Symmetry Breaking from Monopole Condensation in 3d QED

Pierluigi Niro

SISSA (Trieste)

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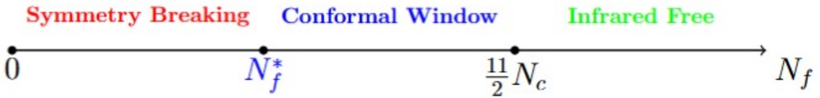
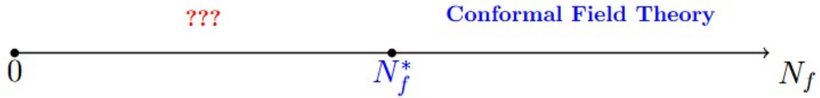
Phase Diagrams and the End of the Conformal Window

- ➤ QCD_4 = 4d $SU(N_c) + N_f$ fundamental Dirac fermions



1. What is N_f^* as a function of N_c ? $N_f^*(3) \in (9, 10)$
2. What happens for $N_f < N_f^*$?
   $SU(N_f)_L \times SU(N_f)_R \times U(1)_V \to SU(N_f)_V \times U(1)_V$ via $\langle \bar{\psi}\psi \rangle$ ✓

- ➤ QED_3 = 3d $U(1)$ + (even) N_f charge-1 Dirac fermions



1. What is N_f^* ? $N_f^* \in (2, 4)$
2. What happens for $N_f < N_f^*$? ✍
   Naively: $SU(N_f) \to SU(N_f/2) \times SU(N_f/2) \times U(1)_f$ via $\langle \bar{\psi}\vec{\sigma}\psi \rangle$ ✗

Preliminaries: $U(1)$ Maxwell Theory in 3d

$$\mathcal{L} = -\frac{1}{4e^2} f^{\mu\nu} f_{\mu\nu}\,, \qquad f = da\,, \qquad [e^2] = 1\,.$$

- Global $U(1)_m$ magnetic symmetry: $\star j_m = \frac{f}{2\pi}$
- Operators with charge q_m are monopoles, disorder point-like operators $\mathcal{M}_{q_m}(x)$ such that $\int_{S^2(x)} f = 2\pi q_m$

Can be dualized to the theory of a compact scalar ('dual photon'):

$$\widetilde{\mathcal{L}} = -\frac{e^2}{8\pi^2}(\partial_\mu \sigma)(\partial^\mu \sigma)\,, \qquad \sigma \sim \sigma + 2\pi\,.$$

- $U(1)_m$ is the shift symmetry of σ: $j_m = \frac{e^2}{(2\pi)^2} d\sigma$
- Monopole operators are $\mathcal{M}_{q_m} = \exp\left(i q_m \sigma\right) \Rightarrow \langle \mathcal{M}_{q_m} \rangle \neq 0$

$U(1)_m$ is spontaneously broken by monopole condensation

3d Coulomb phase = massless photon = S^1 sigma model

Preliminaries: Dirac Fermions in 3d

$$\mathcal{L} = -i\bar{\psi}(\gamma^\mu \partial_\mu - i\gamma^\mu A_\mu + m)\psi\,, \qquad A_\mu = U(1) \text{ global}\,.$$

- $U(1)$ global symmetry: $\psi \to e^{i\alpha}\psi$
- Charge conjugation $\mathcal{C}$: $\psi \to \psi^*$
- Time reversal (at $m = 0$) $\mathcal{T}$: $\psi \to \gamma^0\psi$

$U(1)$ and $\mathcal{T}$ have a mixed 't Hooft anomaly [Niemi, Semenoff; Redlich]

- Can be expressed as a $\theta = \pi$ term via anomaly inflow

$$S_{\text{anomaly}} = \frac{\pi}{8\pi^2}\int_{X_4} dA \wedge dA\,, \qquad \partial X_4 = \mathcal{M}_3$$

- $U(1)$ cannot be gauged and simultaneously preserve $\mathcal{T}$
- Can be canceled for an even number of fermions

Preliminaries: QED_3 at Large (even) N_f

$$\mathcal{L} = -\frac{N_f}{4\lambda} f^{\mu\nu} f_{\mu\nu} - i \sum_{i=1}^{N_f} \bar{\psi}_i \gamma^\mu \left(\partial_\mu - i a_\mu\right) \psi^i \,, \qquad \lambda \equiv e^2 N_f \,.$$

Massless point is protected by $\mathcal{T}$ (assume N_f even)

Exact propagator at large N_f: $\langle a_\mu(p) a_\nu(-p) \rangle = -\frac{i\eta_{\mu\nu}}{N_f} \begin{cases} \lambda/p^2 & \text{UV} \\ 16/|p| & \text{IR} \end{cases}$

$\exists$ non-trivial strongly coupled CFT in the IR, amenable to a systematic $1/N_f$ expansion [Appelquist, Nash, Wijewardhana]

What is the fate of massless QED_3 in the IR?

From bootstrap analysis: strong indication that $\exists$ CFT $\forall N_f \geq 4$ [Chester, Pufu; He, Rong, Su; Albayrak, Erramilli, Li, Poland, Xin]
and that $N_f = 2$ is not a symmetry-preserving CFT [Li]

QED$_3$ with $N_f = 2$: Global Symmetries and Anomalies

$$U(2) = \frac{SU(2)_f \times U(1)_m}{\mathbb{Z}_2}\,, \qquad \mathcal{C}\,, \qquad \mathcal{T}$$

Flavor $SU(2)_f$, Magnetic $U(1)_m$, Quotient by $\mathbb{Z}_2$: $(-\mathbb{I}_2, -1)$

- Naively, $SO(3)_f = SU(2)_f/\mathbb{Z}_2$ (gauge-invariant op's = bosons)
- Non-monopole operators ($q_m = 0$) are in reps of $SO(3)_f$, e.g. $\vec{O} = i\bar{\psi}\vec{\sigma}\psi$ is in the adjoint of $SU(2)_f$
- Monopole operators ($q_m \neq 0$) are in reps of $SU(2)_f$, e.g. $\mathcal{M}^i(x)$ is in the fundamental of $U(2)$ [Borokhov, Kapustin, Wu]

Mixed 't Hooft anomaly between $U(2)$ and $\mathcal{T}$ [Benini, Hsin, Seiberg]

$$S_{\text{anomaly}} = \pi \int_{X_4} c_2(U(2)) = \frac{\pi}{8\pi^2} \int_{X_4} \left[\mathrm{tr}\mathcal{F} \wedge \mathrm{tr}\mathcal{F} - \mathrm{tr}(\mathcal{F} \wedge \mathcal{F})\right]$$

There must be gapless dof's in the IR to match the anomaly

Previous Proposals

1. Condensation of the fermion bilinear $\langle i\bar{\psi}\vec{\sigma}\psi\rangle \neq 0$ [Pisarski] induces SSB of $SO(3)_f \to U(1)_f$ leading to an S^2 sigma model

   What is the fate of $U(1)_m$? Is it broken or unbroken?

   - If unbroken, anomaly for the unbroken $U(1)_f \times U(1)_m$?
   - If broken, by which monopole operator? Where is its NGB?

2. There is an IR fixed point with unbroken $U(2)$ symmetry, possibly described by a self-dual CFT with enhanced $O(4)$ global symmetry [Xu, You; Hsin, Seiberg]

   Assume the bootstrap results and exclude this scenario $\Rightarrow$

   It leads to consider (at least some) symmetry breaking of $U(2)$

Symmetry Breaking Scenario

$\langle \mathcal{M}^i \rangle \neq 0$: Spontaneous Symmetry Breaking $U(2) \to U(1)_{\text{unbroken}}$, via the condensation of the $q_m = 1$ monopole (as Higgsing in SM)

$$\Rightarrow 3 \text{ NGBs parametrizing } U(2)/U(1) = S^3$$

Hopf Map: given $v^2 \equiv \langle \mathcal{M}_i^\dagger \rangle \langle \mathcal{M}^i \rangle$, construct the map $\pi : S^3 \to S^2$

$$\mathcal{M}^i \to \vec{n} = \frac{\mathcal{M}^\dagger \vec{\sigma} \mathcal{M}}{v^2}\,, \qquad \vec{n}^2 = 1$$

Given $\vec{n}$ (triplet of $SU(2)_f$ and singlet of $U(1)_m$), one gets

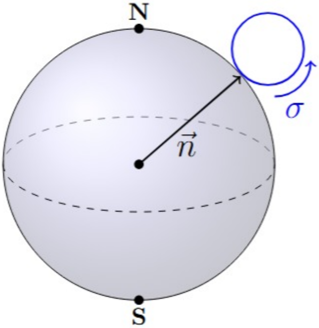
$$\mathcal{M}^i(\vec{n}, \sigma) = v\, \xi^i(\vec{n}) e^{i\sigma}\,, \qquad \sigma \sim \sigma + 2\pi\,, \quad \xi^\dagger \vec{\sigma} \xi = \vec{n}$$

σ parametrizes the S^1 fiber over each point of the S^2 base

$$ds^2(S^3) = R^2 d\vec{n} \cdot d\vec{n} + \frac{e_{\text{eff}}^2}{8\pi^2}(d\sigma - \alpha)^2\,, \qquad \int_{S^2} \frac{d\alpha}{2\pi} = 1$$

3 NGBs: $\vec{n} \in S^2$ (triplet breaking) and $\sigma \in S^1$ (dual photon)

Fermion Bilinear and Small Triplet Mass



Example: $\vec{n} = \pm\hat{z}$, preserving $U(1)_f$

$U(1)_\pm \equiv \frac{1}{2}\left(U(1)_m \pm U(1)_f\right)$

- **N** pole: $\mathcal{M}^i(+\hat{z},\sigma) = ve^{i\sigma}(1\,0)^t$
  $q_+ = 1$ and $q_- = 0$
- **S** pole: $\mathcal{M}^i(-\hat{z},\sigma) = ve^{i\sigma}(0\,1)^t$
  $q_+ = 0$ and $q_- = 1$

Roles of $U(1)_\pm$ reversed $\Rightarrow$ fibration

- Fermion bilinear is aligned with $U(1)_f$ singled out by $\langle\mathcal{M}^i\rangle$:
  $\langle i\bar{\psi}\vec{\sigma}\psi\rangle \xrightarrow{\text{RG}} C\vec{n}$ (no further symmetry breaking)
- Triplet Mass ($\mathcal{T}$-invariant) :
  $\mathcal{L}_{\vec{m}} = i\vec{m}\cdot\bar{\psi}\vec{\sigma}\psi \xrightarrow{\text{RG}} C\vec{m}\cdot\vec{n} \quad \Rightarrow \quad \vec{n} \parallel \vec{m}$
  Mass perturbation selects a single point on S^2: at low energies we get a Coulomb phase, parametrized by the dual photon $\sigma \in S^1$

't Hooft Anomaly Matching

- $\mathcal{C}$ and $\mathcal{T}$ are unbroken ($\mathcal{T}$ follows from Vafa-Witten theorem)
- $\mathcal{T}/U(2)$ anomaly needs to be matched in the S^3 sigma model: it admits a conventional theta term, since $\pi_3(S^3) = \mathbb{Z}$

$$S_\theta = \frac{\theta}{24\pi^2} \int_{\mathcal{M}_3} \mathrm{Tr}\left(U^{-1} dU\right)^3 , \qquad U \in U(2)/U(1)$$

$\mathcal{T}$ allows only $\theta = 0, \pi$: $\theta = \pi$ matches the anomaly

Technically, coupling to $\mathcal{A} \in U(2)$: $S_\theta[\mathcal{A}] = \theta \int_{X_4} c_2(U(2))$

Perturbative Regime: Large Triplet Mass

Couple to $U(1)_f$ with $J_f^\mu = \bar{\psi}\gamma^\mu\sigma_z\psi$ and $U(1)_m$ with $\star J_m = f/2\pi$

$$\mathcal{L} = -\frac{1}{4e^2} f^{\mu\nu} f_{\mu\nu} - i\bar{\psi}_i \left[(\not{\partial} - i\not{a})\delta^i_j - i\not{A}_f (\sigma_z)^i_j \right] \psi^j + \frac{1}{2\pi} da \wedge A_m$$

Fermion charges: $q_g^1 = q_g^2 = 1$, $q_f^1 = -q_f^2 = 1$, $q_m^1 = q_m^2 = 0$

- Add $\vec{m} = m\,\hat{z}$: $\quad \mathcal{L}_{\vec{m}} = im\,\bar{\psi}\sigma_z\psi = im\left(\bar{\psi}_1\psi^1 - \bar{\psi}_2\psi^2\right)$
- Integrate out fermions at $|m| \gg e^2$: Coulomb phase (1 NGB)

$$\mathcal{L}_{IR} = -\frac{1}{4e_m^2} f^{\mu\nu} f_{\mu\nu} + \cdots + \frac{1}{2\pi} da \wedge (A_m + \text{sign}(m) A_f)$$

$$\begin{cases} m > 0 : \text{condensing } \mathcal{M}^1 \text{ and unbroken } U(1)_- \\ m < 0 : \text{condensing } \mathcal{M}^2 \text{ and unbroken } U(1)_+ \end{cases}$$

(same result as $|m| \ll e^2$ in our symmetry-breaking scenario)

Non-Perturbative Bound on Electrical Matter

- CS levels are quantized and cannot vary smoothly at any order in perturbation theory [Coleman, Hill]
- Previous conclusions persist for any $|m| \neq 0$ as long as there is no phase transition (that would make CS levels jump)
- Electrically charged matter would need to become massless to trigger the jump in the levels (or Higgs the gauge group), but
- Non-perturbative bound on electrically charged matter

$$\langle \mathcal{O}_q^\dagger(y)\, e^{iq\int_x^y a}\, \mathcal{O}_q(x)\rangle \leq e^{-|q||m||x-y|} \qquad \forall |m| \neq 0$$

for any fundamental/composite operator $\mathcal{O}$ of gauge charge q

No elementary/bound states become massless! (Coulomb repulsion)

The behavior found for $|m| \gg e^2$ persists smoothly to all $|m| \neq 0$

Non-Perturbative Constraints on Symmetry Breaking

- The Vafa-Witten theorem imposes constraints on the allowed patterns of symmetry breaking in time-reversal invariant theories [Vafa, Witten]
- E.g. applied to massless QCD$_4$ (at $\theta = 0$) it states that time reversal and $SU(N_f)_V \times U(1)_V \subset SU(N_f)_L \times SU(N_f)_R \times U(1)_V$ cannot be spontaneously broken
- Subtlety for massless QED$_3$: naively, $U(1)_f \subset SU(2)_f$ is unbroken, but there is a mixing between $U(1)_f$ and $U(1)_m$

Applying Vafa-Witten arguments to QED$_3$:

1. Time-reversal $\mathcal{T}$ is unbroken
2. If no monopole operator condenses, $U(1)_f \times U(1)_m$ is unbroken
3. If a (q_m, q_f) monopole operator condenses, the linear combination $q_m U(1)_f - q_f U(1)_m$ is unbroken

Extrapolating to $\vec{m} = 0$

$\forall \vec{m} \neq 0$: SSB of the residual symmetry $U(1)_f \times U(1)_m \to U(1)_{\text{unbroken}}$ via monopole condensation of $\mathcal{M}^i = v\,\xi^i(\vec{m})e^{i\sigma}$, with $\xi^\dagger \vec{\sigma}\xi = \vec{m}/|\vec{m}|$ and the low-energy theory is a 3d Coulomb phase (1 NGB)

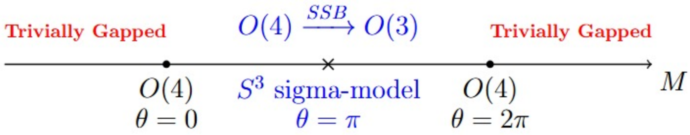
What happens when $\vec{m} = 0$?

1. Gapped Phase ✗
   Incompatible w/ anomaly matching and Coulomb phase
2. Gapless CFT with (at least) $U(2)$ Symmetry ✓
   Implausible (bootstrap results), but realized if $2 > N_f^*$
3. $U(2) \to U(1)_f \times U(1)_m \Rightarrow S^2$ sigma model ✗
   Incompatible w/ anomaly matching: it needs $S^2 \times$ T
   Incompatible w/ Coulomb phase (small $|\vec{m}|$ would lead to T)
4. $U(2) \to U(1)_{\text{unbroken}} \Rightarrow S^3$ sigma model ✓
   Our proposed scenario (see also [Chester, Komargodski]), compatible w/ anomaly matching ($\theta = \pi$) and Coulomb phase

Phase Diagram with Singlet Mass for $N_f = 2$

[Chester, Komargodski] also proposed S^3 sigma-model, motivated by adding the $\mathcal{T}$-odd mass operator to the Lagrangian as $\mathcal{L}_M = iM\bar{\psi}_i\psi^i$

- Integrating out fermions for $|M| \gg e^2$, one gets $U(1)_{\text{sign}(M)}$ pure Chern-Simons theory, which has a single trivially gapped vacuum
- Agreement between anomalous dimensions in the $O(4)$ model (from bootstrap) and in large N_f QED$_3$ extrapolated to $N_f = 2$: $\varphi^q \sim \mathcal{M}_q$
- Massless QED$_3$ cannot flow to standard $O(4)$ model due to anomaly
- Proposal: there are two special points $M \sim \pm e^2$ (related by time reversal), where the theory flows to the $O(4)$ model
- Matching of phases requires S^3 sigma-model at $M = 0$, equipped with a theta term with $\theta = \pi$ for anomaly matching



Outlook

- Symmetry breaking in 3d QED is driven by monopoles, which carry both magnetic and flavor quantum numbers, and not only by fermion bilinears (as in 4d QCD)
- Using Vafa-Witten theorems and 't Hooft anomaly matching, we can uniquely determine the scenario realized for $N_f < N_f^*$ to be SSB
- One of the NGBs is (for any even N_f) the dual photon. Generic SSB pattern is $U(N_f) = SU(N_f) \times U(1)_m \longrightarrow SU(N_f/2) \times SU(N_f/2) \times U(1)_{\text{unbroken}} = SU(N_f/2) \times U(N_f/2)$, s.t. $U(1)_f$ and $U(1)_m$ mix in a broken and unbroken combination

? How to analytically estimate N_f^*? Or at least support the symmetry-breaking scenario for $N_f = 2$?

? What can be said for odd N_f or in presence of CS terms (no time reversal symmetry), or for non-Abelian theories?

? Applications to phases of matter and lattice systems