The $AdS_4 \times \mathbb{CP}^3$ Virasoro-Shapiro Aplitude

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String Theory as a Bridge between Gauge Theory and Quantum Gravity
Sapienza University of Rome
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Based on work with:

Luis F. Alday, Joao A. Silva, Maria Nocchi, Shai M. Chester, De-liang Zhong

Problem:

Formulating string theory on curved backgrounds (with RR flux)

One way forward:

Bootstrap AdS string amplitudes (as worldsheet integrals):

- 4 gravitons, type IIB superstring in $AdS_5 \times S^5$ / $\mathcal{N} = 4$ SYM
  [Alday,TH,Silva;2023]
- 4 gluons, orientifold of type IIB in $AdS_5 \times S^5$ / $\mathcal{N} = 2$ SCFT
  [Alday,Chester,TH,Zhong;2024]

This talk:

- 4 gravitons, type IIA superstring in $AdS_4 \times \mathbb{CP}^3$ / ABJM theory

ABJM theory

- ABJM theory is the AdS/CFT dual of M-theory on $AdS_4 \times S^7$
- ABJM $=$ Chern-Simons matter theory with $U(N)_k \times U(N)_{-k}$ gauge group
- We consider the t'Hooft limit: $N, k \to \infty$ with N/k fixed

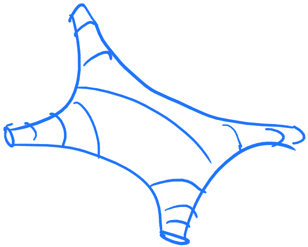
  AdS dual: type IIA strings on $AdS_4 \times \mathbb{CP}^3$

Outline:

1. String amplitudes in flat space
2. Single-valuedness
3. The $AdS_4 \times \mathbb{CP}^3$ Virasoro-Shapiro amplitude
4. High energy limit

1. String amplitudes in flat space

String amplitudes depend on...



... the parameters of the theory:

- g_s = string coupling $\ll 1$
  $\rightarrow$ consider tree level = genus 0
- $\sqrt{\alpha'}$ = string length

... the particles being scattered

- consider 4 gravitons (closed strings) or 4 gluons (open strings)
- momenta p_i in terms of Mandelstams $S + T + U = 0$

$$S \sim \alpha'(p_1 + p_2)^2 \qquad T \sim \alpha'(p_1 + p_3)^2 \qquad U \sim \alpha'(p_1 + p_4)^2$$

- polarizations ϵ_i

Famous string amplitudes

4 gravitons in type IIB superstring: $\mathcal{A} = K_{\text{closed}}(\epsilon_i, p_i) A^{(0)}_{\text{closed}}(S, T)$

Virasoro-Shapiro amplitude

$$A^{(0)}_{\text{closed}}(S, T) = -\frac{\Gamma(-S)\Gamma(-T)\Gamma(-U)}{\Gamma(S+1)\Gamma(T+1)\Gamma(U+1)}$$

4 gluons in type I superstring: $\mathcal{A} = K_{\text{open}}(\epsilon_i, p_i) \left(\text{Tr}(t^{i_1} t^{i_2} t^{i_3} t^{i_4}) A^{(0)}_{\text{open}}(S, T) + \text{permutations} \right)$

Veneziano amplitude

$$A^{(0)}_{\text{open}}(S, T) = -\frac{\Gamma(-S)\Gamma(-T)}{\Gamma(1-S-T)}$$

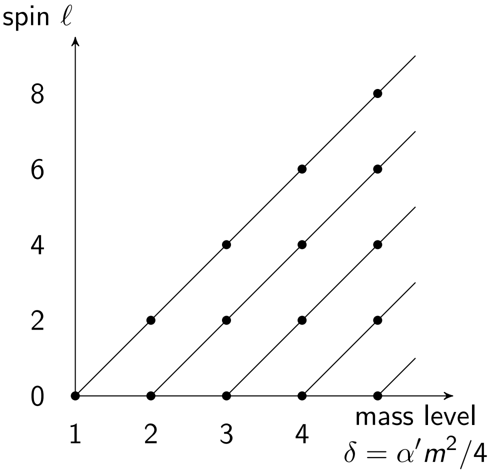
The exchanged massive string spectrum is extracted via the partial wave expansion

$$\lim_{T\to\delta} A^{(0)}(S,T) = \sum_{\ell} \frac{a_{\delta,\ell} P_\ell(\cos\theta)}{T-\delta}$$

It forms linear Regge trajectories.

$$A^{(0)}_{\text{closed}} = -\frac{\Gamma(-S)\,\Gamma(-T)\,\Gamma(-U)}{\Gamma(S+1)\Gamma(T+1)\Gamma(U+1)}$$

Spectrum for $A^{(0)}_{\text{closed}}(S,T)$:



Low energy expansion

Low energy effective action (point particles with derivative interactions)
$\rightarrow$ Low energy expansion ($S \sim T \sim 0 \leftrightarrow$ short strings):

$$A^{(0)}_{\text{closed}}(S,T) = \underbrace{\frac{1}{STU}}_{\text{sugra}} + \underbrace{2\,\zeta(3)}_{R^4} + \underbrace{\zeta(5)\,(S^2+T^2+U^2)}_{D^4R^4} + \underbrace{2\,\zeta(3)^2\,STU}_{D^6R^4} + \dots$$

$$A^{(0)}_{\text{open}}(S,T) = \underbrace{-\frac{1}{ST}}_{\text{SYM}} + \underbrace{\zeta(2)}_{F^4} + \underbrace{\zeta(3)\,(S+T)}_{D^2F^4} + \underbrace{\zeta(4)\,(S^2+\tfrac{1}{4}ST+T^2)}_{D^4F^4} + \dots$$

The LEE of closed string amplitudes contains only odd zeta-values!

This has a deep mathematical reason!
[Stieberger;2013],[Brown,Dupont;Schlotterer,Schnetz;Vanhove,Zerbini;2018]

2. Single-valuedness

Zeta values and polylogarithms

Zeta values are related to polylogs:

$$\left.\begin{array}{lrcl} \text{zeta values:} & \zeta(n) & = & \displaystyle\sum_{k=1}^{\infty} \frac{1}{k^n} \\ \text{polylogarithms:} & \mathrm{Li}_n(z) & = & \displaystyle\sum_{k=1}^{\infty} \frac{z^k}{k^n} \end{array}\right\} \quad \zeta(n) = \mathrm{Li}_n(1)$$

Let's talk about polylogs.

Multiple polylogarithms (MPLs)

Definition ($|z_1 \dots z_r| = r =$ weight) $\quad z_i \in \{0, 1\}$

$$L_{z_1 \dots z_r}(z) = \int_{0 \le t_r \le \dots \le t_1 \le z} \frac{dt_1}{t_1 - z_1} \cdots \frac{dt_r}{t_r - z_r}$$

Properties:

- $\partial_z L_{z_i w}(z) = \dfrac{1}{z - z_i} L_w(z)$
- multi-valued
- holomorphic

Examples:

- $L_{1^p}(z) = \dfrac{1}{p!} \log^p(1 - z)$
- $L_{0^p 1}(z) = -\mathrm{Li}_{p+1}(z)$

Single-valued multiple polylogarithms (SVMPLs)

SVMPLs [Brown;2004]

$$\mathcal{L}_w(z) = \sum_{|w_1|+|w_2|=|w|} c_{w_1 w_2} L_{w_1}(z) L_{w_2}(\bar{z})$$

Properties:

- $\partial_z \mathcal{L}_{z_i w}(z) = \frac{1}{z - z_i} \mathcal{L}_w(z)$
- single-valued
- non-holomorphic

Examples:

- $\mathcal{L}_{1^p}(z) = \frac{1}{p!} \log^p |1 - z|^2$
- $\mathcal{L}_{01}(z) = \mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) - \log(1 - \bar{z}) \log |z|^2$

Single-valued zeta values

Single-valued zeta values [Brown;2013]

$$\zeta_{\mathrm{sv}}(w) \equiv \mathcal{L}_w(1)$$

Subset of the usual multiple zeta values.

In particular

$$\zeta_{\mathrm{sv}}(2n+1) = 2\zeta(2n+1) \qquad \zeta_{\mathrm{sv}}(2n) = 0$$

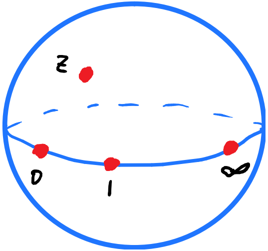
Odd zeta values are single-valued, even ones are not!

Example for multiple zetas (nested sums):

$$\zeta_{\mathrm{sv}}(3,5,3) = 2\zeta(3,5,3) - 2\zeta(3)\zeta(3,5) - 10\zeta(3)^2\zeta(5)$$

Single-valuedness from worldsheet integrals

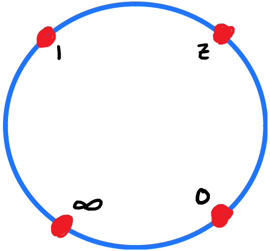
Closed strings



$$A^{(0)}_{\text{closed}} = \frac{1}{U^2}\int dz^2\; |z|^{-2S-2}|1-z|^{-2T-2}$$
$$= \frac{1}{STU} + 2\,\zeta(3) + \ldots$$

Single-valued!

Open strings



$$A^{(0)}_{\text{open}} = -\frac{1}{U}\int_0^1 dz\; z^{-S-1}(1-z)^{-T-1}$$
$$= -\frac{1}{ST} + \zeta(2) + \ldots$$

Not single-valued!

Certain (2-dimensional) integrals preserve single-valuedness!

[Brown,Dupont;Schlotterer,Schnetz;Vanhove,Zerbini;2018]

Strings in (weakly) curved background

Consider curvature corrections to amplitudes: (R = curvature scale)

$$A(S,T) = A^{(0)}(S,T) + \frac{\alpha'}{R^2} A^{(1)}(S,T) + \dots$$

Toy non-linear sigma model:

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \, G_{\mu\nu}(X)$$

$$= S_{\text{flat}} + \frac{1}{R^2} \lim_{q\to 0} \frac{\partial^2}{\partial q^\mu \partial q^\nu} V^{\mu\nu}_{\text{graviton}}(q) + \dots$$

Curved metric expanded around flat space:

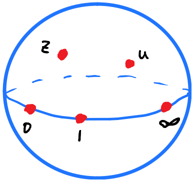
$$G_{\mu\nu}(X) = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{R^2} + \cdots$$

$$h_{\mu\nu} \sim X_\mu X_\nu \sim \lim_{q\to 0} \frac{\partial^2}{\partial q^\mu \partial q^\nu} e^{iq\cdot X}$$

curvature corrections $\sim$ extra soft gravitons

$$A^{(1)}(S,T) \sim \lim_{q\to 0} \frac{\partial^2}{\partial q^\mu \partial q^\nu} \langle V_1 V_2 V_3 V_4 V^{\mu\nu}_{\text{graviton}}(q) \rangle_{\text{flat}}$$

String amplitudes with an extra soft graviton:

$A^{(1)}_{\text{closed}} \sim$  $\sim \int dz^2\, |z|^{-2S-2}|1-z|^{-2T-2} \underbrace{\int d^2u \frac{|u|^{2p_1\cdot q}|1-u|^{2p_3\cdot q}|z-u|^{2p_2\cdot q}}{|u|^2|1-u|^2}}$

In a small q expansion: SVMPLs(z)

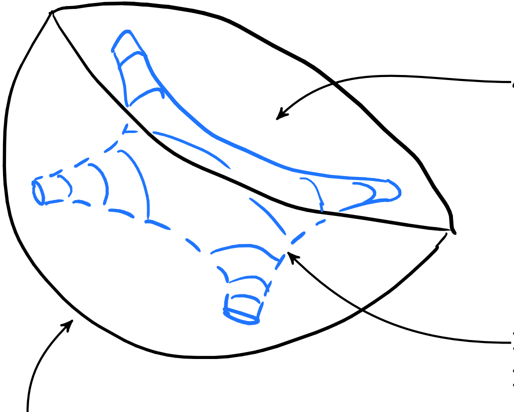
(Similar integrals (and SVMPLs) also appear for strings on $AdS_3 \times \mathcal{M}$ with pure NSNS flux. [Alday,Giribet,TH;2024])

The curvature corrections $A^{(k)}$ should be world-sheet integrals over SVMPLs!

Next we will use AdS/CFT to make this precise!

3. The $AdS_4 \times \mathbb{CP}^3$ Virasoro-Shapiro Aplitude

AdS_4 / CFT_3



4d bulk of AdS:
IIA string theory on $AdS_4 \times \mathbb{CP}^3$

- string coupling g_s
- string length $\sqrt{\alpha'}$
- AdS radius R

2d string worldsheet:
2d CFT???

3d boundary of AdS:
ABJM theory

- Chern-Simons-matter theory
- $U(N)_k \times U(N)_{-k}$ gauge group
- $N \to \infty$ with $\frac{N}{k} \sim \lambda = \frac{R^4}{\alpha'^2}$ fixed

Weakly coupled strings:

$$g_s \ll 1 \quad \Leftrightarrow \quad N \gg 1$$

CFT stress-tensor correlator		AdS graviton amplitude
$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle$	$\longleftrightarrow$ integral transform	$A(S,T)$

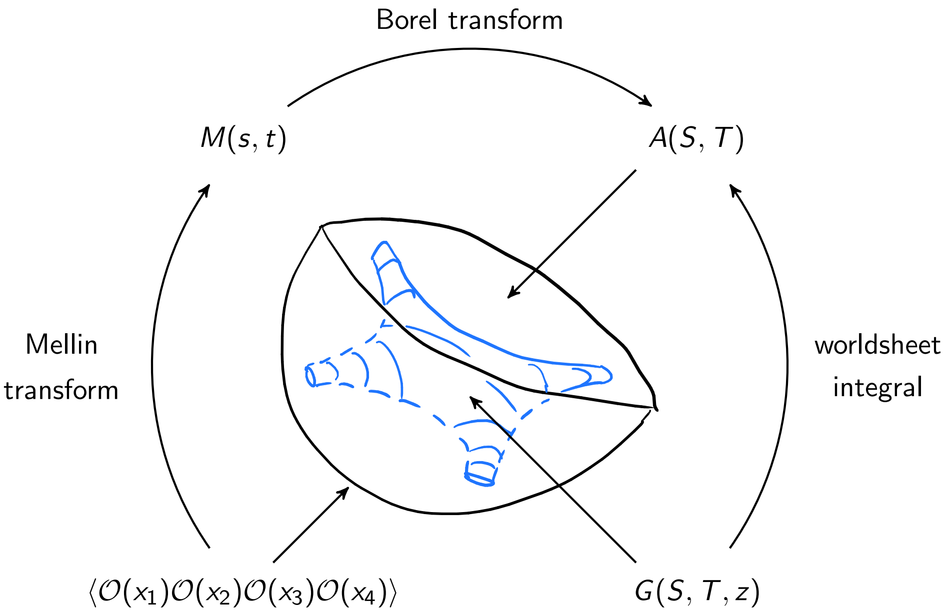
Small curvature expansion:

$$\frac{\alpha'}{R^2} = \frac{1}{\sqrt{\lambda}}$$

$$A(S,T) = A^{(0)}(S,T) + \frac{\alpha'}{R^2} A^{(1)}(S,T) + \left(\frac{\alpha'}{R^2}\right)^2 A^{(2)}(S,T) + \dots$$

$$A^{(0)}(S,T) = \frac{\Gamma(1-S)\Gamma(1-T)\Gamma(1-U)}{\Gamma(1+S)\Gamma(1+T)\Gamma(1+U)} \left(\frac{TU}{S}, \frac{ST}{U}, \frac{SU}{T}, \frac{S}{2}, \frac{U}{2}, \frac{T}{2}\right)$$

Integral transforms



The Mellin transform

Cross-ratios: $U = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$ $V = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$

Mellin transform

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle \propto \int_{-i\infty}^{i\infty} dsdt\, U^s V^t\, \Gamma(s,t)\, M(s,t)$$

$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle$		$M(s,t)$
powers of U, V	$\leftrightarrow$	poles in s, t

Mellin amplitudes share many properties of scattering amplitudes.

Operator product expansion

We can expand $\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle$ into conformal blocks using:

Operator product expansion (OPE)

$$\mathcal{O}(x)\mathcal{O}(0) = \sum_{\mathcal{O}_{\Delta,\ell} \text{ primaries}} C_{\Delta,\ell}\; c_{\Delta,\ell}(x,\partial_y)\mathcal{O}_{\Delta,\ell}(y)\big|_{y=0}$$

OPE data

- $\Delta =$ dimension
- $\ell =$ spin
- $C_{\Delta,\ell} =$ OPE coefficients

$M(s,t)$ has only simple poles, given by [Mack;2009], [Penedones,Silva,Zhiboedov;2019]

Poles and residues of $M(s,t)$

$$M(s,t) \sim \frac{C^2_{\Delta,\ell}Q_{\Delta,\ell,m}(t)}{s-(\Delta-\ell+2m)}$$

Massive string operators

String masses in flat space:

$$m^2 = \frac{4\delta}{\alpha'}\,, \quad \delta = 1, 2, 3, \ldots$$

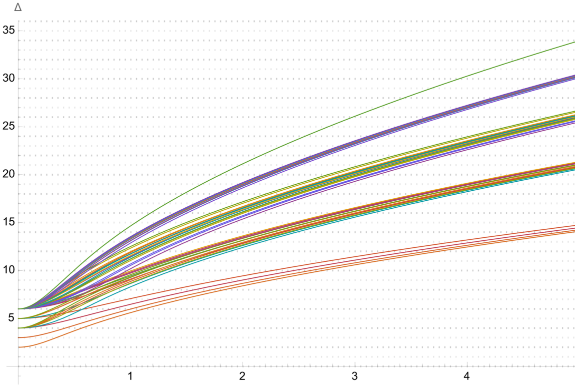
AdS dictionary ($\sqrt{\lambda} = R^2/\alpha' \gg 1$):

$$\Delta(\Delta - d) = R^2 m^2 + O(\lambda^0) = R^2 \frac{4\delta}{\alpha'} + O(\lambda^0)$$

Expanded OPE data:

	$A^{(0)}$		$A^{(1)}$		$A^{(2)}$
$\Delta_{\delta,\ell} =$	$2\sqrt{\delta}\lambda^{\frac{1}{4}}$	+	$\lambda^{-\frac{1}{4}}\Delta^{(1)}_{\delta,\ell}$	+	$\lambda^{-\frac{3}{4}}\Delta^{(2)}_{\delta,\ell}$
$C^2_{\delta,\ell} =$	$C^{2(0)}_{\delta,\ell}$	+	$\lambda^{-\frac{1}{2}}C^{2(1)}_{\delta,\ell}$	+	$\lambda^{-1}C^{2(2)}_{\delta,\ell}$

Integrability:
unprotected operators (Konishi etc.)
from weak to strong coupling:



Plot for $\mathcal{N} = 4$ SYM from [Gromov,Hegedus,Julius,Sokolova;2023].
Analogous ABJM plot does not exist yet.

$\mathcal{N} = 6$ superconformal blocks [Binder,Chester,Jerdee,Pufu;2020]

The 3d $\mathcal{N} = 6$ superconformal blocks are labelled by:

- dimension Δ
- spin ℓ
- discrete symmetries:

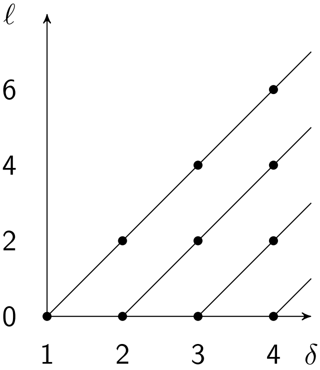
Parity $\mathcal{P}$: $\mathbb{Z}_2$ extending spacetime symmetries $\quad \text{Spin}(3,2) \to \text{Pin}(3,2)$

$\mathcal{Z}$: $\mathbb{Z}_2$ extending R-symmetry group $\quad SO(6) \to O(6)$

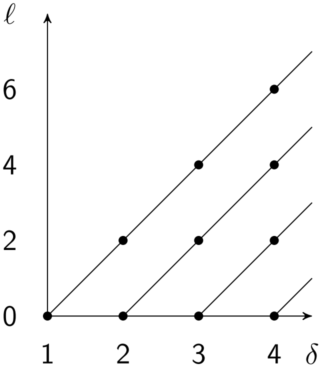
superconf. block	ℓ	$\mathcal{P}$	$\mathcal{Z}$
$\text{Long}_{\Delta,\ell}$	odd	$+$	$+$
$\text{Long}^1_{\Delta,\ell}$	even	$+$	$+$
$\text{Long}^2_{\Delta,\ell}$	even	$-$	$+$
$\text{Long}^3_{\Delta,\ell}$	even, > 0	$-$	$-$

Spectrum from flat space to AdS

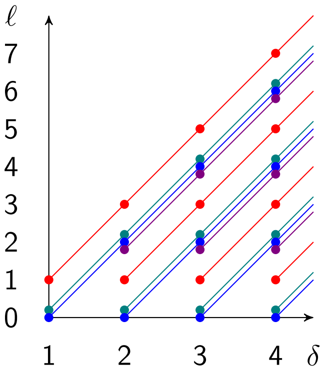
Flat space:



$A^{(0)}(S, T)$ in $\mathcal{N} = 8$ blocks:



$A(S, T)$ in $\mathcal{N} = 6$ blocks:



10d $\mathcal{N} = 2$ long multiplets

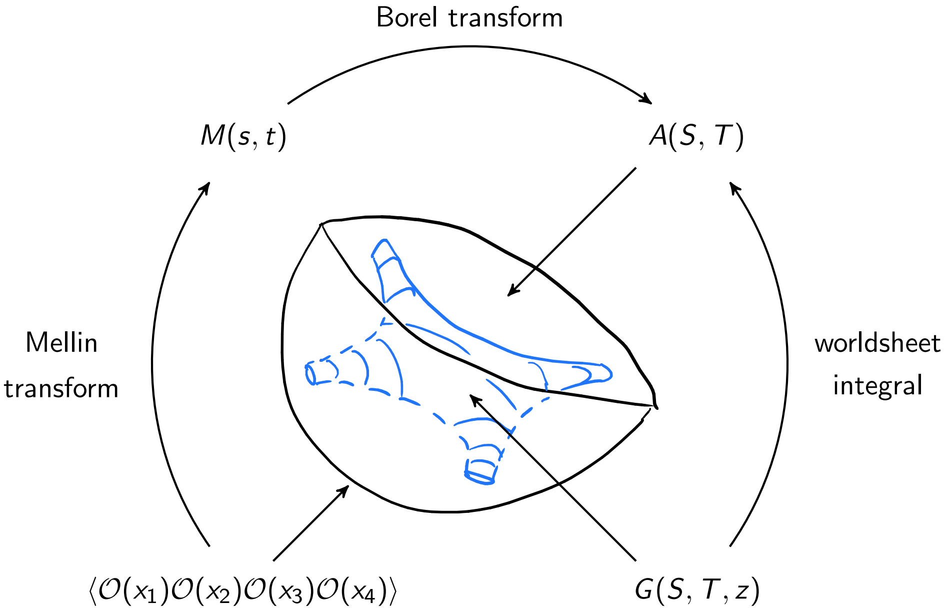
3d $\mathcal{N} = 8$ long multiplets

3d $\mathcal{N} = 6$ long multiplets

flat space limit can be expanded in $\mathcal{N} = 8$ blocks

curvature corrections break $\mathcal{N} = 8$ to $\mathcal{N} = 6$

Integral transforms (again)

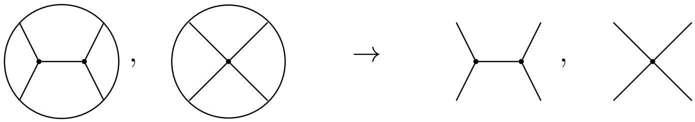


The Borel transform

Borel transform / flat space limit

$$A(S,T) = \lambda^{\frac{3}{2}} \int_{-i\infty}^{i\infty} \frac{d\alpha}{2\pi i} e^{\alpha} \alpha^{-6} M\left(\frac{2\sqrt{\lambda}S}{\alpha}, \frac{2\sqrt{\lambda}T}{\alpha}\right)$$

1. Maps Witten diagrams to Feynman diagrams for $R \to \infty$ [Penedones;2010]



2. Borel summation of the low energy expansion:

$$M(s,t) = \sum_{p,q} \frac{\Gamma(6+p+q)}{\lambda^{\frac{3}{2}}} \left(\frac{s}{2\sqrt{\lambda}}\right)^p \left(\frac{t}{2\sqrt{\lambda}}\right)^q \alpha_{p,q} \quad \Rightarrow \quad A(S,T) = \sum_{p,q} S^p T^q \alpha_{p,q}$$

Pole structure of the AdS amplitude

Borel transform of the OPE leads to:

Pole structure of the AdS amplitude

$$A^{(k)}(S,T) = \frac{R^{(k)}_{3k+1}(T, \text{OPE data})}{(S-\delta)^{3k+1}} + \ldots + \frac{R^{(k)}_{1}(T, \text{OPE data})}{S-\delta} + O((S-\delta)^0)$$

The numerator functions are known explicitly.

Worldsheet ansatz

Ansatz

$$A_i^{(k)}(S,T) = \int d^2z |z|^{-2S-2} |1-z|^{-2T-2} \left(G_{i,1}^{(k)}(z) + |z|^2 G_{i,2}^{(k)}(z) + |1-z|^2 G_{i,3}^{(k)}(z) \right)$$

Assumed properties of $G_{i,j}^{(k)}(S,T,z)$:

- transcendental weight $3k$ (SVMPLs(z), SVMZVs)
- homogeneous degree $2k+2$ polynomial in S, T
- crossing symmetry:

$$A_1(S,T) = A_2(U,T) \qquad A_3(S,T) = A_2(S,U)$$
$$A_4(S,T) = A_5(U,T) \qquad A_6(S,T) = A_5(S,U)$$

Worldsheet ansatz (flat space example)

Ansatz

$$A_i^{(k)}(S,T) = \int d^2z |z|^{-2S-2} |1-z|^{-2T-2} \left(G_{i,1}^{(k)}(z) + |z|^2 G_{i,2}^{(k)}(z) + |1-z|^2 G_{i,3}^{(k)}(z) \right)$$

Assumed properties of $G_{i,j}^{(k)}(S,T,z)$:

- transcendental weight $3k$ (SVMPLs(z), SVMZVs)
- homogeneous degree $2k+2$ polynomial in S, T

Check that it works for flat space:

$$A^{(0)}(S,T) = \frac{\Gamma(1-S)\Gamma(1-T)\Gamma(1-U)}{\Gamma(1+S)\Gamma(1+T)\Gamma(1+U)} \left(\frac{TU}{S}, \frac{ST}{U}, \frac{SU}{T}, \frac{S}{2}, \frac{U}{2}, \frac{T}{2} \right)$$

$$G_{2,1}^{(0)}(S,T) = 0 \qquad G_{2,2}^{(0)}(S,T) = \frac{T^2}{2} \qquad G_{2,3}^{(0)}(S,T) = \frac{S^2}{2}$$

$$G_{5,1}^{(0)}(S,T) = \frac{ST}{2} \qquad G_{5,2}^{(0)}(S,T) = 0 \qquad G_{5,3}^{(0)}(S,T) = 0$$

Solving the constraints (first curvature correction)

Matching the ansatz (82 coefficients)

Worldsheet ansatz

$$G^{(1)}_{i,j}(S,T,z) = \sum_{p,q} c^{(1)}_{i,j,p,q}\left(S^4, S^3T, S^2T^2, ST^3, T^4\right)_p \times \left(\mathcal{L}_{000}(z), \mathcal{L}_{001}(z), \mathcal{L}_{010}(z), \mathcal{L}_{101}(z), \mathcal{L}_{110}(z), \mathcal{L}_{111}(z), \zeta(3)\right)_q$$

and

OPE pole structure

$$A^{(1)}(S,T) = \frac{R^{(1)}_4(T, \text{OPE data})}{(S-\delta)^4} + \ldots + \frac{R^{(1)}_1(T, \text{OPE data})}{S-\delta} + O((S-\delta)^0)$$

fixes all unknowns in both expressions!

Second curvature correction

The ansatz for $A^{(2)}(S, T)$ has 758 rational coefficients.

They are fixed by:

- 749: matching poles with superconformal blocks
- 6: no operator mixing for leading Regge trajectories
- 2: dimension of the first operator (Konishi)
- 1: using one (out of two) localization constraint

Checks

Success! But we made assumptions...

There are direct connections to many other results:

Quantity	Compare with
Wilson coefficients	supersymmetric localization
Conformal dimensions	integrability
High energy limit	classical string scattering in AdS

Let's compare!

Check 1: Low energy expansion

Relates to low energy effective action (SUGRA + derivative interactions)

$$A_i(S,T) = \text{SUGRA} + \sum_{a,b,k=0}^{\infty} \frac{S^a T^b}{\sqrt{\lambda}^k} \alpha_{i,a,b}^{(k)}$$

We get from $A^{(0)}(S,T)$ (flat space), $A^{(1)}(S,T)$, $A^{(2)}(S,T)$ (curvature corrections):

$$A_2(S,T) = \frac{ST}{U} + \frac{1}{\sqrt{\lambda}} \frac{ST+U^2}{6U^2} + \frac{1}{\lambda} \frac{17ST-2U^2}{36U^3} + \ldots \qquad \text{SUGRA}$$

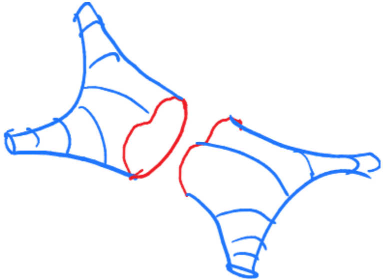
$$+\zeta(3)\left(2S^2T^2 + \frac{1}{\sqrt{\lambda}} \frac{23STU}{3} + \frac{1}{\lambda} \frac{61ST+80U^2}{18} + \ldots \right) \qquad R^4$$

$$+\zeta(5)\left(2S^2T^2(U^2-ST) + \frac{1}{\sqrt{\lambda}} \frac{STU(17ST+118U^2)}{12} + \frac{1}{\lambda} \frac{37U^4+\ldots}{4} + \ldots \right) \qquad D^4R^4$$

$$+\ldots$$

Agrees with localisation! [Binder,Chester,Pufu;2019] Altogether we fully fix D^4R^4.

Check 2: OPE data



We extract the OPE data:

	$A^{(0)}$ data		$A^{(1)}$ data		$A^{(2)}$ data	
$\Delta_{n,\delta,\ell} =$	$2\sqrt{\delta}\lambda^{\frac{1}{4}}$	$+$	$\lambda^{-\frac{1}{4}}\Delta^{(1)}_{n,\delta,\ell}$	$+$	$\lambda^{-\frac{3}{4}}\Delta^{(2)}_{n,\delta,\ell}$	$+\dots$
$C^2_{n,\delta,\ell} =$	$C^{2(0)}_{n,\delta,\ell}$	$+$	$\lambda^{-\frac{1}{2}}C^{2(1)}_{n,\delta,\ell}$	$+$	$\lambda^{-1}C^{2(2)}_{n,\delta,\ell}$	$+\dots$

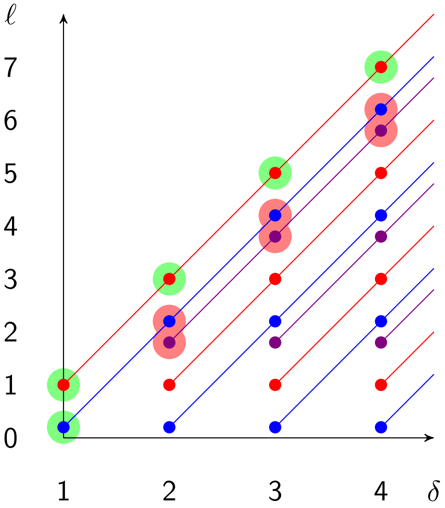
Can we compare with integrability results?

Check 2: Comparing with integrability

superconf. block	ℓ	$\mathcal{P}$	$\mathcal{Z}$
$\text{Long}_{\Delta,\ell}$	odd	$+$	$+$
$\text{Long}^1_{\Delta,\ell}$	even	$+$	$+$
$\text{Long}^2_{\Delta,\ell}$	even	$-$	$+$
$\text{Long}^3_{\Delta,\ell}$	even, > 0	$-$	$-$

(green dot) = dimension matches integrability
[Gromov,Sizov;2014]
[Bombardelli,Cavaglià,Conti,Tateo;2014]

(red dot) = non-degenerate operator without integrability result



4. High energy limit

Why the high energy limit?

What is the next step towards the worldsheet theory?

Flat space [Gross,Mende;1987]:

classical solution (bosonic) of the worldsheet theory	$\rightarrow$	high energy limit ($S, T \rightarrow \infty$) of string amplitudes

An independent way to compute $\lim_{S,T\rightarrow\infty} A(S,T)$, agnostic to many details!

High energy limit via saddle point

The high energy limit of $A^{(0)}(S, T)$ is given by the saddle point $z = \bar{z} = \dfrac{S}{S+T}$

$$\lim_{S,T\to\infty} \int d^2z \, |z|^{-2S} |1-z|^{-2T} \sim e^{-2S \log|\frac{S}{S+T}| - 2T \log|\frac{T}{S+T}|}$$

In AdS the limit can be computed in the same way.

Goal: Compute this exponent from the string action.

Classical solution in flat space

Gross and Mende computed the high energy limit by minimizing the action

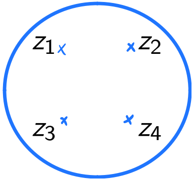
$$\mathcal{S}(X^\mu) = \int d^2\zeta \left(\partial X^\mu(\zeta) \bar{\partial} X_\mu(\zeta) - i \sum_{j=1}^{4} p_j \cdot X(\zeta)\, \delta^{(2)}(\zeta - z_j) \right)$$

EOM: $\partial\bar{\partial} X^\mu = -\frac{i}{2} \sum_j p_j^\mu \delta^{(2)}(\zeta - z_j)$ Virasoro: $\partial X \cdot \partial X = \bar{\partial} X \cdot \bar{\partial} X = 0$

Solution: $X^\mu_{\text{clas}} = -i \sum_j p_j^\mu \log|\zeta - z_j|$

This classical solution gives the correct high energy exponent:

$$\lim_{S,T\to\infty} A^{\text{flat}}(S,T) \sim e^{-\mathcal{S}(X^\mu_{\text{clas}})}\Big|_{z=\frac{S}{S+T}} = e^{-2S\log|\frac{S}{S+T}| - 2T\log|\frac{T}{S+T}|}$$



The AdS path integral

The action for AdS:

$$\mathcal{S}(X,\Lambda) = \int d^2\zeta \left(\partial X^M \bar{\partial} X_M + \Lambda(X^M X_M + R^2) - i \sum_{j=1}^{4} P_j^M X_M \delta^{(2)}(\zeta - z_j) \right)$$

AdS_d is embedded in $\mathbb{R}^{2,d-1} \ni X^M$

$$-R^2 = X^M X_M = -X^0 X^0 + X^\mu X_\mu$$

Eliminate X^0 and expand X^μ around flat space:

$$X^\mu = X_0^\mu + \frac{1}{R^2} X_1^\mu + \ldots \qquad\qquad X_0^\mu = -i \sum_j p_j^\mu \log \left| 1 - \frac{\zeta}{z_j} \right|$$

Equation of motion for X_1^μ:

$$\partial\bar{\partial} X_1^\mu = \partial X_0 \cdot \bar{\partial} X_0 \, X_0^\mu = \frac{i}{4} \sum_{i,j,k} \frac{p_i \cdot p_j}{(\zeta - z_i)(\bar{\zeta} - z_j)} p_k^\mu \log \left| 1 - \frac{\zeta}{z_k} \right|$$

Classical solution in AdS

Equation of motion for X_1^μ:

$$\partial\bar{\partial}X_1^\mu = \partial X_0 \cdot \bar{\partial} X_0 \, X_0^\mu = \frac{i}{8}\sum_{i,j,k} \frac{p_i \cdot p_j}{(\zeta - z_i)(\bar{\zeta} - z_j)} p_k^\mu \mathcal{L}_{z_k}(\zeta)$$

We can "integrate" this using

$$\int d\zeta \frac{\mathcal{L}_w(\zeta)}{\zeta - z_i} \to \mathcal{L}_{z_i w}(\zeta)\,, \qquad \int d\bar{\zeta} \frac{\mathcal{L}_w(\zeta)}{\bar{\zeta} - z_j} \to \mathcal{L}_{w z_j}(\zeta) + \cdots$$

Result:

$$X_{1,\text{clas}}^\mu = \frac{i}{8}\sum_{i,j,k=1}^{4} p_i \cdot p_j \; p_k^\mu \left(\mathcal{L}_{z_i z_k z_j}(\zeta) + \mathcal{L}_{z_k}(z_j)\mathcal{L}_{z_i z_j}(\zeta) - \mathcal{L}_{z_j}(z_k)\mathcal{L}_{z_i z_k}(\zeta)\right)$$

More generally:

$$X_{\text{clas}}^\mu = \mathcal{L}_{|w|=1}(\zeta) + \frac{1}{R^2}\mathcal{L}_{|w|=3}(\zeta) + \frac{1}{R^4}\mathcal{L}_{|w|=5}(\zeta) + \dots$$

Comparison with AdS Virasoro-Shapiro amplitude

$$e^{-\mathcal{S}(X^{\mu}_{\text{clas}})}\Big|_{z=\frac{S}{S+T}} = \exp\left(-SF_1\left(\tfrac{S}{T}\right) - \frac{S^2}{R^2}F_3\left(\tfrac{S}{T}\right) - \frac{S^3}{R^4}F_5\left(\tfrac{S}{T}\right) - O\left(\frac{S^4}{R^6}\right)\right)$$

In the limit $S, T, R \to \infty$ with S/T and S/R fixed, F_5 and further terms vanish!

We successfully compare with AdS Virasoro-Shapiro at the saddle point:

$$e^{-\frac{S^2}{R^2}F_3\left(\frac{S}{T}\right)} = 1 + \frac{1}{R^2}G^{(1)}_{\text{closed}}(z = \tfrac{S}{S+T}) + \frac{1}{R^4}G^{(2)}_{\text{closed}}(z = \tfrac{S}{S+T}) + \dots$$

This implies

$$G^{(2)}_{\text{closed}}(z = \tfrac{S}{S+T}) = \frac{1}{2}\left(G^{(1)}_{\text{closed}}(z = \tfrac{S}{S+T})\right)^2$$

Final result to all orders in S/R:

$$\lim_{S,T,R\to\infty} A^{\text{AdS}}(S,T) = \left(\lim_{S,T\to\infty} A^{\text{flat}}(S,T)\right) e^{-\frac{S^2}{R^2}F_3\left(\frac{S}{T}\right)}$$

Summary: High energy limit

We compared $A(S, T)$ to classical computation a la Gross & Mende.

- Relation to worldsheet action agnostic to fermions and prefactors
- $A(S, T)$ fixed to all orders in S/R

$$\lim_{S,T,R\to\infty} A^{\mathrm{AdS}}(S,T) = \left(\lim_{S,T\to\infty} A^{\mathrm{flat}}(S,T)\right) e^{-\varepsilon_{\mathrm{open/closed}}(S,T)}$$

- The exponent (weight 3 SVMPLs) is the same for all cases studied:

$$\varepsilon^{AdS_4}_{\mathrm{closed}}(S,T) = \varepsilon^{AdS_5}_{\mathrm{closed}}(S,T) = 2\varepsilon^{AdS_5}_{\mathrm{open}}(S/4, T/4)$$

Summary

Checks:

- Low energy expansion
- OPE data for massive strings
- High energy limit

Future directions

- Dimensions for new ABJM operators from integrability
- 1-loop AdS string amplitudes from torus worldsheet integral
- Go beyond the small curvature expansion
- Compute AdS string amplitudes directly from string theory

Thank you!