Continuous TQFTs & non-invertible symmetries

Riccardo Argurio

Université Libre de Bruxelles

String Theory as a Bridge between Gauge Theory and Quantum Gravity
Sapienza University of Rome

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2405.06596 and 2502.today with Adrien Arbalestrier and Luigi Tizzano
2409.11822 with Andrés Collinucci, Giovanni Galati, Ondrej Hulik and Elise Paznokas

Symmetries and Topological Defects

As emphasized in [Gaiotto, Kapustin, Seiberg, Willett, 14], symmetries of a QFT are essentially related to its topological sector

Recall: Noether's theorem associates a continuous symmetry to a conserved current $\partial_\mu j^\mu = 0 \quad \Leftrightarrow \quad d * j = 0$ with j a 1-form

Using j we can define topological symmetry defects:

$$U_\alpha(\Sigma_{d-1}) = e^{i\alpha Q(\Sigma_{d-1})} = e^{i\alpha \int_{\Sigma_{d-1}} *j}$$

Their action on a local operator $\mathcal{O}_q(x)$ of charge q is given by

$$U_\alpha(\Sigma_{d-1})\mathcal{O}_q(x) = e^{iq\alpha}\mathcal{O}_q(x)$$

Generalization to discrete groups and to higher co-dimension:
$\Rightarrow$ Conservation of j is traded for topological nature of U.

Symmetries and Topological Defects

The fusion of overlapping defects represents the multiplication of elements of the symmetry group:

$$U_\alpha(\Sigma_{d-p-1})U_\beta(\Sigma_{d-p-1}) = U_{\alpha+\beta}(\Sigma_{d-p-1})$$

Change the perspective: consider all topological defects of a QFT as implementing symmetries.

Important note: some theories can have topological defects that satisfy non-group-like fusion rules—i.e. they do not allow for an inverse!

- When symmetry defects are themselves charged, it signals the presence of ’t Hooft anomalies

Symmetries and TQFTs

Topological Quantum Field Theories (TQFTs) then play naturally an important role when discussing symmetries

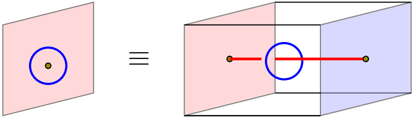
- They can function as simple ‘training ground’ QFTs
- They can dress topological defects
- They can summarize all the symmetry content of a QFT_d by a SymTFT_{d+1} [Freed et al, Kong et al, Apruzzi et al]

A useful tool: the Symmetry TQFT

The symmetries and the anomalies of a QFT are its topological sector
$\Rightarrow$ Encode them into a dynamical TFT in $d+1$. [Freed et al, Kong et al, Apruzzi et al]

This bulk TFT has two boundaries:

- A physical boundary where the local d.o.f. of the QFT live
- A topological boundary where the bulk fields have a set of consistent boundary conditions



The SymTFT can be entirely specified by its extended operators/defects, and which ones can end on the boundaries.

Compact TQFTs and discrete symmetries

A nice set of TQFTs is specified by the fact that they have a global $\mathbb{Z}_N$ or $\mathbb{Z}_N \times \mathbb{Z}_N$ global (higher-form) symmetry. [e.g. Kapustin, Seiberg, 14]

Simple representatives are abelian Chern-Simons in $3d$ and BF-theory in any d:

$$S_{3d} = i\frac{N}{4\pi}\int AdA \qquad , \qquad S_d = i\frac{N}{2\pi}\int AdB$$

Both theories can be formulated with $U(1)$ compact gauge groups. The order of the $\mathbb{Z}_N$ global symmetry is given by the level N.

These theories are equivalently characterized by a finite set of extended operators, with non-trivial braiding/linking

$$W_A^n = e^{in\int A} \,, \quad W_B^m = e^{im\int B} \,, \quad W_A^N = 1 = W_B^N \,, \quad \mathbb{B}(W_A^n, W_B^m) = e^{2\pi i \frac{nm}{N}}$$

Continuous symmetries

A SymTFT$_{d+1}$ is able to describe symmetries of a QFT$_d$ only if they are a subset of its own symmetries.

Then what about physical theories that have continuous symmetries?

- Need TQFTs with continuous global symmetries.

A class of such TQFTs is obtained considering the same CS and BF theories but with $\mathbb{R}$ gauge groups.

Recently introduced in the context of SymTFT by
[Antinucci, Benini, 24; Brennan, Sun, 24] [see also Apruzzi et al, 24; Bonetti et al, 24]

Non-compact/Continuous TQFTs

Simplest: BF theories (3d here for simplicity)

First option: a in $\mathbb{R}$, B in $U(1)$:

$$S = \frac{i}{2\pi}\int adB \qquad \text{with} \qquad \int da = 0 \qquad , \qquad \int dB \in 2\pi\mathbb{Z}$$

Line content is

$$e^{i\frac{\alpha}{2\pi}\int a} \quad \text{with} \quad \alpha \in [0, 2\pi]\ , \quad e^{in\int B} \quad \text{with} \quad n \in \mathbb{Z}\ , \quad \mathbb{B}(W_a^\alpha, W_B^n) = e^{i\alpha n}$$

Global (1-form) symmetry is then $U(1) \times \mathbb{Z}$.

- $\mathbb{R}$ gauge fields can be rescaled $\Rightarrow$ level can be set to unity

Non-compact/Continuous TQFTs

Second option: a and b in $\mathbb{R}$:

$$S = \frac{i}{2\pi}\int adb \qquad \text{with} \qquad \int da = 0 \qquad , \qquad \int db = 0$$

Line content is

$$e^{ir\int a} \quad \text{with} \quad r \in \mathbb{R}\ , \quad e^{is\int b} \quad \text{with} \quad s \in \mathbb{R}\ , \quad \mathbb{B}(W_a^r, W_b^s) = e^{2\pi irs}$$

Global (1-form) symmetry is then $\mathbb{R} \times \mathbb{R}$.

Both options have an infinite number of lines, reflecting the infinite number of elements in their global symmetries

Continuous TQFTs and non-invertible symmetries

We will present two examples where continuous TQFTs are instrumental in establishing non-invertible symmetries:

- Axial symmetry in massless QED: all axial rotations are restored by dressing the defects with a continuous TQFT.

  A similar dressing by compact TQFTs allows to restore only rotations with rational angles. [Choi, Lam, Shao, 22; Cordova, Ohmori, 22]

- T-duality in $2d$ compact boson: the symmetry at self-dual radius becomes non-invertible at any other radius.

  At rational radius, this is related to gauging a $\mathbb{Z}_N \subset U(1)^{\text{winding}}$, implemented through a compact SymTFT.

  [Thorngren, Wang, 21; Choi et al, 22; Kaidi et al, 23]

Using continuous TQFTs: Axial symmetry defects in massless QED

The axial symmetry of massless QED has an ABJ anomaly

$$d \star j_{\text{axial}} = \frac{1}{4\pi^2} dA\, dA$$

The topological defects are

$$\exp\left(i\frac{\alpha}{2}\int \star j_{\text{axial}} - i\frac{\alpha}{8\pi^2}\int AdA\right)$$

but they are not invariant under large $U(1)$ gauge transformations of A unless $\alpha \in 2\pi\mathbb{Z}$.

Bring in a continuous TQFT

[Arbalestrier, RA, Tizzano, 24]

The following 3d TQFT coupled to A is invariant under large $U(1)$ gauge transformations of A:

$$\mathcal{T}^{\alpha}[dA] = \frac{i}{2\pi}\int \Phi dc - \frac{\alpha}{4\pi} c dc + \Phi dA$$

The gauge fields c and Φ are defined on the 3d defect. c is an $\mathbb{R}$ gauge field while Φ is a $U(1)$ gauge field.

- It is well-defined (as a continuous TQFT) for any value of α.
- When integrating out c and ϕ it reproduces the $-i\frac{\alpha}{8\pi^2}\int AdA$ term.

[see Karasik, 22; Garcia-Etxebarria, Iqbal, 22 for a different proposal]

Condensation of $U(1)$

$$\mathcal{T}^\alpha[dA] = \frac{i}{2\pi}\int -\frac{\alpha}{4\pi}cdc + \Phi(dc+dA)$$

- The gauge invariant topological defects are

$$\mathcal{D}_\alpha = \exp\left(i\frac{\alpha}{2}\int \star j_{\text{axial}} + \mathcal{T}^\alpha[dA]\right)$$

They are non-invertible: $\mathcal{D}_\alpha \times \mathcal{D}_{-\alpha} \propto \mathcal{D}_0$

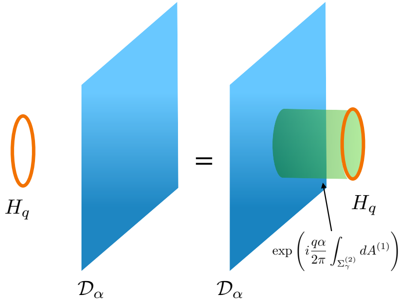
$\mathcal{D}_0$ is a condensation defect for the whole magnetic $U(1)$ 1-form

• Indeed what $\mathcal{T}^\alpha$ does is to (higher) gauge this $U(1)$ on the defect

Relation to [Choi, Lam, Shao, 22; Cordova, Ohmori, 22]:
When $\alpha = 2\pi p/N$: $\mathcal{D}_\alpha \propto \mathcal{D}_0 \times \mathcal{D}_\alpha^{\mathbb{Q}}$ $\Rightarrow$ non-minimal

’t Hooft lines and junctions

The action of $\mathcal{D}_\alpha$ on ’t Hooft lines H_q turns them into non-genuine lines:



Moreover there is a topological junction on the defect given by

$$\exp\left(iq\int_\gamma \Phi + i\frac{q\alpha}{2\pi}\int_{\Sigma_\gamma} dA\right)$$

Continuous SymTFT

The SymTFT for massless QED with $U(1)^{(0)}_{\text{axial}} \times U(1)^{(1)}_{\text{mag}}$ is the following

[Antinucci, Benini, 24]

$$S_{5d} = \frac{i}{2\pi} \int b_3 dA_1 + f_2 dC_2 + \frac{1}{2\pi} A_1 f_2 f_2 + \frac{1}{6\pi} A_1 dA_1 dA_1$$

- It is a continuous TQFT since b_3 and f_2 are $\mathbb{R}$ gauge fields.

While $W_n = \exp\left(in \int A_1\right)$ and $U_\beta = \exp\left(i \frac{\beta}{2\pi} \int f_2\right)$ are genuine, the surface operators built out of b_3 and C_2 are not:

$$\tilde{V}_\alpha = \exp\left(i \frac{\alpha}{4\pi} \int_{\Sigma_3} b_3 - i \frac{\alpha}{8\pi^2} \int_{\Omega_4} f_2 f_2\right)$$

$$\tilde{T}_m = \exp\left(im \int_{\Sigma_2} C_2 - i \frac{m}{\pi} \int_{\Omega_3} A_1 f_2\right)$$

Continuous SymTFT with genuine defects

They can be made genuine for any α by dressing them with continuous TQFTs!

$$V_\alpha = \exp\left(i\frac{\alpha}{4\pi}\int_{\Sigma_3} b_3 + \mathcal{T}^{\alpha}[f_2]\right)$$

and

$$T_m = \exp\left(im\int_{\Sigma_2} C_2 + \mathcal{T}^{2m}_{2d}[A_1, f_2]\right)$$

where

$$\mathcal{T}^{2m}_{2d}[A_1, f_2] = \frac{i}{2\pi}\int_{\Sigma_2} \phi_1 d\Upsilon_0 + 2m\Upsilon_0 f_2 + \phi_1 A_1$$

More general continuous SymTFT

[Arbalestrier, RA, Tizzano, 25]

A more general 4d theory with $U(1)^{(0)}_{\text{axial}} \times U(1)^{(1)}_{\text{mag}}$ symmetry can also have a 2-group $\Rightarrow$ here it will be a higher symmetry structure.

Its SymTFT is

$$S_{5d} = \frac{i}{2\pi} \int b_3 dA_1 + f_2 dC_2 + \frac{k_{\text{ABJ}}}{4\pi} A_1 f_2 f_2 + \frac{k_{2\text{group}}}{4\pi} A_1 dA_1 f_2 + \frac{k_{\text{cubic}}}{12\pi} A_1 dA_1 dA_1$$

In this case, the defects

$$V_\alpha = \exp\left(i \frac{\alpha}{2\pi} \int_{\Sigma_3} b_3 + \mathcal{T}^{\alpha k_{\text{ABJ}}} - i \frac{\alpha k_{2\text{group}}}{8\pi^2} \int_{\Omega_4} f_2 dA_1 \right)$$

are dressed by a continuous TQFT but still non-genuine in the bulk

- They are genuine when pushed to the boundary.

Using continuous TQFTs: Compact boson in 2d

[RA, Collinucci, Galati, Hulik, Paznokas, 24]

The theory

$$S_{2d} = \frac{R^2}{4\pi} \int d\Phi \wedge \star d\Phi \qquad \text{with} \qquad \Phi \sim \Phi + 2\pi$$

has a global 0-form symmetry $U(1)_{\text{momentum}} \times U(1)_{\text{winding}}$ for generic R.
Its SymTFT is of $\mathbb{R} \times \mathbb{R}$ type:

$$S_{3d} = \frac{i}{2\pi} \int b^+ db^-$$

with lines

$$U_x = \exp\left(ix \int b^+\right) \quad , \quad V_y = \exp\left(iy \int b^-\right) \quad \text{with} \quad x, y \in \mathbb{R}$$

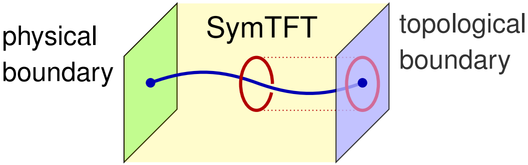
Topological and physical boundaries

The radius is fixed by the b.c. at the topological boundary:

$$\int b^+ \in 2\pi R\mathbb{Z} \qquad , \qquad \int b^- \in 2\pi R^{-1}\mathbb{Z}$$

At the physical boundary one imposes conformal b.c.:

$$b^- = i \star b^+$$



Changing the radius is just changing the boundary conditions in this SymTFT $\Rightarrow$ It is then a topological manipulation in the 2d CFT.

A topological rescaling

Rescaling the radius from R to R' can be made in two steps:

- Gauge $U(1)_{\text{winding}}$ with flat connections

This decompactifies the boson, which now has just an $\mathbb{R}$ momentum symmetry

- Gauge $\mathbb{Z} \subset \mathbb{R}$ with a periodicity given by R'

This recompactifies the boson at the new radius.

Obviously we can take $R' = 1/R$!

T-duality symmetry and condensation defects

Combining such a rescaling from R to $1/R$ with a T-duality, we have a symmetry of the compact boson at any radius R.

Its defects are non-invertible since they involve gaugings on one side of space-time.

- How are these defects encoded in the SymTFT?

They arise as bulk 2d condensation defects for a diagonal $\mathbb{R} \subset \mathbb{R} \times \mathbb{R}$

$$C(T^2_{\alpha,\beta}) = \int_{x,y\in\mathbb{R}} e^{-2\pi i x y} U_y(\beta) V_{-y}(\beta) U_x(\alpha) V_{-x}(\alpha)$$

[See Antinucci et al, 22 and Kaidi et al, 23 for the discrete case]

Outlook

We have seen the use of continuous TQFTs both to dress defects and to define SymTFTs. They are closely tied to the (flat) gauging of continuous symmetries.

Many more situations can be addressed with these tools:
e.g. Maxwell in 4d, Maxwell-Chern-Simons in 5d,...

[Paznokas, 25; Arbalestrier, RA, Tizzano, 25]

Given the abundance of QFTs with continuous global symmetries, the study of infinite collections of topological defects seems unavoidable.

It is then desirable to have a closer look at non-compact TQFTs, in order to establish a more rigorous definition. (Lattice formulation?)

[RA, Galati, Godechal, w.i.p.]

Homework: What are the physical consequences of the ‘irrational’ non-invertible symmetries such as the ones discussed here?

Thank you!