Integrated correlators in a $\mathcal{N} = 2$ SYM theory with fundamental flavors

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Based on: 2407.03509 with Billò, Frau, Lerda, Pini
and other works (2404.03466, 2207.08846,. . .)

Plan

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Motivation

- The analysis of the strong-coupling regime in an interacting theory is a very difficult problem but, when there is a high amount of symmetry, remarkable progress can be made.

- In particular this happens for $\mathcal{N} = 4$ SYM theory, where many exact results have been found over the years, especially in the planar limit

$$N \to \infty \quad \text{and} \quad \lambda \equiv g^2_{YM} N \quad \text{fixed}$$

Less is known about exact results in $4d\ \mathcal{N} = 2$ **gauge theories**

- Main tools: **supersymmetric localization**, holography and integrability.

Simplest case: $\mathcal{N} = 4$ SYM

- We can recast $\mathcal{N} = 4$ SYM as a $\mathcal{N} = 2$ gauge theory
- It is a Lagrangian theory, whose field content is

Vector multiplet

1 vector field A_μ

2 real scalars $\phi, \overline{\phi}$

2 fermions $\psi_\alpha^A \quad (A = 1, 2)$

Hypermultiplet

4 real scalars $q, \tilde{q} + c.c.$

2 fermions λ_α^A

in the adjoint representation of SU(N)

- Some of the simplest operators are single-trace superconformal primaries

$$O_n(x, Y) = \mathrm{tr}\left(\Phi^{I_1}(x_1) \ldots \Phi^{I_n}(x_n)\right) Y_{I_1} \ldots Y_{I_n} \quad I = 1, \ldots, 6$$

These are local, gauge invariant, 1/2-BPS operators

Simplest case: $\mathcal{N} = 4$ SYM

2-point functions

$$\left\langle O_n(x_1, Y_1)\, O_n(x_2, Y_2) \right\rangle = G_n \, \frac{\mathcal{I}_2(Y)}{|x-y|^{2n}}$$

3-point functions

$$\left\langle O_{n_1}(x_1, Y_1)\, O_{n_2}(x_2, Y_2)\, O_{n_3}(x_3, Y_3) \right\rangle = \frac{G_{n_1,n_2,n_3}\, \mathcal{I}_3(Y)}{|x-z|^{n_1+n_2-n_3} |x-z|^{n_1+n_3-n_2} |y-z|^{n_2+n_3-n_1}}$$

$$C_{n_1,n_2,n_3} = \frac{G_{n_1,n_2,n_3}}{\sqrt{G_{n_1}\, G_{n_2}\, G_{n_3}}} \underset{N \to \infty}{\simeq} \boxed{\frac{\sqrt{n_1\, n_2\, n_3}}{N}}$$

they do not depend on the coupling! [Lee, Minwalla, Rangamani, Seiberg, 1998]

Simplest case: $\mathcal{N} = 4$ SYM

What about four-point functions of primary operators?

Conformal invariance **is not sufficient** to fix space-time dependence, much more complicated perturbatively $\Longrightarrow$ difficult to explore high orders and find **exact** results

$$\langle O_2(x_1, Y_1) \dots O_2(x_4, Y_4) \rangle = \boxed{\frac{\mathcal{I}_4(Y)}{|x_{12}|^4 \, |x_{34}|^4}} \times \boxed{\mathcal{T}(u, v; g)}$$

superconformal symmetry

cross-ratios,coupling

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\,, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

It turned out that an efficient strategy to get **exact** results for four-point correlators is to consider integrated four-point functions.

Integrated correlators

$$\int \prod_{i=1}^{4} d^4 x_i \, \mu(\{x_i\}) \, \mathcal{T}(u, v; g) = \partial_\tau \partial_{\bar\tau} \partial_m^2 \log \mathcal{Z}_{\mathcal{N}=2^*} \big|_{m=0}$$

where $\tau = \tau_1 + i\tau_2 \equiv \frac{\theta}{2\pi} + i\frac{4\pi}{g_{YM}^2}$ [Binder, Chester, Pufu, Wang, 2019]

$$\int \prod_{i=1}^{4} d^4 x_i \, \mu'(\{x_i\}) \, \mathcal{T}(u, v; g) = \partial_m^4 \log \mathcal{Z}_{\mathcal{N}=2^*} \big|_{m=0}$$

[Chester, Pufu, 2020]

$\mathcal{N} = 2^*$ $\longrightarrow$ **massive deformation of $\mathcal{N} = 4$ SYM**

$\mu(\{x_i\})$, $\mu'(\{x_i\})$ $\longrightarrow$ fixed by superconformal symmetry

Where does this result come from?

- Consider the $\mathcal{N} = 2$ *flavor current multiplet*: it is made up of **three scalars** $\mathcal{J}^{IJ} = \mathcal{J}^{JI}$ $(I, J = 1, 2)$, **two chiral fermions** $X_{I\alpha}$, **two anti-chiral fermions** $\overline{X}_I^{\dot{\alpha}}$, **two real scalars** K and $\overline{K}$ and **one conserved current** j_μ.
- This multiplet can be realized in terms of the hypermultiplet.
- Then consider supersymmetry-preserving deformations realized by operators of this multiplet.
- In SUSY theories the easiest way of studying action deformations involving a specific multiplet is by coupling it to a background off-shell multiplet. [Festuccia, Seiberg, 2011]

Integrated correlators

- In this case the off-shell multiplet that we can consider is a vector multiplet, so that the deformation becomes

$$\Delta S = \int d^4x \left(A_\mu j^\mu - \phi K - \overline{\phi}\,\overline{K} + D_{IJ}\mathcal{J}^{IJ} + \psi_I X^I + \overline{\psi}_I \overline{X}^I \right)$$

which preserves $\mathcal{N} = 2$ superconformal symmetry.

- Vector multiplet as a background $\Rightarrow$ give its fields expectation values **but** superconformal symmetry is broken **unless** one takes
$A_\mu = \psi^I = \overline{\psi}^I = D_{IJ} = 0\,, \quad \phi = \overline{\phi} = -m$
so that

$$\Delta S \longrightarrow S_m = m \int d^4x \left(K + \overline{K}\right)$$

[Binder, Freedman, Pufu, Zan, 2021]

preserves Poincaré supersymmetries.

Integrated correlators

- One can do the same procedure on the **sphere** $\mathbb{S}^4$. In this case the deformed action is

$$S_m = m \int d^4x \sqrt{g} \left(K + \overline{K} + \frac{i}{R}\mathcal{J} \right) , \quad \mathcal{J} = \mathcal{J}_{11} + \mathcal{J}_{22}$$

If one realizes the fields of the flavor current multiplet in terms of the hypermultiplet fields

$$\mathcal{J} = \mathrm{tr}\, qq + \mathrm{tr}\, \tilde{q}\tilde{q} + \mathrm{tr}\, \overline{q}\, \overline{q} + \mathrm{tr}\, \overline{\tilde{q}}\, \overline{\tilde{q}} \quad \textbf{moment-map} \in \textbf{20'} \text{ of } SU(4)_R$$

$$K = -i\, \mathrm{tr} \lambda_I\, \lambda^I , \qquad \overline{K} = -i\, \mathrm{tr} \overline{\lambda}_I\, \overline{\lambda}^I ,$$

this deformation correponds to the **mass-deformation** of $\mathcal{N} = 4$ SYM, called $\mathcal{N} = 2^*$ SYM theory, obtained by giving a mass to the adjoint hypermultiplet of $\mathcal{N} = 4$.

Integrated correlators

- By taking derivatives with respect to m one gets **integrated correlators** of moment-map operators $\mathcal{J}$ and their descendants K

$$\partial_m^4 \int \mathcal{D}[\text{fields}] e^{-S-S_m}\Big|_{m=0} = \int \mathcal{D}[\text{fields}] e^{-S} \left(\int d^4x \sqrt{g} \left(K + \overline{K} + i\mathcal{J} \right) \right)^4$$

⬇

$$\int \prod_{i=1}^{4} d^4x_i \sqrt{g(x_i)} \left\langle \left(K + \overline{K} + i\mathcal{J} \right)^4 \right\rangle$$

⬇

$$\int \prod_{i=1}^{4} d^4x_i \sqrt{g(x_i)} \left\langle \mathcal{J}^4 - \mathcal{J}^2 \left(K + \overline{K} \right)^2 + \left(K + \overline{K} \right)^4 \right\rangle$$

Integrated correlators

- Exploiting superconformal **Ward identities** one can express these correlators only in terms of 4-point functions of the moment-map operators integrated over a certain measure

$$\int \prod_{i=1}^{4} d^4x_i \, \mu'(\{x_i\}) \left\langle \mathcal{J}\mathcal{J}\mathcal{J}\mathcal{J} \right\rangle$$

$$\mathcal{J} \equiv O_2$$

- The final step is to understand why supersymmetric localization is such a useful tool to evaluate these integrated four-point functions.

Summary of the strategy

Deform the original theory preserving $\mathcal{N} = 2$ supersymmetry

↓

Place the deformed theory on the sphere $\mathbb{S}^4$

↓

Take derivatives with respect to the parameters of the deformation

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Deform the original theory preserving $\mathcal{N} = 2$ supersymmetry

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Take derivatives with respect to the parameters of the deformation

↓

Apply localization to compute the deformed partition function

Localization and matrix model

Localization

Supersymmetric localization maps the computation of the **partition function** $\mathcal{Z}$ of a $4d$ $\mathcal{N}=2$ gauge theory to a **matrix model** on $\mathbb{S}^4$

[Pestun, 2007]

Path integrals $\Longrightarrow$ Finite dimensional integrals

$$\int \mathcal{D}[\text{fields}] e^{-S-S_m} \quad \Longrightarrow \quad \int da\, e^{-\mathrm{tr}\, a^2 - m^2 S_2 - m^4 S_4 + \ldots}$$

where $a \equiv a^b T_b$ are $N \times N$ traceless Hermitian matrices and e.g.

$$S_2 = -\frac{1}{2}\left[\sum_{\ell=1}^{\infty}\sum_{n=0}^{2\ell}(-1)^{\ell+n}\frac{(2\ell+1)!}{n!(2\ell-n)!}\zeta_{2\ell+1}\left(\frac{\lambda}{8\pi^2 N}\right)^{\ell}\mathrm{tr}\, a^{2\ell-n}\,\mathrm{tr}\, a^n\right]$$

Localization

Hence for instance we get

$$\partial_m^2 \log \mathcal{Z}\big|_{m=0} = \int da\, e^{-\mathrm{tr} a^2} S_2 = \langle S_2 \rangle$$

Basic ingredients for the partition function in the **matrix model**: expectation values of multitrace operators in the Gaussian theory for which there exist recursion relations easy to implement in the planar limit.

Take-home message

The computation of the derivatives of the partition function and then of integrated four-point functions of primary operators are reduced to the computation of Gaussian integrals.

Matrix model

This is a very important result because

- Matrix model computations led to exact results both in the **planar** expansion and also for **any** value of N for these kind of correlators in $\mathcal{N} = 4$ SYM.

- In the large-N limit the **strong-coupling** expansions of these integrated four-point functions of superconformal primary operators determine useful constraints on higher derivative corrections of **closed string scattering amplitudes** of 4 massless string states (gravitons and their superpartners).

$\mathcal{N} = 2$ *superconformal gauge theories*

$\mathcal{N} = 2$ superconformal gauge theories

- Much progress has also been made in $\mathcal{N} = 2$ superconformal gauge theories.
- In particular, integrated correlators were studied in a $\mathcal{N} = 2$ SCFT with Sp(N) gauge group, one anti-symmetric hypermultiplet and four fundamental ones with SO(8) flavor symmetry. [Behan, Chester, Ferrero, 2022]
- This theory is dual to N D3 branes, 4 D7 branes, and an $O7$ plane in Type IIB string theory.
- Specifically, in the large-N limit the four-point function of flavour multiplets is dual to the scattering of **SO(8) open** string gluons on $AdS_5 \times \mathbb{S}^3$.
  [Alday, Chester, Hansen, Zhong, 2024]
  [Alday, Hansen, 2024]

The **D** theory

- We consider a $\mathcal{N} = 2$ SCFT, dubbed **D** theory, with **SU(N)** gauge group, **two** anti-symmetric hypers, **four** fundamental and **U(4)** flavour symmetry ($\beta = 0$)
- In Type IIB string theory this model can be engineered with N fractional D3-branes in a $\mathbb{Z}_2$-orbifold probing an O7-orientifold background and with four D7 branes plus their orientifold images.
- Flavour group of the four fundamental hypers
- Gauge group of the D7 branes world-volume theory

→ $U(4)$

- Also in this case the **D7-sector** consists of open string states which in the large-N limit propagate on $AdS_5 \times \mathbb{S}^3$. Among these states there are the **U(4) gluons**.

The **D** theory

- The U(4) gluons are dual to the **moment-map** operators $\mathcal{J}$ belonging to the flavor current multiplet.
- Integrated 4-point functions can be studied exploiting localization

$$\partial_{m_A}\partial_{m_B}\partial_{m_C}\partial_{m_D} \log \mathcal{Z}_{\mathbf{D}^*}\big|_{m=0} = \int \prod_{i=1}^{4} dx_i \; \mu'(\{x_i\}) \; \langle \mathcal{J}^A(x_1) \dots \mathcal{J}^D(x_4) \rangle_{\mathbf{D}}$$

$\mu'(\{x_i\})$ fixed by superconformal symmetry

Holographically similar to Sp(N) theory, but **different** SCFTs. We expect a similar behaviour at **strong coupling** for these correlators, but this is very tough to verify ➔ much more involved **matrix model**!

The massless matrix model

We now have a non-trivial interaction action

$$\mathcal{Z}_{\mathbf{D}} = \int da\, \mathrm{e}^{-\mathrm{tr}\, a^2 - S_{\mathrm{int}}}$$

[Pestun, 2007]

where

$$S_{int} = 4 \sum_{k=1}^{\infty} \Big(-\frac{\lambda}{8\pi^2 N}\Big)^{k+1} (2^{2k}-1) \frac{\zeta_{2k+1}}{k+1}\, \mathrm{tr}\, a^{2k+2} \Longleftarrow \text{like in Sp(N) theory}$$

$$+ \underbrace{2 \sum_{k=1}^{\infty} \sum_{\ell=1}^{k-1} (-1)^k \Big(\frac{\lambda}{8\pi^2 N}\Big)^{k+1} \binom{2k+2}{2\ell+1} \frac{\zeta_{2k+1}}{k+1}\, \mathrm{tr}\, a^{2\ell+1} \mathrm{tr}\, a^{2k-2\ell+1}}_{}$$

$S^{\mathbf{E}}_{int} \Rightarrow \mathcal{N} = 2$ SU(N) SCFT with 1 symm+1 antisymm hypers

Hence with respect to $\mathcal{N} = 4$ we now have to deal with this term which has non-trival dependence on the coupling.

The massless matrix model

Solution: perform the change of basis

$$\mathrm{tr}\, a^k = \left(\frac{N}{2}\right)^{\frac{k}{2}} \sum_{\ell=0}^{\lfloor \frac{k-1}{2} \rfloor} \sqrt{k-2\ell} \begin{pmatrix} k \\ \ell \end{pmatrix} \underbrace{\mathcal{P}_{k-2\ell}}_{\text{orthonormal for } N\to\infty} + \underbrace{\langle \mathrm{tr}\, a^k \rangle_0}_{\text{VEV in free matrix model}}$$

[Beccaria, Billò, Galvagno, Hasan, Lerda, 2020]

so that one gets an **exact** expression for S_{int} for all values of λ

$$S_{int} = -\frac{1}{2} \sum_{k,\ell=1}^{\infty} \mathcal{P}_{2k+1}\, \mathsf{X}_{2k+1,2\ell+1}\, \mathcal{P}_{2\ell+1} - \sum_{k=1}^{\infty} \mathsf{Y}_{2k}\, \mathcal{P}_{2k}$$

$$\mathsf{X}_{k,\ell} = -8(-1)^{\frac{k+\ell+2k\ell}{2}} \sqrt{k\,\ell} \int_0^\infty \frac{dt}{t} \frac{e^t}{(e^t-1)^2} J_k\left(\frac{t\sqrt{\lambda}}{2\pi}\right) J_\ell\left(\frac{t\sqrt{\lambda}}{2\pi}\right)$$

$$\mathsf{Y}_{2k} = (-1)^{k+1}\, 2\sqrt{2k} \int_0^\infty \frac{dt}{t} \frac{e^t}{(e^t+1)^2} J_{2k}\left(\frac{\sqrt{\lambda}\, t}{\pi}\right) - \delta_{k,1} \frac{\sqrt{2}\log 2}{4\pi^2}\, \lambda$$

1- and 2-point functions

This result allows us to find

$$\langle \mathcal{P}_{2n} \rangle_{\mathbf{D}} = \mathsf{Y}_{2n} + \frac{\sqrt{2n}}{2N} \left(\mathsf{Y}^2 - 2\lambda\, \partial_\lambda \mathcal{F}_{\mathbf{E}} \right) + O\left(\frac{1}{N^2}\right)$$

$$\langle \mathcal{P}_{2n} \mathcal{P}_{2m} \rangle_{\mathbf{D}} - \langle \mathcal{P}_{2n} \rangle_{\mathbf{D}} \langle \mathcal{P}_{2n} \rangle_{\mathbf{D}} = \delta_{n,m} + \frac{\sqrt{2n}\sqrt{2m}\ \mathsf{Y}}{N} + O\left(\frac{1}{N^2}\right)$$

with

$$\mathsf{Y} \equiv \sum_{k=1}^{\infty} \sqrt{2k}\, \mathsf{Y}_{2k} = \int_0^{\infty} \frac{dt}{t} \frac{e^t}{(e^t+1)^2} \left[\frac{\sqrt{\lambda}\, t}{\pi} J_1\left(\frac{\sqrt{\lambda}\, t}{\pi}\right) \right] - \frac{\log 2}{2\pi^2} \lambda$$

$$\mathcal{F}_{\mathbf{E}} = \frac{1}{2} \mathrm{tr} \log \left(1 - \mathsf{X}\right) + O\left(\frac{1}{N^2}\right)$$

They will be useful in a moment!

The massive matrix model

We consider a mass-deformation of the **D** theory, giving mass to the **four fundamental** hypers. The small-mass expansion of the massive matrix model in the large-N limit becomes

$$\mathcal{Z}_{\mathbf{D}^*} = \int da \; e^{-\mathrm{tr} a^2} \; e^{-S_{int} - \sum_{i=1}^4 m_i^2 \, S_2 - \sum_{i=1}^4 m_i^4 \, S_4 + O(m^6)}$$

where S_2 and S_4 are single-trace deformations. We can have three different mass combinations for the fourth order derivatives of $\mathcal{F}_{\mathbf{D}^*} = -\log \mathcal{Z}_{\mathbf{D}^*}$

$$-\partial^4_{m_i} \mathcal{F}_{\mathbf{D}^*}\Big|_{m=0} = -24 \left\langle S_4 \right\rangle_{\mathbf{D}} + 12 \left\langle {S_2}^2 \right\rangle_{\mathbf{D}} - 12 \left\langle S_2 \right\rangle^2_{\mathbf{D}}$$

$$-\partial^2_{m_i} \partial^2_{m_j} \mathcal{F}_{\mathbf{D}^*}\Big|_{m=0} = 4 \left\langle {S_2}^2 \right\rangle_{\mathbf{D}} - 4 \left\langle S_2 \right\rangle^2_{\mathbf{D}}$$

$$-\partial_{m_1} \partial_{m_2} \partial_{m_3} \partial_{m_4} \mathcal{F}_{\mathbf{D}^*}\Big|_{m=0} = 0$$

The massive matrix model

We need to compute the r.h.s. of these equations ➔ write S_2 and S_4 in terms of the $\mathcal{P}$ operators. We find **exact** expression in the coupling λ for the first three $1/N$ orders

$$S_4 = -\frac{N}{12}\frac{4\pi}{\sqrt{\lambda}}\,\mathsf{Z}_1^{(3)} - \frac{1}{12}\sum_{k=1}^{\infty}(-1)^k\sqrt{2k}\;\mathsf{Z}_{2k}^{(4)}\,\mathcal{P}_{2k}$$
$$- \frac{1}{24\,N}\left[\frac{\sqrt{\lambda}}{4\pi}\,\mathsf{Z}_1^{(5)} + \frac{1}{6}\Big(\frac{\sqrt{\lambda}}{4\pi}\Big)^2\mathsf{Z}_2^{(6)}\right] + O\Big(\frac{1}{N^3}\Big)$$

$$S_2 = \sum_{k=1}^{\infty}(-1)^k\sqrt{2k}\;\mathsf{Z}_{2k}^{(2)}\,\mathcal{P}_{2k}$$

with

$$\mathsf{Z}_n^{(p)} = \int_0^{\infty}\frac{dt}{t}\,\frac{e^t\,t^{\,p}}{(e^t-1)^2}\,J_n\Big(\frac{\sqrt{\lambda}\,t}{2\pi}\Big)$$

Results

- Weak coupling $\Longrightarrow$ Completely **different** from Sp(N) theory

$$-\partial^4_{m_i}\mathcal{F}_{\mathbf{D}^*}\Big|_{m=0} \underset{\lambda\to 0}{\sim} N\big(12\,\zeta_3 - 60\,\zeta_5\,\hat{\lambda} + 210\,\zeta_7\,\hat{\lambda}^2 + \cdots\big) + \big(54\,\zeta_3^2\,\hat{\lambda}^2 + 360\,\zeta_3\,\zeta_5\,\hat{\lambda}^3 - 4725\,\zeta_3\,\zeta_7\,\hat{\lambda}^4 + \cdots\big) + \frac{1}{N}\big(60\,\zeta_5\,\hat{\lambda} - 525\,\zeta_7\,\hat{\lambda}^2 + 2520\,\zeta_9\,\hat{\lambda}^3 + \cdots\big) + O\Big(\frac{1}{N^2}\Big)$$

- Strong coupling $\Longrightarrow$ **Similar** to Sp(N) theory (overall coefficient)

$$-\partial^4_{m_i}\mathcal{F}_{\mathbf{D}^*}\Big|_{m=0} \underset{\lambda\to \infty}{\sim} \frac{16\pi^2}{\lambda}N + 3\log\lambda + 6\gamma - 6\log(4\pi) - 3\,\zeta_3 + 11 + \frac{3}{4N}\Big(1 - \frac{2\log 2}{\pi^2}\,\lambda\Big) + O\Big(\frac{1}{N^2}\Big)$$

[Billò, Frau, Lerda, Pini, PV, 2024]

These results furnish constraints for the dual gluon amplitudes in AdS

[Alday, Chester, Hansen, Zhong, 2024]
[Alday, Hansen, 2024]

Conclusions and outlook

We studied the derivatives of the free energy of the $\mathbf{D}^*$ theory in the large-N expansion, obtaining exact expressions in λ and derived their strong coupling limit.

- It would be interesting to find a systematic way to compute **higher orders in the 1/N expansion**.
- It would be important to explore the **planar limit at fixed YM coupling**, where the instantons cannot be neglected, to check if provide completion of perturbative results into **modular functions**.
- Extend this approach to other $\mathcal{N} = 2$ superconformal gauge theories and to other kind of integrated correlators (with determinant operators, Wilson loop, ...).

Thanks for your attention!

Backup slides

U(4) flavour group

Let us show how the $\mathbb{Z}_2$-orbifold projection acts on the initial SO(8) gauge group of the eight D7-branes in the orientifold background. Let Λ be a Hermitian anti-symmetric 8×8 Chan-Paton matrix in the $\mathfrak{so}(8)$ algebra. Under the $\mathbb{Z}_2$-orbifold it transforms as

$$\Lambda \to \gamma \Lambda \gamma^{-1} \quad \text{with} \quad \gamma = \begin{pmatrix} 0 & -\mathrm{i}\,\mathbb{1} \\ \mathrm{i}\,\mathbb{1} & 0 \end{pmatrix}$$

[Gimon, Polchinski, 1996]

where we have written the matrix in 4×4 blocks. Thus, Λ is invariant under the orbifold only if it takes the form

$$\begin{pmatrix} \mathrm{A} & \mathrm{i}\,\mathrm{S} \\ -\mathrm{i}\,\mathrm{S} & \mathrm{A} \end{pmatrix} \quad \text{with} \quad \mathrm{A}^t = -\mathrm{A}\,, \quad \mathrm{A}^* = -\mathrm{A}\,, \quad \mathrm{S}^t = \mathrm{S}\,, \quad \mathrm{S}^* = \mathrm{S}$$

Matrices of this form represent the embedding into $\mathfrak{so}(8)$ of a $\mathfrak{u}(4)$ Hermitian matrix $\mathrm{A} + \mathrm{S}$.

U(4) mass combinations

In the $\mathbf{D}^*$ theory we restrict the masses to be along the four Cartan directions of U(4) labeled by $i = 1, \ldots, 4$. To find the U(4) invariant mass combinations, recall that the four Cartan generators $\boldsymbol{\lambda}^i$ in the defining representation of U(4) must be embedded into 8×8 matrices as

$$\begin{pmatrix} 0 & \mathrm{i}\,\boldsymbol{\lambda}^i \\ -\mathrm{i}\,\boldsymbol{\lambda}^i & 0 \end{pmatrix}$$

So we can consider the combination of these embedded Cartan generators

$$M = \begin{pmatrix} 0 & \begin{matrix} \mathrm{i}\,m_1 & 0 & 0 & 0 \\ 0 & \mathrm{i}\,m_2 & 0 & 0 \\ 0 & 0 & \mathrm{i}\,m_3 & 0 \\ 0 & 0 & 0 & \mathrm{i}\,m_4 \end{matrix} \\ \begin{matrix} -\mathrm{i}\,m_1 & 0 & 0 & 0 \\ 0 & -\mathrm{i}\,m_2 & 0 & 0 \\ 0 & 0 & -\mathrm{i}\,m_3 & 0 \\ 0 & 0 & 0 & -\mathrm{i}\,m_4 \end{matrix} & 0 \end{pmatrix}$$

U(4) mass combinations

This matrix satisfies

$$\mathrm{tr} M^{2k+1} = 0 \quad \mathrm{tr} M^{2k} = 2\sum_{i=1}^{4} m_i^{2k} \quad \mathrm{Pfaff}(M) = m_1\, m_2\, m_3\, m_4$$

From this we see that at order 4 in the masses, there are three independent U(4)-invariant structures, which we can take to be

$$\sum_{i=1}^{4} m_i^4 = \frac{1}{2}\,\mathrm{tr} M^4$$

$$\sum_{i<j=1}^{4} m_i^2\, m_j^2 = -\frac{1}{4}\,\mathrm{tr} M^4 + \frac{1}{8}\left(\mathrm{tr} M^2\right)^2$$

$$m_1\, m_2\, m_3\, m_4 = \mathrm{Pfaff}(M)$$

Matrix model **E** theory

At leading order in the large-N expansion

- $$\left\langle P_{2n}\right\rangle_{\mathbf{E}} = -\frac{\sqrt{2k}\,\lambda\,\partial_\lambda \mathcal{F}_{\mathbf{E}}}{N}$$
- $$\left\langle P_{2n} P_{2m}\right\rangle_{\mathbf{E}} = \delta_{n,m}$$
- $$\left\langle P_{2n+1} P_{2m+1}\right\rangle_{\mathbf{E}} = \mathsf{D}_{2n+1,2m+1}$$

$$\mathsf{D}_{n,m} \equiv \left(\frac{1}{1-\mathsf{X}}\right)_{n,m}$$

[Beccaria, Billò, Frau, Lerda, Pini, 2021]

- $$\left\langle P_{2n+1} P_{2m+1} P_{2n+2m+2}\right\rangle_{\mathbf{E}} = \frac{\sqrt{2n+2m+2}}{N}\mathsf{d}_{2n+1}\mathsf{d}_{2m+1}$$

$\mathsf{d}_k = \sum_{k'} \sqrt{k'}\mathsf{D}_{k,k'}$

[Billò, Frau, Lerda, Pini, PV, 2022]

Evaluate VEVs in the **D** theory matrix model

For instance for the 1-point functions

$$\langle \mathcal{P}_{2n} \rangle_{\mathbf{D}} = \frac{\left\langle \mathcal{P}_{2n} \exp\left(\sum_k \mathsf{Y}_{2k}\, \mathcal{P}_{2k} \right) \right\rangle_{\mathbf{E}}}{\left\langle \exp\left(\sum_k \mathsf{Y}_{2k}\, \mathcal{P}_{2k} \right) \right\rangle_{\mathbf{E}}}$$

Expanding in Y_{2k}, we get

$$\langle \mathcal{P}_{2n} \rangle_{\mathbf{D}} = \langle \mathcal{P}_{2n} \rangle_{\mathbf{E}} + \sum_{k=1}^{\infty} \mathsf{Y}_{2k} \, \langle \mathcal{P}_{2n}\, \mathcal{P}_{2k} \rangle_{\mathbf{E}}^{c} + \frac{1}{2} \sum_{k,\ell=1}^{\infty} \mathsf{Y}_{2k}\, \mathsf{Y}_{2\ell} \, \langle \mathcal{P}_{2n}\, \mathcal{P}_{2k}\, \mathcal{P}_{2\ell} \rangle_{\mathbf{E}}^{c} + \dots$$

Same strategy for 2-point functions.

Details on the strong coupling

Let us present an example. $\mathrm{Z}_n^{(p)}$ is defined as

$$\mathrm{Z}_n^{(p)} = \int_0^\infty \frac{dt}{t} \frac{e^t\, t^p}{(e^t-1)^2} J_n\Big(\frac{\sqrt{\lambda}\, t}{2\pi}\Big)$$

for $n \geq 1$ and $p > 1$. In order to study its strong coupling expansion, we use the Mellin-Barnes integral representation of the **Bessel function**

$$J_n(x) = \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i} \frac{\Gamma(-s)}{\Gamma(s+n+1)} \Big(\frac{x}{2}\Big)^{2s+n}$$

and obtain

$$\mathrm{Z}_n^{(p)} = \int_0^\infty \frac{dt}{t} \frac{e^t\, t^p}{(e^t-1)^2} \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i} \frac{\Gamma(-s)}{\Gamma(s+n+1)} \Big(\frac{\sqrt{\lambda}\, t}{4\pi}\Big)^{2s+n}$$

Details on the strong coupling

Evaluating the t-integral, we get

$$Z_n^{(p)} = \int_{-i\infty}^{+i\infty} \frac{ds}{2\pi i} \frac{\Gamma(-s)\,\Gamma(2s+n+p)\,\zeta_{2s+n+p-1}}{\Gamma(s+n+1)} \left(\frac{\sqrt{\lambda}}{4\pi}\right)^{2s+n}$$

When $\lambda \to \infty$ this integral receives contributions from poles on the negative real axis of s. Summing the residues over such poles, one finds

$$Z_n^{(p)} \underset{\lambda\to\infty}{\sim} -\frac{1}{2}\sum_{k=0}^{\infty} \frac{(2k-1)\,B_{2k}}{(2k)!} \frac{\Gamma\left(\frac{n+p}{2}+k-1\right)}{\Gamma\left(\frac{n-p}{2}+2-k\right)} \left(\frac{4\pi}{\sqrt{\lambda}}\right)^{p+2k-2}$$

where B_{2k} are the **Bernoulli numbers**. When n and p are both even or both odd, this asymptotic expansion terminates after a finite number of terms or even disappears as for example in $Z_1^{(5)}$ or $Z_2^{(6)}$.

Strong coupling expansions

The $\log 2$ terms can be removed by introducing a shifted 't Hooft coupling defined as

$$\frac{1}{\lambda'} = \frac{1}{\lambda} + \frac{\log 2}{2\pi^2 N} \; .$$

In terms of λ' we have

$$-\partial^4_{m_i} \mathcal{F}_{\mathbf{D}^*}\Big|_{m=0} \underset{\lambda'\to\infty}{\sim} \frac{16\pi^2}{\lambda'} N + 3\log\lambda' + 3f(N) - 8\log 2 + 3\,\zeta_3$$

$$-\partial^2_{m_i}\partial^2_{m_j} \mathcal{F}_{\mathbf{D}^*}\Big|_{m=0} \underset{\lambda'\to\infty}{\sim} \log\lambda' + f(N)$$

where

$$f(N) = 2\gamma - 2\log(4\pi) - 2\,\zeta_3 + \frac{11}{3} + \frac{1}{4N} + O\Big(\frac{1}{N^2}\Big)$$