DT invariants of local CY 3-folds from Galois coverings of BPS quivers

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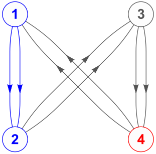
Towards BPS spectra from 5d BPS quivers?

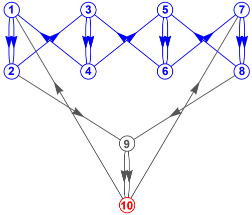
5d BPS quivers can be quite involved. Direct computation of spectra not practical.

For instance, the 5d SCFT that flows to 5d $SU(p)_q$ arises at the $Y^{p,q}$ singularity:

"5d $SU(2)_0$":



"5d $SU(5)_0$":



- For 4d $\mathcal{N} = 2$ SQFTs with finite chambers: mutation method. [Alim *et al.*, 2011]
- 5d BPS quivers never have finite chambers – infinite spectrum (KK towers).
- Much recent progress was made using **discrete symmetries** and collimation chambers. [Bonelli, Del Monte, Tanzini, 2020; Longhi, 2021; Del Monte, Longhi, 2021; Bridgeland, Del Monte, Giovenzana, 2024]
- Building on this body of work, we find new relations between 5d BPS quivers, and between their BPS spectra, using the concept of **Galois cover** first introduced in the physics literature (for 4d BPS quivers) in [Cecotti, Del Zotto, 2015].

Galois covers of 5d BPS quivers

Galois covers of BPS quivers

Consider a quiver $\mathcal{Q} = (Q_0, Q_1)$ with $|Q_0| = M$ nodes.

Let $\mathbb{G}$ be a finite abelian group. For definiteness, we can take $\mathbb{G} = \mathbb{Z}_n$.

Assign to every arrow $a \in Q_1$ a $\mathbb{G}$ grading:

$$\deg(a) = g_{(a)} \in \mathbb{G} \ .$$

[Cecotti, Del Zotto, 2015; cite mathematicians here...]

The **Galois covering quiver $\tilde{\mathcal{Q}} = (\tilde{Q}_0, \tilde{Q}_1)$ for this $\mathbb{G}$ grading** is obtained as follows:

- For every node $(i) \in Q_0$, we have $|\mathbb{G}|$ nodes (a ‘block’ of nodes)

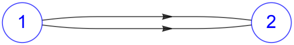
$$(i, g) \in \tilde{Q}_0 \ , \qquad \forall g \in \mathbb{G} \ .$$

- For every arrow $[a : (i) \to (j)]$, we have an arrow

$$[\tilde{a} : (i, g) \to (j, g')] \quad \text{if } g' - g = \deg(a) \ .$$

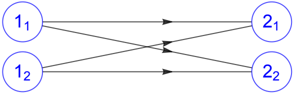
Galois covers of BPS quivers

Example 1: $\mathbb{G} = \mathbb{Z}_2$, 2-cover of the Kronecker quiver $\mathcal{Q} = K(2)$: [Gaiotto, Moore, Neitzke, 2008] [Cecotti, Del Zotto, 2015]



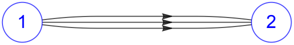
BPS quiver of pure $\mathcal{N} = 2$ SYM $SU(2)$.

$$\deg(a_0, a_1) = (0, 1)$$

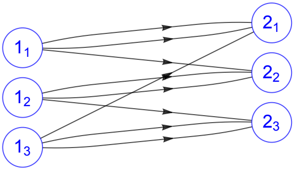


BPS quiver of $\mathcal{N} = 2$ SYM $SU(2)$, $N_f = 2$.

Example 2: $\mathbb{G} = \mathbb{Z}_3$, 3**-cover of** $\mathcal{Q} = K(3)$**.**

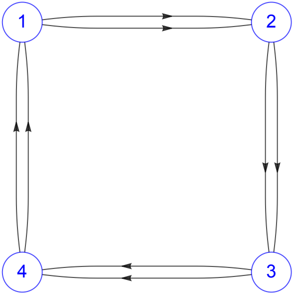


$$\deg(a_0, a_1, a_2) = (0, 0, 1)$$



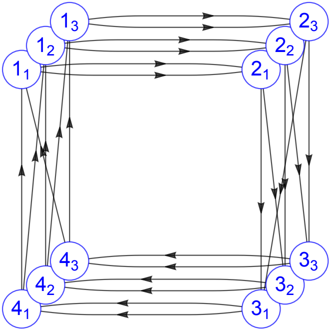
Galois covers of 5d BPS quivers

Example 3: $\mathbb{Z}_n$ covers of 5d $SU(2)_0$ gives the 5d $SU(2n)_0$ theory ($\mathbf{X} = C(Y^{2n,0})$):



$$\deg(a_0, a_1, b_0, b_1, c_0, c_1, d_0, d_1)$$

$$= (0, 0, 0, 1, 0, 0, 0, 1)$$

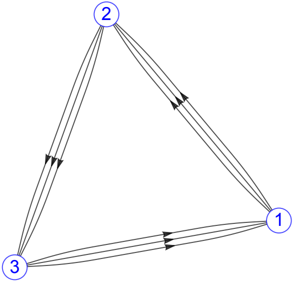


3-cover: 5d $SU(6)_0$.

Note: the superpotential is lifted as well, so this is a genuine Galois covering.

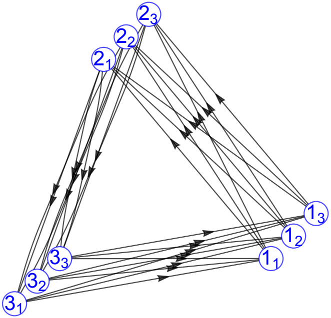
Galois covers of 5d BPS quivers

Example 4: $\mathbb{Z}_3$ covers of local $\mathbb{P}^2$ (E_0 SCFT) gives local $dP_6 = \mathrm{Bl}_6(\mathbb{P}^2)$:



$$\deg(a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2)$$

$$= (0, 1, 2, 0, 1, 2, 0, 1, 2)$$



3-cover: 5d E_6 ($SU(2)_0$, $N_f = 5$.

Galois covering functors

The inverse of the Galois cover is the map:

$$F \ : \ \tilde{\mathcal{Q}} \longrightarrow \mathcal{Q} \cong \tilde{\mathcal{Q}}/\mathbb{G} \ .$$

Roughly, it is the gauging of a discrete symmetry $\mathbb{G}$ of the quiver $\tilde{\mathcal{Q}}$. This discrete symmetry permutes nodes and arrows and leaves the superpotential invariant.

There exists induced actions on quiver representations:

$$F_\lambda : \tilde{\mathcal{Q}}\text{−rep} \longrightarrow \mathcal{Q}\text{−rep} \ , \qquad\qquad F^\lambda : \mathcal{Q}\text{−rep} \longrightarrow \tilde{\mathcal{Q}}\text{−rep} \ .$$

In particular, the push-down functor relates the quiver ranks

$$\gamma = \sum_i N_i \gamma_{(i)} \ , \qquad\qquad \tilde{\gamma} = \sum_i \sum_g N_{i,g} \tilde{\gamma}_{(i,g)} \ .$$

by summing the ranks of a ‘block’ in the $\tilde{\mathcal{Q}}$ quiver:

$$F_\lambda(\tilde{\gamma}) = \gamma \ , \qquad \text{with} \quad N_i = \sum_g N_{i,g} \ .$$

$\mathbb{G}$-action on the Higgs branch and fixed loci

Given the $\mathbb{G}$-grading of the quiver $\mathcal{Q}$, we can define a $T = U(1)$ action on the Higgs branch of the SQM at fixed quiver rank,

$$\mathcal{M}_\gamma^{\mathcal{Q}} \cong \{X \,|\partial_X W = 0\}^{ss}/\mathrm{GL}_\gamma \ , \qquad T \ : \ \mathcal{M}_\gamma \to \mathcal{M}_\gamma \ ,$$

Consider the isomorphism

$$\mathbb{G} \overset{\cong}{\to} \hat{\mathbb{G}} \equiv \mathrm{Hom}(\mathbb{G}, U(1)) \ : \ g \mapsto \hat{g} \ .$$

For definiteness, consider

$$\mathbb{G} = \mathbb{Z}_n = \{k \in \mathbb{Z} \mod n\} \ , \qquad g = k \mapsto \hat{g} = e^{\frac{2\pi i k}{n}} \ .$$

We then have the circle action on the moduli space of quiver representations

$$T \ : \ \mathcal{M}_\gamma^{\mathcal{Q}} \to \mathcal{M}_\gamma^{\mathcal{Q}}$$

induced by the grading of the arrows:

$$T \ : \ X \to \hat{g}_{(a)} X \ , \qquad \text{if } \mathfrak{R}[a] = X \text{ and } \deg(a) = g_{(a)} \ .$$

$\mathbb{G}$-action on the Higgs branch and fixed loci

The set of fixed loci of the T action is deeply related to the moduli spaces of the Galois covering quiver.

Consider all ranks $\tilde{\gamma} = (N_{i,g})$ of $\tilde{\mathcal{Q}}$ that are projected to the ranks $\gamma = (N_i)$ by the pull-down functor:

$$N_i = \sum_{g \in \mathbb{G}} N_{i,g}$$

Then the fixed locus are in 1-to-$|\mathbb{G}|$ correspondence with the moduli spaces

$$\mathcal{M}_{\tilde{\gamma}}^{\tilde{\mathcal{Q}}} \qquad \text{such that} \quad F_\lambda(\tilde{\gamma}) = \gamma \ .$$

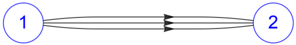
- This holds assuming some 'nice enough' properties – e.g. $\mathcal{M}_\gamma$ smooth, compact.
- The FI terms of $\tilde{\mathcal{Q}}$ are fined-tuned to preserve the $\mathbb{G}$ symmetry

$$\xi_{i,g} = \xi_i \quad \forall g \in \mathbb{G}$$

(Potential issues when we land on walls of marginal stability.)

$\mathbb{G}$-action on the Higgs branch and fixed loci

Example: 3-**cover of** $\mathcal{Q} = K(3)$, $\deg(a_0, a_1, a_2) = (0, 0, 1)$.



Take $\gamma = (N_i) = (1, 1)$. Then

$$\mathcal{M}^{\mathcal{Q}}_{\gamma=(1,1)} = \mathbb{P}^2 \cong \{[X_0, X_1, X_2]\} \ .$$

We have

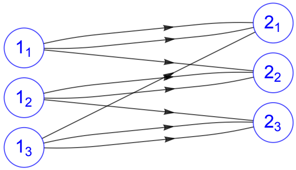
$$T \ : \ [X_0, X_1, X_2] \to [X_0, X_1, \lambda X_2]$$

Fixed loci:

$$\{[X_0, X_1, 0]\} \cong \mathbb{P}^1$$

$$\{[0, 0, X_2]\} \cong \mathbb{P}^0 = \mathrm{pt}$$

Let $\tilde{\gamma} = (N_{1_1}, N_{1_2}, N_{1_3}; N_{2_1}, N_{2_2}, N_{2_3})$.



Then for $\tilde{\gamma} = (1, 0, 0; 1, 0, 0)$ (and 2 permutations) we have

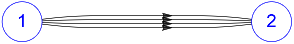
$$\mathcal{M}^{\tilde{\mathcal{Q}}}_{\tilde{\gamma}} \cong \mathbb{P}^1 \ .$$

For $\tilde{\gamma}' = (1, 0, 0; 0, 1, 0)$ (and 2 permutations) we have

$$\mathcal{M}^{\tilde{\mathcal{Q}}}_{\tilde{\gamma}'} \cong \mathbb{P}^0 \ .$$

$\mathbb{G}$-action on the Higgs branch and fixed loci

Example: 2**-cover of** $\mathcal{Q} = K(4)$, $\deg(a_0, a_1, a_2, a_3) = (0, 0, 1, 1)$.



Take $\gamma = (N_i) = (2, 1)$. Then

$$\mathcal{M}^{\mathcal{Q}}_{\gamma=(1,1)} = \mathrm{Gr}(2,4) \cong \{[X_0, X_1, X_2, X_3]\} \ .$$

We have

$$T \ : \ [X_0, X_1, X_2, X_4] \to [X_0, X_1, \lambda X_2, \lambda X_3]$$

Fixed loci:

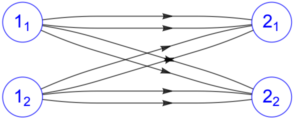
$$\{[X_0, X_1, 0, 0]\} \cong \mathbb{P}^0$$

$$\{[0, 0, X_1, X_2]\} \cong \mathbb{P}^0$$

and

$$\{\begin{pmatrix} X_0^1 & X_1^1 & 0 & 0 \\ 0 & 0 & X_2^2 & X_3^2 \end{pmatrix}\} \cong \mathbb{P}^1 \times \mathbb{P}^1$$

Let $\tilde{\gamma} = (N_{1_1}, N_{1_2}; N_{2_1}, N_{2_2})$.



Then for $\tilde{\gamma} = (2, 0; 1, 0)$ and for $\tilde{\gamma} = (2, 0; 0, 1)$ (and permutations) we have

$$\mathcal{M}^{\tilde{\mathcal{Q}}}_{\tilde{\gamma}} \cong \mathbb{P}^0 \ .$$

For $\tilde{\gamma}' = (1, 1; 1, 0)$ (and permutation) we have

$$\mathcal{M}^{\tilde{\mathcal{Q}}}_{\tilde{\gamma}'} \cong \mathbb{P}^1 \times \mathbb{P}^1 \ .$$

Relations between BPS indices

For any T-action on a variety $\mathcal{M}$ which has fixed loci $F_s \subset \mathcal{M}$, we have

$$\chi(\mathcal{M}) = \sum_s \chi(F_s) \ ,$$

e.g. by Atiyah-Bott localisation.

Given the relations just discussed, we therefore obtain relations between the BPS indices induced by the Galois covering functor:

$$\Omega_Q(\gamma) = \frac{1}{|\mathbb{G}|} \sum_{\tilde{\gamma} \mid F_\lambda(\tilde{\gamma})=\gamma} (-1)^{\mathrm{vdim}(\mathcal{M}_{\tilde{\gamma}})-\mathrm{vdim}(\mathcal{M}_\gamma)} \Omega_{\tilde{Q}}(\tilde{\gamma})$$

and for the motivic DT invariants:

$$\Omega_Q(\gamma; y) = \frac{1}{|\mathbb{G}|} \sum_{\tilde{\gamma} \mid F_\lambda(\tilde{\gamma})=\gamma} (-y)^{\mathrm{vdim}(\mathcal{M}_{\tilde{\gamma}})-\mathrm{vdim}(\mathcal{M}_\gamma)} y^{\lambda_{\tilde{\gamma}}} \Omega_{\tilde{Q}}(\tilde{\gamma}; y)$$

- These can be checked experimentally in a large number of examples.
- There are various subtleties whenever the moduli spaces are “too singular”. (Under investigation.)

Galois covers as orbifolds $\mathbf{X}/\mathbb{G}$

Consider 5d BPS quivers related to a singularity $\mathbf{X}$:

$$\mathbf{X} \quad \longleftrightarrow \quad D_{S^1}\mathcal{T}^{\rm 5d}_{\mathbf{X}} \quad \longleftrightarrow \quad \mathcal{Q}_{\mathbf{X}}$$

We make the following **conjecture:** the $\mathbb{G}$-Galois covering of $\mathcal{Q}_{\mathbf{X}}$ is always the fractional brane quiver for some $\mathbb{G}$-orbifold of $\mathbf{X}$:

$$\mathcal{Q}_{\mathbf{X}} = \tilde{\mathcal{Q}}_{\mathbf{Y}}/\mathbb{G} \qquad \Leftrightarrow \qquad \mathbf{Y} = \mathbf{X}/\mathbb{G}$$

Example: The 5d $SU(2n)_0$ BPS quiver is the n-cover of the 5d $SU(2)_0$ BPS quiver:

$$\mathbf{Y} = C(Y^{2n,0}) = \mathbf{X}/\mathbb{Z}_n = C(Y^{2,0})/\mathbb{Z}_n$$

- For toric singularities and their toric abelian orbifolds, one can prove this using brane tiling techniques. [Hanany, Herzog, Vegh, 2006]
- Not all orbifolds arise in this way. (We want a ‘block quiver structure’.)
- Proof of general conjecture an interesting challenge.