Source Multipoles and Energy-Momentum Tensors for Spinning Black Holes and Other Compact Objects

String Theory as a Bridge between Gauge Theory and Quantum Gravity

Claudio Gambino, University of Rome “La Sapienza”

Based on

[**CG**, Pani, Riccioni, *2403.16574*]
[Bianchi, **CG**, Pani, Riccioni, *2412.01771*]
[**CG**, *2502.XXXXX*]

SAPIENZA UNIVERSITÀ DI ROMA
INFN Istituto Nazionale di Fisica Nucleare

Outline

- Scattering Amplitudes for Classical Gravity
- Momentum-Space Formalism Inspired by Amplitudes
- Gravitational Multipoles in Higher Dimensions
- Stress Multipoles
- Black Hole Sources
- Kerr Mimickers
- Conclusions

Scattering Amplitudes for Classical Gravity

On-Shell

- Gauge independent
- Higher order calculations
- Four dimensions

Off-Shell

- Gauge dependent
- Control over degrees of freedom
- Higher dimensions

KMOC & Eikonal Phase

Gravitational Observables

Gauge dependent quantities e.g. the metric

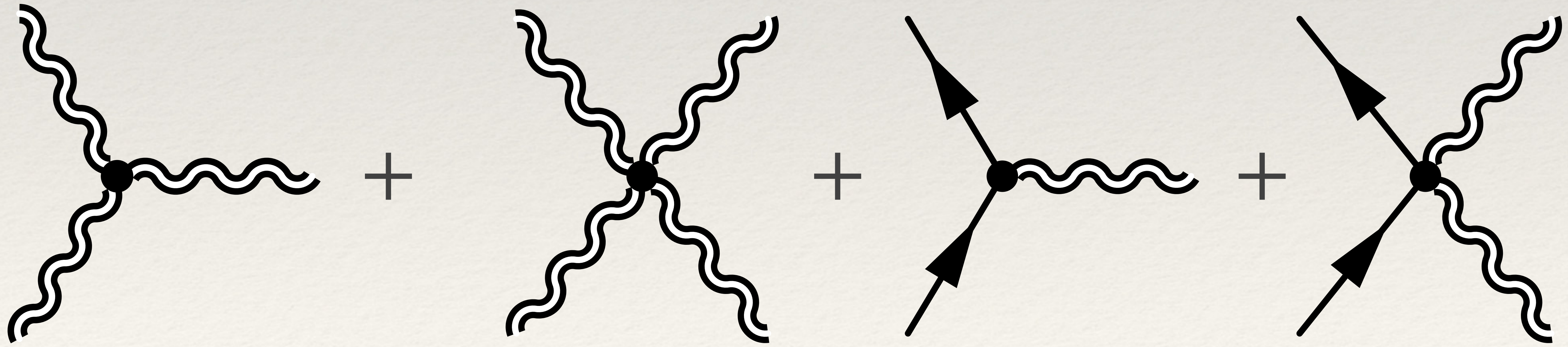
[Bjerrum-Bohr, Damgaard, Festuccia, Plantè, Vanhove, *1806.04920*]
[Kosower, Maybee, O'Connell, *1811.10950*]
[Buonanno, Khalil, O'Connell, Roiban, Solon, Zeng, *2204.05194*]

Spinning Off-Shell Amplitudes

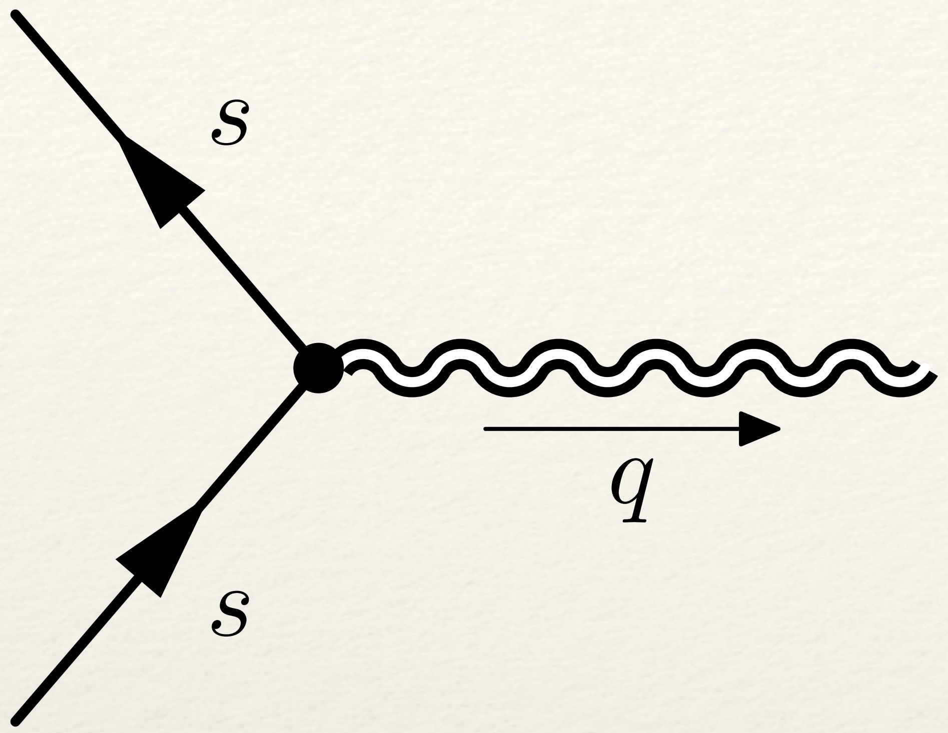
$$\kappa^2 = 32\pi G$$

$$S = \int d^{d+1}x \left(-\frac{2}{\kappa^2}\sqrt{-g}R + \mathcal{L}_m(\Phi_s, g_{\mu\nu}) + \mathcal{L}_{GF}(g_{\mu\nu}) \right)$$

Canonical quantization of gravity: $g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}$

$S \rightarrow$  $+ \cdots$

[Donoghue, *gr-qc/9405057*]



$\mu\nu = (\tau_{\Phi^2 h})^{\mu\nu}_{a,b}(q)$

As in any gauge theory, the gauge boson emission is associated with the conserved current of the theory

$$-\frac{i\kappa}{2}(2m)^{\epsilon}\, T_{\mu\nu}(q)\delta_{\sigma\sigma'} = {}^{a}\langle p_2; s, \sigma' \,|\, (\tau_{\Phi^2 h})^{a,b}_{\mu\nu} \,|\, p_1; s, \sigma\rangle^{b} = \hat{\tau}^{\mu\nu}_{\Phi^2 h}(q, S)\delta_{\sigma\sigma'} + O(\hbar)$$

$\epsilon = 1$ **for bosons**

$\epsilon = 0$ **for fermions**

Dressed Vertex

Quantum Corrections

[Bern, Luna, Roiban, Shen, Zeng, *2005.03071*]

From the action it is possible to derive the 2 *massive - 1 graviton* dressed vertex

Spin-1 →

Energy-Momentum Tensor at quadrupole order

Gauged fixed Stationary Source

$$T^{\mu\nu}(q) = m\, u^\mu u^\nu \left(1 + F_{2,1}\left(-q\cdot S\cdot S\cdot q \right) \right) + m\, F_{2,2}(S\cdot q)^\mu (S\cdot q)^\nu$$

$$q\cdot S\cdot S\cdot q \equiv q^\mu S_\mu{}^\nu S_\nu{}^\sigma q_\sigma$$

$$u^\mu = \delta^\mu_0$$

$$S^{\mu\nu} u_\nu = q^\mu u_\mu = 0$$

[**CG**, Pani, Riccioni, *2403.16574*]

- $S^{\mu\nu} = J^{\mu\nu}/m$ **is the anti-symmetric spin-densitity tensor**

e.g. $d = 3$ → $S^{ij} = \begin{pmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \varepsilon^{ijk} s_k$

- **The action from which the EMT is derived is a low-energy/long-range effective action**

$$T^{\mu\nu}(q) \to T^{\mu\nu}(q) + O(q^2)$$

→ **Local terms**

- **Valid in arbitrary spacetime dimensions**

Inspired by Amplitudes

By writing the most generic conserved rank-2 tensor up to local terms, we can generalize the momentum-space version of the linearized EMT to every order in the angular momentum expansion.

- **The EMT is built out of m, $S^{\mu\nu}$ and u^μ**
- $q_\mu T^{\mu\nu}(q) = O(q^2)$
- **Localized matter sources**

We define the gravitational *Form Factors* $F_{n,i}$

$$\zeta = -q \cdot S \cdot S \cdot q$$

$$T^{\mu\nu}(q) = m\, u^\mu u^\nu \left(1 + \sum_{n=1}^{+\infty} F_{2n,1} \zeta^n \right) + m \sum_{n=0}^{+\infty} F_{2n+2,2} (S \cdot q)^\mu (S \cdot q)^\nu \zeta^n$$

$$+ \frac{i}{2} m \left(u^\mu (S \cdot q)^\nu + u^\nu (S \cdot q)^\mu \right) \left(1 + \sum_{n=1}^{+\infty} F_{2n+1,3} \zeta^n \right) + \textbf{Local Terms}$$

$F_{0,1} = F_{1,3} = 1$ → **Mass & Spin normalized to their ADM value**

$F_{0,2} = 0$ → **No Stress Monopole**

Why momentum space?

- We can write a closed-form expression for the EMT at every order in the angular momentum expansion
- Separation of scales
- The same expression is valid in arbitrary spacetime dimensions

Once the EMT is defined, we can compute the linearized induced metric

Linearized Metric : $$h_{\mu\nu}(x) = \frac{\kappa}{2}\int \frac{d^d q}{(2\pi)^d}\frac{e^{-iq\cdot x}}{q^2} P_{\mu\nu,\rho\sigma} T^{\rho\sigma}(q)$$

Propagator in some gauge

Gravitational Multipoles in Higher Dimensions

[Heynen, Mayerson, *2312.04352*]
[Bianchi, **CG**, Pani, Riccioni, *2412.01771*]

Mass Multipoles : $$g_{00} = -1 + 4\frac{d-2}{d-1}\sum_{\ell=0}^{+\infty}\frac{Gm\rho(r)}{r^\ell}\mathbb{M}^{(\ell)}_{A_\ell}N_{A_\ell} + \cdots$$

Current Multipoles : $$g_{0i} = 2(d-2)\sum_{\ell=0}^{+\infty}\frac{Gm\rho(r)}{r^\ell}\mathbb{J}^{(\ell)}_{i,A_\ell}N_{A_\ell} + \cdots$$

$$N_{A_\ell} = \frac{x_{a_1}\cdots\, x_{a_\ell}}{r^\ell}$$

Stress Multipoles : $$g_{ij} = \delta_{ij} + 4\frac{d-2}{d-1}\sum_{\ell=0}^{+\infty}\frac{Gm\rho(r)}{r^\ell}\tilde{\mathbb{G}}^{(\ell)}_{ij,A_\ell}N_{A_\ell} + \cdots$$

Actual definition of Stress Multipole Tensor → $$\mathbb{G}^{(\ell)}_{ij,A_\ell} = \tilde{\mathbb{G}}^{(\ell)}_{ij,A_\ell} + \frac{1}{2}\delta_{ij}\left(\mathbb{M}^{(\ell)}_{A_\ell} - \tilde{\mathbb{G}}^{(\ell)}_{kk,A_\ell}\right)$$

For a metric with a sufficiently rapid fall-off behaviour, the harmonic gauge is ACMC-∞

[Mayerson, *2210.05687*]

[Geroch, '70]
[Hansen, '74]
[Thorne, '80]
[Gursel, '83]

↓

We can read the gravitational multipoles à la Thorne from the linearized metric described in terms of Form Factors and compare them with the previous expression

$$\mathbb{M}^{(2\ell)}_{A_{2\ell}} = \frac{(d+4\ell-4)!!}{(d-2)!!}(-1)^\ell\Big(F_{2\ell,2}+(d-2)F_{2\ell,1}\Big)(-S\cdot S)_{A_{2\ell}}\Big|_{\mathrm{STF}}$$

$$\mathbb{J}^{(2\ell+1)}_{i,A_{2\ell+1}} = \frac{(d+4\ell-2)!!}{(d-2)!!}(-1)^\ell F_{2\ell+1,3}\, S_{ia_1}(-S\cdot S)_{A_{2\ell}}\,|_{\mathrm{ASTF}}$$

$$\mathbb{G}^{(2\ell)}_{ij,A_{2\ell}} = (d-1)\frac{(d+4\ell-4)!!}{(d-2)!!}(-1)^\ell F_{2\ell,2}\, S_{ia_1}S_{ja_2}(-S\cdot S)_{A_{2\ell-2}}\,|_{\mathrm{RSTF}}$$

- **Multipoles are normalized such that $\mathbb{M}^{(0)} = 1$ & $\mathbb{J}^{(1)}_{ia_1} = S_{ia_1}$**

- **Mass & Stress multipoles are vanishing for odd powers of the spin and Current multipoles are vanishing for even powers**

$$\mathbb{M}^{(2\ell+1)}_{A_{2\ell+1}} = 0 \qquad \mathbb{J}^{(2\ell)}_{i,A_{2\ell}} = 0 \qquad \mathbb{G}^{(2\ell+1)}_{ij,A_{2\ell+1}} = 0$$

- ***STF*, *ASTF* & *RSTF* stand for different symmetries that gravitational multipole tensors need to respect**

- **Stress multipoles are vanishing in four-dimensional spacetimes!**

What are the Stress multipoles ?

We just defined source mutlipoles!

Stress Multipole Moments

In the original Thorne formalism Stress multipoles are not present, indeed they are vanishing in $d = 3$

However $d = 3$ is special since we can define a spin vector

$$S^{ij} = \varepsilon^{ijk} s_k \longrightarrow \quad S_{ia_1} S_{ja_2} \Big|_{\mathrm{RSTF}}^{d=3} = 0$$

Since Stress multipoles are vanishing, stress form factors are redundant and only the combination $F_{2\ell,1} + F_{2\ell,2}$ is physical!

In $d = 3$ the stress form factor $F_{2\ell,2}$ becomes a gauge degree of freedom

Source Multipole Moments

We have established a 1-to-1 correspondence between form factors and gravitational multipoles.

Hence, since the form factors enter in the EMT, it means that we are now able to read gravitational multipoles directly from the source that generates the gravitational field, in complete analogy to Newtonian gravity.

$$T_{\mu\nu} \longleftrightarrow g_{\mu\nu}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

Form Factors $\longleftrightarrow$ Gravitational Multipoles

Newtonian Gravity

$$\Phi(\vec{x}) = -G \sum_{\ell,m} \frac{1}{r^{\ell+1}} \frac{4\pi}{2\ell+1} I_{\ell m} \, Y_{\ell m}(\theta, \varphi)$$

$$I_{\ell m} = \int \epsilon(\vec{x}) \; r^{\ell} Y_{\ell m}(\theta, \varphi) d^3 x$$

General Relativity

$$g_{00} = -1 + 4\frac{d-2}{d-1} \sum_{\ell=0}^{+\infty} \frac{Gm\rho(r)}{r^{\ell}} \mathbb{M}^{(\ell)}_{A_\ell} N_{A_\ell} + \cdots$$

$$m\left(1 + \sum_{n=1}^{+\infty} F_{2n,1} \zeta^n\right) + \cdots = \int d^d x e^{-iq\cdot x} T^{00}(x)$$

Multipole Moments of Myers-Perry BHs

The Myers-Perry solution is the higher dimensional generalization of the Kerr metric

Myers-Perry Form Factors

$$F_2^{(d)}(\zeta) = -\frac{1}{2}\zeta\,\mathcal{Z}_1^{(d)}(\zeta) \qquad F_3^{(d)}(\zeta) = \mathcal{Z}_0^{(d)}(\zeta)$$

$$F_1^{(d)}(\zeta) = F_2^{(d)}(\zeta) + F_3^{(d)}(\zeta)$$

$$\mathcal{Z}_n^{(d)}(\zeta) = \Omega(d)\ \zeta^{-\frac{d-2}{2}} J_{n+\frac{d-2}{2}}\left(\frac{d-1}{2}\zeta\right)$$

$$F_i(\zeta) = \sum_{n=0}^{+\infty} F_{n,i}\ \zeta^n$$

[Myers, Perry, ’86]
[Bianchi, **CG**, Pani, Riccioni, *2412.01771*]

Black Hole Sources

$$T^{\mu\nu}(q) = m\, u^\mu u^\nu F_1^{(d)}(\zeta) + m\frac{F_2^{(d)}(\zeta)}{\zeta^2}(S\cdot q)^\mu (S\cdot q)^\nu + \frac{i}{2} m \Big(u^\mu (S\cdot q)^\nu + u^\nu (S\cdot q)^\mu \Big) F_3^{(d)}(\zeta)$$

$$\zeta = \sqrt{\sum_k q_{\perp,k}^2 a_k^2}$$

→

k runs over the number of the angular momenta

&

$$q_{\perp,k} = q_{x,k}^2 + q_{y,k}^2$$

Replacing the MP form factors and evaluating the Fourier transform of the EMT we can derive the matter distribution sourcing rotating BHs!

In the Kerr case

$$T^{\mu\nu}(q)\Big|_{d=3} = m\, u^\mu u^\nu \left(\cos\zeta - F_2^{(3)}(\zeta)\right) + m\frac{F_2^{(3)}(\zeta)}{\zeta^2}(S\cdot q)^\mu (S\cdot q)^\nu$$

$$-\frac{i}{2}m\left(u^\mu (S\cdot q)^\nu + u^\nu (S\cdot q)^\mu\right)\frac{\sin\zeta}{\zeta}$$

In $d = 3$ the stress form factor is a gauge parameter and can be suitably chosen without changing the multipolar structure of the source

EMT in Position Space $\quad T_{\mu\nu}(x) = \int \frac{d^d q}{(2\pi)^d} e^{-iq\cdot x} T_{\mu\nu}(q)$

Israel Source

A reasonable choice is to set $F_2^{(d=3)} = 0$, from which the Kerr EMT reads

$$T^{00}(q) = m\ \cos aq_{\perp} \qquad T^{0i}(q) = -\frac{i}{2}m(s\times q)^i\ \frac{\sin aq_{\perp}}{aq_{\perp}} \qquad T^{ij}(q) = 0$$

Performing the Fourier transform we recover the Israel source of Kerr BHs, describing a superluminal rotating disk of radius $\rho = a$ with negative energy density

$$T^{00}(x) = -\frac{m}{2\pi}\delta(z)\frac{a}{(a^2-\rho^2)^{3/2}}\Theta(a-\rho) \qquad T^{0i}(x) = \frac{m}{4\pi}\delta(z)\frac{(\hat{s}\times r)^i}{(a^2-\rho^2)^{3/2}}\Theta(a-\rho)$$

[Israel, ’70]

MP Source for $d = 4$

For MP the stress form factor is not redundant and has to be taken into account

Myers-Perry Energy Density

$$T^{00}(x) = \frac{m}{(2\pi)^2}\left(\frac{4}{9}A_1 + \frac{2}{3}A_0\right)$$

$$A_1 = \frac{4}{3}\delta\left(a_1^2\rho_2^2 + a_2^2\rho_1^2 - (\tfrac{3}{2}a_1a_2)^2\right)\Theta(\tfrac{3}{2}a_1 - \rho_1)\Theta(\tfrac{3}{2}a_2 - \rho_2)$$

$$A_0 = \frac{1}{2}\left(\frac{4}{3}\frac{\pi}{a_2}\delta(y_1)\delta(x_1)\delta(\tfrac{3}{2}a_2 - \rho_2) + \frac{4}{3}\frac{\pi}{a_1}\delta(y_2)\delta(x_2)\delta(\tfrac{3}{2}a_1 - \rho_1)\right)$$

The mass-energy distribution is singular for $a_1^2\rho_2^2 + a_2^2\rho_1^2 = (\tfrac{3}{2}a_1a_2)^2$, and vanishing everywhere else, describing a 3-ellipsoid embedded in $\mathbb{R}^4$ of semi-axis $\rho_1 = \frac{3}{2}a_1$ and $\rho_2 = \frac{3}{2}a_2$

- **The Israel EMT corresponds to the "pressureless" case of** $F_2^{(d=3)}(\zeta) = 0$.
- **Both Kerr and MP sources reproduce the structure of curvature singularities of the full non-linear solution:**

$$T^{00}\Big|_{d=even} \propto \frac{m}{(2\pi)^{\frac{d}{2}}} \frac{1}{\prod_k a_k^2} \delta\left(\frac{\rho_k^2}{a_k^2} - (\frac{d-1}{2})^2\right) \prod_k \Theta(\frac{d-1}{2} a_k - \rho_k) + \cdots$$

$$T^{00}\Big|_{d=odd} \propto \frac{m}{(2\pi)^{\frac{d-1}{2}}} \frac{\delta(z)}{\prod_k a_k^2} \delta\left(\frac{\rho_k^2}{a_k^2} - (\frac{d-1}{2})^2\right) \prod_k \Theta(\frac{d-1}{2} a_k - \rho_k) + \cdots$$

- **A tight relation between multipoles and curvature singularities is suggested.**
- **The minimal EMT we built leads to lower dimensional sources.**

Black Hole Mimickers

We now have a multipole-based framework to build black hole mimickers!

$$T^{\mu\nu}(q) = m\ u^\mu u^\nu F_1^{(d)}(\zeta) K_1(q^2) + m \frac{F_2^{(d)}(\zeta)}{\zeta^2} K_2(q^2) (S\cdot q)^\mu (S\cdot q)^\nu$$

$$-\frac{i}{2} m \Big(u^\mu (S\cdot q)^\nu + u^\nu (S\cdot q)^\mu \Big) F_3^{(d)}(\zeta) K_3(q^2)$$

K_i : **Structure Functions**
(they do not modify the multipoles)

Setting $K_i(q^2) = 1$ means to consider point-like fundamental objects, while a non-trivial choice gives an <u>internal smeared structure</u>

[CG, *2502.XXXXX*]

The Idea: **fix Kerr form factors and a non-trivial structure function that leads to a physically reasonable source. By construction this will be a BH mimicker with the exact same multipolar structure of Kerr.**

Gaussian Structure Functions

$$K_i(q^2) = e^{-q^2 R_i^2} = 1 - q^2 R_i^2 + \frac{q^4 R_i^4}{2} + \cdots$$

Gaussian-Smeared Israel Source

$$T^{00}(q) = m \ \cos(aq_\perp) e^{-q^2 R_1^2} \qquad T^{0i}(q) = -\frac{i}{2} m (s \times q)^i \frac{\sin(aq_\perp)}{aq_\perp} e^{-q^2 R_3^2} \qquad T^{ij}(q) = 0$$

In cylindrical coordinates (t, ρ, ϕ, z) it is possible to analytically express the gaussian-smeared Israel source

$$\mathcal{I}_c(\rho, z; R_c) = m\frac{e^{-\frac{z^2}{4R_c^2}}}{8\pi^{3/2}R_c^3}\sum_{n=0}^{+\infty}(-1)^n\frac{n!}{(2n)!}\left(\frac{a^2}{R_c^2}\right)^n {}_1F_1\left(n+1; 1; -\frac{\rho^2}{4R_c^2}\right)$$

$$\mathcal{I}_s(\rho, z; R_s) = m\frac{e^{-\frac{z^2}{4R_s^2}}}{8\pi^{3/2}R_s^3}\sum_{n=0}^{+\infty}(-1)^n\frac{n!}{(2n+1)!}\left(\frac{a^2}{R_s^2}\right)^n {}_1F_1\left(n+1; 1; -\frac{\rho^2}{4R_s^2}\right)$$

$$T_I^{00}(x) = \mathcal{I}_c(\rho, z; R_1) \qquad T_I^{0i}(x) = \frac{1}{2}(s \times \partial_x)^i\, \mathcal{I}_s(\rho, z; R_3)$$

$$T_I^{ij}(x) = 0$$

Source Phenomenology

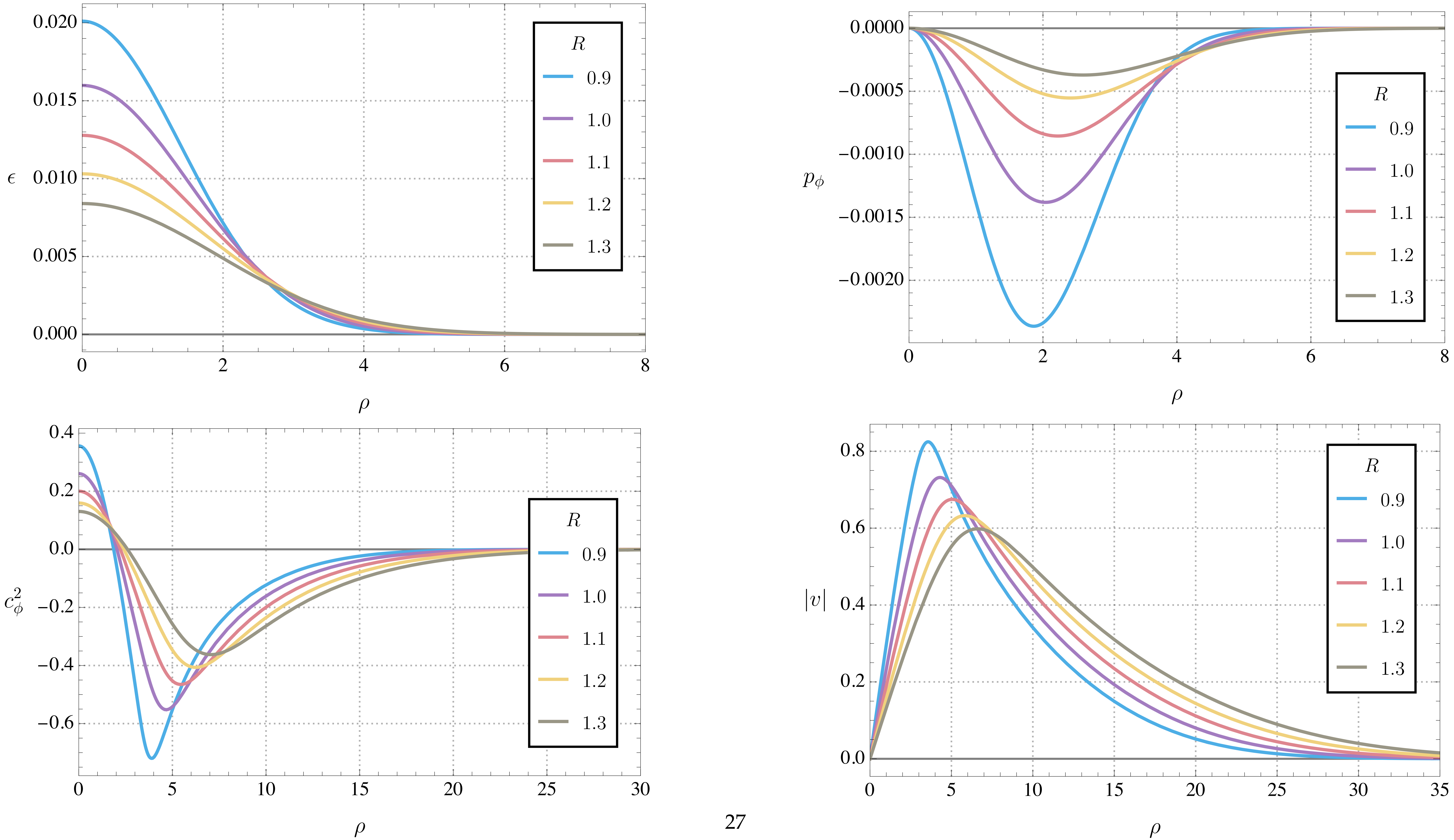
Rotating Anisotropic Fluid

$$T^{\mu\nu} = \epsilon\; u^\mu u^\nu + p_\phi\; l_\phi^\mu l_\phi^\nu$$

$$R_1 = R$$
$$R_3 = \alpha R$$

- **Positive Energy Conditions:** $\epsilon \geq 0$ & $\xi_\phi = \epsilon + p_\phi \geq 0$
- **Causality Conditions:** $0 \leq |v| = |\rho\Omega| < 1$ & $0 \leq c_n^2 = \partial p_n / \partial\epsilon < 1$
- **Real-Valued threshold:** $0 < \alpha < 1$

$z = 0 \qquad a = 0.8 \qquad \alpha = 0.99$ **[CG, 2502.XXXXX]**



Non-Perturbative Generalization in the Static Limit

$$T^{\mu\nu} = \Big(\epsilon(r) + p(r)\Big)u^\mu u^\nu + p(r)g^{\mu\nu} \xrightarrow{G \to 0} \tilde{T}^{\mu\nu} = \epsilon_0(r)u^\mu u^\nu$$

$$\epsilon_0(r) = m\frac{e^{-\frac{r^2}{4R^2}}}{8\pi^{3/2}R^3}$$

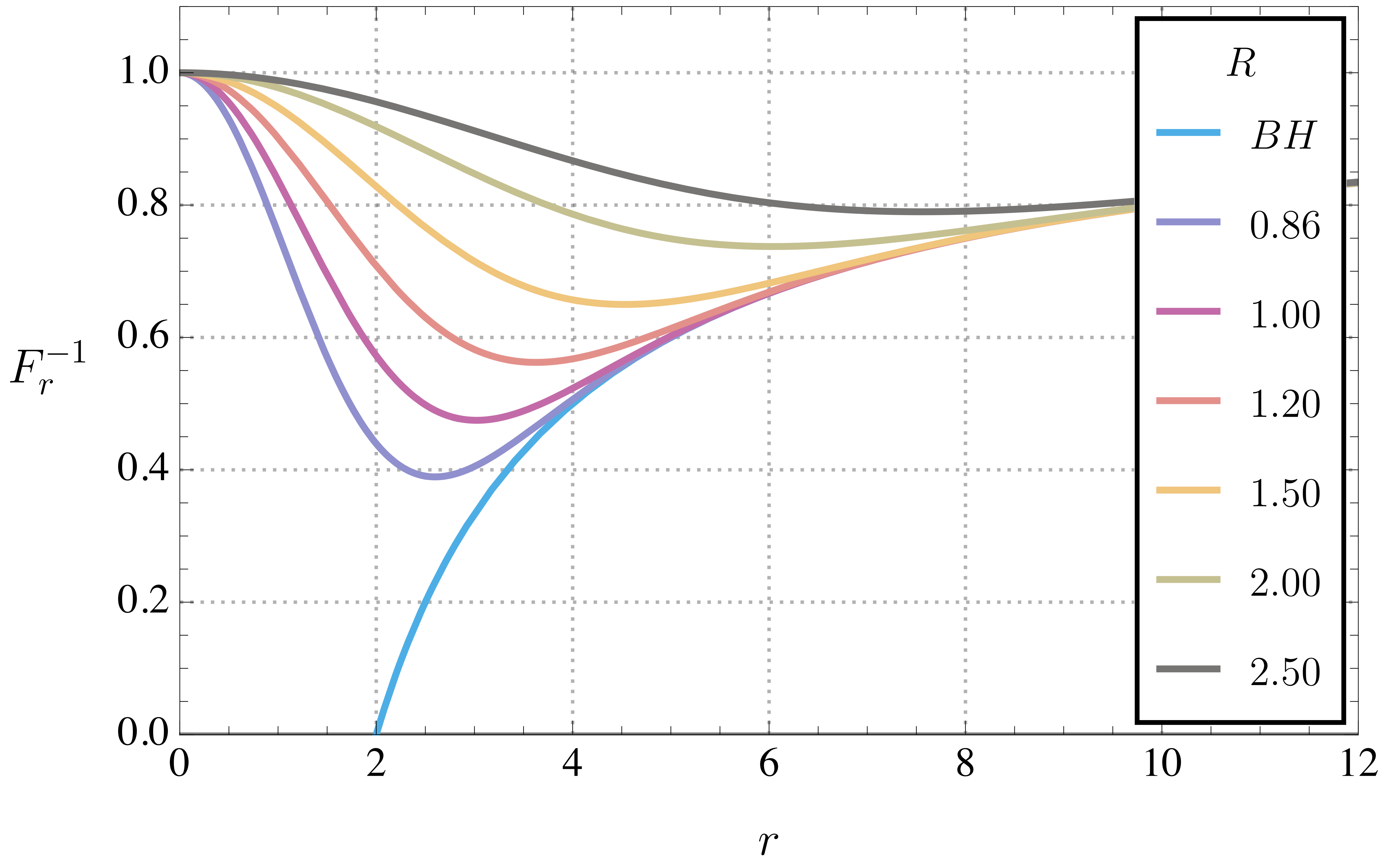
In the static limit $a = 0$ the problem is reduced to solve the TOV equations for a gaussian energy-density function.

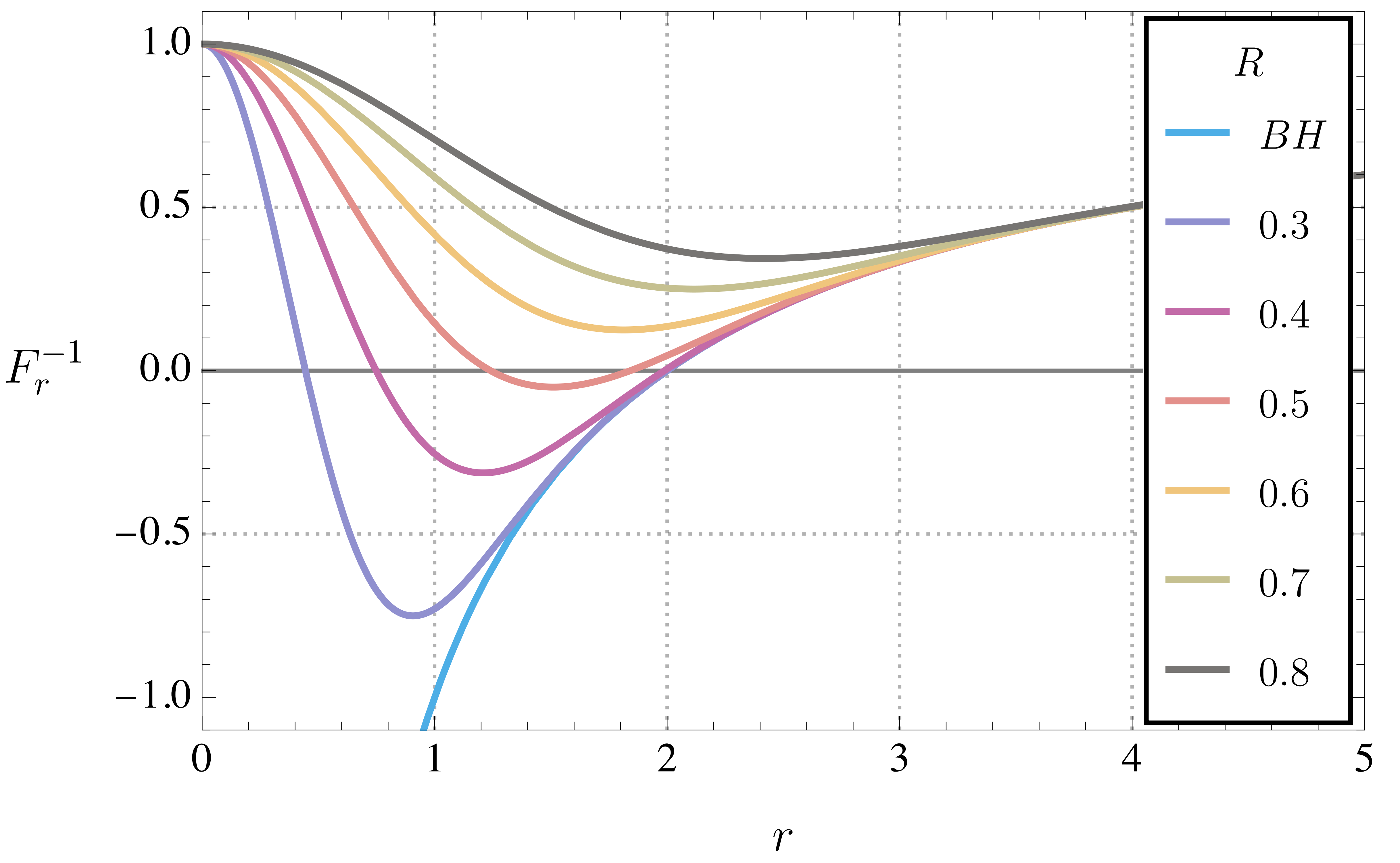
Instead of fixing the EOS we impose: $\epsilon(r) = \epsilon_0(r)$

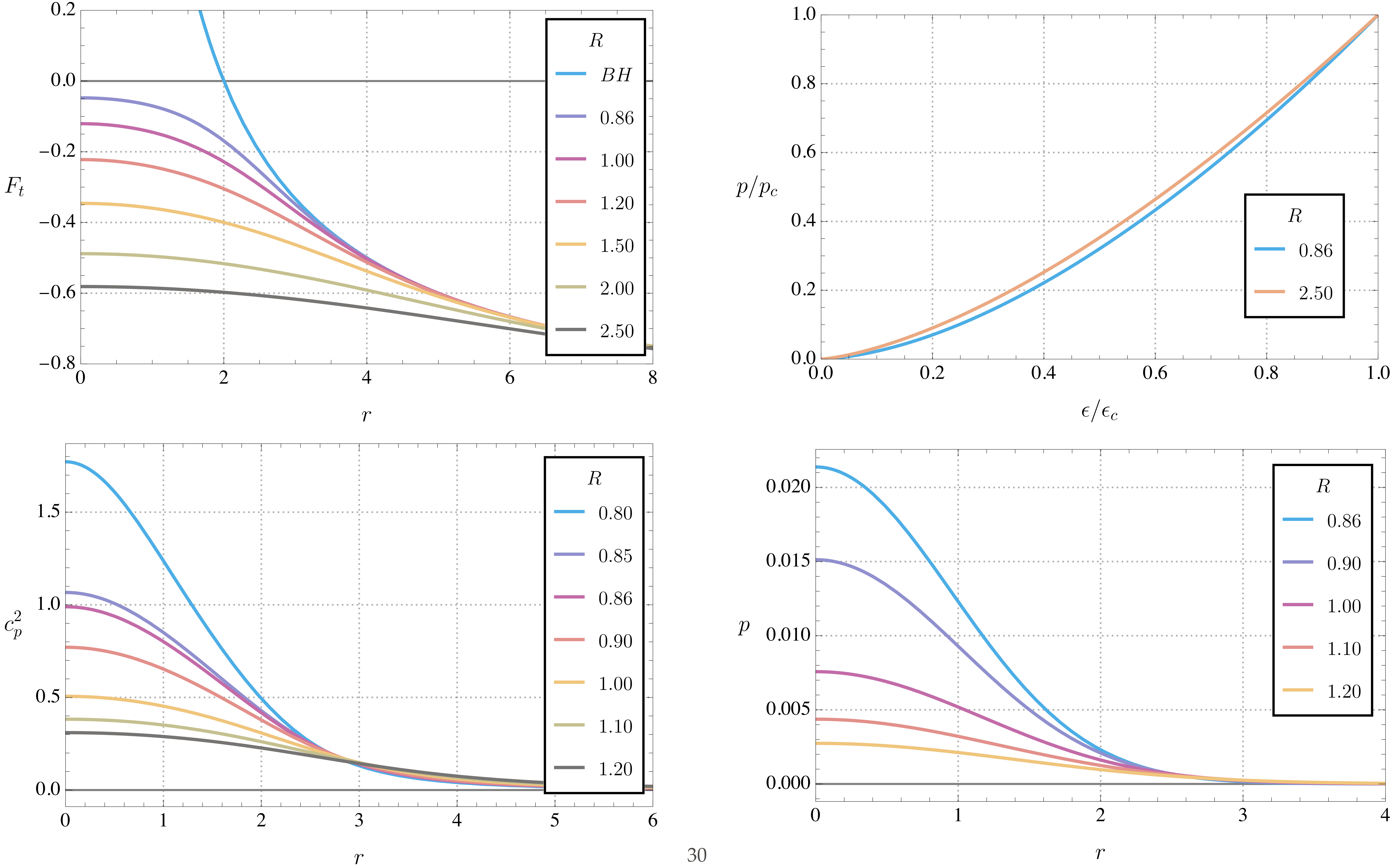
$$ds^2 = F_t(r)dt^2 + F_r(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad F_r(r) = \left(1 - \frac{2GM(r)}{r}\right)^{-1}$$

Analytic solution of the TOV

$$M(r) = m\mathrm{Erf}\left(\frac{r}{2R}\right) - \frac{mr}{R\sqrt{\pi}}e^{-\frac{r^2}{4R^2}}$$







F_t
r
R
BH
0.86
1.00
1.20
1.50
2.00
2.50
p/p_c
ϵ/ϵ_c
R
0.86
2.50
c_p^2
r
R
0.80
0.85
0.86
0.90
1.00
1.10
1.20
p
r
R
0.86
0.90
1.00
1.10
1.20

Conclusions

- **Inspired by scattering amplitudes we were able to define gravitational form factors through a momentum-space approach.**
- **We generalized the definition of gravitational multipoles in arbitrary dimensions and established a 1-to-1 relation with form factors defining the source multipoles in a relativistic context.**
- **For rotating black holes we found a closed-form expression for the form factors and derived the matter source inducing black hole metrics.**
- **Giving a non-trivial internal structure to the source we were able to build a physically reasonable EMT sourcing black hole mimickers.**

Next Steps…

- **Generalize the gaussian-smeared Israel source to non-perturbative level.**
- **Test other structure functions to get different mimicker models and regular black hole solutions.**
- **Generalize the definition of form factors to fundamental multipoles.**
- **Investigate the nature of form factors computing gravitational scattering amplitudes.**

Thank You!