The 1,2,3,4,.. of applying the period geometry of CY-n folds to Feynman integrals

Pollica Calabi-Yau Workshop 2025

Albrecht Klemm, University of Sheffield

3rd of June 2025

University of Sheffield

LEVERHULME TRUST

MODERN FOUNDATIONS of SCATTERING AMPLITUDES

Based on work with

Kilian Bönisch, Mathias Driesse, Claude Duhr, Janis Dücker, Fabian Fischbach, Florian Loebbert, Gustav Mogull, Christoph Nega, Jan Plefka, Julian Piribauer, Franzika Porkert, Reza Safari, Benjamin Sauer, Lorenzo Tancredi, Johann Usovitch

[1]=JHEP 4 (20) 88, [2]=JHEP 4 (20) 88, [3]=JHEP 9 (22) 156, [4]=PRL 130 (2023) and [5]=JHEP 3 (24) 177, [6]= JHEP 03 (24) 177, [7]= JHEP 07 (24) 8, [8]= Phys.Rev.D 109 (24) 12, 124046, [9]=Nature **641** (8063) 15 May 2025, [10]=Calabi-Yau Period Geometry and Restricted Moduli in Type II Compactifications arXiv:2503.00272

CFT applications, Yangian Integrability applications, Blackhole scattering applications

Overview

The Period Motive of Calabi-Yau manifolds that we know in string theory especially the B-model approach to mirror symmetry and the B-type topological string has new applications to

Overview

The Period Motive of Calabi-Yau manifolds that we know in string theory especially the B-model approach to mirror symmetry and the B-type topological string has new applications to

I. Evaluation of higher loop corrections to Quantum Field Theory, for the new precision test of the Standard Model in precisions at future collider experiments CERN ....

Overview

The Period Motive of Calabi-Yau manifolds that we know in string theory especially the B-model approach to mirror symmetry and the B-type topological string has new applications to

I. Evaluation of higher loop corrections to Quantum Field Theory, for the new precision test of the Standard Model in precisions at future collider experiments CERN ....

II. Amplitude evaluations in systems with Yangian integrable symmetries, like 4d N=4 Super-Yang-Mills theory and Fishnet Theories.

Overview

The Period Motive of Calabi-Yau manifolds that we know in string theory especially the B-model approach to mirror symmetry and the B-type topological string has new applications to

I. Evaluation of higher loop corrections to Quantum Field Theory, for the new precision test of the Standard Model in precisions at future collider experiments CERN ....

II. Amplitude evaluations in systems with Yangian integrable symmetries, like 4d N=4 Super-Yang-Mills theory and Fishnet Theories.

III. Post Minkowskian (PM) Worldline Quantum Field Theory approximation to General Relativity to prediction the gravitational wave forms in black hole scattering/mergers detected by LIGO,....

Introduction perturbative QFT

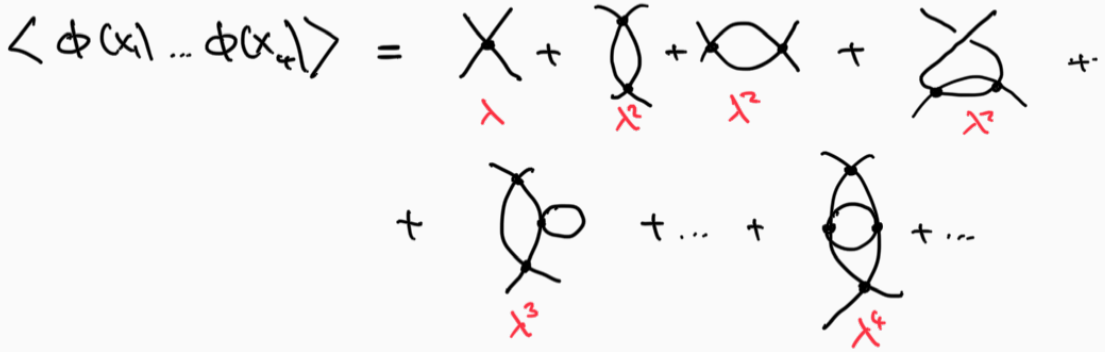
$$Z[J] = \int \mathcal{D}\phi \exp\left[\frac{i}{\hbar}\int \mathrm{d}^D x(\mathcal{L} + J\phi)\right] \ .$$

E.g. with $\mathcal{L} = \int \mathrm{d}^D x \left[\frac{1}{2}(\partial_u \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4\right]$.

All physical correlators are of the form

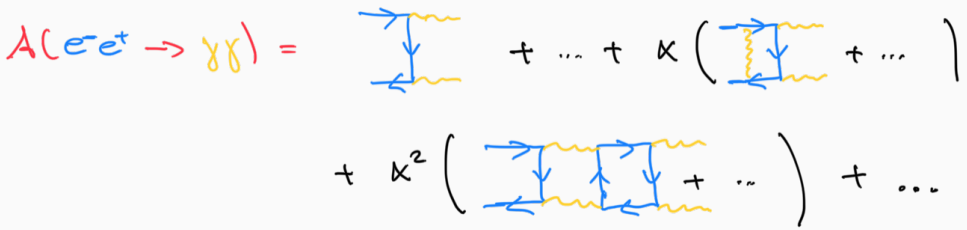
$$\langle \phi(x_1)..\phi(x_n)\rangle = Z[J]^{-1}\left(\frac{\delta}{\delta J(x_1)}\right)..\left(\frac{\delta}{\delta J(x_n)}\right) Z[J]\Big|_{J=0}$$

In interacting theories $\lambda \neq 0$ this is expanded asymptotically in Feynman graphs

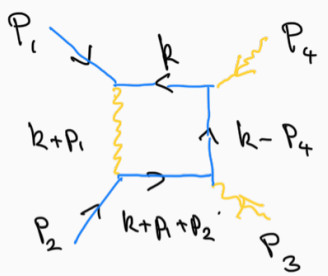


Introduction perturbative QFT

Realistic theories: Probability for e^- e^+ to annihilate to two photons $P(e^-e^+ \to \gamma\gamma) \sim |\mathcal{A}(e^-e^+ \to \gamma\gamma)|^2$, $\alpha \sim \frac{1}{137}$



Scalar part e.g. for e.g. the box integral I: Propagators $\frac{1}{q^2 - m^2 + i \cdot 0}$



$\sum_{i=1}^{4} p_i = 0$ momentum conservation

$$\sim \int \frac{d^D k}{(k^2-m^2)\,(k+p_1)^2\,((k+p_1+p_2)^2-m^2)\,((k-p_4)^2-m^2)}$$

$D = D_0 - 2\epsilon$, $I = \sum_{k=-n}^{\infty} I_k \epsilon^n$ with I_k functions of masses and Lorentz invariant products of the external momenta that we need to know!

Emerging relation Feyman Integrals and Periods

Feynman integrals $\Leftrightarrow$ Periods of algebraic varities

Planar Feynman graph	Max. Cut Integrals	Period - Geometry
1-loop	rational functions	Pts in Fano 1-fold

Emerging relation Feyman Integrals and Periods

Feynman integrals $\Leftrightarrow$ Periods of algebraic varities

Planar Feynman graph	Max. Cut Integrals	Period - Geometry
1-loop	rational functions	Pts in Fano 1-fold
2-loop	elliptic functions	families of elliptic curve

Emerging relation Feyman Integrals and Periods

Feynman integrals $\Leftrightarrow$ Periods of algebraic varities

Planar Feynman graph	Max. Cut Integrals	Period - Geometry
1-loop	rational functions	Pts in Fano 1-fold
2-loop	elliptic functions	families of elliptic curve
3-loop	fullfil 3 ord. hom diff eqs.	families of K3
4-loop	fullfil 4 ord. hom diff eqs.	families of CY-3-fold
$\vdots$	$\vdots$	$\vdots$

Emerging relation Feyman Integrals and Periods

Feynman integrals $\Leftrightarrow$ Periods of algebraic varities

Planar Feynman graph	Max. Cut Integrals	Period - Geometry
1-loop	rational functions	Pts in Fano 1-fold
2-loop	elliptic functions	families of elliptic curve
3-loop	fullfil 3 ord. hom diff eqs.	families of K3
4-loop	fullfil 4 ord. hom diff eqs.	families of CY-3-fold
$\vdots$	$\vdots$	$\vdots$

The full Feynman integral has boundaries: Periods integrals are replaced by chain integrals; hom. diff. eqs. are replaced by inhom. diff. eqs.

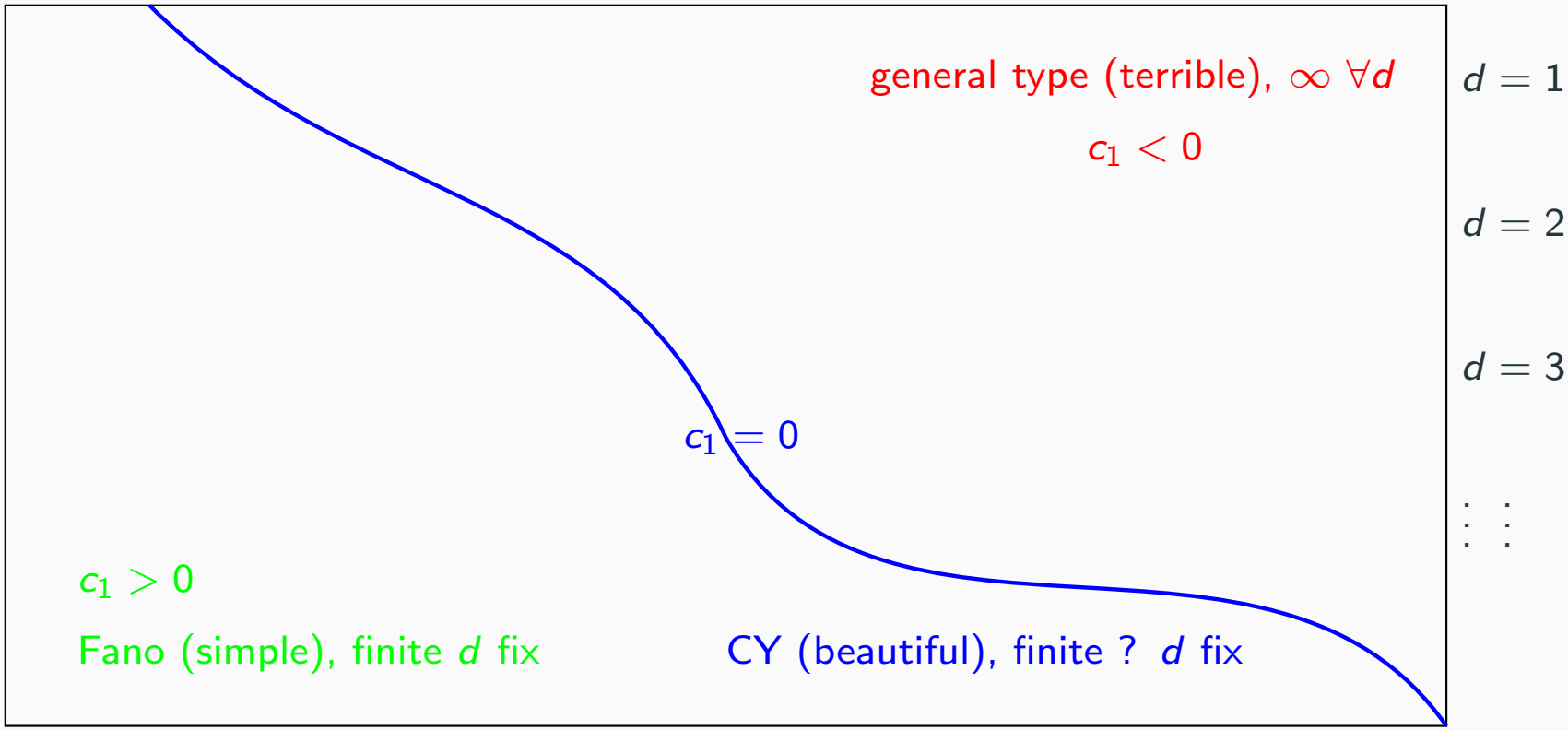
Emerging relation Feyman Integrals and Periods

Feynman integrals $\Leftrightarrow$ Periods of algebraic varities

Planar Feynman graph	Max. Cut Integrals	Period - Geometry
1-loop	rational functions	Pts in Fano 1-fold
2-loop	elliptic functions	families of elliptic curve
3-loop	fullfil 3 ord. hom diff eqs.	families of K3
4-loop	fullfil 4 ord. hom diff eqs.	families of CY-3-fold
$\vdots$	$\vdots$	$\vdots$

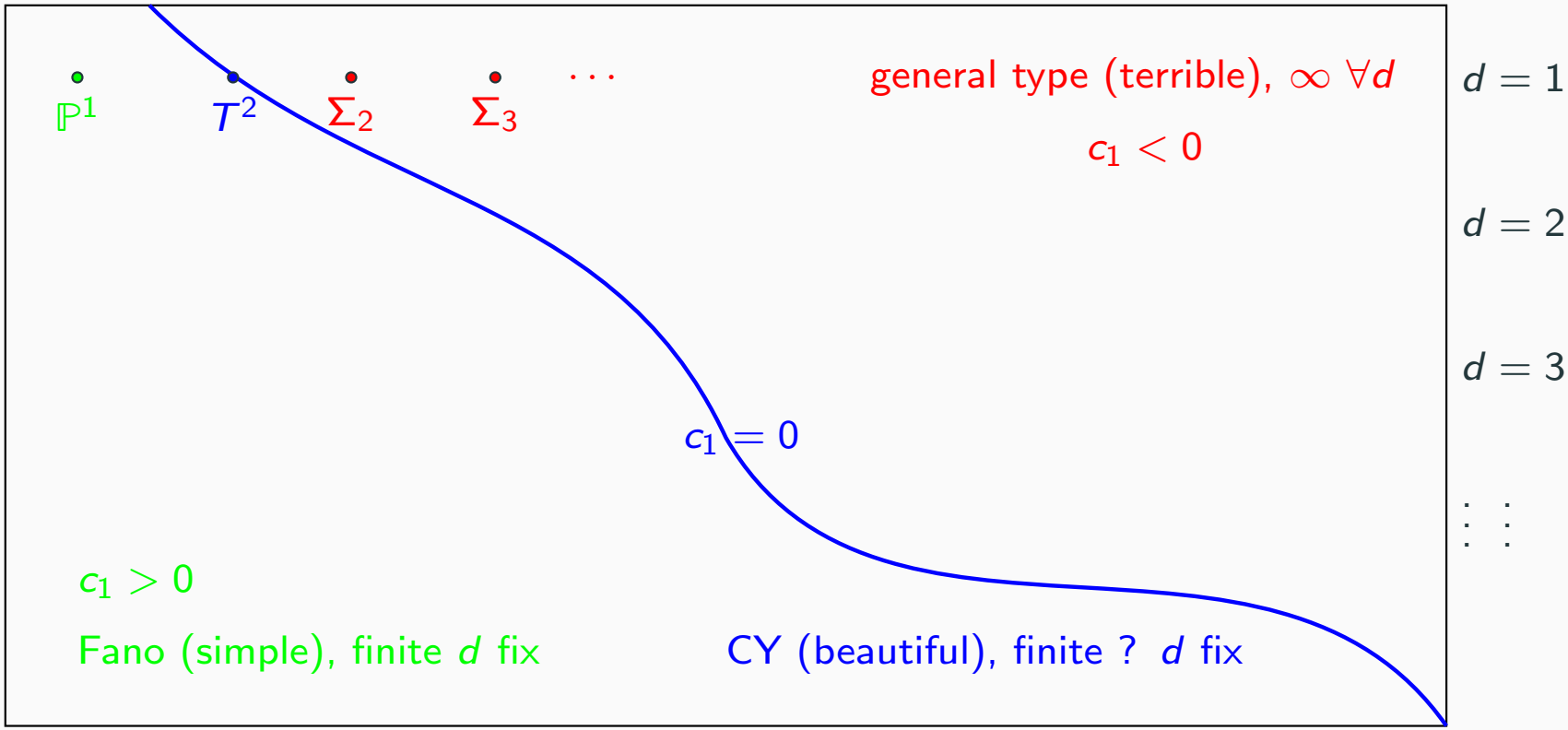
The full Feynman integral has boundaries: Periods integrals are replaced by chain integrals; hom. diff. eqs. are replaced by inhom. diff. eqs. I. Gel'fand, S. Bloch, P. Vanhove, M.Kerr, C. Duran, S. Weinzierl, F. Brown, O. Schnetz, J. Bourjaily, A. Mc Leod, M. Hippel, M. Wilhelm, J. Broedel, L Trancredi, S. Müller-Stach, ... + 248 cits. in [3]

Kodaira map of algebraic varieties

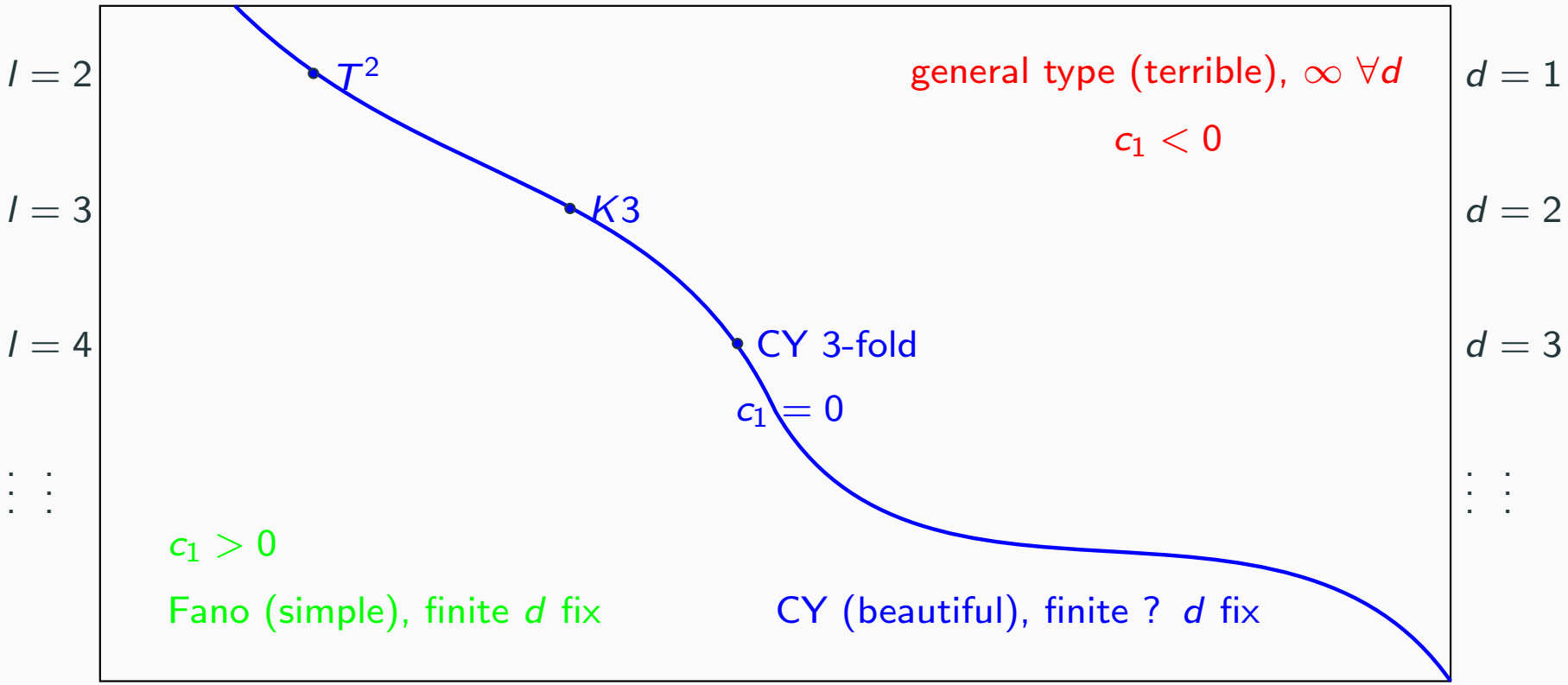


Kodaira map of algebraic varieties

$l = 0 \quad l = 1 \quad l = 2 \quad l = 3 \quad \cdots$
$g = 0 \quad g = 1 \quad g = 2 \quad g = 3 \quad \cdots$



Kodaira map of algebraic varieties



Dictionary Feynman graphs/amplitudes and geometry

Perturbative QFT	Geometry X	Differential eq.	Arithmetic Geometry
maximal cut Feynman integral	Period integral $\underline{\Pi}$ (ϵ-deformed)	Homogeneous Gauss Manin $(d - A(z))\underline{\Pi} = 0$	Motive defined by l-adic coh $H^k_{et}(\overline{X}, \mathbb{Q}_l)$
	↺ Monodromy group $\in \Gamma(\mathbb{Z})$; irreducible ?		↺ Galois group $\mathrm{Gal}(\overline{K}/K)$ irreducible ?
actual Feynman integral	Chain integral (ϵ-deformed)	Inhomogeneous Gauss Manin connection $(d - A(z))\underline{\Pi} = B(z)$	Extended motive

Gauss Manin connection and sub sectors

One way to get the Gauss-Manin connection and the inhomogeneous term is to use the integration by by parts relations IBP relation between so called master integrals. Consider l-loop Feynman integrals in general dimensions $D \in \mathbb{R}_+$ of the form

$$I_{\underline{\nu}}(\underline{z}, D) := \int \prod_{r=1}^{l} \frac{\mathrm{d}^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{p} \frac{1}{D_j^{\nu_j}} \tag{1}$$

$D_j = q_j^2 - m_j^2 + i \cdot 0$ for $j = 1, \ldots, p$ are the propagators, q_j is the jth momenta through D_j, $m_j^2 \in \mathbb{R}_+$ are masses, $i \cdot 0$ indicates the choice of contour/branchcut in $\mathbb{C}$. Subject to momentum conservation the q_j are linear in the external momenta $p_1, \ldots, p_E$, $\sum_{i=j}^{E} p_j = 0$ and the loop momenta k_r. We defined $\epsilon := \frac{D_0 - D}{2}$.

Master Integrals and integration by parts relations

The Feynman integral depends besides D on dot products of p_i and the masses m_j^2, written compactly in a vector $\underline{w} = (w_1, \ldots, N) = (p_{i_1} \cdot p_{i_2}, m_j^2)$ and dimensional analysis of $I_{\underline{\nu}}$ shows that it depends only on the ratios of two parameters x_i, we chose

$$z_k := \frac{w_k}{w_N} \qquad \text{for } 1 \leq k < N$$

and label now the parameters of the integrals $I_{\underline{\nu}}$ by the dimensionless parameters $\underline{z}$.

Master Integrals and integration by parts relations

The propagator exponents and $D \in \mathbb{Z}$ span a lattice $(\underline{\nu}, D) \in \mathbb{Z}^{p+1}$. The $I_{\underline{\nu}}(\underline{z}, D)$ are called master integrals.

Master Integrals and integration by parts relations

The propagator exponents and $D \in \mathbb{Z}$ span a lattice $(\underline{\nu}, D) \in \mathbb{Z}^{p+1}$. The $I_{\underline{\nu}}(\underline{z}, D)$ are called master integrals.

The integration by parts (IBP) identities

$$\int \prod_{r=1}^{l} \frac{\mathrm{d}^D k_r}{i\pi^{\frac{D}{2}}} \frac{\partial}{\partial k_k^\mu} \left(q_l^\mu \prod_{j=1}^{p} \frac{1}{D_j^{\nu_j}} \right) = 0 \ .$$

relate the master integrals with different exponents $\underline{\nu}$.

Master Integrals and integration by parts relations

The propagator exponents and $D \in \mathbb{Z}$ span a lattice $(\underline{\nu}, D) \in \mathbb{Z}^{p+1}$. The $I_{\underline{\nu}}(\underline{z}, D)$ are called master integrals.

The integration by parts (IBP) identities

$$\int \prod_{r=1}^{l} \frac{\mathrm{d}^D k_r}{i\pi^{\frac{D}{2}}} \frac{\partial}{\partial k_k^\mu} \left(q_l^\mu \prod_{j=1}^{p} \frac{1}{D_j^{\nu_j}} \right) = 0 \ .$$

relate the master integrals with different exponents $\underline{\nu}$.

There is a finite region in the lattice $\mathbb{Z}^{p+1}$ that contains all relevant master integrals. I.e. in those master integrals one can express all derivatives w.r.t. the z_k as a linear combination rational coefficients by the IBP relations.

Master Integrals and integration by parts relations

These master integrands from the graph cohomology. In simple cases this corresponds to the cohomology $H^{g_l}(M_{g_l}, \mathbb{Z})$ of a CY M_{g_l}, whose dim g_l growth with the loop order.

.

Master Integrals and integration by parts relations

These master integrands from the graph cohomology. In simple cases this corresponds to the cohomology $H^{g_l}(M_{g_l}, \mathbb{Z})$ of a CY M_{g_l}, whose dim g_l growth with the loop order. The integration by parts relations correspond to the Griffith reduction formula.

.

Master Integrals and integration by parts relations

These master integrands from the graph cohomology. In simple cases this corresponds to the cohomology $H^{g_l}(M_{g_l}, \mathbb{Z})$ of a CY M_{g_l}, whose dim g_l growth with the loop order.
The integration by parts relations correspond to the Griffith reduction formula.
A complete set of IBP relations corresponds to the complete Picard Fuchs ideal of Gauss-Manin connection for the period integrals.

Master Integrals and integration by parts relations

These master integrands from the graph cohomology. In simple cases this corresponds to the cohomology $H^{g_l}(M_{g_l}, \mathbb{Z})$ of a CY M_{g_l}, whose dim g_l growth with the loop order.
The integration by parts relations correspond to the Griffith reduction formula.
A complete set of IBP relations corresponds to the complete Picard Fuchs ideal of Gauss-Manin connection for the period integrals.

Among the elements in the lattice $\mathbb{Z}^p$ and, in particular, for the master integrals one can define sectors and a semi-ordering on the latter by defining a map

$$\underline{\nu} \mapsto \underline{\vartheta}(\underline{\nu}) =: (\theta(\nu_j))_{1 \leq j \leq p} \ .$$

where θ is the Heaviside step function. The semi-ordering is then defined by $\underline{\vartheta}(\underline{\nu}) \leq \underline{\vartheta}(\underline{\tilde{\nu}})$, iff $\theta(\nu_j) \leq \theta(\tilde{\nu}_j)$, $\forall j$. This defines an inclusive order on subgraphs with less propagators and therefore simpler topology.

IBP relation summary:

The IBP relations characterise a suitable finite set of master integrals

$$I_{\underline{\nu}}(\underline{x}, D) := \int \prod_{r=1}^{l} \frac{\mathrm{d}^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{p} \frac{1}{D_j^{\nu_j}} \ ,$$

with $D_j = q_j^2 - m_j^2 + i \cdot 0$ for $j = 1, \ldots, p$ propagators and $(\underline{\nu}, D)$ in a finite region in $\mathbb{Z}^{p+1}$, by a first order Gauss Manin connection

$$\boxed{d\underline{I}(\underline{x}, \epsilon) = \mathbf{A}(\underline{x}, \epsilon)\underline{I}(\underline{x}, \epsilon)}$$

$\epsilon = (D_{cr} - D)/2$.

Master Integral Basis Change possibly to canonical form

$$\underline{I}(\underline{x}, \epsilon) \to \underline{I}^{better}(\underline{z}(x); \epsilon) = R_0(\underline{z}(x); \epsilon)\underline{I}(\underline{z}(x); \epsilon)$$

$$\mathbf{A}(\underline{z}; \epsilon)^{better} = [R_0(\underline{z}; \epsilon)\mathbf{A} + dR_0(\underline{z}; \epsilon)]R_0(\underline{z}; \epsilon)^{-1}$$

Master Integral Basis Change possibly to canonical form

$$\underline{I}(\underline{x},\epsilon) \to \underline{I}^{better}(\underline{z}(x);\epsilon) = R_0(\underline{z}(x);\epsilon)\underline{I}(\underline{z}(x);\epsilon)$$

$$\mathbf{A}(\underline{z};\epsilon)^{better} = [R_0(\underline{z};\epsilon)\mathbf{A} + dR_0(\underline{z};\epsilon)]R_0(\underline{z};\epsilon)^{-1}$$

$$\mathbf{A}(\underline{z};\epsilon)^{best} = [R_n(\underline{z};\epsilon)\mathbf{A}^{n-1} + dR_n(\underline{z};\epsilon)]R_n(\underline{z};\epsilon)^{-1} = \epsilon A(\underline{z})$$

Master Integral Basis Change possibly to canonical form

$$\underline{I}(\underline{x},\epsilon) \to \underline{I}^{better}(\underline{z}(x);\epsilon) = R_0(\underline{z}(x);\epsilon)\underline{I}(\underline{z}(x);\epsilon)$$

$$\mathbf{A}(\underline{z};\epsilon)^{better} = [R_0(\underline{z};\epsilon)\mathbf{A} + dR_0(\underline{z};\epsilon)]R_0(\underline{z};\epsilon)^{-1}$$

$$\mathbf{A}(\underline{z};\epsilon)^{best} = [R_n(\underline{z};\epsilon)\mathbf{A}^{n-1} + dR_n(\underline{z};\epsilon)]R_n(\underline{z};\epsilon)^{-1} = \epsilon A(\underline{z})$$

$$\left[d_z - \epsilon \begin{pmatrix} \mathbf{0} & & & \mathbf{0} & & & \mathbf{0} & & \\ * & \dots & * & A^1_{11} & \cdots & A^1_{1r_1} & & & \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \mathbf{0} & \\ * & \dots & * & A^1_{r_1} & \cdots & A^1_{r_1 r_1} & & & \\ & \vdots & & & \ddots & & & & \\ * & \dots & * & & & & A^n_{11} & \cdots & A^n_{1r_1} \\ \vdots & \ddots & \vdots & & \mathbf{0} & & \vdots & \ddots & \vdots \\ * & \dots & * & & & & A^n_{r_2} & \cdots & A^n_{r_n r_n} \end{pmatrix} \right] \begin{pmatrix} I^{sub} \\ \Pi^1_1 \\ \vdots \\ \Pi^1_{r_1} \\ \vdots \\ \Pi_{n_1} \\ \vdots \\ \Pi_{n_r} \end{pmatrix}^{best} = 0$$

The blocks

Here $A_{ij}^k(z)$ are $d\log(\mathrm{alg}(z))$ and the $*$ are rational functions in z.

The blocks

Here $A_{ij}^{k}(z)$ are $d\log(\mathrm{alg}(z))$ and the $*$ are rational functions in z.

l-loop blocks in this improved IBP describe period (chain) integrals in the sense of Kontsevich and Zagier.

The blocks

Here $A_{ij}^k(z)$ are $d\log(\mathrm{alg}(z))$ and the $*$ are rational functions in z.

l-loop blocks in this improved IBP describe period (chain) integrals in the sense of Kontsevich and Zagier.

They can represent the inhomogenous Gauss-Manin connection of singular Calabi-Yau manifolds M_{g_l}.

The blocks

Here $A_{ij}^k(z)$ are $d\log(\mathrm{alg}(z))$ and the $*$ are rational functions in z.

l-loop blocks in this improved IBP describe period (chain) integrals in the sense of Kontsevich and Zagier.

They can represent the inhomogenous Gauss-Manin connection of singular Calabi-Yau manifolds M_{g_l}.

The inhomogenous parts $*$ come from simpler topology subgraphs.

The blocks

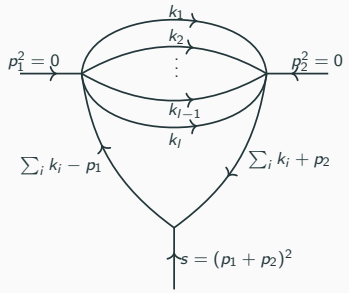
Here $A_{ij}^k(z)$ are $d\log(\mathrm{alg}(z))$ and the $*$ are rational functions in z.

l-loop blocks in this improved IBP describe period (chain) integrals in the sense of Kontsevich and Zagier.

They can represent the inhomogenous Gauss-Manin connection of singular Calabi-Yau manifolds M_{g_l}.

The inhomogenous parts $*$ come from simpler topology subgraphs.

Example [5]: The (l+1)-loop ice-cone graph contains the l-loop banana graph,



with the (chain) periods integrals of a CY (l+1)-fold, as subgraph in two ways (cutting left or right lower propagator). This leads to two CY (l+1)-fold blocks

The ϵ factorized form for the 5PM 1SF correction [8][9]

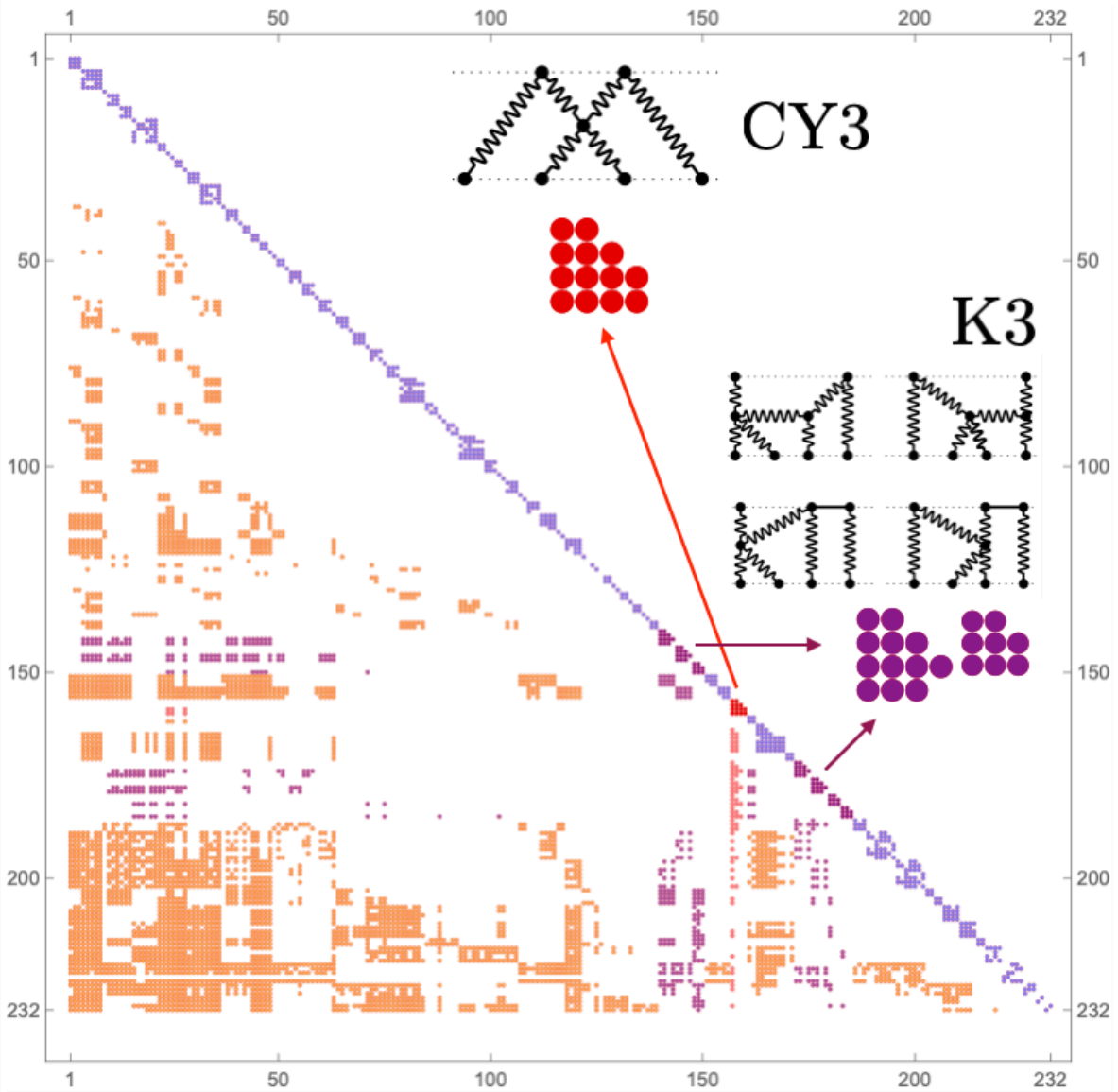


Figure 1: Non-zero entries of the 232×232 differential equation matrix

The use of the ϵ factorized form

$$\nabla_\epsilon I(z,\epsilon) = (d_z - \epsilon \mathbf{A}(\mathbf{z}))I(z,\epsilon) = 0$$

Suppose we have boundary conditions $I(z_0, 0) = I_0$ then we can write the solution vector $I(z,\epsilon)$ as path ordered exponential

$$I(z,\epsilon) = \mathbb{P}\exp[\epsilon \int_{z_0}^{z} \mathbf{A}(\mathbf{z})dz] I_0$$

in terms of iterated integrals in the form

$$I(z,\epsilon) = \left[\mathbf{1} + \epsilon \int_{z_0}^{z} \mathbf{A}(\mathbf{z}')dz' + \epsilon^2 \int_{z_0}^{z} \int_{z_0}^{z'} \mathbf{A}(\mathbf{z}')dz' \mathbf{A}(\mathbf{z}'')dz'' + \ldots\right] I_0$$

For examples for the 1 loop diagrams this formalism gives rise to the *function class* of multiple polylogarithms. For higher loop amplitudes this class generalized to iterated Calabi-Yau periods.

Definition of compact Calabi-Yau (CY) n - folds

A Calabi -Yau n-fold M is a compact *complex manifold* of complex dimension n that

Definition of compact Calabi-Yau (CY) n - folds

A Calabi -Yau n-fold M is a compact *complex manifold* of complex dimension n that

ω) is a Kähler manifold, i.e. has a Kähler $(1,1)$-form ω,

Definition of compact Calabi-Yau (CY) n - folds

A Calabi -Yau n-fold M is a compact *complex manifold* of complex dimension n that

ω) is a Kähler manifold, i.e. has a Kähler $(1,1)$-form ω,

Ω) and has unique no-where vanishing holomorphic $(n,0)$-form Ω.

Definition of compact Calabi-Yau (CY) n - folds

A Calabi -Yau n-fold M is a compact *complex manifold* of complex dimension n that

ω) is a Kähler manifold, i.e. has a Kähler $(1,1)$-form ω,

Ω) and has unique no-where vanishing holomorphic $(n,0)$-form Ω.

The latter condition is equivalent to

1) the canonical class is trivial $K_M = c_1(T_M) = 0$,
2) given a Kähler class, $\exists$ metric g with $R_{i\bar{j}}(g) = 0$,
3) the holonomy of the metric g is $(\subset SU(n))$ $SU(n)$,

Definition of compact Calabi-Yau (CY) n - folds

A Calabi -Yau n-fold M is a compact *complex manifold* of complex dimension n that

ω) is a Kähler manifold, i.e. has a Kähler $(1,1)$-form ω,

Ω) and has unique no-where vanishing holomorphic $(n,0)$-form Ω.

The latter condition is equivalent to

1) the canonical class is trivial $K_M = c_1(T_M) = 0$,
2) given a Kähler class, $\exists$ metric g with $R_{i\bar{j}}(g) = 0$,
3) the holonomy of the metric g is $(\subset SU(n))$ $SU(n)$,

Remarks: CY n-fold are generalisations of elliptic curves

- CY 1-fold is an elliptic curve, say $y^2 = x(x-1)(x-z)$ with Ω given by $\frac{dx}{y}$ and $\omega = \frac{dx}{y} \wedge \frac{d\bar{x}}{\bar{y}}$ is its volume form.

Definition of compact Calabi-Yau (CY) n - folds

A Calabi -Yau n-fold M is a compact *complex manifold* of complex dimension n that

ω) is a Kähler manifold, i.e. has a Kähler $(1,1)$-form ω,

Ω) and has unique no-where vanishing holomorphic $(n,0)$-form Ω.

The latter condition is equivalent to

1) the canonical class is trivial $K_M = c_1(T_M) = 0$,
2) given a Kähler class, $\exists$ metric g with $R_{i\bar{j}}(g) = 0$,
3) the holonomy of the metric g is $(\subset SU(n))$ $SU(n)$,

Remarks: CY n-fold are generalisations of elliptic curves

- CY 1-fold is an elliptic curve, say $y^2 = x(x-1)(x-z)$ with Ω given by $\frac{dx}{y}$ and $\omega = \frac{dx}{y} \wedge \frac{d\bar{x}}{\bar{y}}$ is its volume form.
- We use $SU(n)$ rather then $\subset SU(n)$ to avoid trivial products of lower CY n-folds in the generalisation.

General properties of Calabi-Yau n-fold fold families

Theorem Tian/Todorov: The complex moduli space $\mathcal{M}_{cs}(M)$ of a CY n-fold M is parametrized for by $h^{n-1,1} = \dim_{\mathbb{C}}(H^{n-1,1}(M))$ globally unobstructed complex deformation parameters z, i.e. is a manifold of complex dimension $h^{n-1,1} =: r$ (E and K_3 are special).

General properties of Calabi-Yau n-fold fold families

Theorem Tian/Todorov: The complex moduli space $\mathcal{M}_{cs}(M)$ of a CY n-fold M is parametrized for by $h^{n-1,1} = \dim_{\mathbb{C}}(H^{n-1,1}(M))$ globally unobstructed complex deformation parameters z, i.e. is a manifold of complex dimension $h^{n-1,1} =: r$ (E and K_3 are special).

General properties of Calabi-Yau n-fold fold families

Theorem Tian/Todorov: The complex moduli space $\mathcal{M}_{cs}(M)$ of a CY n-fold M is parametrized for by $h^{n-1,1} = \dim_{\mathbb{C}}(H^{n-1,1}(M))$ globally unobstructed complex deformation parameters z, i.e. is a manifold of complex dimension $h^{n-1,1} =: r$ (E and K_3 are special).

Application: The complex moduli dependent period integrals on CY n-fold families generalize elliptic functions. They are identified for important examples with the maximal cut Feynman higher-loop integrals, where the complex moduli z are identified with the scale invariant physical parameters e.g. $z_i = p^2/m_i^2, \ldots$.

Periods on Calabi-Yau n-folds

Periods integrals

$$\Pi_{ij}(\underline{z}) = \int_{\Gamma_i} \gamma^j(\underline{z})$$

define a non-degenerate pairing between between (middle) homology and (middle) cohomology well defined by the theorem of Stokes:

$$\Pi : H_n(M_n, \mathbb{Z}) \times H^n(M_n, \mathbb{Q}) \to \mathbb{C}\,.$$

It is possible and often natural to consider extensions of $\mathbb{Z}$. There is an intersection pairing

$$\Sigma : H_n(M_n, \mathbb{Z}) \times H_n(M_n, \mathbb{Z}) \to \mathbb{Z},$$

that can be made in particular integral. If n is odd Σ is antisymmetric and can be made symplectic. If n is even Σ is a symmetric on the even self dual lattice $H_n(M_n, \mathbb{K})$. E.g. for $K3$ $b_2 = 22$ and $\sigma = b_2^+ - b_2^- = \frac{1}{3}\int_{M_2} c_1^2 - 2c_2 = -16$ hence b_2 has signature $(3, 19)$ and is $E_8(-1)^{\oplus 2} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus 3}$.

If n is odd we fix can integral symplectic basis $\underline{\Gamma} = \{A_I, B^I\}$, $I = 0, \ldots, r$ with $\mathrm{Span}_{\mathbb{Z}}(\underline{\Gamma}) = H_n(W, \mathbb{Z})$ and

$$A_I \cap A_J = B^I \cap B^J = 0, \quad A_I \cap B^J = -B^J \cap A_I = \delta_I^J.$$

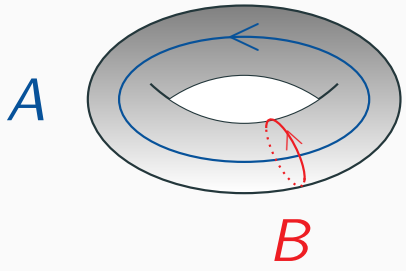
It is clearly defined up only to an $\mathrm{Sp}(b_n(M), \mathbb{Z})$ choice.

If n is odd we fix can integral symplectic basis $\underline{\Gamma} = \{A_I, B^I\}$, $I = 0, \ldots, r$ with $\mathrm{Span}_{\mathbb{Z}}(\underline{\Gamma}) = H_n(W, \mathbb{Z})$ and

$$A_I \cap A_J = B^I \cap B^J = 0, \quad A_I \cap B^J = -B^J \cap A_I = \delta_I^J.$$

It is clearly defined up only to an $\mathrm{Sp}(b_n(M), \mathbb{Z})$ choice.

Exp: Calabi-Yau 1-fold: $p_3 = wy^2 - x(x-w)(x-wz) = 0 \subset \mathbb{P}^2$

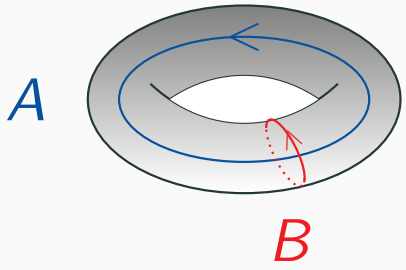


If n is odd we fix can integral symplectic basis $\underline{\Gamma} = \{A_I, B^I\}$, $I = 0, \ldots, r$ with $\mathrm{Span}_{\mathbb{Z}}(\underline{\Gamma}) = H_n(W, \mathbb{Z})$ and

$$A_I \cap A_J = B^I \cap B^J = 0, \quad A_I \cap B^J = -B^J \cap A_I = \delta_I^J.$$

It is clearly defined up only to an $\mathrm{Sp}(b_n(M), \mathbb{Z})$ choice.

Exp: Calabi-Yau 1-fold: $p_3 = wy^2 - x(x-w)(x-wz) = 0 \subset \mathbb{P}^2$



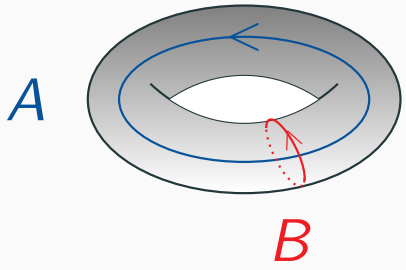
$$\Omega(z) = \oint \frac{2dx \wedge dy}{p_3} = \frac{dx}{y}, \; \partial_z \Omega(z) \sim \frac{xdx}{y}$$

If n is odd we fix can integral symplectic basis $\underline{\Gamma} = \{A_I, B^I\}$, $I = 0, \ldots, r$ with $\mathrm{Span}_{\mathbb{Z}}(\underline{\Gamma}) = H_n(W, \mathbb{Z})$ and

$$A_I \cap A_J = B^I \cap B^J = 0, \quad A_I \cap B^J = -B^J \cap A_I = \delta_I^J.$$

It is clearly defined up only to an $\mathrm{Sp}(b_n(M), \mathbb{Z})$ choice.

Exp: Calabi-Yau 1-fold: $p_3 = wy^2 - x(x-w)(x-wz) = 0 \subset \mathbb{P}^2$



$$\Omega(z) = \oint \frac{2dx \wedge dy}{p_3} = \frac{dx}{y}, \; \partial_z \Omega(z) \sim \frac{xdx}{y}$$

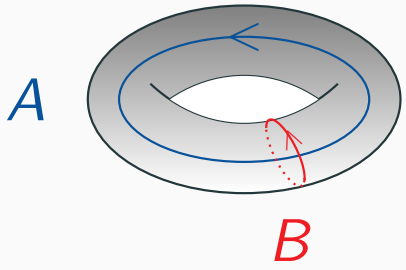
$E_1(z) = \oint_A \Omega$, $E_2(z) = \oint_B \Omega$ Elliptic integrals.

If n is odd we fix can integral symplectic basis $\underline{\Gamma} = \{A_I, B^I\}$, $I = 0, \ldots, r$ with $\mathrm{Span}_{\mathbb{Z}}(\underline{\Gamma}) = H_n(W, \mathbb{Z})$ and

$$A_I \cap A_J = B^I \cap B^J = 0, \quad A_I \cap B^J = -B^J \cap A_I = \delta_I^J.$$

It is clearly defined up only to an $\mathrm{Sp}(b_n(M), \mathbb{Z})$ choice.

Exp: Calabi-Yau 1-fold: $p_3 = wy^2 - x(x-w)(x-wz) = 0 \subset \mathbb{P}^2$



$$\Omega(z) = \oint \frac{2dx \wedge dy}{p_3} = \frac{dx}{y}, \; \partial_z \Omega(z) \sim \frac{xdx}{y}$$

$$E_1(z) = \oint_A \Omega, \; E_2(z) = \oint_B \Omega \quad \text{Elliptic integrals.}$$

Well studied in part because they solve Keplers problem

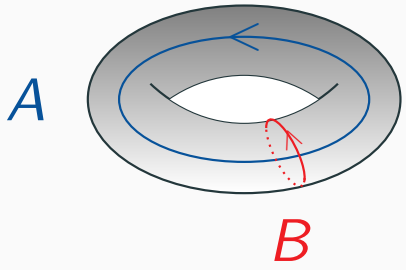
Periods annihilated by Picard-Fuchs (1881) 2cd order linear operator $L^{(2)}$.

If n is odd we fix can integral symplectic basis $\underline{\Gamma} = \{A_I, B^I\}$, $I = 0, \ldots, r$ with $\mathrm{Span}_{\mathbb{Z}}(\underline{\Gamma}) = H_n(W, \mathbb{Z})$ and

$$A_I \cap A_J = B^I \cap B^J = 0, \quad A_I \cap B^J = -B^J \cap A_I = \delta_I^J.$$

It is clearly defined up only to an $\mathrm{Sp}(b_n(M), \mathbb{Z})$ choice.

Exp: Calabi-Yau 1-fold: $p_3 = wy^2 - x(x-w)(x-wz) = 0 \subset \mathbb{P}^2$



$$\Omega(z) = \oint \frac{2dx \wedge dy}{p_3} = \frac{dx}{y}, \; \partial_z \Omega(z) \sim \frac{xdx}{y}$$

$$E_1(z) = \oint_A \Omega, \; E_2(z) = \oint_B \Omega \quad \text{Elliptic integrals.}$$

Well studied in part because they solve Keplers problem

Periods annihilated by Picard-Fuchs (1881) 2cd order linear operator $L^{(2)}$.

$$\mathcal{L} \int_\Gamma \Omega = \left[(1-z)\partial_z^2 + (1-2z)\partial_z - \frac{1}{4} \right] \int_\Gamma \Omega = 0 \; .$$

Calabi-Yau periods

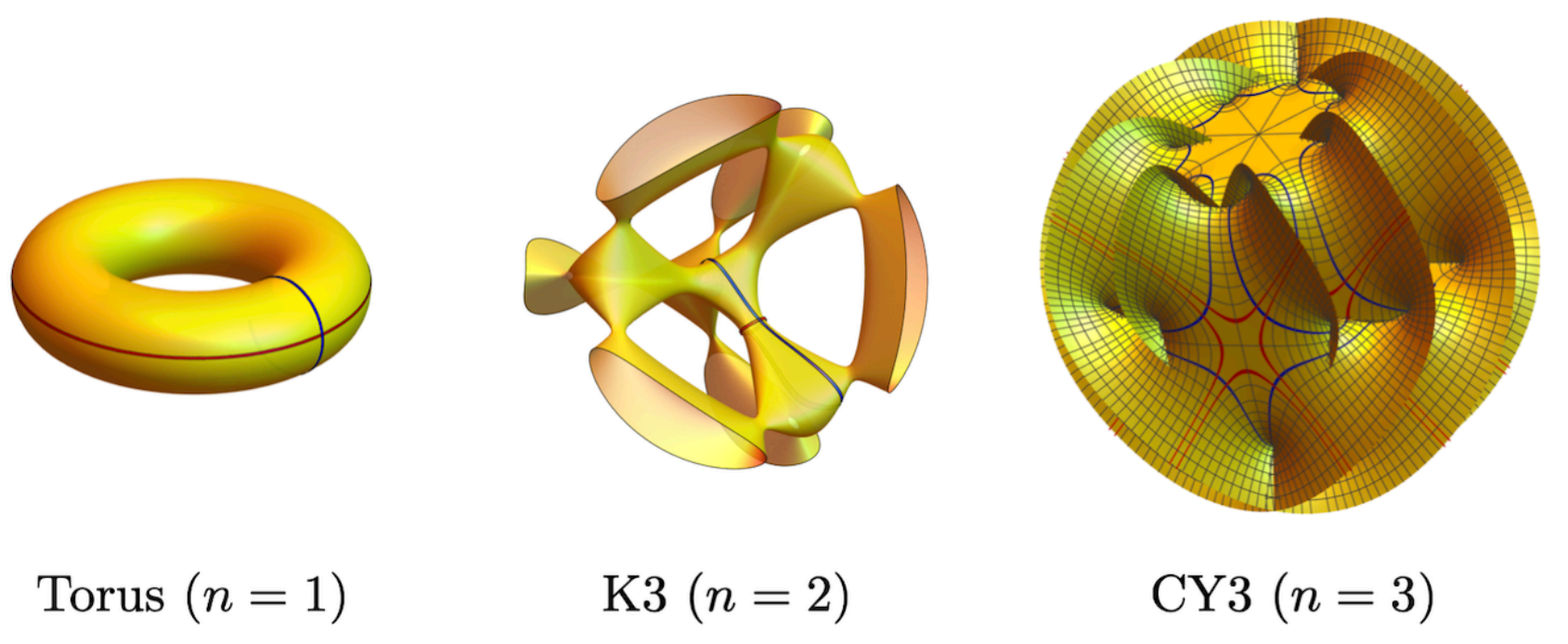


Figure 2: Three Calabi-Yau n-folds: the torus, K3 surface and a projection of the CY3 contributing to the radiated energy at G^5. The red lines are one-dimensional projections of the integration cycles defining the corresponding periods. These periods depend on the so-called modulus of the CY parametrizing its shape and yield the master integrals in our problem.

Period geometry on CY n-fold

The main constraints which govern the period geometry of CY-folds are the Riemann bilinear relations

$$e^{-K} = i^{n^2} \int_{M_n} \Omega \wedge \bar{\Omega} > 0 \tag{2}$$

defining the real positive exponential of the Kähler potential $K(z)$ for the Weil-Peterssen metric $G_{i\bar{j}} = \partial_{z_i} \bar{\partial}_{\bar{z}_{\bar{j}}} K(z)$ on $\mathcal{M}_{cs}(M_n)$.

Period geometry on CY n-fold

The main constraints which govern the period geometry of CY-folds are the Riemann bilinear relations

$$e^{-K} = i^{n^2} \int_{M_n} \Omega \wedge \bar{\Omega} > 0 \tag{2}$$

defining the real positive exponential of the Kähler potential $K(z)$ for the Weil-Peterssen metric $G_{i\bar{\jmath}} = \partial_{z_i} \bar{\partial}_{\bar{z}_{\bar{\jmath}}} K(z)$ on $\mathcal{M}_{cs}(M_n)$. As well as from relations on holomorphic bilinears and their derivatives that follow from Griffiths transversality

$$\int_{M_n} \Omega \wedge \underline{\partial}^k_{I_k} \Omega = \begin{cases} 0 & \text{if } k < n \\ C_{I_n}(\underline{z}) \in \mathbb{Q}[\underline{z}] & \text{if } k = n \ . \end{cases} \tag{3}$$

Here $\underline{\partial}^k_{I_k} \Omega = \partial_{z_{I_1}} \ldots \partial_{z_{I_k}} \Omega \in F^{n-k} := \bigoplus_{p=0}^{k} H^{n-p,p}(W)$ are sections of holomorphic sub bundles $F^n \subset F^{n-k} \subset \ldots \subset F^1 \subset F^0$ determining the Hodge filtration of the Hodge bundle F^0.

Simple consequences of Griffith transversality

Consider the Picard-Fuchs operator

$$\mathcal{L}^{(n+1)}(z) = \sum_{i=0}^{n+1} a_i(z)\partial_z^i \quad \text{with} \quad a_{n+1} = 1$$

of a one parameter Calabi-Yau n-fold with middle Hodge structure of type $\underbrace{1, 1, \ldots, 1}_{n+1}$.

Simple consequences of Griffith transversality

Consider the Picard-Fuchs operator

$$\mathcal{L}^{(n+1)}(z) = \sum_{i=0}^{n+1} a_i(z)\partial_z^i \quad \text{with} \quad a_{n+1} = 1$$

of a one parameter Calabi-Yau n-fold with middle Hodge structure of type $\underbrace{1, 1, \ldots, 1}_{n+1}$. Define the adjoint operator

$$\mathcal{L}^{(n+1)\vee}(z) = \sum_{i=0}^{n+1} (-\partial_z)^i a_i(z)\,.$$

Simple consequences of Griffith transversality

Consider the Picard-Fuchs operator

$$\mathcal{L}^{(n+1)}(z) = \sum_{i=0}^{n+1} a_i(z)\partial_z^i \quad \text{with} \quad a_{n+1} = 1$$

of a one parameter Calabi-Yau n-fold with middle Hodge structure of type $\underbrace{1, 1, \ldots, 1}_{n+1}$. Define the adjoint operator

$$\mathcal{L}^{(n+1)\vee}(z) = \sum_{i=0}^{n+1} (-\partial_z)^i a_i(z)\,.$$

Griffith transversality implies that the operator is essentially selfadjont

$$\mathcal{L}^{(n+1)} C_{\underline{z}} = (-1)^{n+1} C_{\underline{z}} \mathcal{L}^{(n+1)\vee}$$

Simple consequences of Griffith transversality

Consider the Picard-Fuchs operator

$$\mathcal{L}^{(n+1)}(z) = \sum_{i=0}^{n+1} a_i(z)\partial_z^i \quad \text{with} \quad a_{n+1} = 1$$

of a one parameter Calabi-Yau n-fold with middle Hodge structure of type $\underbrace{1, 1, \ldots, 1}_{n+1}$. Define the adjoint operator

$$\mathcal{L}^{(n+1)\vee}(z) = \sum_{i=0}^{n+1} (-\partial_z)^i a_i(z)\,.$$

Griffith transversality implies that the operator is essentially selfadjont

$$\mathcal{L}^{(n+1)} C_{\underline{z}} = (-1)^{n+1} C_{\underline{z}} \mathcal{L}^{(n+1)\vee}$$

Here $C_{\underline{z}}(z) = C_{\underbrace{z \ldots z}_{n}}(z)$ from (**??**) fulfils the differential equation

$$C'_{\underline{z}} = -\frac{2}{n+1} a_n C_{\underline{z}} \tag{4}$$

Simple consequences of Griffith transversality

The normalization for the solution to (**??**) is given by the classical intersection κ. E.g. for quintic CY 3 fold one gets

$$C_{zzz} = \frac{5}{z^3(1-5^5 z)}$$

Simple consequences of Griffith transversality

The normalization for the solution to (**??**) is given by the classical intersection κ. E.g. for quintic CY 3 fold one gets

$$C_{zzz} = \frac{5}{z^3(1-5^5 z)}$$

Equivalently self-adjointness implies

$$\sum_{j=k}^{n+1} \binom{j}{k} \left\{ \frac{C_{\underline{z}}^{(j-k)}}{C_{\underline{z}}} a_j + (-1)^{n+j} a_j^{(j-k)} \right\} = 0\,,$$

e.g. for n=3

$$a_3^3 + 4a_3'' + 6a_3 a_3' + 8a_1 - 4a_2 a_3 - 8a_2' = 0\,.$$

https://cycluster.mpim-bonn.mpg.de/aesz.html

Simple consequences of Griffith transversality

The normalization for the solution to (**??**) is given by the classical intersection κ. E.g. for quintic CY 3 fold one gets

$$C_{zzz} = \frac{5}{z^3(1-5^5 z)}$$

Equivalently self-adjointness implies

$$\sum_{j=k}^{n+1} \binom{j}{k} \left\{ \frac{C_{\underline{z}}^{(j-k)}}{C_{\underline{z}}} a_j + (-1)^{n+j} a_j^{(j-k)} \right\} = 0\,,$$

e.g. for n=3

$$a_3^3 + 4a_3'' + 6a_3 a_3' + 8a_1 - 4a_2 a_3 - 8a_2' = 0\,.$$

For a review on 1-par CY 3-fold operators see Duco van Straten https://arxiv.org/abs/1704.00164

https://cycluster.mpim-bonn.mpg.de/aesz.html

Wronskian, Griffith transversality and iterated integrals

Let denote ϖ_k $k = 0, \ldots, n$ some basis in the kernel of $\mathcal{L}^{(n+1)}$ and the $(n+1) \times (n+1)$ Wronskian

$$W = ||\partial_z^i \varpi_k||$$

Wronskian, Griffith transversality and iterated integrals

Let denote ϖ_k $k = 0, \ldots, n$ some basis in the kernel of $\mathcal{L}^{(n+1)}$ and the $(n+1) \times (n+1)$ Wronskian

$$W = ||\partial_z^i \varpi_k||$$

Then Griffith-transversality (**??**) implies that the inverse of the Wronskian is up rational factors Z linear in (derivatives) of periods

$$W^{-1} = \Sigma W^T Z^{-1} \tag{5}$$

were Z^{-1} is obviously rational ($C = C_{zzz}$, C', C'', and a rational)

$$Z^{-1} = \frac{(2\pi i)^3}{C} \begin{pmatrix} 0 & \frac{C''}{C} - 2\frac{C'}{C} + \frac{a_2}{a_4} & -\frac{C'}{C} & 1 \\ 2\frac{C'}{C} - \frac{C''}{C} - \frac{a_2}{a_4} & 0 & -1 & 0 \\ \frac{C'}{C} & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Griffith transversality, iterated integrals and weight fitrations

Feynman integrals solve inhomogeneous Gauss-Manin connections $(d_z - A(z))\underline{\Pi} = B(z)$. Here the inhomogeneity $B(z)$ is determined by the subgraphs.

Griffith transversality, iterated integrals and weight fitrations

Feynman integrals solve inhomogeneous Gauss-Manin connections $(d_z - A(z))\underline{\Pi} = B(z)$. Here the inhomogeneity $B(z)$ is determined by the subgraphs.

By the variation of constant methods, the particular solution is an iterated integral of the form

$$\varpi^{(p)}(z) = \sum_k \varpi_k \int^z d\tilde{z} W_k W^{-1} \tag{6}$$

where by (**??**) W^{-1} is linear $\partial_z^i \varpi_I$.

Griffith transversality, iterated integrals and weight fitrations

Feynman integrals solve inhomogeneous Gauss-Manin connections $(d_z - A(z))\underline{\Pi} = B(z)$. Here the inhomogeneity $B(z)$ is determined by the subgraphs.

By the variation of constant methods, the particular solution is an iterated integral of the form

$$\varpi^{(p)}(z) = \sum_k \varpi_k \int^z d\tilde{z} W_k W^{-1} \tag{6}$$

where by (**??**) W^{-1} is linear $\partial_z^i \varpi_l$.

Feynman integrals are chain integrals with physical singularities, the Landau poles (momenta become on-shell) the infrared and ultraviolet divergences and they have a transcendental weight filtration.

Griffith transversality, iterated integrals and weight fitrations

Feynman integrals solve inhomogeneous Gauss-Manin connections $(d_z - A(z))\underline{\Pi} = B(z)$. Here the inhomogeneity $B(z)$ is determined by the subgraphs.

By the variation of constant methods, the particular solution is an iterated integral of the form

$$\varpi^{(p)}(z) = \sum_k \varpi_k \int^z d\tilde{z} W_k W^{-1} \tag{6}$$

where by (**??**) W^{-1} is linear $\partial_z^i \varpi_l$.

Feynman integrals are chain integrals with physical singularities, the Landau poles (momenta become on-shell) the infrared and ultraviolet divergences and they have a transcendental weight filtration. Therefore they fit into various variation of mixed Hodge structures. Eq. (**??**) and $Z \in \mathbb{Q}[\underline{z}]$ is necessary for the compatibility of the transcendental weight filtration.

Monodromy and iteration over dimensions

Let Λ_n be the solution space of a Calabi-Yau operator $\mathcal{L}^{n+1}$, viewed as a monodromy representation.

Monodromy and iteration over dimensions

Let Λ_n be the solution space of a Calabi-Yau operator $\mathcal{L}^{n+1}$, viewed as a monodromy representation.

Call $\mathrm{Sym}^k(\mathcal{L}^{(n+1)})$ the operator which are solve by the kth symmetric product of periods and $\wedge^k(\mathcal{L}^{(n+1)})$ the operator that is solved by the k minor W_{I_k,J_k} of W.

Monodromy and iteration over dimensions

Let Λ_n be the solution space of a Calabi-Yau operator $\mathcal{L}^{n+1}$, viewed as a monodromy representation.

Call $\mathrm{Sym}^k(\mathcal{L}^{(n+1)})$ the operator which are solve by the kth symmetric product of periods and $\wedge^k(\mathcal{L}^{(n+1)})$ the operator that is solved by the k minor W_{I_k,J_k} of W.

Since z is monodromy invariant we can see these operations as building new monodromy representations specified by Yang tablaux from old.

Monodromy and iteration over dimensions

Let Λ_n be the solution space of a Calabi-Yau operator $\mathcal{L}^{n+1}$, viewed as a monodromy representation.

Call $\mathrm{Sym}^k(\mathcal{L}^{(n+1)})$ the operator which are solve by the kth symmetric product of periods and $\wedge^k(\mathcal{L}^{(n+1)})$ the operator that is solved by the k minor W_{I_k,J_k} of W.

Since z is monodromy invariant we can see these operations as building new monodromy representations specified by Yang tablaux from old.

One parameter K3 operators $\mathcal{L}^3(K3)$ of type $(1,1,1)$ are always symmetric squares the ones of elliptic curves E

$$\mathcal{L}^3(K3) = \mathrm{Sym}^2(\mathcal{L}^2(E))$$

Monodromy and iteration over dimensions

One parameter CY 4-fold operators $\mathcal{L}^5(M_4)$ of type $(1,1,1,1,1)$ up to an $\mathbb{Z}_2$ choice related to picking a spin representation given by

$$\mathcal{L}^5(M_4) = \wedge^2(\mathcal{L}^4(M_3))$$

where M_3 is a CY 3-fold.

Monodromy and iteration over dimensions

One parameter CY 4-fold operators $\mathcal{L}^5(M_4)$ of type $(1,1,1,1,1)$ up to an $\mathbb{Z}_2$ choice related to picking a spin representation given by

$$\mathcal{L}^5(M_4) = \wedge^2(\mathcal{L}^4(M_3))$$

where M_3 is a CY 3-fold.

If the monodromies in $O(5,\mathbb{Z})$ and $Sp(4,\mathbb{Z})$ are irreducible one understand entirely from representation theory

$$\wedge^2\Lambda_3 = \mathbf{5} \oplus \mathbf{1}.$$

Monodromy and iteration over dimensions

One parameter CY 4-fold operators $\mathcal{L}^5(M_4)$ of type $(1,1,1,1,1)$ up to an $\mathbb{Z}_2$ choice related to picking a spin representation given by

$$\mathcal{L}^5(M_4) = \wedge^2(\mathcal{L}^4(M_3))$$

where M_3 is a CY 3-fold.

If the monodromies in $O(5,\mathbb{Z})$ and $Sp(4,\mathbb{Z})$ are irreducible one understand entirely from representation theory

$$\wedge^2\Lambda_3 = \mathbf{5} \oplus \mathbf{1}.$$

Slightly more complicated one has that quite generally

$$\mathcal{L}^4(M_3) = \mathrm{Hadamard}^{(2)}\mathcal{L}^2(E)\ .$$

E.g. in the 5PM 1SF calculation all CY operators are obtainable from $\mathcal{L}^2(y^2 = x(x-1)(x-z))$ by the above constrs! In which sense do we need Higher dimensional CY period geometries?

A hint from Topological string theory

CY 3-folds are special for topological field theory as $F_g \neq 0 \ \forall g$ due to $\dim_{vir} \overline{M}_g(hol.\ maps, M_3) = 0$ for all g.

A hint from Topological string theory

CY 3-folds are special for topological field theory as $F_g \neq 0 \ \forall g$ due to $\dim_{vir} \overline{M}_g(hol.\ maps, M_3) = 0$ for all g.

Using direct integration the F_g are solved in terms of rational functions $F_g(S^{zz}, \tilde{S}^z, S, \mathbb{Q}[\underline{z}])$ in terms of the propagators.

A hint from Topological string theory

CY 3-folds are special for topological field theory as $F_g \neq 0 \ \forall g$ due to $\dim_{vir} \overline{M}_g(hol.\ maps, M_3) = 0$ for all g.

Using direct integration the F_g are solved in terms of rational functions $F_g(S^{zz}, \tilde{S}^z, S, \mathbb{Q}[\underline{z}])$ in terms of the propagators. The latter are given by $\wedge^2 \mathcal{L}^4(M_3)$, i.e periods of CY 4folds as

$$S^{zz} = -\frac{1}{C}\left(\frac{w_{0,2}}{w_{0,1}} - q^z_{zz}\right), \qquad \tilde{S}^z = -\frac{1}{C}\left(\frac{w_{1,2}}{w_{0,1}} - q_{zz}\right),$$
$$\tilde{S} = -\frac{1}{2C}\left(\frac{w_{1,3}}{w_{0,1}} - \partial_z q_{zz} - q_{zz} q^z_{zz}\right) + \frac{\partial_z C}{2C^2}\left(\frac{w_{1,2}}{w_{0,1}} - q_{zz}\right) - \frac{q^z_z}{2}\ . \tag{7}$$

with $w_{i,j}$ sub determinants of the Wronskian and $C, q_* \in \mathbb{Q}[\underline{z}]$ Gu, Kashani-Poor, Marino, AK, e-Print: 2305.19916.

A hint from Topological string theory

CY 3-folds are special for topological field theory as $F_g \neq 0 \ \forall g$ due to $\dim_{vir} \overline{M}_g(hol.\ maps, M_3) = 0$ for all g.

Using direct integration the F_g are solved in terms of rational functions $F_g(S^{zz}, \tilde{S}^z, S, \mathbb{Q}[\underline{z}])$ in terms of the propagators. The latter are given by $\wedge^2 \mathcal{L}^4(M_3)$, i.e periods of CY 4folds as

$$S^{zz} = -\frac{1}{C}\left(\frac{w_{0,2}}{w_{0,1}} - q^z_{zz}\right), \qquad \tilde{S}^z = -\frac{1}{C}\left(\frac{w_{1,2}}{w_{0,1}} - q_{zz}\right),$$
$$\tilde{S} = -\frac{1}{2C}\left(\frac{w_{1,3}}{w_{0,1}} - \partial_z q_{zz} - q_{zz} q^z_{zz}\right) + \frac{\partial_z C}{2C^2}\left(\frac{w_{1,2}}{w_{0,1}} - q_{zz}\right) - \frac{q^z_z}{2}\ . \tag{7}$$

with $w_{i,j}$ sub determinants of the Wronskian and $C, q_* \in \mathbb{Q}[\underline{z}]$ Gu, Kashani-Poor, Marino, AK, e-Print: 2305.19916.

Maybe the topological string is an explanation for the two conundrums. Why do higher dim CY n-folds instead of higher genus Riemann surfaces appear.

Further Consequences of Griffith transversality:

Griffith-transversality (**??**) implies that the Gauss-Manin connection can be brought into a canonical form in inhomogeneous coordinates

$$\partial_{t_*^i} \begin{pmatrix} \mathcal{V}_0 \\ \mathcal{V}_j \\ \mathcal{V}^j \\ \mathcal{V}^0 \end{pmatrix} = \begin{pmatrix} 0 & \delta_{ik} & 0 & 0 \\ 0 & 0 & C_{ijk} & 0 \\ 0 & 0 & 0 & \delta_i^j \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{V}_0 \\ \mathcal{V}_k \\ \mathcal{V}^k \\ \mathcal{V}^0 \end{pmatrix} .$$

Further Consequences of Griffith transversality:

Griffith-transversality (**??**) implies that the Gauss-Manin connection can be brought into a canonical form in inhomogeneous coordinates

$$\partial_{t_*^i} \begin{pmatrix} \mathcal{V}_0 \\ \mathcal{V}_j \\ \mathcal{V}^j \\ \mathcal{V}^0 \end{pmatrix} = \begin{pmatrix} 0 & \delta_{ik} & 0 & 0 \\ 0 & 0 & C_{ijk} & 0 \\ 0 & 0 & 0 & \delta_i^j \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{V}_0 \\ \mathcal{V}_k \\ \mathcal{V}^k \\ \mathcal{V}^0 \end{pmatrix} .$$

As a consequence the Wronskian can be split into a semi simple and a nilpotent part

$$W = W_{sem} \cdot W_{nil} .$$

This splitting is essential to achieve the ϵ-factorised form.

Calabi-Yau motives:

The periods of Calabi-Yau varieties and their extensions (inhomogeneous solutions) evaluate Feynman integrals. Different Calabi-Yau varieties have the same periods structures.

Calabi-Yau motives:

The periods of Calabi-Yau varieties and their extensions (inhomogeneous solutions) evaluate Feynman integrals. Different Calabi-Yau varieties have the same periods structures. Therefore we abstract the essential data of the period $\Pi(z)$ into the notion of a Calabi-Yau period motive with the properties

(a) $\Pi(z)$ is restricted by the real Griffiths bilinear relations defining positivity of volumes and the holomorphic Griffiths transversality conditions.

(b) $\vec{\Pi}(z)$ is a flat section of the Hodge bundle over the moduli space and fulfils a first-order homogeneous differential equation $\nabla_{GM}\Pi = (\partial_z - N(z))\Pi = 0$, or equivalently a set of higher order homogenous differential equations $\mathcal{L}^{(k)}\vec{\Pi} = 0$. The higher-order operators $\mathcal{L}^{(k)}(z, \partial_z)$ generate the Picard-Fuchs differential ideal.

Calabi-Yau motives:

(c) There is a $\mathbb{Z}[\alpha]$-integer intersection form Σ with entries $\Sigma_{ab} = \Gamma_a \cap \Gamma_b$, which is anti-symmetric and symplectic for n odd with signature $\left(\frac{b_n}{2}, \frac{b_n}{2}\right)$, and for n even it is symmetric of a signature $(m, b_n - m)$ determined by the Hirzebruch signature index.

(d) Flat sections of the Hodge bundle are determined by their monodromies M_γ for loops γ around special divisors of $\mathcal{M}_{cs}(M)$, that for a choice of basis $\Gamma_a \in H_n(M, \mathbb{Z}[\alpha])$ generate the monodromy group $\Gamma_M \subset \mathrm{Sp}(b_n, \mathbb{Z}[\alpha])$ for n odd and $\Gamma_M \subset O(\Sigma, \mathbb{Z}[\alpha])$ for n even. In particular, $\vec{\Pi}(z)$ defines a representation of Γ_M.

Dictionary so far

	$l \times l$-**block in** ϵ **form**	**Calabi-Yau (CY) geometry**
1	Maximal cut integrals in D_{cr} dimensions	$(n, 0)$-form periods of CY manifolds or CY motives
2	Dimensionless ratios $z_i = {m_i}^2/p^2$	Unobstructed compl. moduli of M_n, or equi'ly Kähler moduli of the mirror W_n
3	Integration-by-parts (IBP) reduction	Griffiths reduction method
4	Integrand-basis for maximal cuts of of master integrals in D_{cr}	Middle (hyper) cohomology $H^{g_l}(M_{g_l})$ M_{g+l}
5	Complete set of differential operators in ϵ-form	Inhomogeneous Picard-Fuchs ops or Gauss-Manin (GM) connection

Regular tilings and Calabi-Yau motives

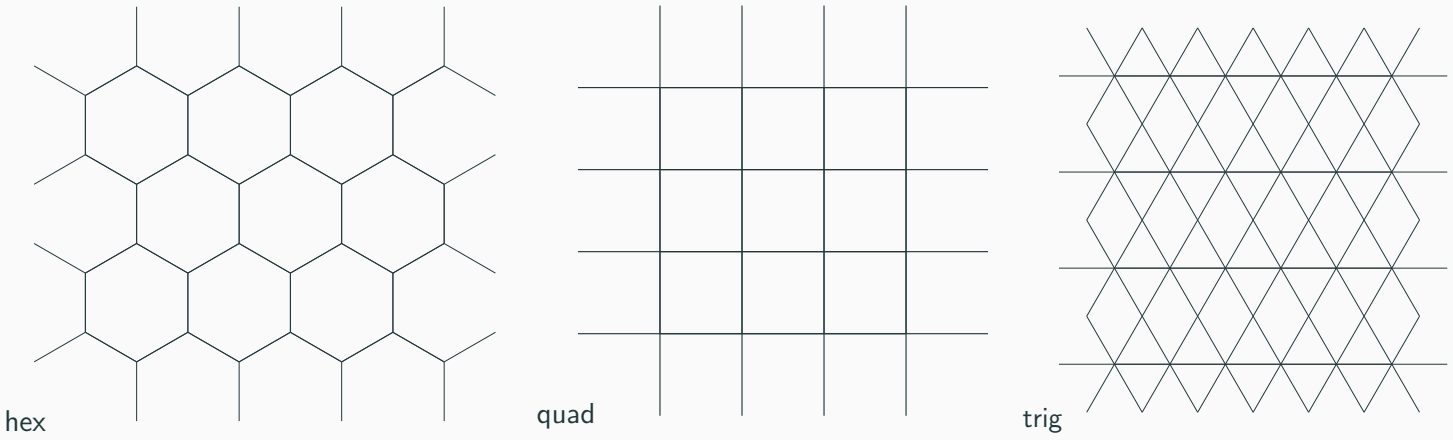


Figure 3: The three regular tilings of the plan with vertices of valence $\nu = 3, 4, 6$ respectively.

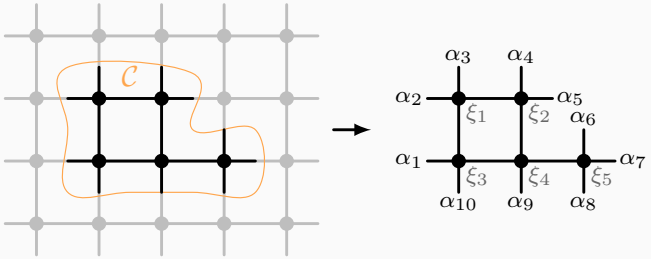


Figure 4: Ten-point five-loop fishnet integral cut out of a square tiling of the plane.

Regular tilings and Calabi-Yau motives

To obtain a graph G consider a convex closed oriented curve $\mathcal{C}$ that cuts edges of the tiling and does not pass to vertices. To each vertex inside the curve $\mathcal{C}$ we associate a $\mathbb{P}^1$ with homogeneous coordinates $[x_i : u_i]$, $i = 1, \ldots, l$ over which we want to integrate with the measure

$$\mathrm{d}\mu_i = u_i \mathrm{d}x_i - x_i \mathrm{d}u_i \ . \tag{8}$$

To the end point of each cut edge outside $\mathcal{C}$ we associate a parameter $a_j \in \mathbb{C}$, $j = 1, \ldots, r$. The graph is constructed by the l vertices with propagators

$$P^I_{ij} = \frac{1}{(x_i - x_j)^{w_{ij}}}, \qquad P^E_{ij} = \frac{1}{(x_i - a_j)^{w_{ij}}} \ . \tag{9}$$

To be conformal in D dimension the weights of propagators incident to each vertex V_i has to fullfill

$$\sum w_{ij} = D \tag{10}$$

Regular tilings and Calabi-Yau motives

We deal mainly with $D = 2$ and choose the propagator weights all equal $w_{ij} = w = 2/\nu(V)$, where $\nu(V)$ is the valence of the vertices, i.e. for the hexagonal tiling we have $w = \frac{2}{3}$, for the quartic tiling $w = \frac{1}{4}$ amd for the trigonal tiling $w = \frac{1}{3}$.

To the hexagonal and the quartic lattice we can associate an in general singular l-dimensional Calabi-Yau variety M_l as the $d = 3$ or $d = 2$ fold cover

$$W = \frac{y^d}{d} - P([\underline{x} : \underline{u}]; \underline{a}) = 0 \tag{11}$$

over the base $B = (\mathbb{P}^1)^l$ branched at

$$P([\underline{x} : \underline{w}]; \underline{a}) = \prod_{ij}(u_j x_i - x_j u_i) \prod_{ij}(x_i - a_j u_i) = 0 \ , \tag{12}$$

respectively. The orders of the covering automorpishm exchanging the sheets will play a crucial role in the following geometric analysis.

Regular tilings and Calabi-Yau motives

Note that (**??**) defines a Calabi-Yau manifold, because the canonical class of the base is with H_i the hyperplane class of the i'th $\mathbb{P}^1$ given by

$$K_B = 2 \bigoplus_{i=1} H_i, \tag{13}$$

and the Calabi-Yau condition ensuring $K_{M_l} = 0$

$$\frac{d}{d-1} K_B = [P([\underline{x} : \underline{u}]; \underline{a})] = \nu \bigoplus_{i=1} H_i \tag{14}$$

is true with $d = 3, 2$ as $\nu = 3, 4$ for graphs from the hexagonal and the quartic tiling, respectively.

Regular tilings and Calabi-Yau motives

Another way of stating this is that the periods over the unique holomorphic (ℓ,0)-form, given by the Griffiths residuum form Ω

$$\Pi_G = \int_C \Omega = \int_C \frac{1}{2\pi i} \oint_\gamma \frac{dy \prod_{i=1}^{l} d\mu_i}{W} = \int_C \frac{\prod_{i=1}^{l} d\mu_i}{\partial_y W} = \int_C \frac{\prod_{i=1}^{l} d\mu_i}{P^{\frac{d-1}{d}}} = \int_C \prod_{ij} P_{ij}^{I} \prod_{ij} P_{ij}^{E} \prod_{i=1}^{l} d\mu_i \,, \tag{15}$$

are well defined. The significance for the application is that these period integrals over cycles $C \in H_l(M_l, \mathbb{Z})$ are building blocks for the amplitudes.

$$I_G = \int_C \Omega = \int \sqrt{\left| \prod_{ij} P_{ij}^{I} \prod_{ij} P_{ij}^{E} \right|^2} \prod_{i=1}^{l} d\mu_i \wedge d\bar{\mu}_i \,, \tag{16}$$

Regular tilings and Calabi-Yau motives

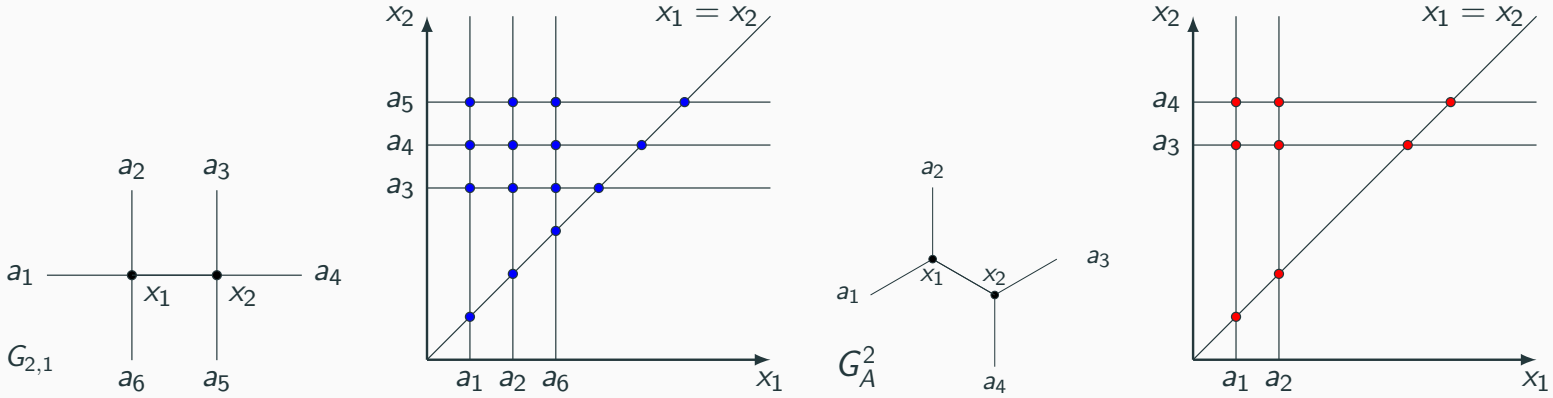


Figure 5: Singularities of the $K3$ denoted for the valence 4 graph $M_{G_{1,2}}$ and the valence 3 graph $M_{G_A^2}$. Note that 3 of the a_i can be set to $0, 1, \infty$ by a diagonal $PSL(2, \mathbb{C})$ acting on the projective plane in which the a_i lie

Regular tilings and Calabi-Yau motives

Claim 1: To each graph G we can associate a Calabi-Yau variety X whose periods determine I.

Regular tilings and Calabi-Yau motives

Claim 1: To each graph G we can associate a Calabi-Yau variety X whose periods determine I.

Claim 2: Each I gives rise to a Calabi-Yau motive with integer symmetry (l even) or antisymmetric (l odd) intersection form Σ, a point of maximal unipotent monodromy and a period vector $\Pi(\underline{z}) = \int_{\Gamma_i} \Omega$ with $\Gamma_i \in H_l(W^{(m,n)}, \mathbb{Z})$. The Feynman amplitude is given near the Mum points by the quantum volume of the mirror

$$I = i^{l^2} \Pi^\dagger \Sigma \Pi = e^{-K(\underline{z},\bar{\underline{z}})} = Vol_q(M^{(m,n)})$$

and globally by analytic continuation of the periods. Here $M^{(m,n)}$ is the mirror of $W^{(m,n)}$.

Regular tilings and Calabi-Yau motives

Claim 3: There exist an integrable conformal fishnet theories (CFNT) developed first (Gürdogan, Kazakov 2015) as deformation of $N = 4$ $SU(N_c)$ SYM theory. Let X, Z be $SU(N_c)$ matrix fields then the Lagrangian is

$$\mathcal{L}_{FN} = N_c \mathrm{tr}\left(-\partial_\mu X \partial^m u \bar{X} - \partial_\mu Z \partial^m u \bar{Z} + \xi^2 X Z \bar{X} \bar{Z}\right)$$

Each $I_{m,n}$ integral is an amplitude in the CFNT, i.e. $I_{m,n}(\underline{z})$ has to be single valued i.e. a Bloch Wigner dilogarithm or in the $D = 2$ case e^{-K}.

Regular tilings and Calabi-Yau motives

Claim 3: There exist an integrable conformal fishnet theories (CFNT) developed first (Gürdogan, Kazakov 2015) as deformation of $N = 4$ $SU(N_c)$ SYM theory. Let X, Z be $SU(N_c)$ matrix fields then the Lagrangian is

$$\mathcal{L}_{FN} = N_c \mathrm{tr}\left(-\partial_\mu X \partial^m u \bar{X} - \partial_\mu Z \partial^m u \bar{Z} + \xi^2 X Z \bar{X} \bar{Z}\right)$$

Each $I_{m,n}$ integral is an amplitude in the CFNT, i.e. $I_{m,n}(\underline{z})$ has to be single valued i.e. a Bloch Wigner dilogarithm or in the $D = 2$ case e^{-K}.

The factorisation of the amplitudes of the integrable system subject to the Yang-Baxter relations imply many non-trivial relations for he periods of the $W^{(m,n)}$. E.g. we the one parameter specialisation the periods of $W^{(n,m)}$ are $(m \times m)$ minors of the periods $W_l^{(1,m+m)}$ etc.

Regular tilings and Calabi-Yau motives

Claim 4: ($Y(SO(3,1)) = Y(Sl(2,\mathbb{R})) \oplus \overline{Y(Sl(2,\mathbb{R}))}$.) The holomorphic Yangian generated by the algebra

$$
\begin{aligned}
P_j^\mu &= -i\partial_{a_j}^\mu, & K_j^\mu &= -2ia_j^\mu(a_j^\nu \partial_{a_j,\nu} + \Delta_j) + ia_j^2 \partial_{a_j}^\mu \\
L_j^{\mu\nu} &= i(a_j^\mu \partial_{a_j}^\nu - a_j^\nu \partial_{a_j}^\mu), & D_j &= -i(a_j^\mu \partial_{a_j,\mu}),
\end{aligned}
$$

Regular tilings and Calabi-Yau motives

Claim 4: ($Y(SO(3,1)) = Y(Sl(2,\mathbb{R})) \oplus \overline{Y(Sl(2,\mathbb{R}))}$.) The holomorphic Yangian generated by the algebra

$$\begin{aligned} P_j^\mu &= -i\partial_{a_j}^\mu, & K_j^\mu &= -2ia_j^\mu(a_j^\nu\partial_{a_j,\nu} + \Delta_j) + ia_j^2\partial_{a_j}^\mu \\ L_j^{\mu\nu} &= i(a_j^\mu\partial_{a_j}^\nu - a_j^\nu\partial_{a_j}^\mu), & D_j &= -i(a_j^\mu\partial_{a_j,\mu}), \end{aligned}$$

in differentials w.r.t. to the external position, generates together with the permutation symmetries of the latter a differential ideal that annihilates the $I(\underline{z})$ and is *equivalent* to the Picard-Fuchs differential ideal that describes the variation of the Hodge structure in the middle cohomology of X and annihilated the periods of Ω.

Regular tilings and Calabi-Yau motives

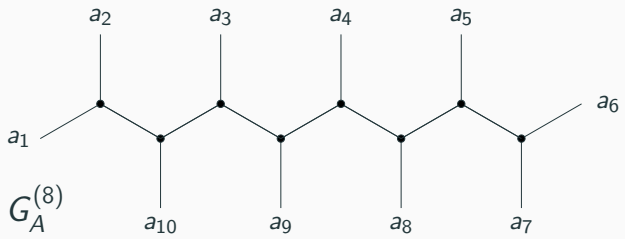


Figure 6: The $G_A^{(8)}$ graph. The A series starts from even dimensional Calabi-Yau spaces

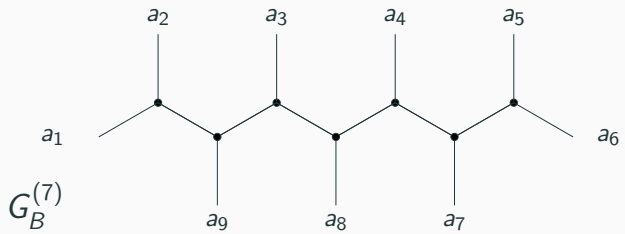


Figure 7: The $G_B^{(7)}$ graph. The B series starts from odd dimensional Calabi-Yau spaces

Regular tilings and Calabi-Yau motives

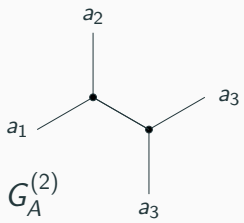


Figure 8: The $G_A^{(2)}$ graph and its transformation to a genus 2 Picard curve

$$y^3 = (x - a_1)(x - a_2)(x - a_3)^2(x - a_4)^2$$

Regular tilings and Calabi-Yau motives

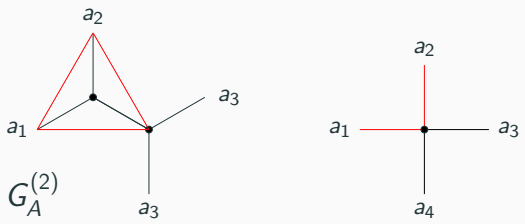


Figure 8: The $G_A^{(2)}$ graph and its transformation to a genus 2 Picard curve

$$y^3 = (x - a_1)(x - a_2)(x - a_3)^2(x - a_4)^2$$

Regular tilings and Calabi-Yau motives

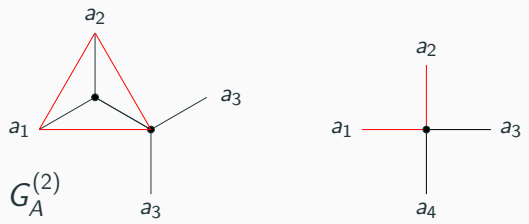


Figure 8: The $G_A^{(2)}$ graph and its transformation to a genus 2 Picard curve

$$y^3 = (x - a_1)(x - a_2)(x - a_3)^2(x - a_4)^2$$

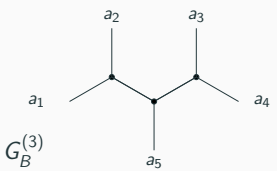


Figure 9: The $G_B^{(3)}$ graph and its transformation to a genus Picard curve

$$y^3 = (x - a_1)(x - a_2)(x - a_3)(x - a_4)(x - a_5)^2$$

Regular tilings and Calabi-Yau motives

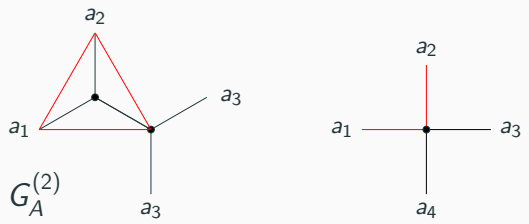


Figure 8: The $G_A^{(2)}$ graph and its transformation to a genus 2 Picard curve

$$y^3 = (x - a_1)(x - a_2)(x - a_3)^2(x - a_4)^2$$

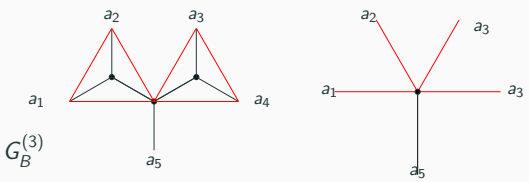


Figure 9: The $G_B^{(3)}$ graph and its transformation to a genus Picard curve

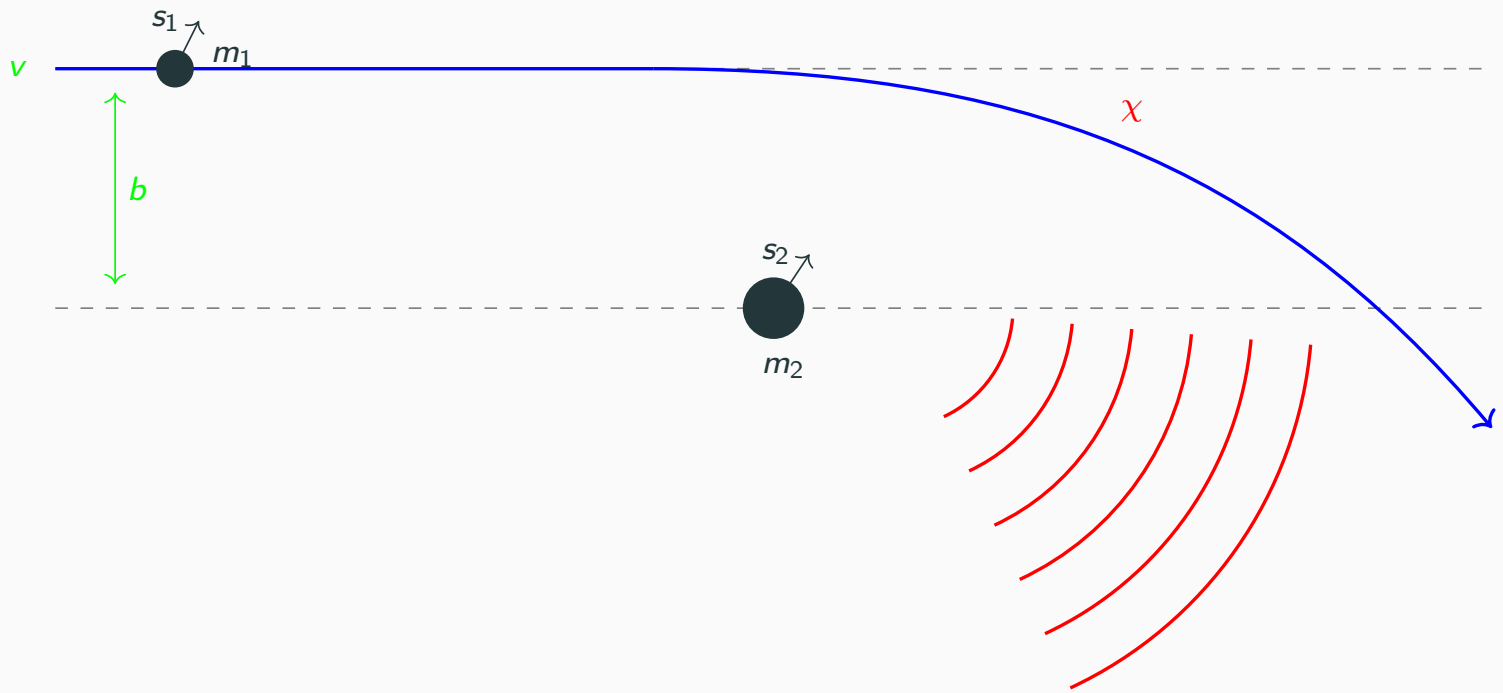
$$y^3 = (x - a_1)(x - a_2)(x - a_3)(x - a_4)(x - a_5)^2$$

Worldline QFT approach to General Relativity

Figure 10: Gravitational wave forms in Blackhole scattering

Worldline QFT approach to General Relativity

Treat scattering of two black holes (BH) as starter to the description of BH mergers as the main sources for gravitational waves detected at LIGO, ... in a perturbative expansion to gain some analytic control as inThibault Damour's effective one-body approach.



Worldline QFT approach to General Relativity

The action for the scattering process Kälin, Porto '06, Dlapa, Kälin, Liu, Porto '12, Jakobsen, Mogull, Plefka, Steinhoff '21,'23,'24

$$S = -\sum_{i=1}^{2} m_i \int \mathrm{d}\tau \left[\frac{1}{2} g_{\mu\nu} \dot{x}_i^\mu \dot{x}_i^\nu\right] + \frac{1}{16\pi G} \int \mathrm{d}^4 x \sqrt{-g} R(g) + S_{\mathrm{g.f.}}^{\mathrm{Donder}}$$

is expanded in Post Minkowskian (PM) approximation in the Worldline Quantum Field Theory (WQFT) approach around the non-interacting background configurations

$$x_i^\mu = b_i^\mu + v_i^\mu \tau + z_i^\mu(\tau)\,, \quad g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{32\pi G}\, h_{\mu\nu}(x)\ .$$

The dynamical fields z_i^μ and $h_{\mu\nu}$ are integrated out in the path integral. $\langle x_i^\mu \rangle$ fulfills the classical e.o.m.

Worldline Quantum Field Theory approach to General Relativity

The goal is to calculate from the initial data: the impact parameter $b^\mu = b_1^\mu - b_2^\mu$ and the incoming velocities v_1, v_2 the physical quantity of interest, which is the radiation induces change in the momentum say $\Delta p_1^\mu = m_1 \int \mathrm{d}\tau \ddot{x}(\tau)$ of the first particle.

In the PM approximation the latter can be expanded in the gravitational coupling G

$$\Delta p_1^\mu = \sum_{n=1}^{\infty} G^n \Delta p^{(n)\,\mu}(x) \ .$$

At each order the contributions $\Delta p^{(n)\,\mu}(x)$ are calculated in the WQFT approach in the Swinger-Keldysh in-in formalism in terms of a Feynman graph expansion with retarded propagators. Here $x = \gamma - \sqrt{\gamma^2 - 1}$ with γ the Lorentz factor of the relative velocities is the only parameter.

Worldline QFT approach to General Relativity

In the 4PM approximation the Feynman integral in the 1SF sector

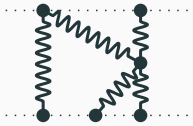
(K3.0)

involve symmetric products of elliptic function, which are periods of the $K3$

$$Y^2 = X(X-1)(X-x)Z(Z-1)(Z-1/x)\,.$$

Worldline QFT approach to General Relativity

In the 4PM approximation the Feynman integral in the 1SF sector



(K3.0)

involve symmetric products of elliptic function, which are periods of the $K3$

$$Y^2 = X(X-1)(X-x)Z(Z-1)(Z-1/x)\,.$$

In the 5PM approximation we find in [8] that in the 5PM approximation the following graphs in the 1SF sector

(CY3)

Worldline QFT approach to General Relativity

The corresponding smooth CY three-fold one-parameter complex family $x = (2\psi)^{-8}$, can be defined as resolution of four symmetric quadrics

$$x_j^2 + y_j^2 - 2\psi x_{j+1} y_{j+1} = 0,\ j \in \mathbb{Z}/4\mathbb{Z}$$

in the homogeneous coordinates x_i, y_j, $j = 0, \ldots, 3$ of $\mathbb{P}^7$.

Worldline QFT approach to General Relativity

The corresponding smooth CY three-fold one-parameter complex family $x = (2\psi)^{-8}$, can be defined as resolution of four symmetric quadrics

$$x_j^2 + y_j^2 - 2\psi x_{j+1} y_{j+1} = 0,\ j \in \mathbb{Z}/4\mathbb{Z}$$

in the homogeneous coordinates x_i, y_j, $j = 0, \ldots, 3$ of $\mathbb{P}^7$. Its periods fulfill the hypergeometric ${}_4F_3\left(\begin{smallmatrix}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\\ 1,1,1\end{smallmatrix}|z\right)$ Picard-Fuchs differential operator

$$\mathcal{L}_1^{(4)} = \theta^4 - 2^8 z(\theta + \frac{1}{2})^4$$

Worldline QFT approach to General Relativity

The corresponding smooth CY three-fold one-parameter complex family $x = (2\psi)^{-8}$, can be defined as resolution of four symmetric quadrics

$$x_j^2 + y_j^2 - 2\psi x_{j+1} y_{j+1} = 0,\ j \in \mathbb{Z}/4\mathbb{Z}$$

in the homogeneous coordinates x_i, y_j, $j = 0, \ldots, 3$ of $\mathbb{P}^7$. Its periods fulfill the hypergeometric ${}_4F_3\left(\begin{smallmatrix}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\\ 1,1,1\end{smallmatrix}|z\right)$ Picard-Fuchs differential operator

$$\mathcal{L}_1^{(4)} = \theta^4 - 2^8 z(\theta + \frac{1}{2})^4$$

The periods of the above $K3$ and CY threefold fullfill the differential equation determine all special functions that are necessary to solve for $\Delta p^{(5)\,\mu}(x)$ and the radiative energy in the 1SF sector.

Worldline QFT approach to General Relativity

The corresponding smooth CY three-fold one-parameter complex family $x = (2\psi)^{-8}$, can be defined as resolution of four symmetric quadrics

$$x_j^2 + y_j^2 - 2\psi x_{j+1} y_{j+1} = 0,\ j \in \mathbb{Z}/4\mathbb{Z}$$

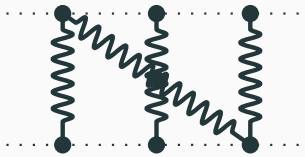
in the homogeneous coordinates x_i, y_j, $j = 0, \ldots, 3$ of $\mathbb{P}^7$. Its periods fulfill the hypergeometric ${}_4F_3\left(\begin{smallmatrix}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\\ 1,1,1\end{smallmatrix}|z\right)$ Picard-Fuchs differential operator

$$\mathcal{L}_1^{(4)} = \theta^4 - 2^8 z(\theta + \frac{1}{2})^4$$

The periods of the above $K3$ and CY threefold fullfill the differential equation determine all special functions that are necessary to solve for $\Delta p^{(5)\,\mu}(x)$ and the radiative energy in the 1SF sector. Note that all solutions are either symmetric or Hadamard products of the second oder Legendre differential equation.

Beukers K3

In the 5PM 2SF sector there is a graph



(K3')

that leads [8] to period integrals on the Beukers K3

$$1-(1-XY)Z-zXYZ(1-X)(1-Y)(1-Z)=0$$

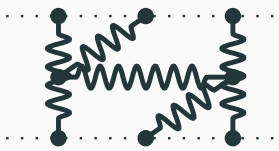
with Picard-Fuchs differential operator

$$L_A^{(3)}=\theta^3+z^2(\theta+1)^3-z(2\theta+1)\left(17\theta^2+17\theta+5\right)$$

explaining geometrically Apery's proof of the irrationality of $\zeta(3)$.

Another CY 3-fold

The graph



(CY3')

was identified using its Baikov integral representation as period integrals of the CY 3-fold

$$P = t^2(WX + Y^2)^2(W + Z)^2(X + Z)^2$$

$$+ 2^6(1 + t)(WXY)^2 Z(W + X + Z)$$

by Frellesvig, Morales, Wilhelm PRL '24. It has Picard-Fuchs operator $t = x^1 - 1$, $z = 2^{-8}x^4$

$$\mathcal{L}_2^{(4)} = \theta^4 - 2^{30}z^3(\theta + \frac{1}{2})^4 - 2^4 z(192\theta^4 + 128\theta^3 + 112\theta^2$$

$$+48\theta + 7) + 2^{14}z^2(192\theta^4 + 256\theta^3 + 208\theta^2 + 64\theta + 7) \ .$$

Results in the 5PM Sector

The important improvement is that physical quantities as the deflections angle and *gravitational waveform* can now be calculated *analytically* using iterated integrals of Calabi-Yau 3-fold periods, K3-periods and Polylogarithms.

Results in the 5PM Sector

The important improvement is that physical quantities as the deflections angle and *gravitational waveform* can now be calculated *analytically* using iterated integrals of Calabi-Yau 3-fold periods, K3-periods and Polylogarithms.

Iterated integrals of Calabi-Yau n-fold periods generalise in a precise sence, the well-known Polylogarithm that occur generically at one loop order of perturbation theory and in the 5PM calculation in most of the in sub sectors.

ϵ-factorisation

To derive the ϵ-factorization we split the periods matrix into its semi-simple and unipotent part. The unipotent part satisfies

$$(\mathrm{d} - A^{\mathsf{u}}(x))W^{\mathsf{u}}(x) = 0\,, \quad A^{\mathsf{u}}(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & Y_1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $A^{\mathsf{u}}(x)$ is nilpotent, i.e. $(A^{\mathsf{u}}(x))^4 = 0$.

ϵ-factorisation

To derive the ϵ-factorization we split the periods matrix into its semi-simple and unipotent part. The unipotent part satisfies

$$(\mathrm{d} - A^{\mathsf{u}}(x))W^{\mathsf{u}}(x) = 0\,, \quad A^{\mathsf{u}}(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & Y_1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $A^{\mathsf{u}}(x)$ is nilpotent, i.e. $(A^{\mathsf{u}}(x))^4 = 0$. In the process of obtain the ϵ-form by rotating the initial masters integrals with the semi simple matrix, we introduce new transcendental functions, which are iterated integrals the CY period $\varpi_i(\mathrm{x}')$

ϵ-factorisation

To derive the ϵ-factorization we split the periods matrix into its semi-simple and unipotent part. The unipotent part satisfies

$$(\mathrm{d} - A^{\mathrm{u}}(x))W^{\mathrm{u}}(x) = 0\,, \quad A^{\mathrm{u}}(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & Y_1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $A^{\mathrm{u}}(x)$ is nilpotent, i.e. $(A^{\mathrm{u}}(x))^4 = 0$. In the process of obtain the ϵ-form by rotating the initial masters integrals with the semi simple matrix, we introduce new transcendental functions, which are iterated integrals the CY period $\varpi_i(\mathrm{x}')$ e.g.

$$G_1(\mathrm{x}) = -\int_0^{\mathrm{x}} \frac{24576\mathrm{x}'\left(1+256\mathrm{x}'^4\right)}{(1-256\mathrm{x}'^4)^2} \frac{\varpi_0(\mathrm{x}')^2}{\alpha_1(\mathrm{x}')} \mathrm{d}\mathrm{x}'\,,$$

$$G_3(\mathrm{x}) = \int_0^{\mathrm{x}} \frac{\mathrm{x}'}{1-256\mathrm{x}'^4} \frac{G_1(\mathrm{x}')\alpha_1(\mathrm{x}')^2}{\varpi_0(\mathrm{x}')^2} \mathrm{d}\mathrm{x}'\,.$$

Here $\alpha_1 = (\theta\varpi_1/\varpi_0)^{-1}$ and $t = \varpi_1/\varpi_0$ is the affine coordinate related to the mirror map.

Scattering angle versus impact parameter up to 5PM 1SF order

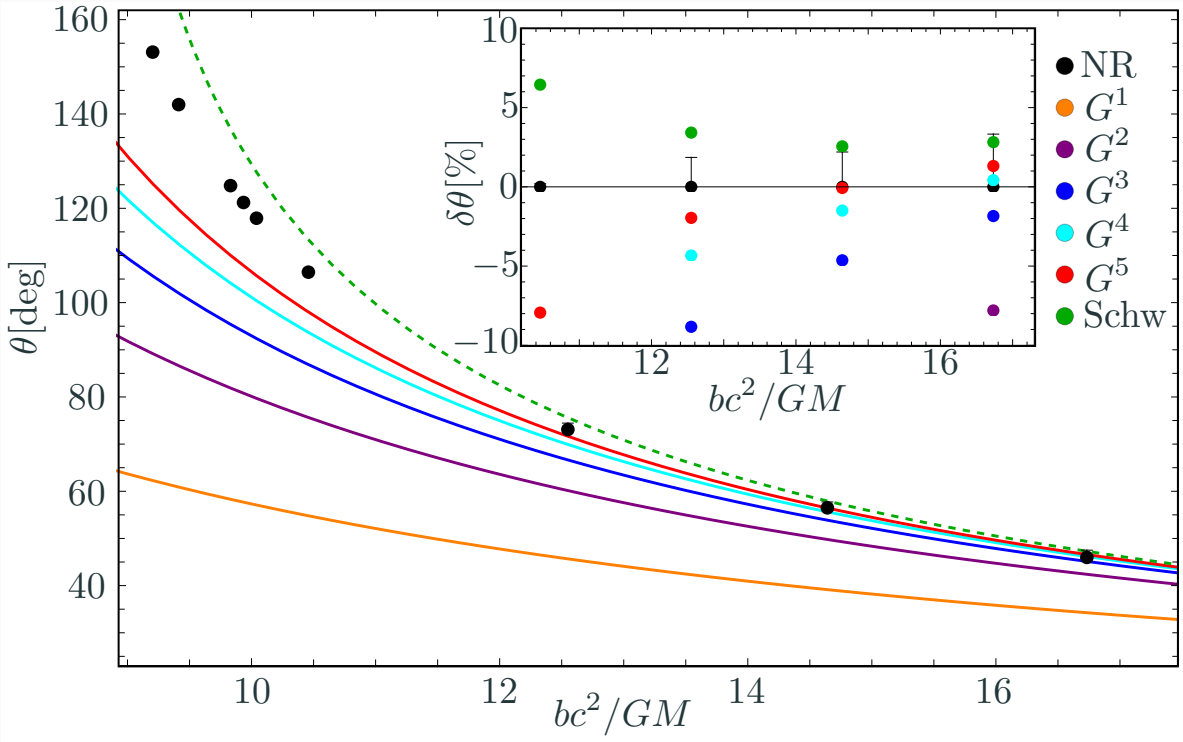


Figure 11: Scattering angle versus impact parameter

Radiation loss versus velocity

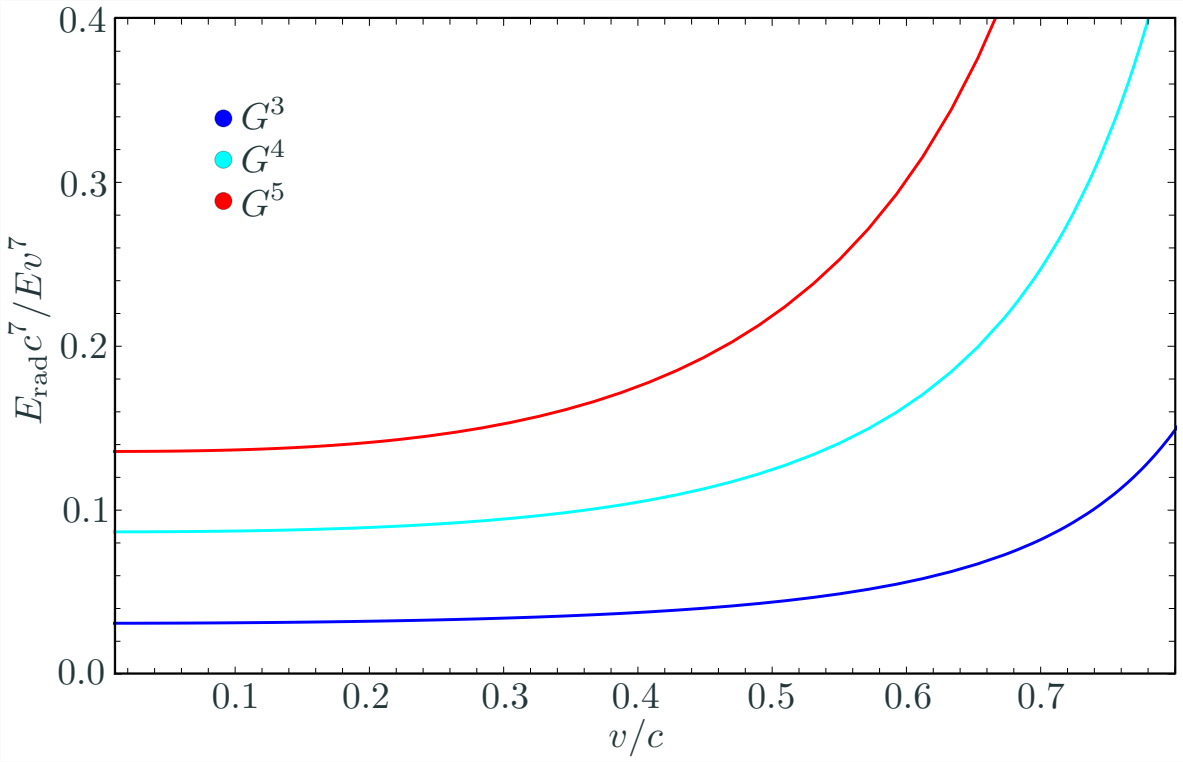
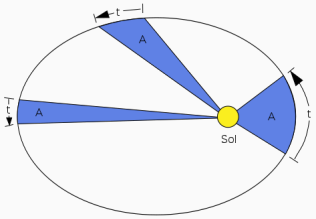
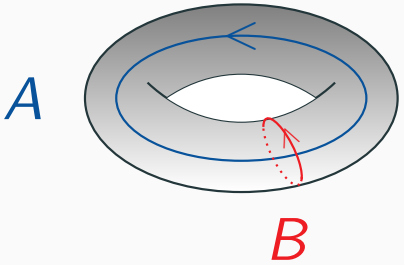


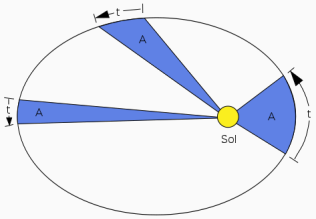
Figure 12: Radiation loss versus velocity

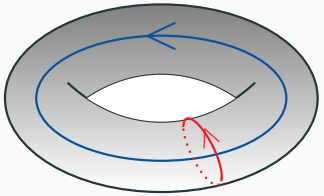
Conclusion and Outlook





Conclusion and Outlook



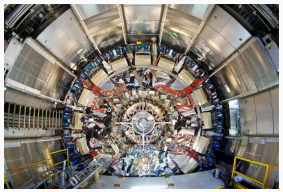


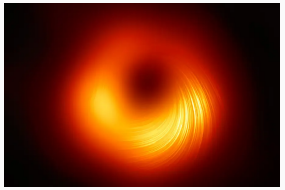
A

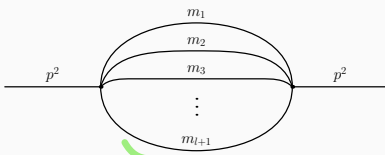
B

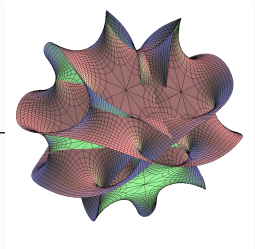
Physics

Mathematics

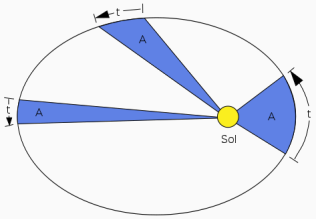


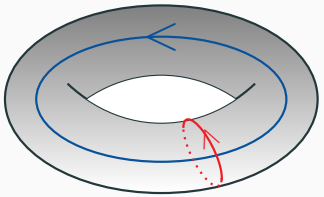






Conclusion and Outlook





A

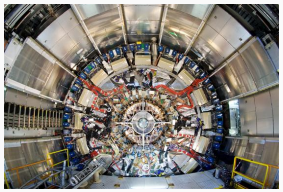
B

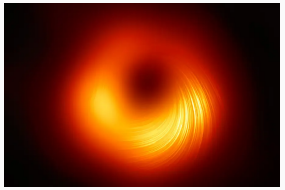
String Theory

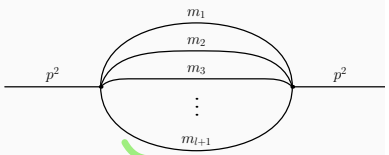
Physics

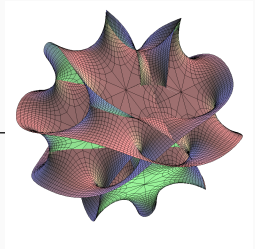
Mathematics

Enumerative Geometry

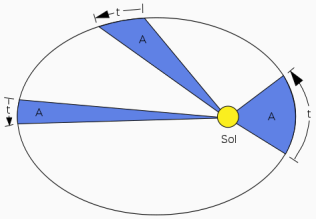


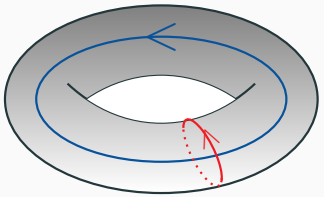






Conclusion and Outlook





A

B

String Theory

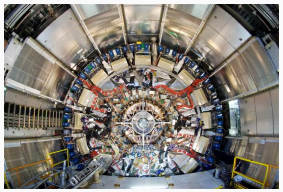
Black Hole Physics

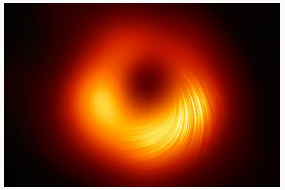
Physics

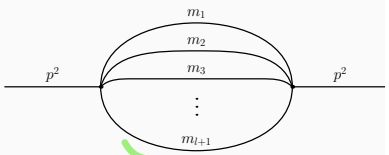
Mathematics

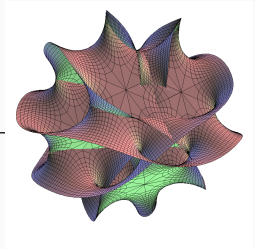
Enumerative Geometry

Arithmetic Geometry









(Relative) Calabi-Yau periods via Symanzik representation

The GM identification would be of limited use if there would not be direct ways to associate the block with a geometry X. E.g. in the Symanzik representation the contribution of an l-loop graph yields an integral with a rational integrand defined by the graph polynomials $\mathcal{U}(\underline{x})$ and $\mathcal{F}(\underline{x}, \underline{p}, \underline{m})$, $\underline{p}$ independent momenta, $\underline{m}$ masses

$$I_{\sigma_{n-1}}(\underline{p}, \underline{m}) = \int_{\sigma_{n-1}} \prod_i x_i^{\nu_i - 1} \frac{\mathcal{U}^{\omega - \frac{D}{2}}}{\mathcal{F}^{\omega}} \mu_{n-1}$$

(Relative) Calabi-Yau periods via Symanzik representation

The GM identification would be of limited use if there would not be direct ways to associate the block with a geometry X. E.g. in the Symanzik representation the contribution of an l-loop graph yields an integral with a rational integrand defined by the graph polynomials $\mathcal{U}(\underline{x})$ and $\mathcal{F}(\underline{x}, \underline{p}, \underline{m})$, $\underline{p}$ independent momenta, $\underline{m}$ masses

$\omega = \sum_{i=1}^{n} \nu_i - lD/2$, l # of loops

n # of edges, ν_i their multiplicity D space time dim

$$I_{\sigma_{n-1}}(\underline{p}, \underline{m}) = \int_{\sigma_{n-1}} \prod_i x_i^{\nu_i - 1} \frac{\mathcal{U}^{\omega - \frac{D}{2}}}{\mathcal{F}^{\omega}} \mu_{n-1}$$

μ_{n-1} measure on $\mathbb{P}^{n-1}$

$\sigma_{n-1} = \{[x_1 : \ldots : x_n] \in \mathbb{P}^{n-1} | x_i \in \mathbb{R}_{\geq 0} \, \forall \, 1 \leq i \leq n\}$ an open domain.

(Relative) Calabi-Yau periods via Symanzik representation

The GM identification would be of limited use if there would not be direct ways to associate the block with a geometry X. E.g. in the Symanzik representation the contribution of an l-loop graph yields an integral with a rational integrand defined by the graph polynomials $\mathcal{U}(\underline{x})$ and $\mathcal{F}(\underline{x}, \underline{p}, \underline{m})$, $\underline{p}$ independent momenta, $\underline{m}$ masses

$\omega = \sum_{i=1}^{n} \nu_i - lD/2$, l # of loops

n # of edges, ν_i their multiplicity D space time dim

$$I_{\sigma_{n-1}}(\underline{p}, \underline{m}) = \int_{\sigma_{n-1}} \prod_i x_i^{\nu_i - 1} \frac{\mathcal{U}^{\omega - \frac{D}{2}}}{\mathcal{F}^{\omega}} \mu_{n-1}$$

μ_{n-1} measure on $\mathbb{P}^{n-1}$

$\sigma_{n-1} = \{[x_1 : \ldots : x_n] \in \mathbb{P}^{n-1} | x_i \in \mathbb{R}_{\geq 0} \, \forall \, 1 \leq i \leq n\}$ an open domain.

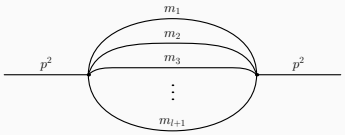
$D = D_{cr} - 2\epsilon$, $I = \sum_{k=-n}^{\infty} I_k \epsilon^n$ with I_k functions of masses and Lorentz invariant products of the external momenta.

Feyman graphs and (relative) Calabi-Yau periods

E.g. for the banana banana graph integrals in critical dimension $D_{cr} = 2$:

Feyman graphs and (relative) Calabi-Yau periods

E.g. for the banana banana graph integrals in critical dimension $D_{cr} = 2$:

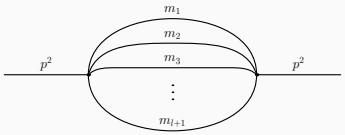


the Symanzik parametrisation leads directly to period integrals in a Calabi-Yau geometry footnotesize(with $t = \frac{p^2}{\mu^2}$, $\xi_i = \frac{m_i}{\mu}$ $(z_i = \frac{m_i^2}{p^2})$).

$$I_{\sigma_l} = \int_{\sigma_l} \frac{\mu_l}{\mathcal{F}(t, \xi_i; x)} = \int_{\sigma_l} \frac{\mu_l}{\left(t - \left(\sum_{i=1}^{l+1} \xi_i^2 x_i\right)\left(\sum_{i=1}^{l+1} x_i^{-1}\right)\right) \prod_{i=1}^{l+1} x_i}$$

Feyman graphs and (relative) Calabi-Yau periods

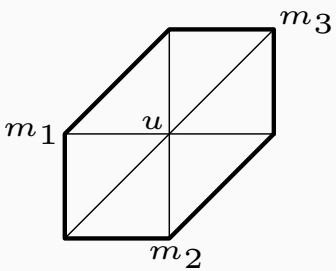
E.g. for the banana banana graph integrals in critical dimension $D_{cr} = 2$:

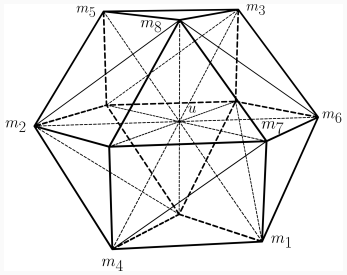


the Symanzik parametrisation leads directly to period integrals in a Calabi-Yau geometry footnotesize(with $t = \frac{p^2}{\mu^2}$, $\xi_i = \frac{m_i}{\mu}$ $(z_i = \frac{m_i^2}{p^2})$).

$$I_{\sigma_l} = \int_{\sigma_l} \frac{\mu_l}{\mathcal{F}(t, \xi_i; x)} = \int_{\sigma_l} \frac{\mu_l}{\left(t - \left(\sum_{i=1}^{l+1} \xi_i^2 x_i\right)\left(\sum_{i=1}^{l+1} x_i^{-1}\right)\right) \prod_{i=1}^{l+1} x_i}$$

as the Newton polytopes of $\mathcal{F}$ is reflexive, hence $\mathcal{F} = 0$ defines a Calabi-Yau manifold.





For a better representation of the geometry

Consider the complete intersection of two polynomials of degree $(1, \ldots, 1)$ in the cartesian product of $(\mathbb{P}^1)'s$

$$\mathbb{P}_{l+1} := \otimes_{i=1}^{l+1} \mathbb{P}^1_{(i)} \,.$$

Such a complete intersection manifold in a product of manifolds is denoted in short as

$$M^{\mathrm{CI}}_{l-1} = \left(\begin{array}{c} \mathbb{P}^1_{(1)} \\ \vdots \\ \mathbb{P}^1_{(l+1)} \end{array} \middle\| \begin{array}{cc} 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{array} \right) \subset \left(\begin{array}{c} \mathbb{P}^1_{(1)} \\ \vdots \\ \mathbb{P}^1_{(l+1)} \end{array} \middle\| \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} \right) =: F_l \subset \mathbb{P}_{l+1} \,.$$

GKZ system for the complete intersection geometry

$$
\begin{aligned}
P_1 \;&= a_0\, w_0^{(1)} + \sum_{m=1}^{l+1} a_{2m-1}\, w_m^{(1)} = a_0 \prod_{k=1}^{l+1} x_1^{(k)} + \sum_{m=1}^{l+1} a_{2m-1}\; x_2^{(m)} \prod_{k\neq m}^{l+1} x_1^{(k)} \\
P_2 \;&= \tilde{a}_0\, w_0^{(2)} + \sum_{m=1}^{l+1} a_{2m}\, w_m^{(2)} = \tilde{a}_0 \prod_{k=1}^{l+1} x_2^{(k)} + \sum_{m=1}^{l+1} a_{2m}\; x_1^{(m)} \prod_{k\neq m}^{l+1} x_2^{(k)} \;.
\end{aligned}
$$

On these parameters $a_i, \tilde{a}_i$ in the canonical representation

$$
\int_\Gamma \Omega(\underline{z}) = \int_\Gamma \frac{1}{(2\pi i)^r} \oint_{S_1^1} \oint_{S_2^1} \frac{\wedge_{i=1}^m \mu_{n_i}}{P_1 P_2} \;,
$$

of the periods integrals, the $(\mathbb{C}^*)^{l+1}$-scaling symmetries

$$
\begin{aligned}
\ell^{(1)} &= \quad (-1,-1;1,1,0,0,\cdots,0,0,0,0) \\
\ell^{(2)} &= \quad (-1,-1;0,0,1,1,\cdots,0,0,0,0) \\
&\vdots \\
\ell^{(l)} &= \quad (-1,-1;0,0,0,0,\cdots,1,1,0,0) \\
\ell^{(l+1)} &= \quad (-1,-1;0,0,0,0,\cdots,0,0,1,1)
\end{aligned}
$$

Advantages of the geometric representation

act and yield $(l+1)$ second order GKZ operators in the Batyrev large radius coordinates $z_k = \prod_{i=1}^{2(l+2)} a_i^{\ell_i^{(k)}} / (a_0 \tilde{a}_0)$, $k = 1, \ldots, l+1$.

Advantages of the geometric representation as(complete intersection) Calabi-Yau manifold

Advantages of the geometric representation

act and yield $(l+1)$ second order GKZ operators in the Batyrev large radius coordinates $z_k = \prod_{i=1}^{2(l+2)} a_i^{\ell_i^{(k)}} / (a_0 \tilde{a}_0)$, $k = 1, \ldots, l+1$.

Advantages of the geometric representation as(complete intersection) Calabi-Yau manifold The GKZ system in the yields the period integrals $\underline{\Pi}$ and near the point of maximal unipotent monodromy $z_i = 0$ in a canonical integral basis w.r.t. to the global monodromy using the $\hat{\Gamma}$-class formalism.

Advantages of the geometric representation

act and yield $(l+1)$ second order GKZ operators in the Batyrev large radius coordinates $z_k = \prod_{i=1}^{2(l+2)} a_i^{\ell_i^{(k)}} / (a_0 \tilde{a}_0)$, $k = 1, \ldots, l+1$.

Advantages of the geometric representation as(complete intersection) Calabi-Yau manifold The GKZ system in the yields the period integrals $\underline{\Pi}$ and near the point of maximal unipotent monodromy $z_i = 0$ in a canonical integral basis w.r.t. to the global monodromy using the $\hat{\Gamma}$-class formalism. The Feynman graph is a geometric period.

Advantages of the geometric representation

act and yield $(l+1)$ second order GKZ operators in the Batyrev large radius coordinates $z_k = \prod_{i=1}^{2(l+2)} a_i^{\ell_i^{(k)}} / (a_0 \tilde{a}_0)$, $k = 1, \ldots, l+1$.

Advantages of the geometric representation as(complete intersection) Calabi-Yau manifold The GKZ system in the yields the period integrals $\underline{\Pi}$ and near the point of maximal unipotent monodromy $z_i = 0$ in a canonical integral basis w.r.t. to the global monodromy using the $\hat{\Gamma}$-class formalism. The Feynman graph is a geometric period. E.g. for the banana graph it is the highest logarithmic one inducing the Seidel Thomas twist at the nearest conifold, when the momenta go on-shell.

Advantages of the geometric representation

act and yield $(l+1)$ second order GKZ operators in the Batyrev large radius coordinates $z_k = \prod_{i=1}^{2(l+2)} a_i^{\ell_i^{(k)}} / (a_0 \tilde{a}_0)$, $k = 1, \ldots, l+1$.

Advantages of the geometric representation as(complete intersection) Calabi-Yau manifold The GKZ system in the yields the period integrals $\underline{\Pi}$ and near the point of maximal unipotent monodromy $z_i = 0$ in a canonical integral basis w.r.t. to the global monodromy using the $\hat{\Gamma}$-class formalism. The Feynman graph is a geometric period. E.g. for the banana graph it is the highest logarithmic one inducing the Seidel Thomas twist at the nearest conifold, when the momenta go on-shell.

Once the analytic continuation of $\underline{\Pi}$ to the critical divisors in the discriminate locus is known they can be calculated to very high precision everywhere in the physical parameter space in extremely short time.