The $i\varepsilon$-Prescription for String Amplitudes and Regularized Modular Integrals

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String Theory as a Bridge Between Gauge Theory and Quantum Gravity, 17 February 2025

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Correlation functions

$$\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\ldots\mathcal{O}_n(x_n)\rangle$$

and amplitudes

$$\langle \mathbf{p}'_1 \ldots \mathbf{p}'_{n'}|S|\mathbf{p}_1 \ldots \mathbf{p}_n\rangle$$

are at the heart of quantum field theory.

Large effort to include all perturbative and non-perturbative effects, and to increase n.

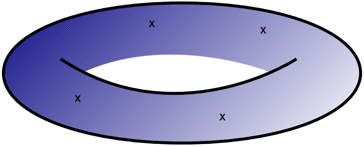
Motivation to study theories where such effects can be included.

This talk:
Explain various techniques for the explicit evaluation of amplitudes of closed and open strings. Some of the techniques have been used earlier in the context of quantum black holes, and topological gauge theory.

Based on:

- 2411.02517 with Z.-Z. Wang
- 1901.03540 with G. Korpas, G. W. Moore, I. Nidaiiev

Amplitudes in String Theory



Σ_1

In string theory, one typically considers an Euclidean worldsheet theory, for example 24 compact bosons $I, J = 1, \ldots, 24$,

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \, g_{IJ} \, \partial_\mu X^I \partial^\mu X^J$$

The n-point, g-loop contribution to a space-time amplitude corresponds to the integral a correlation functions of the worldsheet theory over the moduli space $\mathcal{M}_g$ of genus g Riemann surfaces Σ_g,

$$\mathcal{A}(k_1, k_2, \ldots)_g = \int_{\mathcal{M}_g} d\Omega \int_{\Sigma_g} (\prod_j d^2\sigma_j) \, \langle V(k_1, \sigma_1) V(k_2, \sigma_2) \ldots \rangle_{\Sigma_g}$$

Example 1: one-loop, closed string contribution to the vacuum energy of the bosonic string in 26 dimensions.

Partition function:

$$Z(\tau, \bar{\tau}) = \frac{1}{y^{12}} \frac{1}{|\eta(\tau)|^{48}}, \qquad \tau = x + iy,$$

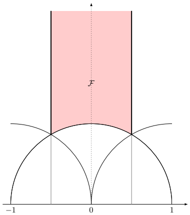
with $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$.

The worldsheet theory is (super)conformal, such that correlation functions transform in a canonical way under large coordinate deformations.

Indeed Z is invariant under

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

Moduli space for Σ_1:
$\mathcal{M}_1 = \mathbb{H}/SL(2,\mathbb{Z}) = \mathcal{F}_\infty$

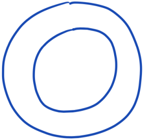


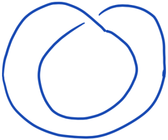
The amplitude reads

$$\mathcal{A}_0 = \int_{\mathcal{F}} i d\tau \wedge d\bar{\tau} \, \frac{1}{y^{14}} \frac{1}{|\eta(\tau)|^{48}}$$

Example 2: one-loop, closed string contribution to the vacuum energy of the bosonic string in 26 dimensions.

The amplitude receives a contribution from the annulus and Möbius strip.

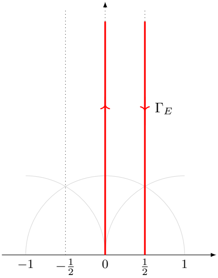




The combined amplitude can be expressed as

$$A_{0,E} = -i \int_{\Gamma(E)} d\tau \, \frac{1}{\eta(\tau)^{24}},$$

where $\Gamma(E)$ is the contour depicted in the figure.



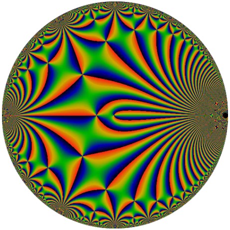
Modular forms

⇒ Modular forms are crucial ingredients, so let me review the relevant aspects on the next few slides.

A modular form for $SL(2,\mathbb{Z})$ is a function $f : \mathbb{H} \to \mathbb{C}$, such that

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^w f(\tau) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

We can consider f as function on the unit disc using the change of variables $q = e^{2\pi i\tau}$



Phase plot of modular invariant j-function.

More generally, we can consider a vector-valued modular forms $\vec{f} = (f_1, \ldots, f_d)$, which transforms as

$$f_\mu\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^w \sum_{\nu=1}^{d} M_\mu^\nu(\gamma)\, f_\nu(\tau), \qquad \gamma \in SL(2, \mathbb{Z})$$

Modular forms often contain interesting arithmetic information.

Example 1: Let $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty}(1 - q^n)$, which is a modular form of weight 1/2 and multiplier system $\xi(\gamma)$, then

$$\begin{aligned}\frac{1}{\eta(\tau)} &= q^{-1/24} \sum_{n=0}^{\infty} p(n)\, q^n \\ &= q^{1/24}(1 + q + 2\,q^2 + 3\,q^3 + \cdots + 190\,569\,292\,q^{100} + \ldots)\end{aligned}$$

with the coefficients $p(n)$ the number of partitions of the integer n.

Example 2: The Jacobi theta series

$$\vartheta_2(\tau) = \sum_{n\in\mathbb{Z}+1/2} q^{n^2/2}, \quad \vartheta_3(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2}, \quad \vartheta_4(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/2}$$

form a three-dimensional vector-valued modular form, with multiplier system generated by

$$M(T) = \begin{pmatrix} e^{\pi i/4} & & \\ & & 1 \\ & 1 & \end{pmatrix}, \qquad M(S) = e^{-\pi i/4} \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$$

Hardy-Ramanujan-Rademacher Circle Method

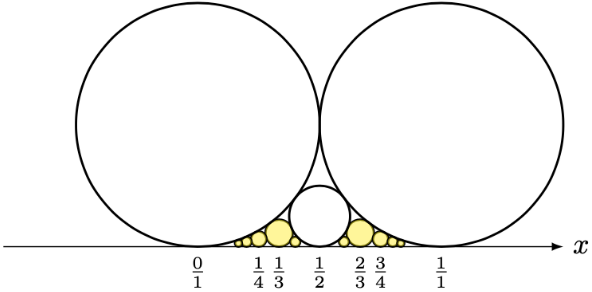
General Fourier expansion:

$$f_\mu(\tau) = \sum_{m \in \mathbb{N}} F_\mu(m - \Delta_\mu)\, q^{m-\Delta_\mu},$$

with coefficients

$$F_\mu(m - \Delta_\mu) = \int_{Yi}^{1+Yi} f_\mu(\tau)\, q^{-m+\Delta_\mu}\, d\tau$$

The contour is judiciously chosen in terms of Ford circles:



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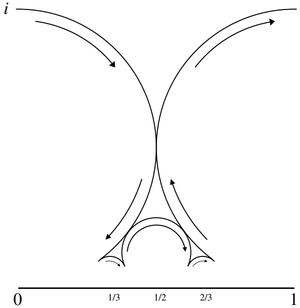


Gives an exact expression in terms of polar coefficients for $w \leq 0$,

$$F_\mu(m-\Delta_\mu) = 2\pi \sum_{n-\Delta_\nu<0} F_\nu(n-\Delta_\nu) \sum_{c=1} \frac{1}{c} K_c(m-\Delta_\mu, n-\Delta_\nu) \times \left(\frac{|n-\Delta_\nu|}{m-\Delta_\mu}\right)^{\frac{1-w}{2}} I_{1-w}\left(\frac{4\pi}{c}\sqrt{(m-\Delta_\mu)|n-\Delta_\nu|}\right).$$

with K_c the Kloosterman sum

$$K_c(\delta_\mu, \delta_\nu) = i^{-w} \sum_{\substack{-c\leq d<0 \\ (c,d)=1}} M^{-1}(\gamma)^{\nu}{}_{\mu} \exp\left[2\pi i \left(\delta_\nu \frac{a}{c} + \delta_\mu \frac{d}{c}\right)\right]$$

and I_s the modified Bessel function of the first kind. Each term in the sum with labelled by (c, d), is the contribution from the Ford circle anchored at $-d/c$.

$\Rightarrow$ Useful in conformal field theory and quantum gravity. Cardy (1986), Strominger, Vafa (1996), Dijkgraaf, Maldacena, Moore, Verlinde (2000), ...

We will in fact be interested in the case $m - \Delta_\mu = 0$,

$$F_\mu(0) = 2\pi i^{-w} \sum_{n-\Delta_\nu<0} \frac{(2\pi|n-\Delta_\nu|)^{1-w}}{\Gamma(2-w)} F_\nu(n-\Delta_\nu) \times \sum_{c=1}^{\infty} \sum_{\substack{-c\leq d<0 \\ (c,d)=1}} c^{w-2} M^{-1}(\gamma)^{\nu}{}_{\mu} \exp\left[2\pi i \delta_\nu \frac{a}{c}\right] \tag{1}$$

The Kloosterman sum reduces to a so-called Ramanujan sum.

Integrals over $\mathcal{M}_g$ are complicated in general. Their structure simplifies in non-compact directions, where the Σ_g degenerates

The contribution of such a region of $\mathcal{M}_g$ takes the schematic form

$$\sum_n \int_{T_0}^{\infty} dt\, t^{-s}\, e^{-E_n t}, \qquad (2)$$

with the sum being over states which propagate along the tube.

The integral over t is similar to the Schwinger parametrization in Euclidean QFT

$$\frac{1}{p^2 + m^2} = \int_0^{\infty} dt_E\, e^{-t_E (p^2 + m^2)},$$

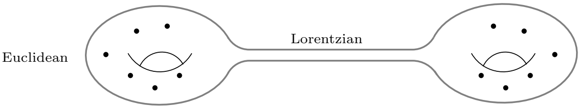
The integrals for string amplitudes are often badly divergent due to negative energy modes of the worldsheet theory.

We can take inspiration from the $i\varepsilon$-prescription in Lorentzian signature:

$$\frac{-i}{p^2 + m^2 - i\epsilon} = \int_0^\infty dt_L \, e^{-it_L(p^2+m^2-i\epsilon)},$$

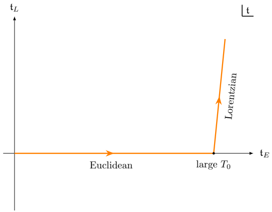
which converges independent of the sign of $p^2 + m^2$.

$i\varepsilon$ **prescription for string amplitudes:** treat the tube in Lorentzian signature and the "interior" of $\mathcal{M}_g$ in Euclidean signature. Berera (1994), D'Hoker, Phong (1994), Witten (2013), Sen (2016),...



$\Rightarrow$ Analytically continue the integration variable t to $t = t_E + it_L$, and replace the integral (2) by

$$\int_{T_0}^{i\infty} dt\, t^{-s} e^{-E_n t}$$



This is convergent, since typically $s > 0$.

More formally, one complexifies $\mathcal{M}_g$, or actually the Teichmüller space, and chooses an appropriate integration cycle in this space.

Eberhardt-Mizera (2022) evaluate the 1-loop amplitude in this way and obtain

$$\mathcal{A}^{i\varepsilon} \approx 58\,798.14 + 196\,620.04\, i,$$

This results was obtained as well using a different regularization motivated by BRST invariance in topological quantum field theory.

Korpas, JM, Moore, Nidaiev (2019)

See also Baccianti, Chandra, Eberhardt, Hartman, Mizera (2025) for a derivation using the circle method.

Modular integral

We will now consider the alternative regularization.

Bringmann, Diamantis, Ehlen (2016); Korpas, JM, Moore, Nidaiev (2019)

For closed string torus amplitudes, we are naturally led to integrals of the form

$$\mathcal{I}_f = \int_{\mathcal{F}_\infty} d\tau \wedge d\bar{\tau}\, y^{-s} f(\tau, \bar{\tau})$$

with

$$f(\tau, \bar{\tau}) = \sum_{m \geq -\Delta, n \geq -\bar{\Delta}} c(m, n)\, q^m \bar{q}^n$$

with $m - n \in \mathbb{Z}$

Widely used, eg also:

- as inner-products for modular forms Petersson (1950),...
- partition functions of topological quantum field theory

  Moore, Witten (1997), Losev, Nekrasov, Shatashvili (1997),...

Improper integral

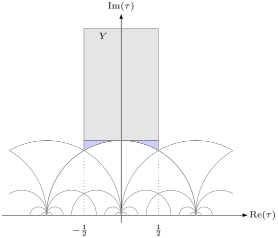
Consider a single term in q-expansion

$$L_{m,n,s} = \int_{\mathcal{F}_\infty} d\tau \wedge d\bar{\tau}\, y^{-s}\, q^m \bar{q}^n$$

Integral is improper since $\mathcal{F}_\infty$ is non-compact and

$$\lim_{\tau \to i\infty} y^{-s} q^m \bar{q}^n = \infty, \qquad if\ m + n < 0$$

Renormalization for $n \geq 0$



Regularize by truncating $\mathcal{F}_\infty$ to $\mathcal{F}_Y$

- $L_{m,n,s}(Y) = \int_{\mathcal{F}_Y} d\tau \wedge d\bar{\tau}\, y^{-s}\, q^m \bar{q}^n$
- $\mathcal{F}_Y$ splits in $\mathcal{F}_1$ + rectangle

$$\Longrightarrow L_{m,n,s}(Y) = L_{m,n,s}(1) - 2i\delta_{m,n} \int_1^Y dy\, y^{-s} e^{-4\pi y n}.$$

$n > 0$: $L^r_{m,n,s} = \lim_{Y\to\infty} L_{m,n,s}(Y)$ is finite

$n = 0$: We have $\int_1^Y dy\, y^{-s} = \frac{1}{s-1}(1 - Y^{1-s})$

Renormalize by substracting divergent terms for $Y \to \infty$:

$$L^r_{m,0,s} = L_{m,0,s}(1) - \begin{cases} 2i\,\delta_{m,0}\,\frac{1}{s-1}, & \text{for } s \neq 1 \\ 0, & \text{for } s = 1 \end{cases}$$

Dixon, Kaplunovsky, Louis (1991); Harvey, Moore (1995); Borcherds (1998)

Renormalization for Generic n

Regularize with regulator Y:

$$L_{m,n,s}(Y) = L_{m,n,s}(1) - 2i\,\delta_{m,n}\int_1^Y dy\, y^{-s} e^{-4\pi yn}.$$

To identify the divergence note:

$$\int_1^Y dy\, y^{-s}\, e^{-4\pi my} = E_s(4\pi m) - Y^{1-s} E_s(4\pi mY)$$

with $E_s(z)$ the *generalized exponential integral*,

$$E_s(z) = \begin{cases} z^{s-1}\int_z^\infty e^{-t}\, t^{-s}\, dt, & \text{for } z \in \mathbb{C}^* \\ \dfrac{1}{s-1}, & \text{for } z = 0,\ s \neq 1 \\ 0, & \text{for } z = 0,\ s = 1 \end{cases}$$

For $s > 1$, E_s has a branch cut for $z \in \mathbb{R}^-$, with imaginary part is given by $\mathrm{Im}(E_s(-x)) = \frac{\pi x^{s-1}}{\Gamma(s)}$.

Renormalize by substracting divergent terms for $Y \to \infty$:

$$
\begin{aligned}
L^r_{m,n,s} &= \lim_{Y\to\infty} \left[L_{m,n,s}(Y) - 2i\,\delta_{m,n}\,Y^{1-s}\,E_s(4\pi m Y) \right] \\
&= L_{m,n,s}(1) - 2i\,\delta_{m,n}\,E_s(4\pi m)
\end{aligned}
$$

We arrive at the renormalized modular integral:

$$
\mathcal{I}^r_f = \lim_{Y\to\infty} \left[\mathcal{I}_f(Y) - 2i \sum_{m \gg -\infty} c(m,m)\,Y^{1-s} E_s(4\pi m Y) \right]
$$

Bringmann, Diamantis, Ehlen (2016)

Evaluation of $\mathcal{I}_f^r$

Evaluation of $\mathcal{I}_f^r$ is particularly elegant if

$$y^{-s} f(\tau,\bar\tau) = \partial_{\bar\tau}\widehat{h}(\tau,\bar\tau)$$

with

$$\widehat{h}(\tau,\bar\tau) = h(\tau) + 2^s \int_{-\bar\tau}^{i\infty} \frac{f(\tau,-v)}{(-i(v+\tau))^s}\, dv$$

and

$$h(\tau) = \sum_{\substack{m\gg-\infty\\ m\in\mathbb{Z}}} d(m)\, q^m$$

then

$$\mathcal{I}_f^r = d(0)$$

This was important to demonstrate $\langle\{\mathcal{Q},\mathcal{O}\}\rangle^r = 0$ in topologically twisted quantum field theory.

Korpas, JM, Moore, Nidaiev (2019)

Proof of the Equivalence with $i\varepsilon$-prescription

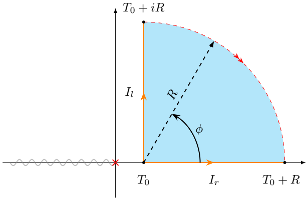
Comparing the integrals beyond $y = T_0$, one finds that the equivalence reduces to the equality

$$\lim_{R\to\infty} J_l(m,R), \quad \text{and} \quad \lim_{R\to\infty} J_r(m,R),$$

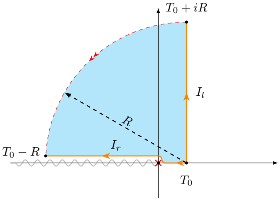
with

$$J_l(m,R) = \int_{T_0}^{T_0+iR} dy\, y^{-s}\, e^{-4\pi m y}, \qquad J_r(m,R) = \int_{T_0}^{T_0+\mathrm{sgn}(m)R} dy\, y^{-s}\, e^{-4\pi m y},$$

where $\mathrm{sgn}(m) = 1$ for $m \geq 0$, and $\mathrm{sgn}(m) = -1$ for $m < 0$. Their equality follows from a contour deformation:



$m \geq 0$



$m < 0$

Two point closed string amplitude

Two point amplitude $\mathcal{A}_2(s)$ with Mandelstam variable $s = -k^2 = 1$.

Stieberger (2023)

$$\mathcal{A}_2(s=1) = -\int_{\mathcal{F}} d\tau \wedge d\bar{\tau} \int_{T^2} dz \wedge d\bar{z}\, e^{2sG(z,\bar{z},\tau,\bar{\tau})}$$
$$= 2i \int_{\mathcal{F}} \frac{d\tau \wedge d\bar{\tau}}{y^{9/2}} \left(\left| \frac{\vartheta_3(2\tau)}{\eta(\tau)^6} \right|^2 + \left| \frac{\vartheta_2(2\tau)}{\eta(\tau)^6} \right|^2 \right)$$

The real part of the amplitude contributes to the mass shift, while the imaginary part contributes to the decay width.

The regularized integral evaluates to

$$\mathcal{A}_2^{\mathrm{r}}(1) \approx 27.85 + 59.37\, i.$$

with $64\pi^2/105$ the exact value for the imaginary part.

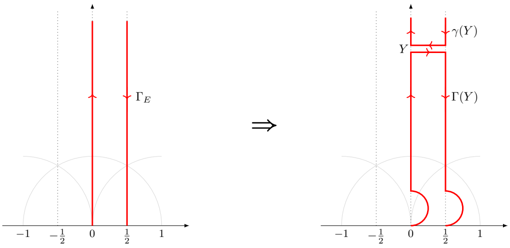
$i\varepsilon$-prescription for open string amplitudes

Recall

$$A_{0,E} = -i \int_{\Gamma(E)} d\tau \, \frac{1}{\eta(\tau)^{24}}$$

The $i\varepsilon$-prescription suggests to deform the contour for $\mathrm{Re}(\tau) = 0$ $\mathrm{Im}(\tau) < 2\pi/T_0$ by $\tau = 2\pi i/(T_0 + it)$, $t \in [0, \infty)$, and similarly for $\mathrm{Re}(\tau) = 1/2$. We further split of the region for $\mathrm{Im}(\tau) > T_0$.

This deforms the contour to:



Eberhardt-Mizera (2022)

Then we can evaluate the contour integral for $\Gamma(T_0)$ explicitly as

$$A^{i\varepsilon}_{0,\Gamma(T_0)} = -12i \sum_{\substack{n\in\mathbb{Z}\\ n\geq -1}} F(n)\left(\frac{\delta_{n,\mathrm{odd}}}{\pi n} e^{-2\pi n\tilde{T}_0} + \tilde{T}_0^{13}\, E_{14}(nT_0) - \frac{(-1)^n}{2}(2\tilde{T}_0)^{13}\, E_{14}\left(\frac{n}{4}T_0\right)\right).$$

with $F(n)$ the Fourier coefficients of η^{-24}, $\tilde{T}_0 = 2\pi/T_0$ and E_s the generalized exponential integral.

- The rhs is independent of T_0. One finds numerically

$$A^{i\varepsilon}_0 \approx 4.436903 \times 10^{-6} + 1.467444 \times 10^{-3}\, i$$

  Convergence of numerical evaluation is sensitive to T_0. Imaginairy part is readily determined in closed form.
- For appropriate choice of T_0, convergence is fast. For example, for $n < 76$ and $T_0 = 4\pi$, the accuracy is 10^{-71}.
- The alternative regularization $\mathcal{A}^r_{\Pi(T_0)}$ equals $A^{i\varepsilon}_{0,\Gamma(\tilde{T}_0)}$, and both prescriptions are thus equivalent.

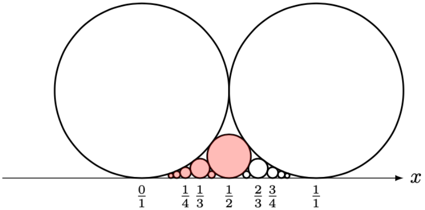
Generalization to other integrands, not necessarily a modular form for $SL(2,\mathbb{Z})$ is straightforward:

$$\int_{\Gamma(T_0)} d\tau\, f(\tau) = \frac{1}{2}F_\infty(0) + i\sum_n F_\infty(n)\frac{\delta_{n,\mathrm{odd}}}{\pi n}e^{-2\pi n\tilde{T}_0} + i\sum_{\ell=1,2}\sum_n(-1)^{\ell-1}F_\ell(n)\,\tilde{T}_0^{1-w}\,E_{2-w}(nT_0),$$

with F_∞, F_1 and F_2 the Fourier coefficients near $i\infty$, 0 and 1/2 respectively.

Circle Method for Open Amplitudes

Eberhardt-Mizera (2022) propose an interesting further contour deformation, namely to integrate over Ford circle anchored in $(0, 1/2]$:



As a result, the amplitude follows from the restriction of the rhs for $F_\mu(0)$ to the integers (d, c) with $-d/c \in (0, 1/2]$.

Specialization of that formula (1) expresses the amplitude as

$$A_{0,\Gamma_\infty} = -i\frac{(2\pi)^{14}}{\Gamma(14)}\mathcal{G}_{14}$$

with

$$\mathcal{G}_{14} = \sum_{c=1}^{\infty} c^{-14} \sum_{\substack{-c/2\le d<0 \\ (d,c)=1}} e^{-2\pi i a/c}, \qquad ad = 1 \mod c.$$

This is a "partial" Ramanujan sum since the sum does not run over $d = 0, \ldots, c-1$. We can determine the real part in closed form, while a closed form does not seem to be available for the imaginary part.

Numerical values of the two regularizations agree, which provides a non-trivial check.

Type I RR-Sector Contribution to Vacuum Energy

For gauge group $SO(n)$, the sum of the annulus, Möbius strip and Klein bottle contributions takes the form

$$A^{\mathrm{I}} = \frac{1}{2^{10}} \int_0^\infty dy\,(n \mp 32)^2\,16 + (n \pm 32)^2\,256\,e^{-2\pi y} + O(e^{-4\pi y}).$$

For $n = 32$, this simplifies to

$$A^{\mathrm{I}} = -2i \int_{\Gamma_E} f(\tau), \qquad f(\tau) = \frac{\vartheta_2(\tau)^4}{\eta(\tau)^{12}}$$

Numerical evaluation of the regularized amplitude reads

$$A^{\mathrm{I,r}}_{\Gamma(T_0)} \approx 0.020705983 + 0.011576613\,i,$$

with the exact value $\pi^6/120 - 8$ for the imaginary part.

Circle Method

f is an element of a 3-dimensional vector-valued modular form.
The matrices M_μ^ν are non-trivial.

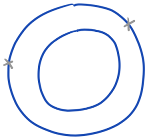
The analysis gives

$$A^{\mathsf{I}}_{\Gamma_\infty} = \frac{2\,\pi^6}{\Gamma(6)} \sum_{\substack{c \text{ odd} \\ c>0}} \frac{1}{c^6} \sum_{\substack{-c/2 \le d < 0 \\ (c,d)=1}} (-1)^a \, e^{-\pi i a/c},$$

and matches with the previous evaluation.

Planar 2-Point Amplitude for Type I String Theory

The 2-point amplitude gives the one-loop mass renormalization and decay rate.
The amplitude reads for generic $s = -k^2$,



$$A^{\mathrm{I}}_{2,\Gamma}(s) = -i \int_\Gamma d\tau \int_0^1 dz \left(\frac{\vartheta_1(\tau, z)}{\eta(\tau)^3} \right)^{2s},$$

For example, for $s = 1$ this evaluates to

$$A^{\mathrm{I}}_{2,\Gamma}(1) = -i \int_\Gamma d\tau \frac{\vartheta_2(2\tau)}{\eta(\tau)^6}$$

Evaluated on the $\Gamma(T_0)$ contour, this evaluates to

$$A^{\mathrm{I}}_{2,\Gamma(T_0)}(1) = 0.003303550 - 0.130275973\, i.$$

Imaginary part $\pi^4/112 - 1$. After adding the contribution from $\gamma(T_0)$, $\mathrm{Im} > 0$.

The Circle Method gives

$$-2\pi i \frac{(\pi/2)^{7/2}}{\Gamma(9/2)} \mathcal{G}_{9/2}$$

with

$$G_{9/2} = \sum_{c=1}^{\infty} c^{-9/2} \sum_{\substack{-\frac{c}{2} \le d < 0 \\ (c,d)=1}} \chi_{d,c}$$

with the summand $\chi_{d,c}$ defined as

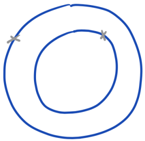
$$\chi_{d,c} = \begin{cases} \omega_{a+c/2,c}^{2}\, \omega_{2a,c}^{-1}\, \omega_{a,c}^{-6}, & c = 2 \mod 4, \\ \frac{1}{\sqrt{2}}\, \omega_{a,4c}^{2}\, \omega_{a,2c}^{-1}\, \omega_{a,c}^{-6}, & c = 1 \mod 2, \\ 0, & c = 0 \mod 4, \end{cases}$$

and $\omega_{d,c}$ given in terms of the Dedekind sum $s(d,c)$, $\omega_{d,c} = e^{\pi i\, s(d,c)}$. It agrees with the mentioned value.

Non-Planar 2-Point Amplitude for Type I String Theory

The non-planar 2-point amplitude reads

$$A_2^{\mathrm{I,np}}(s=1) = -i \int_{\Gamma'_E} d\tau \int_0^1 dz \left(\frac{\vartheta_4(\tau, z)}{\eta(\tau)^3} \right)^2 ,$$



where Γ'_E is the contour running from 0 to $i\infty$, and then from $2 + i\infty$ to 2.

$$\Rightarrow A_{2,\Gamma'}^{\mathrm{I,np}}(s=1) \approx -1.79524856 + 1.85541126\, i$$

The exact value of the imaginary part is $\frac{2}{105}\pi^4$.

Circle Method

This amplitude reads

$$A_2^{\mathrm{I,np}}(s=1) - i\,\frac{\pi^{9/2}}{2^{5/2}\,\Gamma(9/2)}\,G_{9/2}^{\mathrm{np}}$$

with

$$G_{9/2}^{\mathrm{np}} = \sum_{c=1}^{\infty} \frac{1}{c^{9/2}} \sum_{\substack{-2c\le d<0\\(c,d)=1}} \chi_{d,c}^{\mathrm{np}},$$

with

$$\chi_{d,c}^{\mathrm{np}} = \begin{cases} \omega_{4a,c}^{-2}\omega_{2a,c}^{5}\omega_{a,c}^{-8}\,e^{\pi i\frac{d}{2c}}, & c=0 \mod 4,\\ 0, & c=2 \mod 4,\\ \frac{1}{\sqrt{2}}\,\omega_{a,4c}^{-2}\,\omega_{a,2c}^{5}\,\omega_{a,c}^{-8}\,e^{\pi i\frac{d}{2c}}, & c=1 \mod 2, \end{cases}$$

Summary:
We have seen various techniques with which one-loop string amplitudes can be explicitly evaluated, ie using the $i\varepsilon$-prescription, exponential integrals and the Circle Method.

Outlook:
There are various aspects to consider going forward, including:

- mass renormalization/decay rate
- Optical theorem
- Regge trajectory
- higher point functions

Thank you!