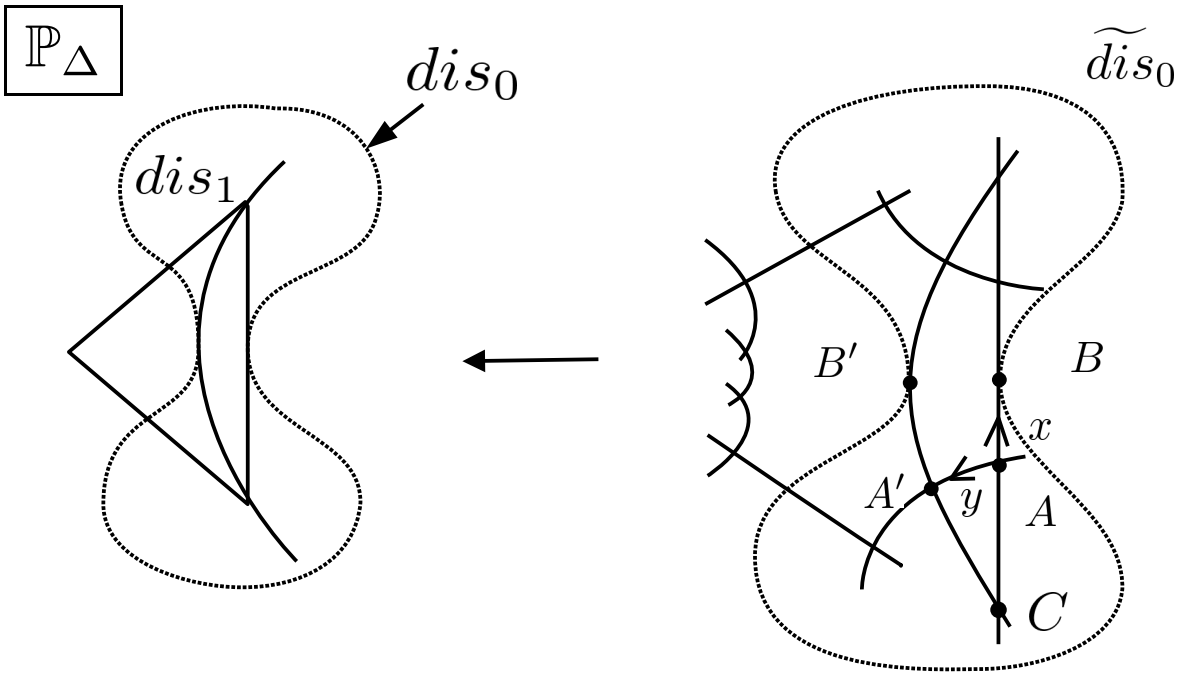
$$\mathcal{D}_1 w(x,y) = 4(1-4x)(1+4x)\theta_x^2 + \ldots\ldots = 0 \qquad \text{(2nd order)}$$

$$\mathcal{D}_2 w(x,y) = q_{03}(x,y)\theta_y^3 + q_{12}(x,y)\theta_x\theta_y^2 \ldots\ldots = 0 \quad \text{(3rd order)}$$

1. Period integrals are restricted GKZ hypergeometric series.

2. PF equations are the same for $\mathfrak{X}_{\mathbb{Z}_8}/\widetilde{G}_0$ and $\mathfrak{X}_{\mathbb{Z}_8\times\mathbb{Z}_8}/\widetilde{G}_0$.

Global description of the family $\mathfrak{X}_{\mathbb{Z}_8}/\tilde{G}_0$ and $\mathfrak{X}_{\mathbb{Z}_8\times\mathbb{Z}_8}/\tilde{G}_0$ over $\mathbb{P}_\Delta$:

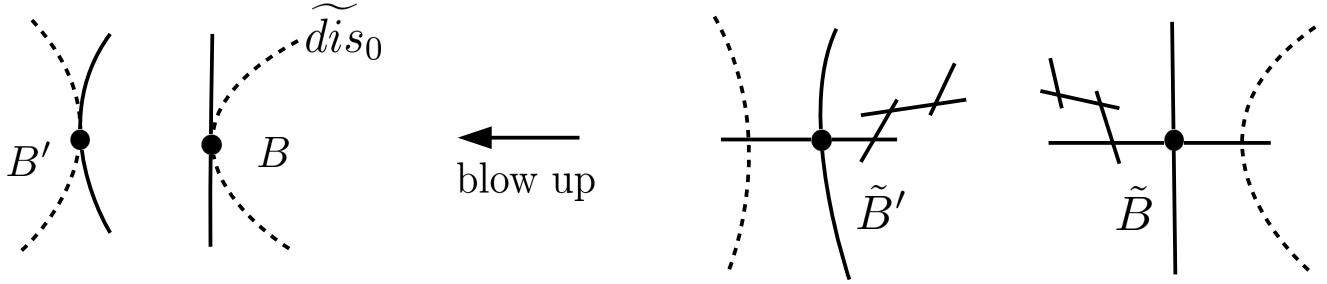


After the blowing up, we find mirror corresp. as follows:

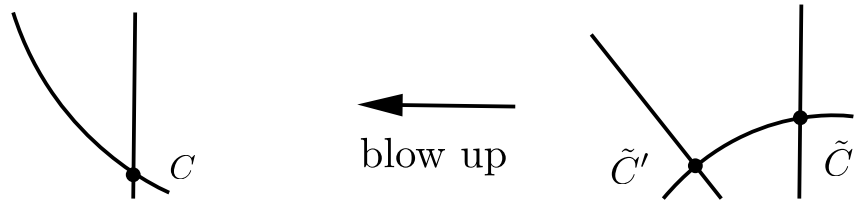
$$p_{\tilde{B}} \leftrightarrow X_{\mathbb{Z}_8\times\mathbb{Z}_8} \qquad p_{\tilde{B}'} \leftrightarrow X'_{\mathbb{Z}_8\times\mathbb{Z}_8}$$
$$p_A \leftrightarrow X \qquad p_{A'} \leftrightarrow X'$$
$$p_{\tilde{C}} \leftrightarrow X_{\mathbb{Z}_8} \qquad p_{\tilde{C}'} \leftrightarrow X'_{\mathbb{Z}_8}$$

Further blowing-ups:

(1)



(2)



Looking into more details, i.e., **integral strucures**, we can group the boundary points as follows:

$p_{\tilde{B}}\ (\leftrightarrow X_{\mathbb{Z}_8\times\mathbb{Z}_8})$ $\quad$ $p_{\tilde{B}'}\ (\leftrightarrow X'_{\mathbb{Z}_8\times\mathbb{Z}_8})$

$p_A\ (\leftrightarrow X)$ $\quad$ $p_{A'}\ (\leftrightarrow X')$

$p_{\tilde{C}}\ (\leftrightarrow X_{\mathbb{Z}_8}),$ $\quad$ $p_{\tilde{C}'}\ (\leftrightarrow X'_{\mathbb{Z}_8})$

The PF equations are the same,
but integral structures are different!

We can identify the 1st group with

$$X/\mathbb{Z}_8 \longrightarrow \mathfrak{x}_{\mathbb{Z}_8}/\widetilde{G}_0 \downarrow \mathbb{P}_\Delta$$

We identify the 2nd group with

$$X/\mathbb{Z}_8\times\mathbb{Z}_8 \longrightarrow \mathfrak{x}_{\mathbb{Z}_8\times\mathbb{Z}_8}/\widetilde{G}_0 \downarrow \mathbb{P}_\Delta$$

$g \geq 1$ **calculations tell us which one corresponds to which.**

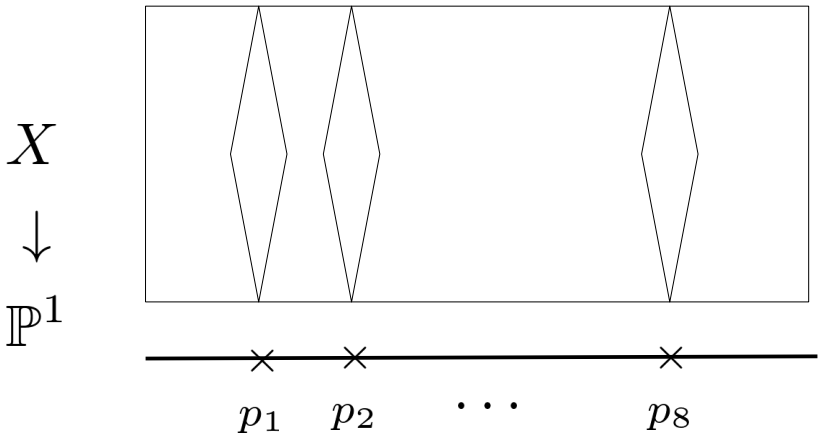
We thus verifiy the mirror symmetry (conjectured by GP)

$$X \quad \overset{\text{mirror dual}}{\longrightarrow} \quad X/\mathbb{Z}_8$$

$$X/\mathbb{Z}_8\times\mathbb{Z}_8 \quad \overset{\text{mirror dual}}{\longleftarrow}$$

§6. Gromow-Witten invariants from the boundary point p_A

Geometry of X ($\leftrightarrow p_A$)



(i) General fibers are (1,8) abelian surfaces

(ii) $\boxed{A_{\mathrm{sing},k} = \bigcup_{x \in E_k} \langle x, x + e_k \rangle}$ $(k = 1, \cdots, 8)$

translation scrolles of E_k with $e_k \in E_k$.

E_k : $\mathcal{H}_8$-inv. ellipic normal curve

Proposition. $\mathrm{Pic}(Z)_{/tor} = \mathbb{Z}H_X \oplus \mathbb{Z}A_X$, $H_2(X, \mathbb{Z})_{/tor} = \mathbb{Z}\sigma \oplus \mathbb{Z}\ell$

$\rightarrow$ we write $\beta = n\sigma + m\ell$

- BPS numbers from p_A

n \ m	0	1	2	3	4	5	6	7	8	
0	0	0	0	0	0	0	0	0	0	
1	64	512	2816	12288	46464	157696	493056	1441792	3989568	$\leftarrow n = 1$
2	0	0	4096	98304	1220608	10813440	76775424	464322560	2480783360	$\leftarrow n = 2$
3	0	0	2816	195072	6301056	124829696	1772620032	19764707328	183168532288	$\leftarrow n = 3$
4	0	0	0	98304	10567680	478740480	13238665216	261369036800	4018366742528	$\leftarrow n = 4$
5	0	0	0	12288	6301056	728901120	40797528064	1437499588608	36413468765248	
6	0	0	0	0	1220608	478740480	58763759616	3812602150912	160955539341312	
:	:	:	:	:	:	:	:	:	:	

- Counting functions

Definition. Let $N_0(\beta)$ be the GW invariants of class $\beta \in H_2(X,\mathbb{Z})$.
We define counting functions of n-sections by

$$Z_{0,n}(q) = \sum_{\beta.A=n} N_0(\beta) q^{\beta.H} \qquad \left(N_0(\beta) = \sum_{k|\beta} \frac{1}{k^3} n_0(\beta/k) \right)$$

We read from the table that

$$\boxed{Z_{0,1}(q) = \frac{64}{\bar{\eta}(q)^8}} \qquad \left(Z_{0,1}(q) = \frac{E_4(q)}{\bar{\eta}(q)^{12}} \text{for a rational elliptic surface} \right)$$

Observation (HT'21). Counting functions are written by quasi-modular forms;

$$\boxed{Z_{0,n}(q) = P_{0,n}(E_2, E_4, E_6) \left(\frac{64}{\bar{\eta}(q)^8} \right)^n} \quad (\bar{\eta}(q) := \prod_{n\geq 1} (1-q^n)\ )$$

$$P_{0,1} = 1, \quad P_{0,2} = \frac{1}{4608}(8\,E_2^2 + E_4),$$

$$P_{0,3} = \frac{1}{2654208}(14E_2^4 + 7E_2^2E_4 + E_4^2 + 2E_2E_6),$$

$$P_{0,4} = \frac{1}{2^{26}3^7}(3008E_2^6 + 2808E_2^4E_4 + 1128E_2^2E_4^2 + 125E_4^3 + 1120E_2^3E_6 + 528E_2E_4E_6 + 31E_6^2),$$

$$\cdots .$$

• ***S*-duality?** (A. Bak *"The spectral construction for a (1,8)-polarized family of abelian varieties"*, arXiv:0903.5488)

$$X \quad \overset{FM}{\longrightarrow} \quad \overset{FM}{\longleftarrow} \quad X/\mathbb{Z}_8 \times \mathbb{Z}_8$$

$$\mathrm{Pic}(X) = \mathbb{Z}H \oplus \mathbb{Z}A \qquad \mathrm{Pic}(X/\mathbb{Z}_8 \times \mathbb{Z}_8) = \mathbb{Z}\tilde{H} \oplus \mathbb{Z}\tilde{A}$$

$$H_2(X,\mathbb{Z}) = \mathbb{Z}\ell \oplus \mathbb{Z}\sigma \qquad H_2(X/\mathbb{Z}_8 \times \mathbb{Z}_8, \mathbb{Z}) = \mathbb{Z}C_E \oplus \mathbb{Z}\tilde{\sigma}$$

$$Z^X_{0,1}(q) = \frac{64}{\bar{\eta}(q)^8}$$

$(\beta = \sigma + m\,\ell)$

counting sections

?

"counting rk=1 sheaves"

?

"counting rk=1 sheaves"

$$Z^{X/\mathbb{Z}_8\times\mathbb{Z}_8}_{0,1}(q) = \frac{1}{\bar{\eta}(q^8)^8}$$

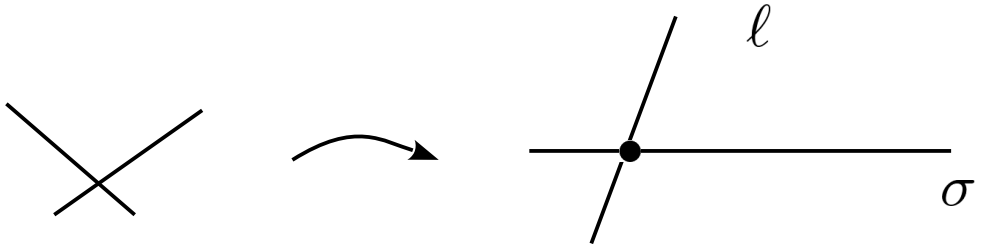
$(\beta = \tilde{\sigma} + m\,\pi_*\ell, \quad \pi_*\ell = 8C_E)$

counting sections

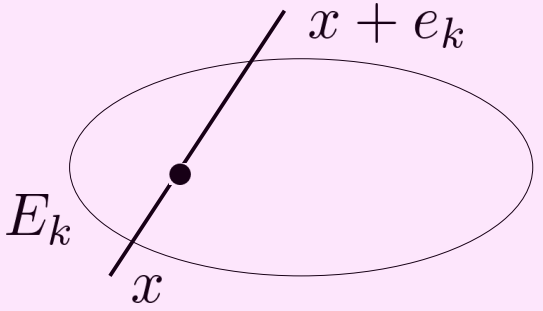
Counting $g = 0$ curves on X by $Z^X_{0,1}(q) = \frac{64}{\bar{\eta}(q)^8}$

$$\frac{1}{\bar{\eta}(q)^8} = 1 + 8q + 44q^2 + 192q^3 + 726q^4 + 2464q^5 + \cdots$$

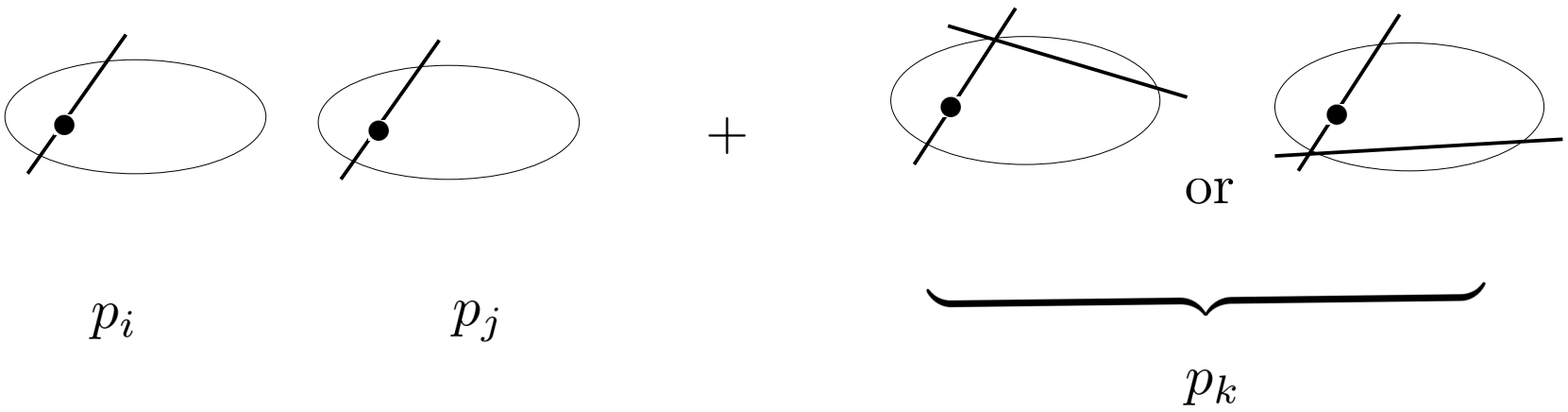
- $N_0(\sigma + \ell) = 8$



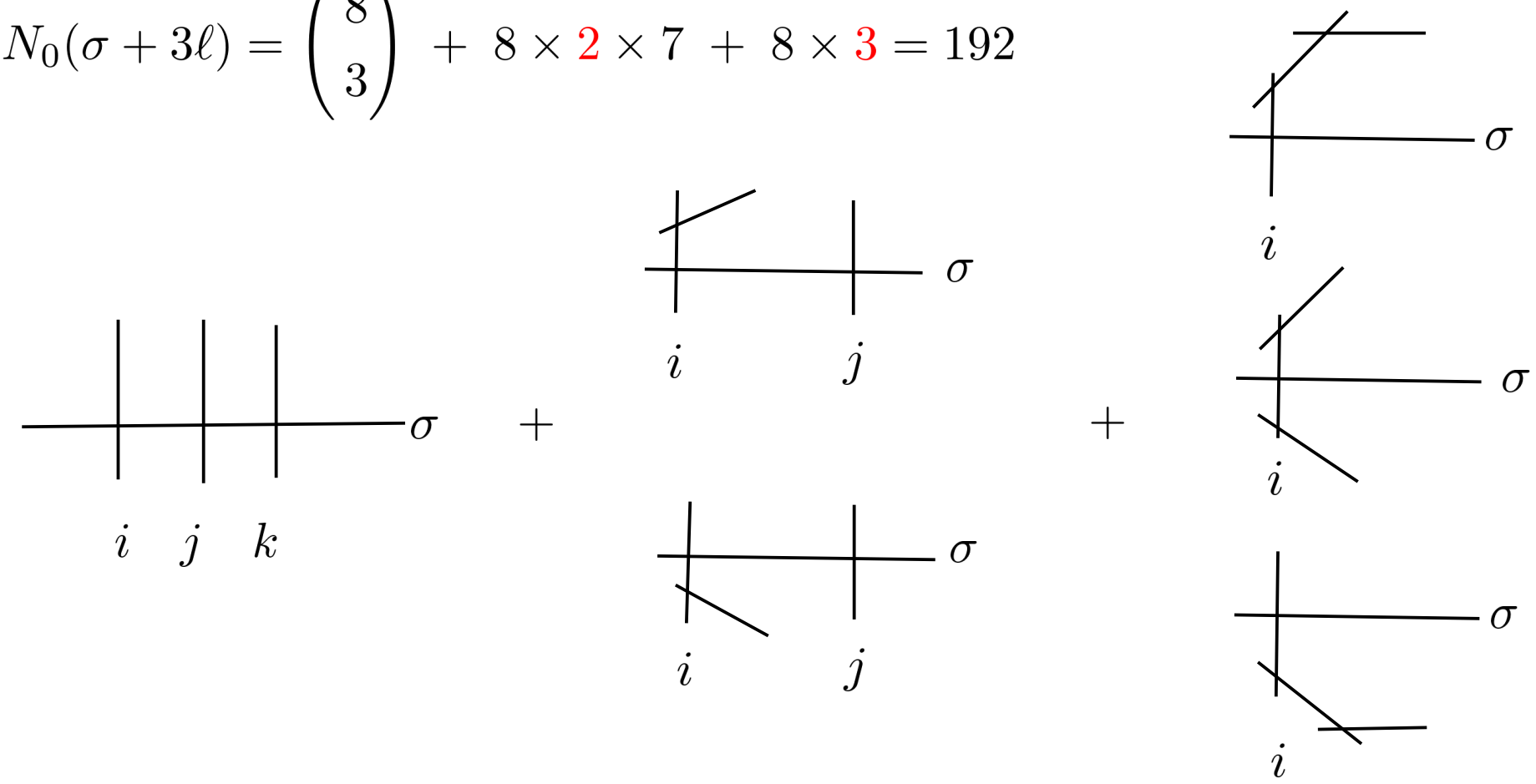
$$A_{\mathrm{sing},k} = \bigcup_{x \in E_k} \langle x, x + e_k \rangle$$



- $N_0(\sigma + 2\ell) = \binom{8}{2} + 8 \times 2 = 44$



- $N_0(\sigma + 3\ell) = \binom{8}{3} + 8 \times 2 \times 7 + 8 \times 3 = 192$



- $N_0(\sigma + 4\ell) = \binom{8}{4} + \binom{8}{2} \times 2^2 + 8 \times 2 \times \binom{7}{2} + 8 \times 3 \times 7 + 8 \times 4 = 718$

$\neq 726$!!

$\frac{1}{\bar{\eta}(q)^8}$ can be explained by
considering **admissible** (multiple-)lines (M.Gross, 2023)

Summary and Questions

(1) We can observe similar modular properties for other boundary points, and also for higher genus GW invariants $g \leq 2$.

(2) The results remind us Modular anomaly equation (H-Saito-Takahashi '98) for rational elliptic surfaces. S-duality for abelian fibration!?

(3) How do the SYZ, Gross-Siebert programs work for X?

(4) ...

More details are reported in the paper "*Mirror symmetry of Calabi-Yau manifolds fibered by (1,8)-polarized abelian surfaces*", Commun. Number Theory Phys. 16 (2022), no. 2, 215–298.

Also, Mathematica codes for detailed computations are available at https://pc1.math.gakushuin.ac.jp/~hosono/MathCodes