Backreaction of Fluxes on Calabi-Yau Metrics

Fabian Ruehle (Northeastern University & IAIFI)

Pollica Workshop “Calabi - Yau Manifolds”
June 02 - June 06, 2025

Based on

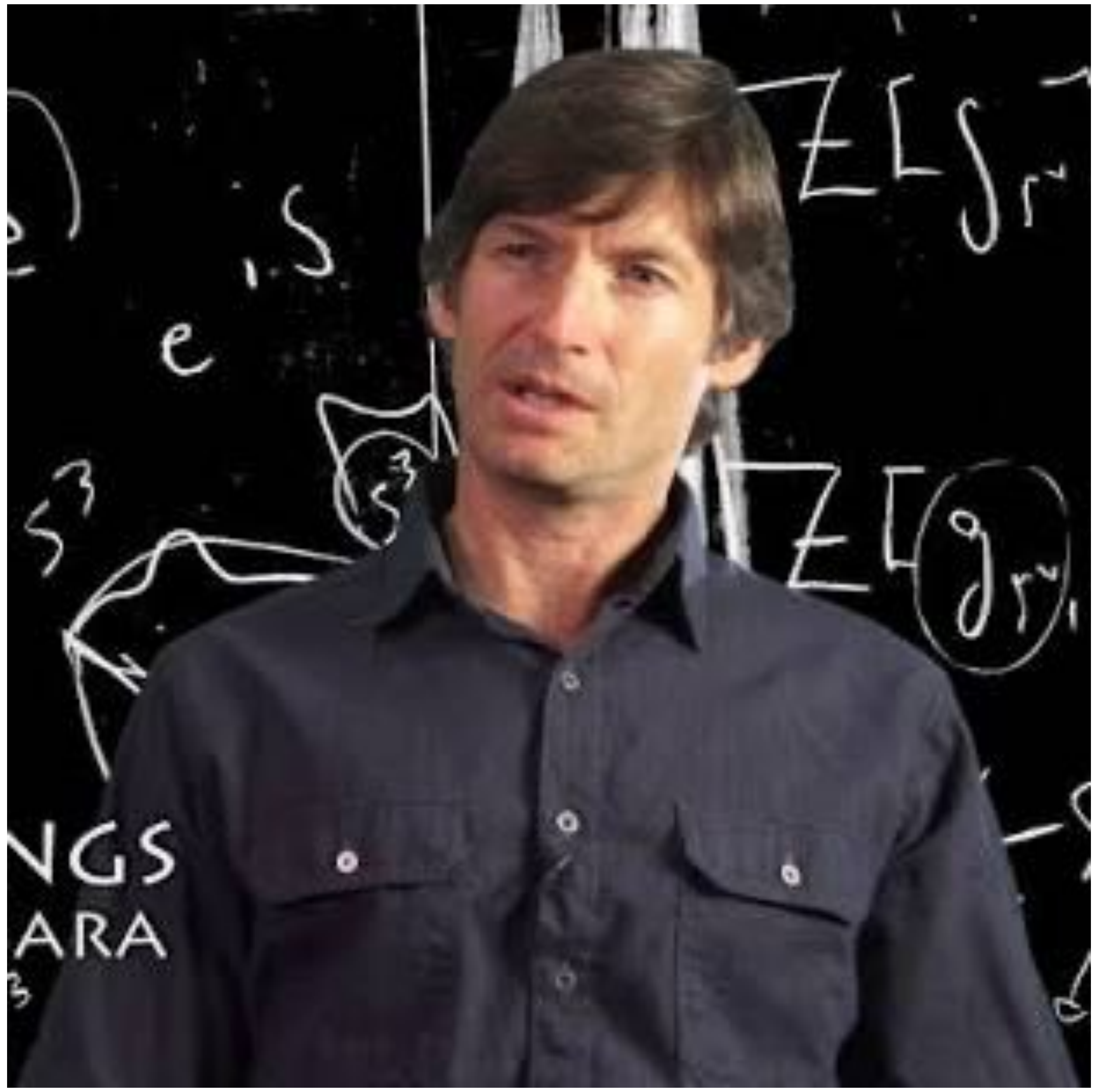
[Anderson, Gerdes, Gray, Krippendorf, Raghuram, FR: 2012.04656]

[Larfors, Lukas, Ruehle, Schneider: 2111.01436 and 2205.13408]

[S Lust, FR, S Schreyer: 25xx.xxxxx]

Outline

1. GKP Setup
2. Intro to Neural Networks
3. CY Point Sampling
4. Putting together all ingredients
5. Conclusions



1. GKP Setup

Gidding-Kachru-Polchinski Mechanism

- GKP described moduli stabilization in Type IIB Flux Compactifications
- The 10D metric takes the form

$$ds_{10}^2 = \underbrace{e^{2A(y)}}_{\text{Warp factor}} \eta_{\mu\nu} dx^\mu dx^\nu + \underbrace{e^{-2A(y)}}_{\text{Warp factor}} \underbrace{g_{ij}^{\rm CY}}_{\text{CY metric}}(y) dy^i dy^j$$

- To stabilize CS moduli, turn on non-trivial 5-form flux

$$\tilde{F}_5 = (1 + \star)\, d\alpha \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

- and non-trivial ISD 3-form flux

$$G_3 = F_3 - \tau H_3 \qquad \rightarrow \qquad \star_6 G_3 = iG_3$$

Gidding-Kachru-Polchinski Mechanism

- The 5-form flux and the warp factor are related by

$$e^{4A} = \alpha \quad \rightarrow \quad -\tilde{\nabla}^2 \left(\mathrm{e}^{-4A}\right) = \frac{G_{ijk}\,\bar{\tilde{G}}^{ijk}}{12\mathrm{Im}\tau} + 2\kappa_{10}^2 T_3 \tilde{\rho}_3^{\mathrm{loc}}$$

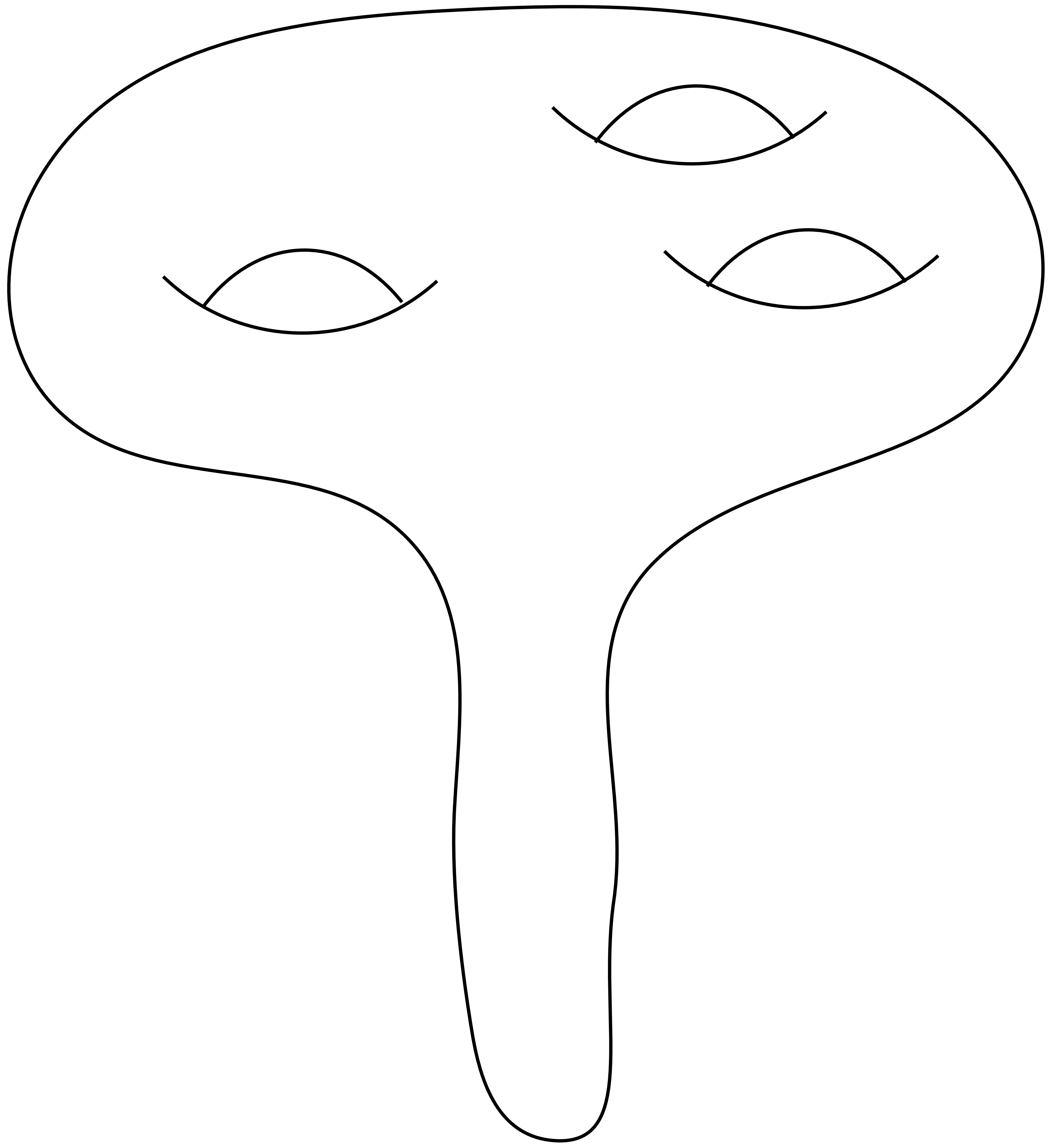
Here, $\tilde{\rho}_3^{\mathrm{loc}} = \sum_i N_i \frac{1}{\sqrt{\tilde{g}}}\delta^6(y - y_i)$ describes local D3 brane sources (e.g. O-planes)

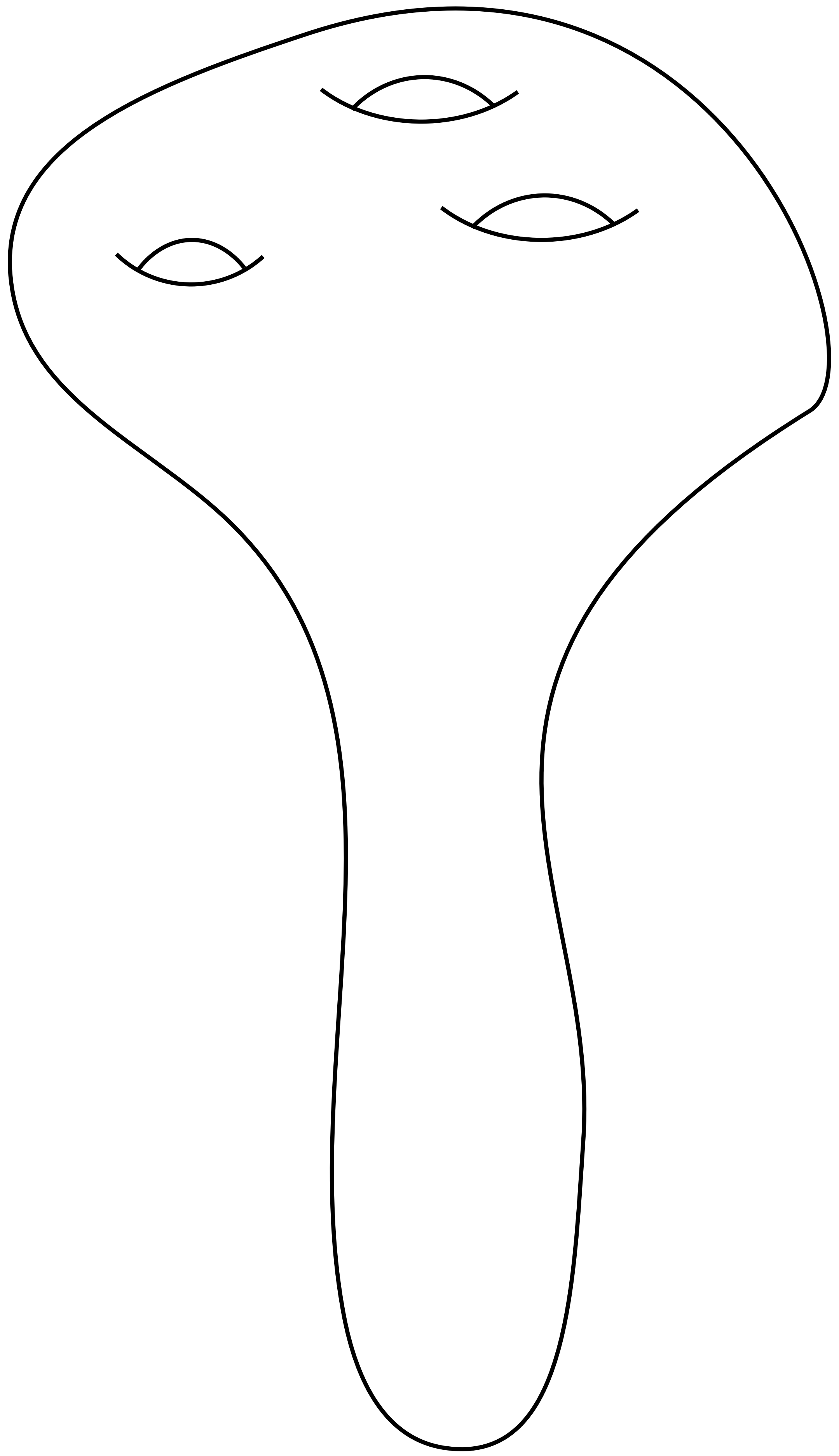
- Imposing F-flatness gives the equations

$$0 = D_\tau W = \frac{1}{\overline{\tau} - \tau}\int_X \Omega \wedge \overline{G}_3 = \Pi^T \eta \left(\mathrm{F}_3 - \overline{\tau}\mathrm{H}_3\right) \quad \Longleftrightarrow \quad \tau = \frac{\overline{\Pi}^T \eta \mathrm{F}_3}{\overline{\Pi}^T \eta \mathrm{H}_3},$$

$$0 = D_i W = \int_X D_i\Omega \wedge G_3 = D_i\Pi^T \eta \left(\mathrm{F}_3 - \tau\mathrm{H}_3\right)$$

Singular Bulk Problem and Warping





Road Map

Find g^{CY}: Find Ricci-flat metric (solve MA-equation)

Find F_3, G_3 (calculate periods, solve F-term EoM)

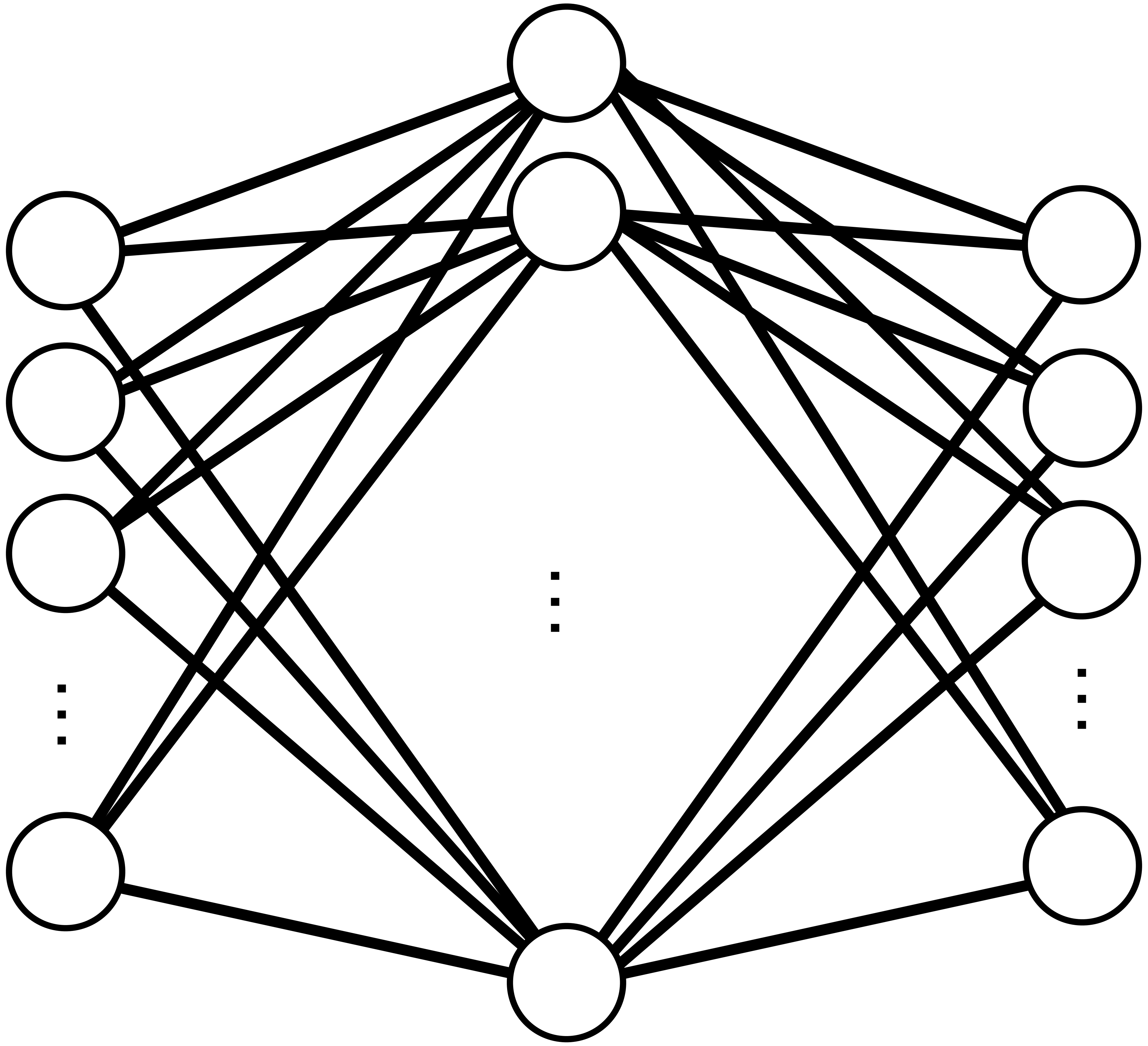
$$D_i W = \int_X D_i \Omega \wedge G_3 = D_i \Pi^T \eta \left(\mathrm{F}_3 - \tau \mathrm{H}_3\right)$$

Find (2,1)-forms that are harmonic w.r.t. g^{CY}

(solve a set of coupled 1st order PDEs that involve the CY metric)

Solve the Laplace eqn with source for the warp factor (involves the CY metric, the harmonic forms, flux quanta, point in CS moduli space):

$$-\tilde{\nabla}^2 \left(\mathrm{e}^{-4A}\right) = \frac{G_{ijk}\, \tilde{\bar{G}}^{ijk}}{12 \mathrm{Im}\tau} + 2\kappa_{10}^2 T_3 \tilde{\rho}_3^{\mathrm{loc}}$$

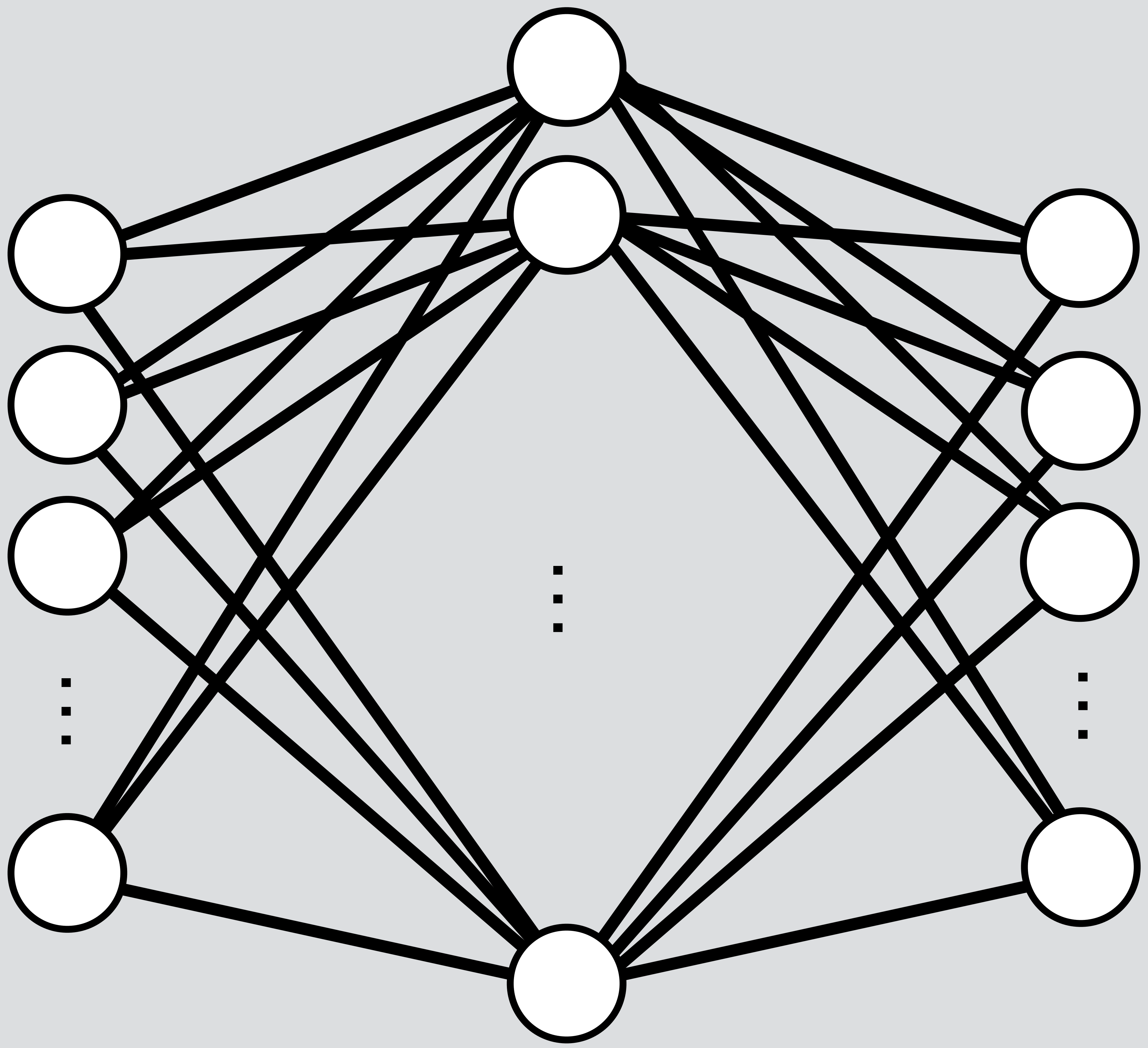


2. Intro to NNs

Neural Networks

- NNs parameterize maps $f : \mathbb{R}^{n_{\text{in}}} \to \mathbb{R}^{n_{\text{out}}}$ as compositions of affine maps $f_i : \mathbb{R}^{n_{i-1}} \to \mathbb{R}^{n_i}$ and component-wise non-linear functions $\sigma_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$
- The NN is trained to approximate the desired function by updating the parameters of the affine maps. Typically linear methods (gradient descent) are used to find a optimum in parameter space
- The Universal Approximation Theorem ensures that any function (with mild assumptions) can be written that way in the infinite parameter limit

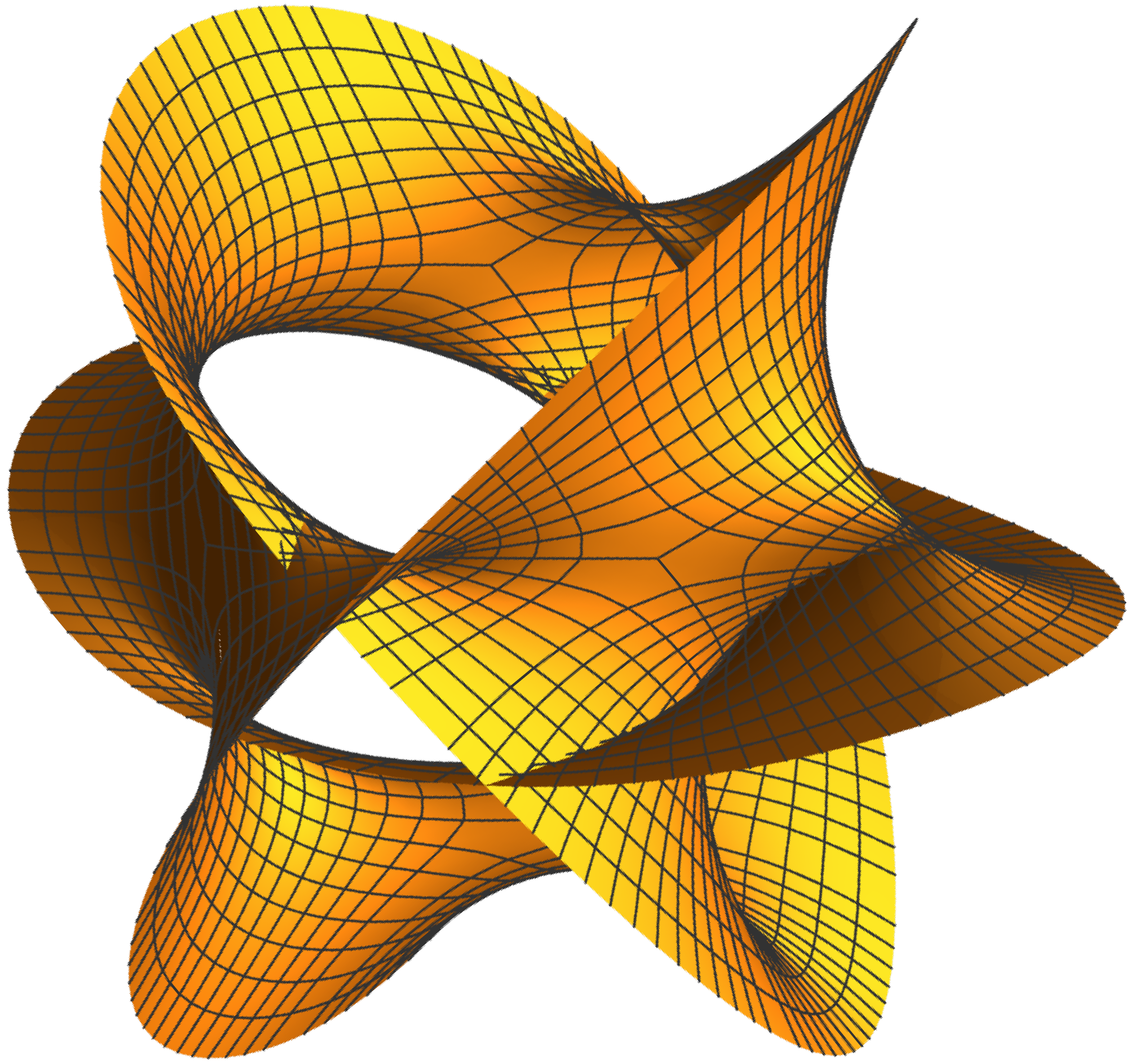
  [Cybenko `89; …; Kidger, Lyons `19]

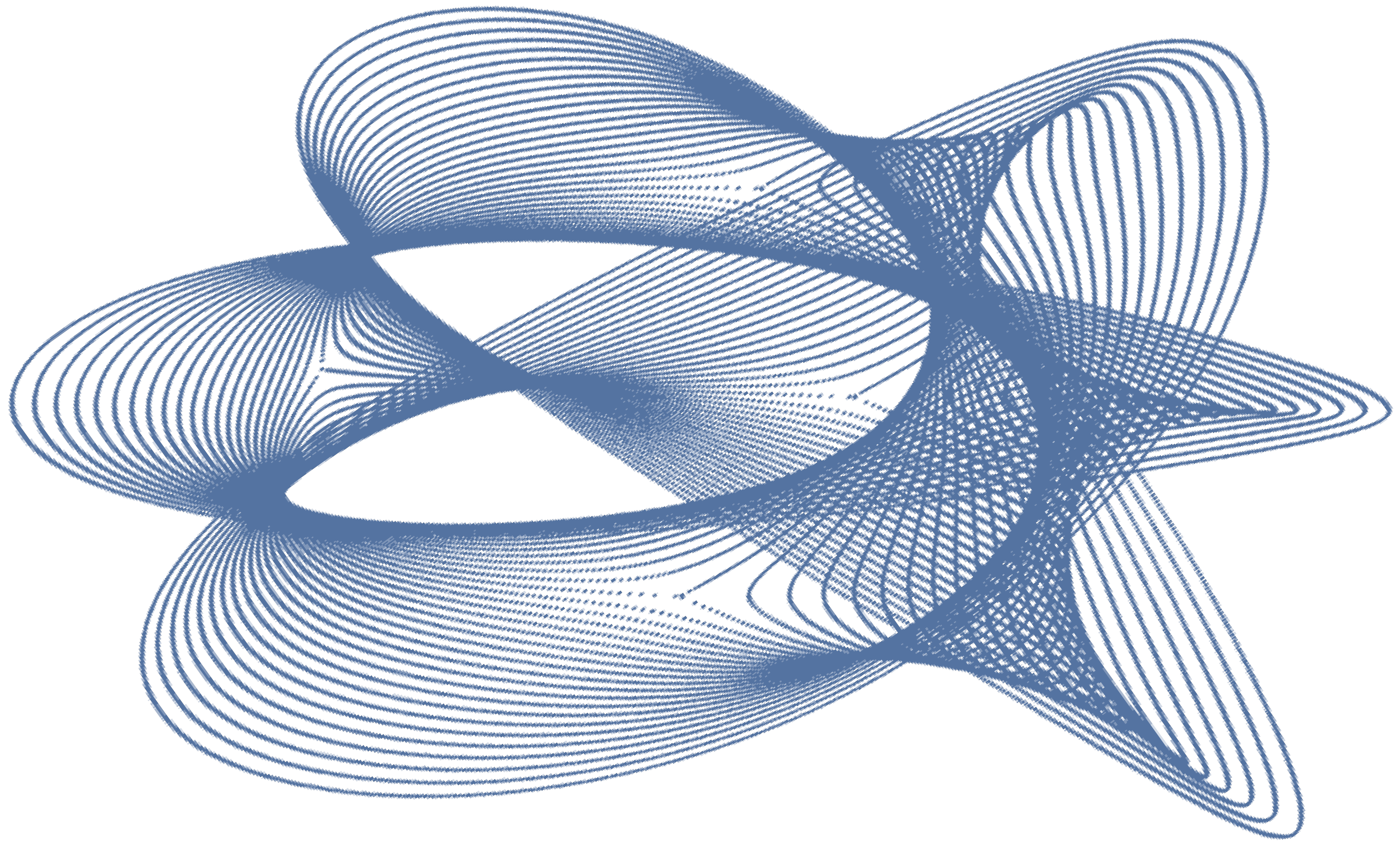


$$z_\alpha^{\text{out}} = \sigma_{\text{out}}(W_\alpha^{\beta,\text{out}} \sigma_1(W_\beta^{\gamma,\text{in}} z_\gamma^{\text{in}} + b_\beta^{\text{in}}) + b_\alpha^{\text{out}})$$

NNs for PDEs

- In the perhaps best-known applications, NNs are trained in a supervised fashion: They get a bunch of images together with labels that describe the images, and they learn from this combination to distinguish cats and dogs
- For PDE's we proposed three methods to use NN:
    - Neural PDE: You obtain an approximate solution on a coarse set of points using classical techniques (finite elements, Donaldson's algorithm) and then train NNs to regress/interpolate these solutions
    - Physics-Informed NN (PINN): Let the NN be the function that enters in the PDE. Take derivatives of the NN, insert it in the PDE, and update its weights until the PDE is satisfied
    - Neural operators: Like above, but in addition to the independent variables w.r.t. which derivatives are taken, also supply parameters (like CY moduli) which the NN learns to regress on





3. CY Point Sampling

Finding points on the CY

- We want a uniform sample of points on the CY (w.r.t. some known metric)
- To obtain points on such CYs, a (too naive) approach would be to take D-K random points in the toric ambient space and solve the equations for the remaining K
- This is too naive, since we will not know how the points will be distributed after restriction to the Calabi-Yau
- Instead, we use a theorem by Shiffman and Zelditch which allows us to find points whose distribution follows a measure that can be constructed from an FS metric

[Shiffman,Zelditch `98]

Sampling points

- Find a basis of Kähler cone generators $J^{(\alpha)} = \sum_i c_i^{(\alpha)} D_i\,,\ \alpha = 1, \ldots, h^{1,1}(\mathcal{A})$
- Calculate a basis of sections $s_j^{(\alpha)} = \prod_{i=1}^{n} x_i^{\langle v_i, w_j \rangle + c_i^{\alpha}}$ in terms of the toric coordinates x_i
- Define maps Φ^{α} from the toric coordinates into $\mathbb{P}H^0(J^{(\alpha)}, \mathcal{A})$(nef divisors in toric varieties are base point free)

$$\Phi_\alpha : \quad [x_0 : x_1 : \ldots] \quad \rightarrow \quad [s_0^{(\alpha)} : s_1^{(\alpha)} : \ldots : s_{r_\alpha}^{(\alpha)}]$$

- One can endow $\mathbb{P}H^0(J^{(\alpha)}, \mathcal{A})$ with an FS metric obtained from the Kähler potential $K^{(\alpha)} = \log \sum_{i,j=0}^{r_\alpha} \bar{s}_{\bar{i}}^{(\alpha)} H^{\bar{i}j} s_j^{(\alpha)}$, H hermitian (we take $H = \mathbb{1}$)

Sampling points - Random points with known distribution

- Construct random sections $S_j^{(\alpha)} = \sum_{j=0}^{r_\alpha} a_j^{(\alpha)} s_j^\alpha$ with $a_j^{(\alpha)} \sim \mathcal{N}(0,1)$
- By a theorem due to Shiffman and Zelditch, the zeros of such sections are distributed w.r.t. the Fubini-Study measure on $\mathbb{P}H^0(J^{(\alpha)}, \mathcal{A})$
- Restrict points to the CY by intersecting D-K random sections with the K equations defining the CY
- The points obtained this way are then distributed w.r.t. the measure

$$dA = \bigwedge_{\alpha=1}^{D-K} \Phi_\alpha^*(J^\alpha)$$

[Shiffman,Zelditch `98; Douglas,Karp,Lukic,Reinbacher `06; Braun,Brelidze,Douglas,Ovrut `08]

- An improved point sampling that leads to a better distribution of points was proposed by Keller and Lukic. Instead of fixing the matrix H , one constructs a family of metrics such that the new points are distributed in previously under-sampled regions [Keller,Lukic `09]

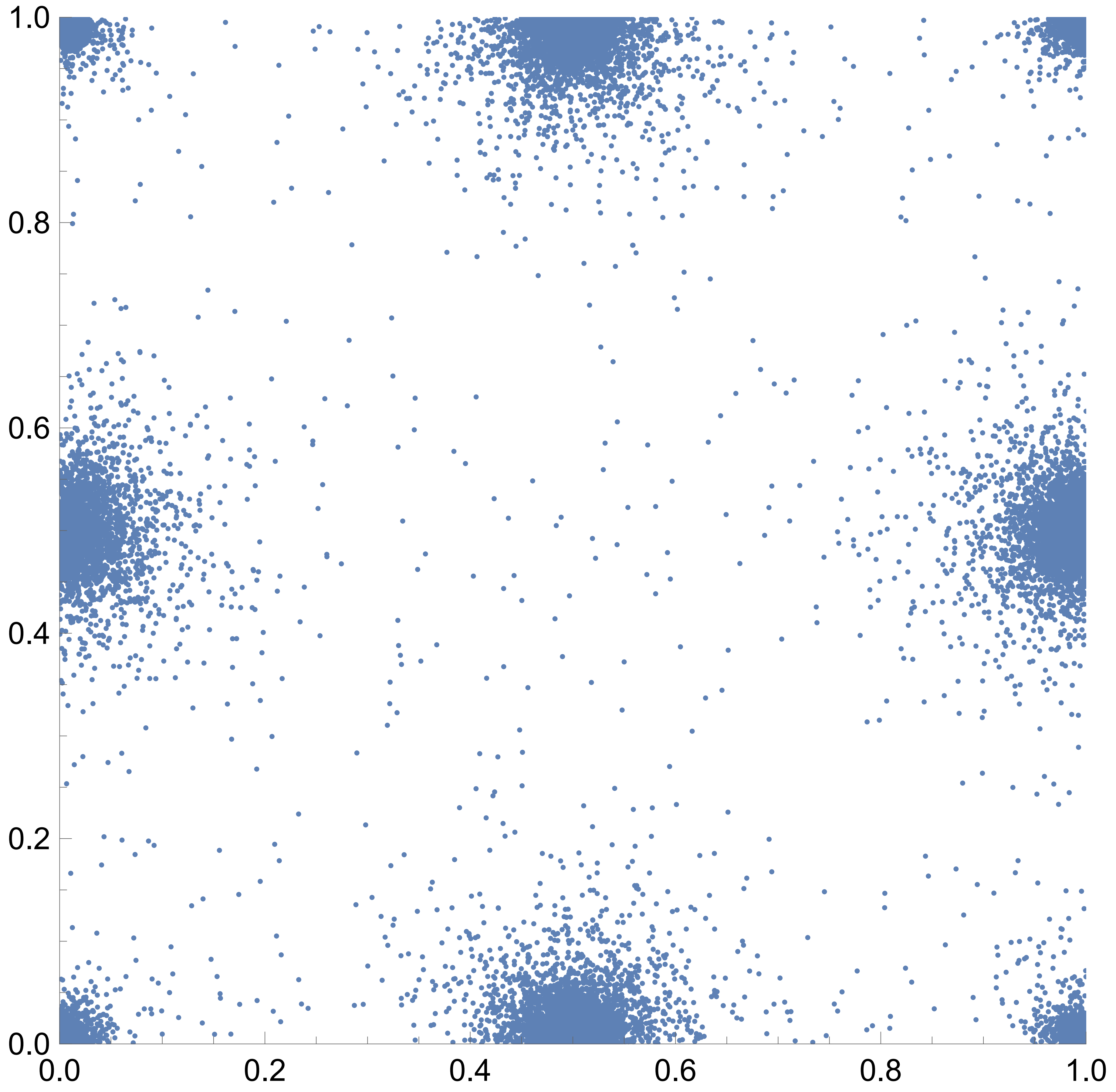
Sampling points - Random points with known distribution

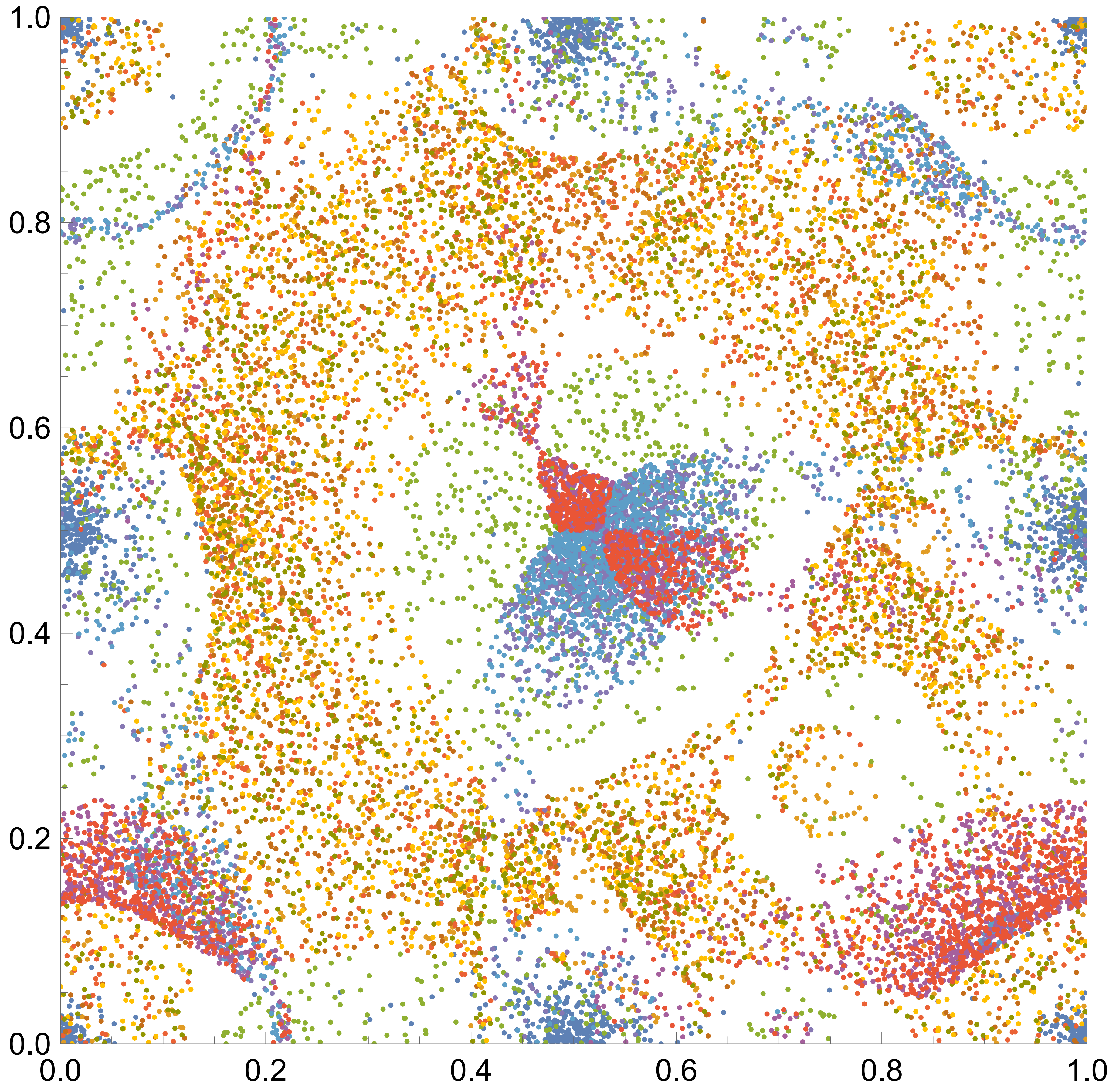
- While we know the measure now, we introduced two complications
  - The CY is written in terms of x_a and not in terms of $s_j^{(i)}$
  - There are more sections than toric coordinates $\Rightarrow$ relations among sections
- This requires some computational algebraic geometry (find the primary decomposition of the defining ideal of the non-CICY toric variety or identifying a primitive basis for a certain kernel of a matrix over the integers)
- To illustrate the improved point sampling, we sample from the cubic in $\mathbb{P}^2$

  $$z_2^2 z_0 - 4z_1^3 + g_2 z_1 z_0^2 + g_3 z_0^3 = 0 \quad \to \quad z_0 = 1\,, \quad z_1 = x = \wp(w), \quad z_2 = y = \wp'(w)$$

  The Eisenstein series have been chosen such that $\tau = i$
- We then map the points in $\mathbb{P}^2$ into the fundamental domain by numerically inverting the Weierstrass $\wp$ function

Sampling points - Random points with known distribution





Cubic, 20k points, sampled with 1 and 11 regions

Find g^{CY}: Find Ricci-flat metric (solve MA-equation)

Find F_3, G_3 (calculate periods, solve F-term EoM)

$$D_i W = \int_X D_i\Omega \wedge G_3 = D_i \Pi^T \eta\,(\mathrm{F}_3 - \tau \mathrm{H}_3)$$

Find (2,1)-forms that are harmonic w.r.t. g^{CY}

(solve a set of coupled 1st order PDEs that involve the CY metric)

Solve the Laplace eqn with source for the warp factor (involves the CY metric, the harmonic forms, flux quanta, point in CS moduli space):

$$-\tilde{\nabla}^2\left(\mathrm{e}^{-4A}\right) = \frac{G_{ijk}\,\tilde{\bar{G}}^{ijk}}{12\mathrm{Im}\tau} + 2\kappa_{10}^2 T_3 \tilde{\rho}_3^{\mathrm{loc}}$$

4. Putting together all ingredients

1.) Approximating the CY Metric

- We want to think of the metric as a map

$$\mathcal{M}_{\mathrm{K}} \times \mathcal{M}_{\mathrm{CS}} \times X \quad \xrightarrow{f} \quad \{\text{Hermitian } d \times d \text{ matrix}\}$$

- Several conceivable ways to approximate the metric (most boost from FS metric)

Name	Ansatz
Free	$g_{\mathrm{pr}} = g_{\mathrm{NN}}$
Additive	$g_{\mathrm{pr}} = g_{\mathrm{FS}} + g_{\mathrm{NN}}$
Multiplicative, element-wise	$g_{\mathrm{pr}} = g_{\mathrm{FS}} + g_{\mathrm{FS}} \odot g_{\mathrm{NN}}$
Multiplicative, matrix	$g_{\mathrm{pr}} = g_{\mathrm{FS}} + g_{\mathrm{FS}} \cdot g_{\mathrm{NN}}$
ϕ-model	$g_{\mathrm{pr}} = g_{\mathrm{FS}} + \partial\bar{\partial}\phi$

- For some approximations, we need to impose conditions on top of the MA equation, e.g. that the metric is Kahler, that the Kahler class is not changed, and/or that it is well-defined on patch transitions

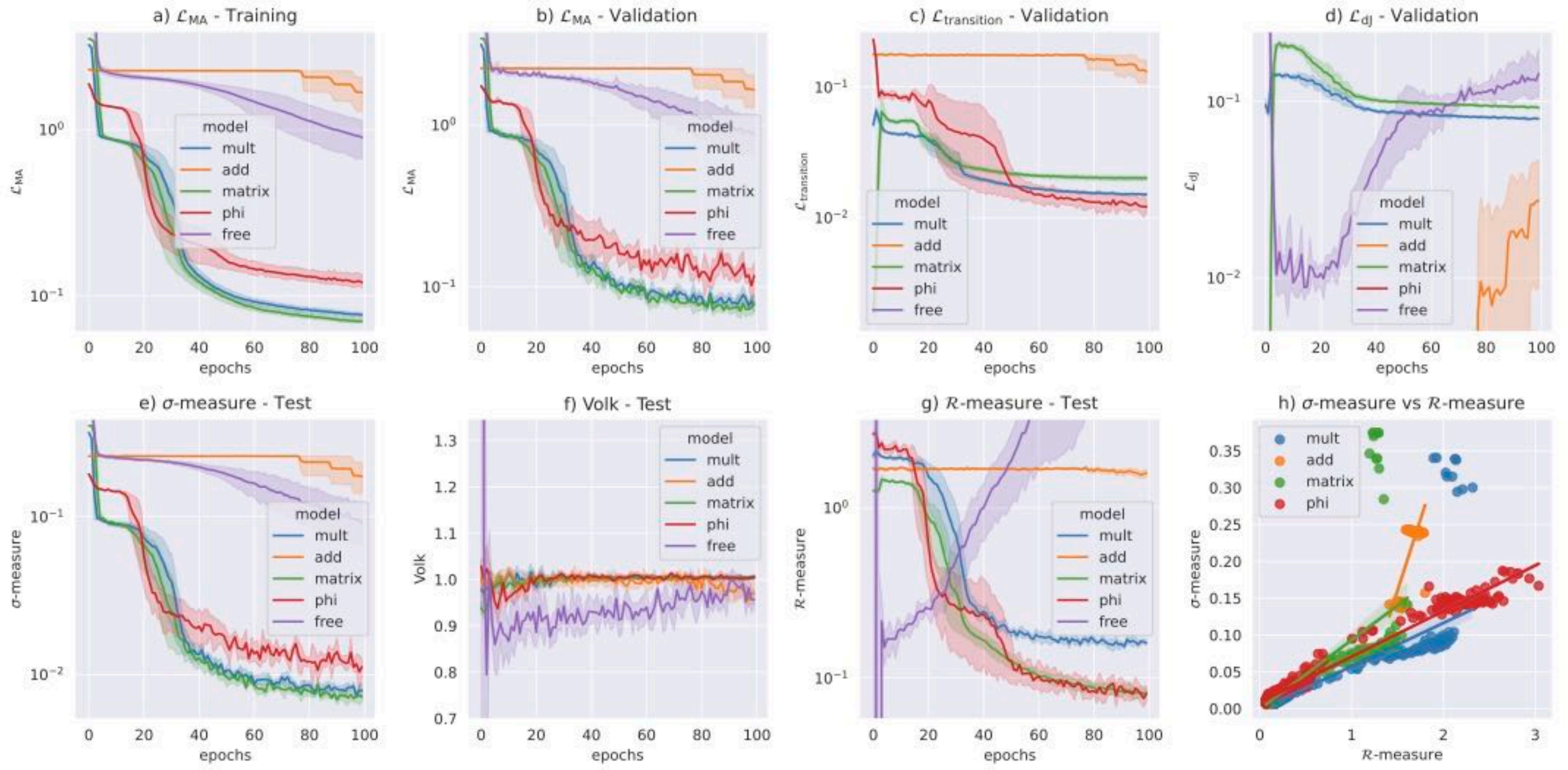
1.) Approximating the CY Metric

- Total loss $\mathcal{L} = \alpha_1 \mathcal{L}_{\text{MA}} + \alpha_2 \mathcal{L}_{\text{dJ}} + \alpha_3 \mathcal{L}_{\text{transition}} + \alpha_4 \mathcal{L}_{\text{Ricci}} + \alpha_5 \mathcal{L}_{\text{Kclass}}$
- Monge-Ampere loss $\mathcal{L}_{\text{MA}} = \left|\left| 1 - \frac{1}{\kappa} \frac{\det g_{\text{pr}}}{\Omega \wedge \bar{\Omega}} \right|\right|_n$
- Kähler loss $\mathcal{L}_{\text{dJ}} = \sum_{ijk} ||\text{Re}\, c_{ijk}||_n + ||\text{Im}\, c_{ijk}||_n \,, \qquad c_{ijk} = \partial_k g_{i\bar{j}} - \partial_i g_{k\bar{j}}$
- Transition loss $\mathcal{L}_{\text{transition}} = \frac{1}{d} \sum_{\mathcal{U},\mathcal{V}} \left|\left| g_{\text{pr}}^{\mathcal{V}} - T_{\mathcal{U}\mathcal{V}} \cdot g_{\text{pr}}^{\mathcal{U}} \cdot (T_{\mathcal{U}\mathcal{V}})^{\dagger} \right|\right|_n$
- Ricci loss (redundant since equivalent to MA loss) $\mathcal{L}_{\text{Ricci}} = ||R||_n = \left|\left| \partial\bar{\partial} \ln \det g_{\text{pr}} \right|\right|_n$
- Kähler class loss $\mathcal{L}_{\text{Kclass}} = \frac{1}{h^{1,1}(X)} \sum_{\alpha=1}^{h^{1,1}(X)} \left|\left| \mu_t(\mathcal{O}_X(e_\alpha)) - \int_X J_{\text{pr}}^2 \wedge F_{\text{FS},\alpha} \right|\right|_n$

1.) Approximating the CY Metric

- For CYs, the ϕ NN usually works best
- By construction, the NN is Kähler and well-defined at transitions
- However, we still need to ensure that ϕ is a section of $\mathcal{O}$, ie, has weight 0
- Instead of inputting the CY coordinates, we can input ratios of sections, which automatically ensure the correct transformation of ϕ . This is known as feature engineering [Berglund,Butbaia,Hubsch,Jejjala,Mayorga Pena,Mishra,Tan `22]
- We will use the simplest example (one-parameter quintic) since the project already has a lot of moving pieces. Want to generalize to CYs with orientifolds after

1.) Approximating the CY Metric



Fermat Quintic, 50k points, trained for a few minutes

[Larfors,Lukas,FR,Schneider `22]

2.) Finding the fluxes

- The solution of the Picard-Fuchs System for the quintic is well-known

[Candelas,De La Ossa,Green,Parkes `91]

- We expand the periods up to order 50 and search for solutions to the F-flatness constraints for up to 2 flux quanta

$$0 = D_\tau W = \frac{1}{\bar{\tau} - \tau} \int_X \Omega \wedge \overline{G}_3 = \Pi^T \eta \left(\mathrm{F}_3 - \bar{\tau}\mathrm{H}_3\right) \quad \Longleftrightarrow \quad \tau = \frac{\overline{\Pi}^T \eta \mathrm{F}_3}{\overline{\Pi}^T \eta \mathrm{H}_3} \, ,$$

$$0 = D_i W = \int_X D_i \Omega \wedge G_3 = D_i \Pi^T \eta \left(\mathrm{F}_3 - \tau \mathrm{H}_3\right)$$

- We find a solution close to the conifold ($\psi = 1$)

$$F_3 = (3, 2, -3, -1) \, , \;\; H_3 = (1, 2, 1, 0) \quad \rightarrow \quad N_{\text{flux}} = 8 \, , \;\; \psi = 1.035 - 0.017i \, , \;\; \tau = -0.333 + 3.598i$$

2.) Finding the fluxes

- Close to the conifold singularity, numerical precision suffers
- To estimate the impact, we compute the Euler number
  - The Euler number comes out only within 15% of the true value, both when using the CY metric and the pullback of the FS metric (so the latter does not involve a NN)
  - In contrast, away from the conifold, the Euler number is correct within the 1% range
- Since the error clearly comes from the numerical integration, we checked with improved point sampling, which reduces the error from 15% to 2%!

3.) Approximating the Harmonic (2,1)-forms

- A procedure to obtain a harmonic basis of (2,1)-forms for the FS metric has been worked out previously: [Candelas,De La Ossa `91]
  - The (2,1)-forms can be constructed from variations of the (3,0)-forms

$$\chi_{\mu\nu\bar{\rho},I} = \frac{1}{2}\Omega_{\mu\nu\sigma}\partial_{\bar{\rho}}m^{\sigma}{}_{a}q_I^a$$

(2,1)-forms, $I = 1, \ldots, h^{21}$

(3,0)-form

Extrinsic curvature

CS deformation of defining polynomials

  - The extrinsic curvature can be computed from pullbacks and metric derivatives

$$\partial_{\bar{\mu}}m_a^{\nu} = X_{\bar{\mu}}{}^{\nu}{}_a q_I^a = -H_{a\bar{b}}\,\hat{g}^{\bar{\nu}\nu}\,\frac{\partial\bar{z}^{\bar{A}}}{\partial\bar{x}^{\bar{\nu}}}\frac{\partial\bar{z}^{\bar{B}}}{\partial\bar{x}^{\bar{\mu}}}\left(\frac{\partial^2\bar{p}^{\bar{b}}}{\partial\bar{z}^{\bar{A}}\partial\bar{z}^{\bar{B}}} - \hat{\Gamma}^{\bar{C}}_{\bar{A}\bar{B}}\frac{\partial\bar{p}^{\bar{b}}}{\partial\bar{z}^{\bar{C}}}\right)q_I^a\,, \qquad H^{a\bar{b}} = \hat{g}^{A\bar{B}}\frac{\partial p^a}{\partial z^A}\frac{\partial\bar{p}^{\bar{b}}}{\partial\bar{z}^{\bar{B}}}\,,$$

3.) Approximating the Harmonic (2,1)-forms

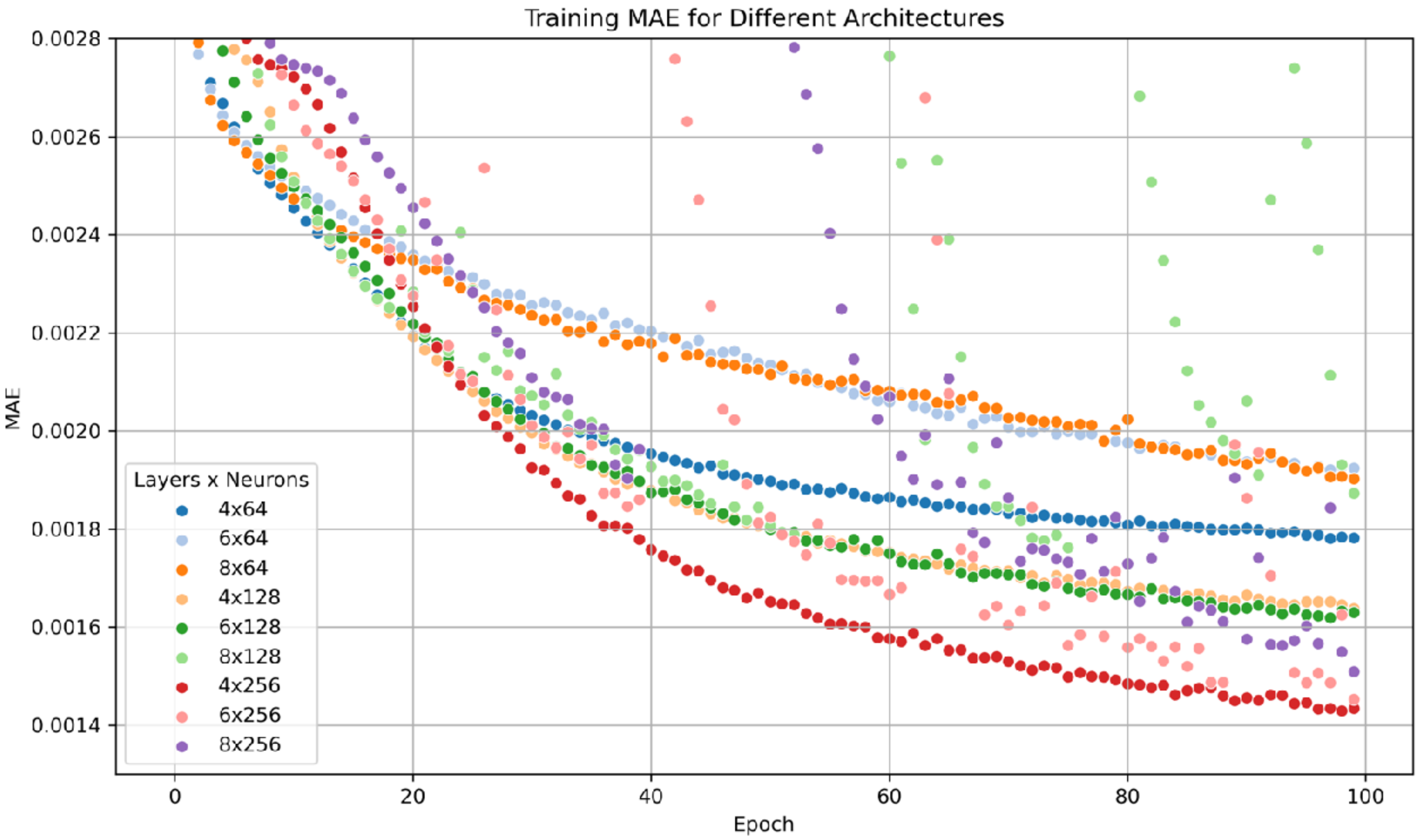
- We then learn a correction $\Delta m^{\mu}{}_{a}$ s.t. $\widetilde{m}^{\mu}{}_{a} = m^{\mu}{}_{a} + \Delta m^{\mu}{}_{a}$ is harmonic wrt the CY metric
- We impose $\bar{\partial}\chi_I = \partial * \chi_I = 0$:

$$\bar{\partial}\chi_I = 0 \longleftrightarrow \partial_{[\bar{m}u} X_{\bar{n}u]}{}^{\rho}{}_{I} = 0\,, \quad \partial * \chi_I = 0 \ \longleftrightarrow \ \xi_{[\alpha\beta\nu]\bar{\nu},I} = \frac{1}{3}\big(\xi_{\alpha\beta\nu\bar{\nu},I} + \xi_{\nu\alpha\beta\bar{\nu},I} + \xi_{\beta\nu\alpha\bar{\nu},I}\big) = 0$$

$$\xi_{\alpha\beta\nu\bar{\nu},I} = \partial_{\nu}\left(\Omega_{\alpha\beta\mu} g^{\bar{\mu}\mu} g_{\sigma\bar{\nu}} X_{\bar{\mu}}{}^{\sigma}{}_{a} q^{a}_{I}\right)$$

- Thus the loss of the second NN is determined from derivatives of the metric NN

3.) Approximating the Harmonic (2,1)-forms



Close-to-singular Quintic, 100k points, standard point sampling

4.) Approximating the Warp Factor

- This is still work in progress
- We will set up a NN that learns the solution A to the Laplace Equation

$$-\tilde{\nabla}^2\left(\mathrm{e}^{-4A}\right)=\frac{G_{ijk}\,\tilde{\bar{G}}^{ijk}}{12\mathrm{Im}\tau}+2\kappa_{10}^2T_3\tilde{\rho}_3^{\mathrm{loc}}$$

- For now, we will have to put in the source term by hand (since the one-parameter quintic does not have an orientifold involution), or accept that the solution is only correct within the order of the source term.

Outlook

- Want to study more realistic setups that allow orientifold involution
- For CICYs in projective spaces, orientifolds have been classified [Carta,Moritz,Westphal `20]
- For numerical control over the metric, want small h^{11}
- For moduli stabilization (solve the PF equation, find harmonic forms), want small h^{21}
- The manifolds with the smallest h^{11} and h^{21} are larger codimension, which means that their mirror is a CI in toric varieties

Conclusions

1

GKP

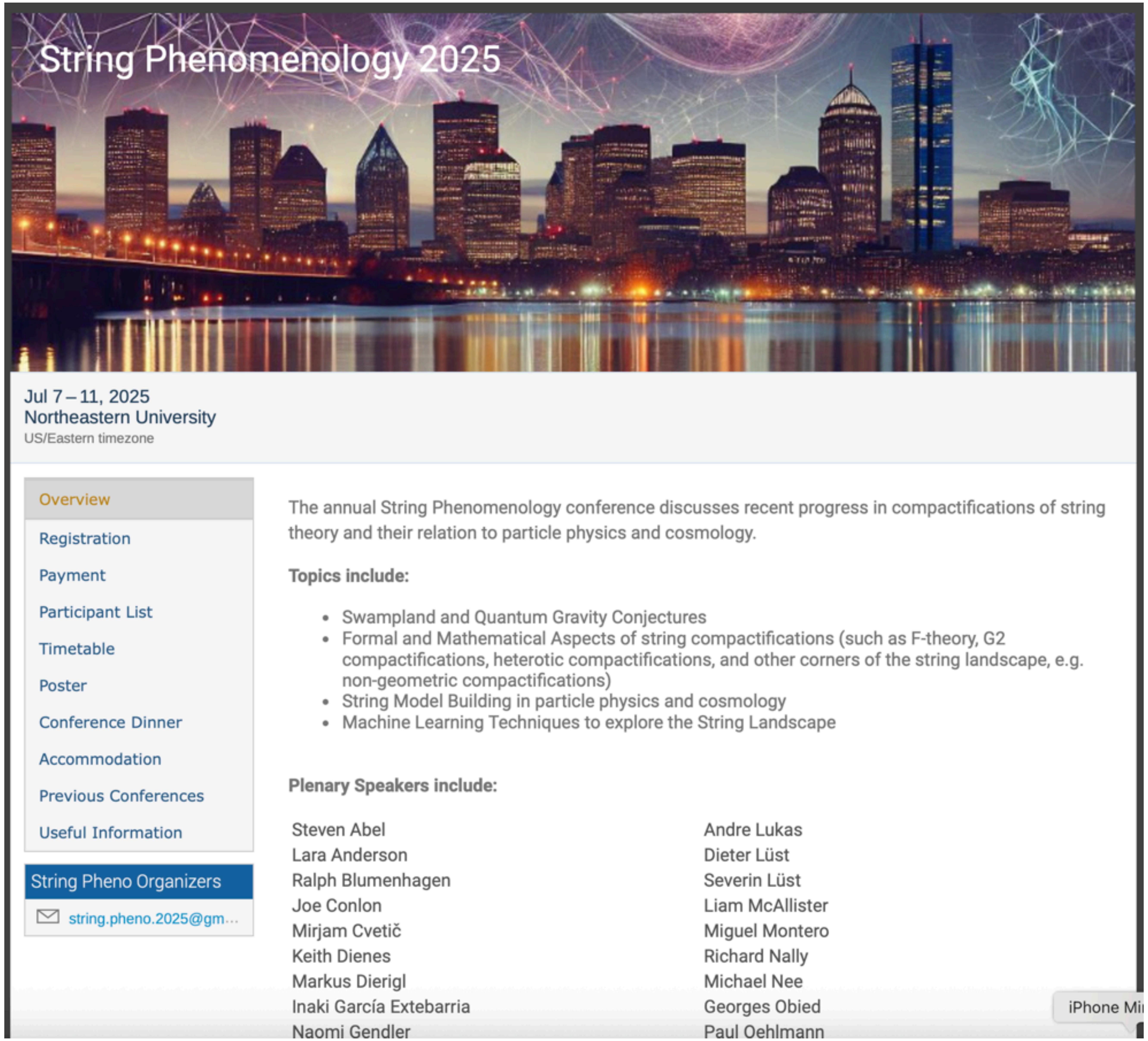
- Approximate CY metric with a NN
- Using the CY metric, approximate correction to harmonic (2,1)-forms by a NN
- Using the CY metric and harmonic forms, approximate solution to the warp factor with a NN
- Stabilize near conifold $\Rightarrow$ CY nearly singular $\Rightarrow$ improve point sampling for better numerics

2

Results

- CY metric approximation gains multiple orders of magnitude
- Accuracy to harmonic (2,1)-form approximation shows improvement by 60%
- Result for warp factor pending $\Rightarrow$ stay tuned
- Looking for good, more realistic models


String Pheno 2025
Invited Speakers
Abel
Anderson
Blumenhagen
Conlon
Cvetič
Dienes
Dierigl
García Extebarria
Gendler
Goodsell
Grimm
Hebecker
Heckman
Jejjala
H Kim
M Kim
Knapp
Krippendorf
Larfors
Lee
Lukas
D Lüst
S Lüst
McAllister
Montero
Nally
Nee
Obied
Oehlmann
Ooguri
Quevedo
Shiu
Silverstein*
Valenzuela
van de Heisteeg
Wang
Weigand
Westphal
Yonekura
https://indico.cern.ch/e/sp2025
Northeastern University, Boston, MA, USA
July 07 – 11, 2025
Rafael Álvarez-García, Jim Halverson, Thomas Harvey, Fabian Ruehle, Washington Taylor, Cumrun Vafa


String Phenomenology 2025
Jul 7 – 11, 2025
Northeastern University
US/Eastern timezone
Overview
Registration
Payment
Participant List
Timetable
Poster
Conference Dinner
Accommodation
Previous Conferences
Useful Information
String Pheno Organizers
string.pheno.2025@gm...
The annual String Phenomenology conference discusses recent progress in compactifications of string theory and their relation to particle physics and cosmology.
Topics include:
Swampland and Quantum Gravity Conjectures
Formal and Mathematical Aspects of string compactifications (such as F-theory, G2 compactifications, heterotic compactifications, and other corners of the string landscape, e.g. non-geometric compactifications)
String Model Building in particle physics and cosmology
Machine Learning Techniques to explore the String Landscape
Plenary Speakers include:
Steven Abel
Lara Anderson
Ralph Blumenhagen
Joe Conlon
Mirjam Cvetič
Keith Dienes
Markus Dierigl
Inaki García Extebarria
Naomi Gendler
Andre Lukas
Dieter Lüst
Severin Lüst
Liam McAllister
Miguel Montero
Richard Nally
Michael Nee
Georges Obied
Paul Oehlmann

Conclusions

1

GKP

- Approximate CY metric with a NN
- Using the CY metric, approximate correction to harmonic (2,1)-forms by a NN
- Using the CY metric and harmonic forms, approximate solution to the warp factor with a
- Stabilize near conifold $\Rightarrow$ CY nearly singular $\Rightarrow$ improve point sampling for better

2

Results

- CY metric approximation gains multiple orders of magnitud
- Accuracy to harmonic (2,1)-form approximation show
- Result for warp factor pending $\Rightarrow$ stay tuned
- Looking for good, more realistic models

Thank you!