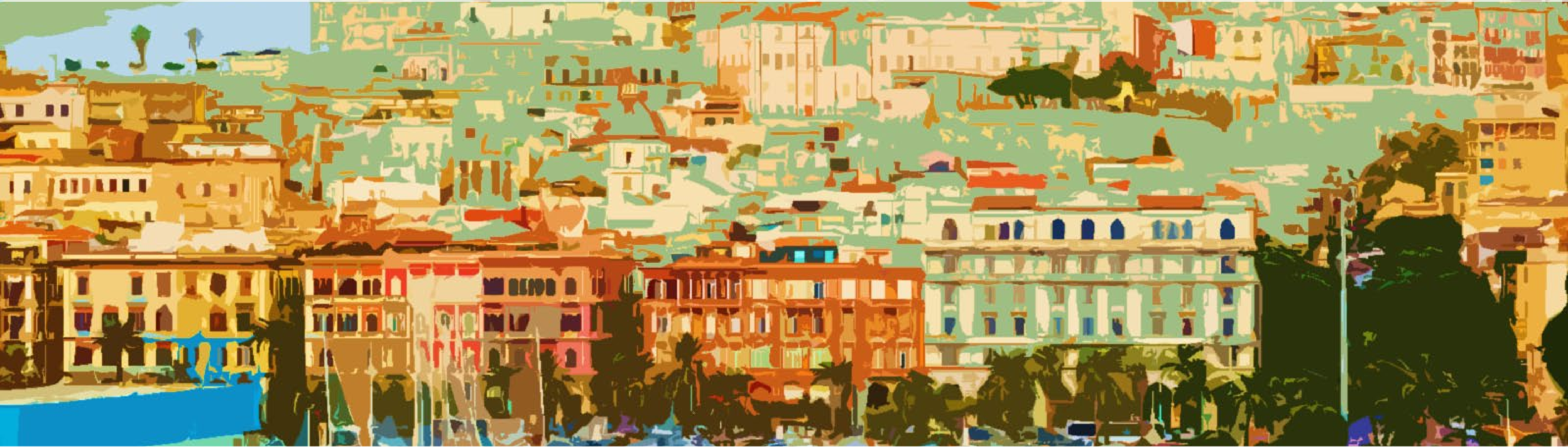
ET Technology School

13/12/2024 - Cagliari, Italy



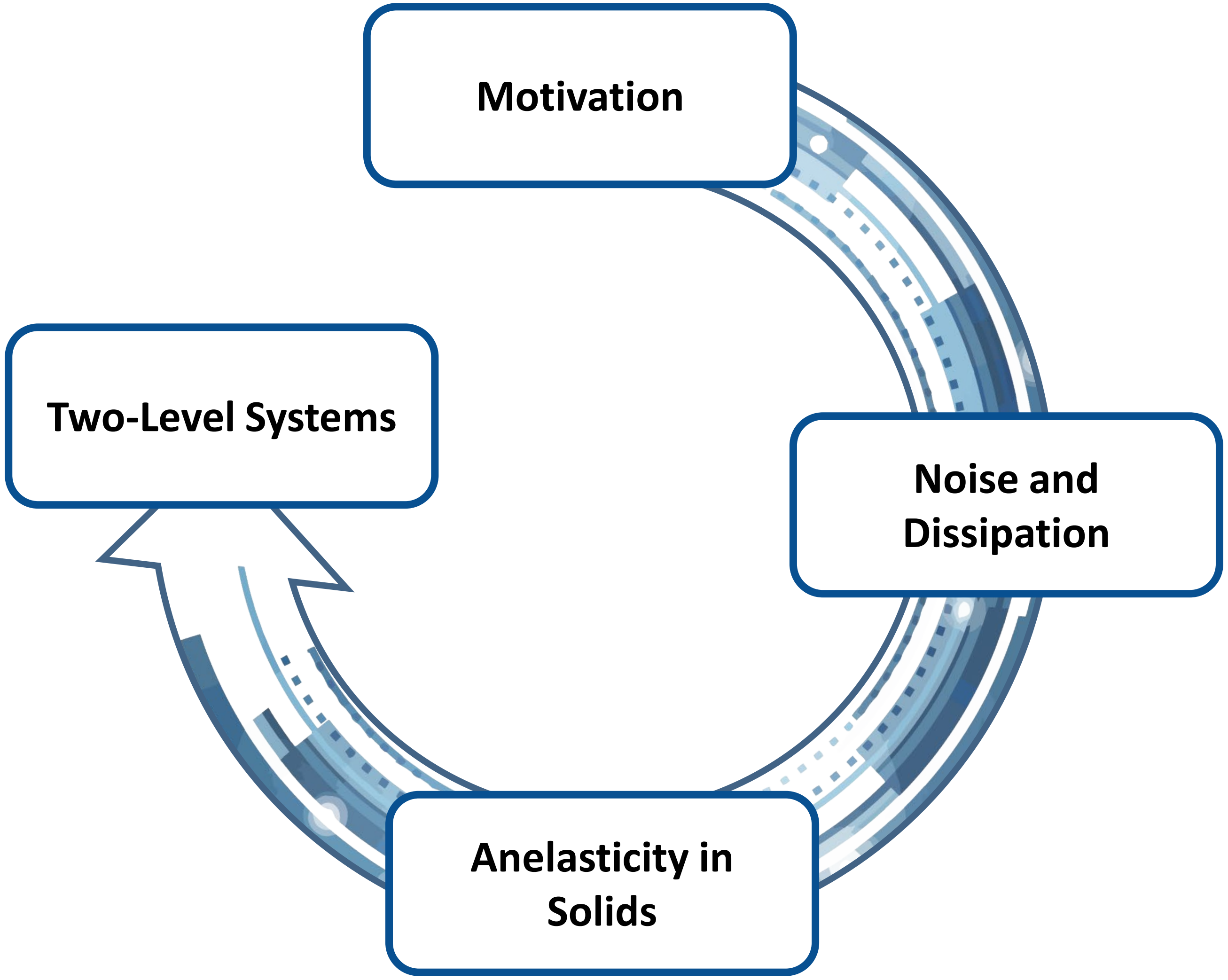
Thermal Noise pt. 1

Alex Amato

Nikhef

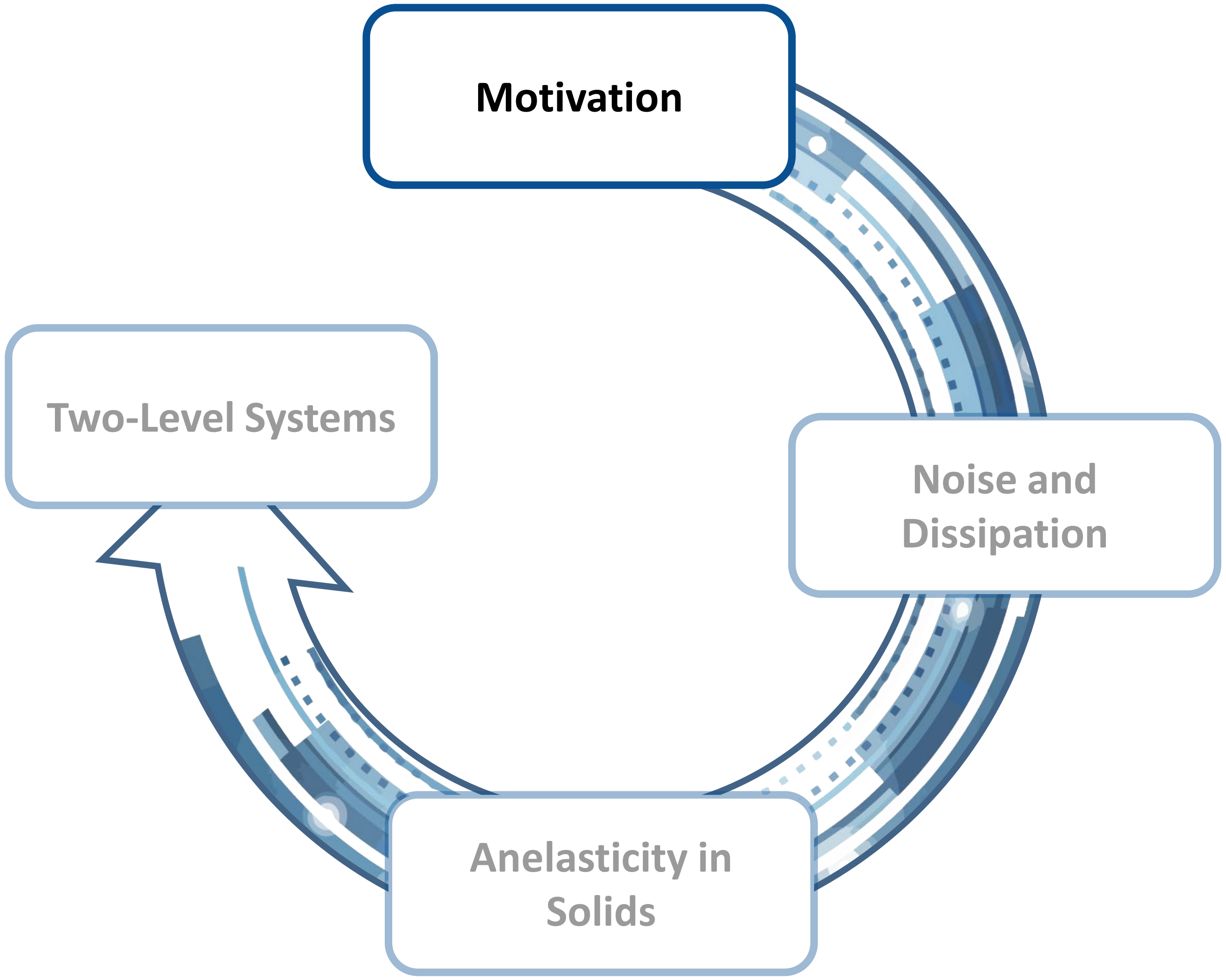
Maastricht University

Overview



Motivation
Noise and Dissipation
Anelasticity in Solids
Two-Level Systems

Motivation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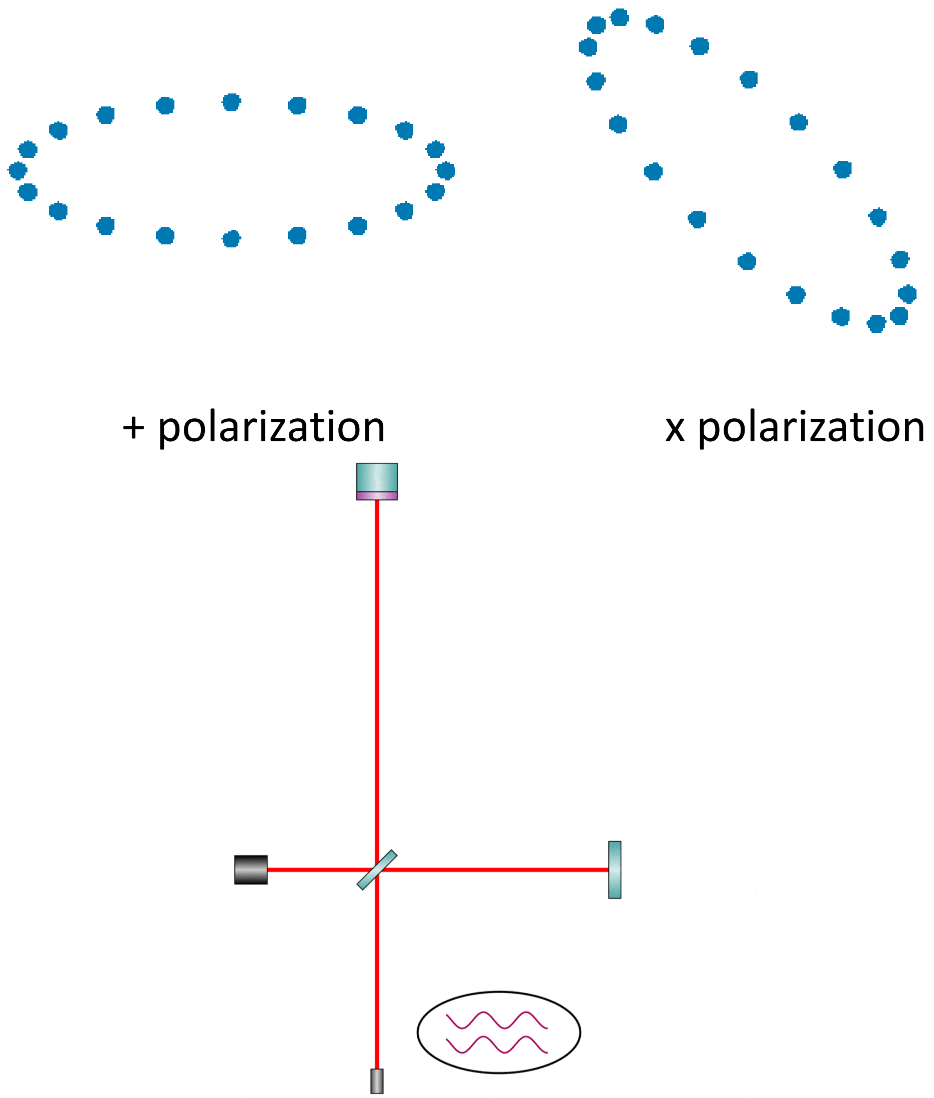
Test Mass of Gravitational-Wave Detectors



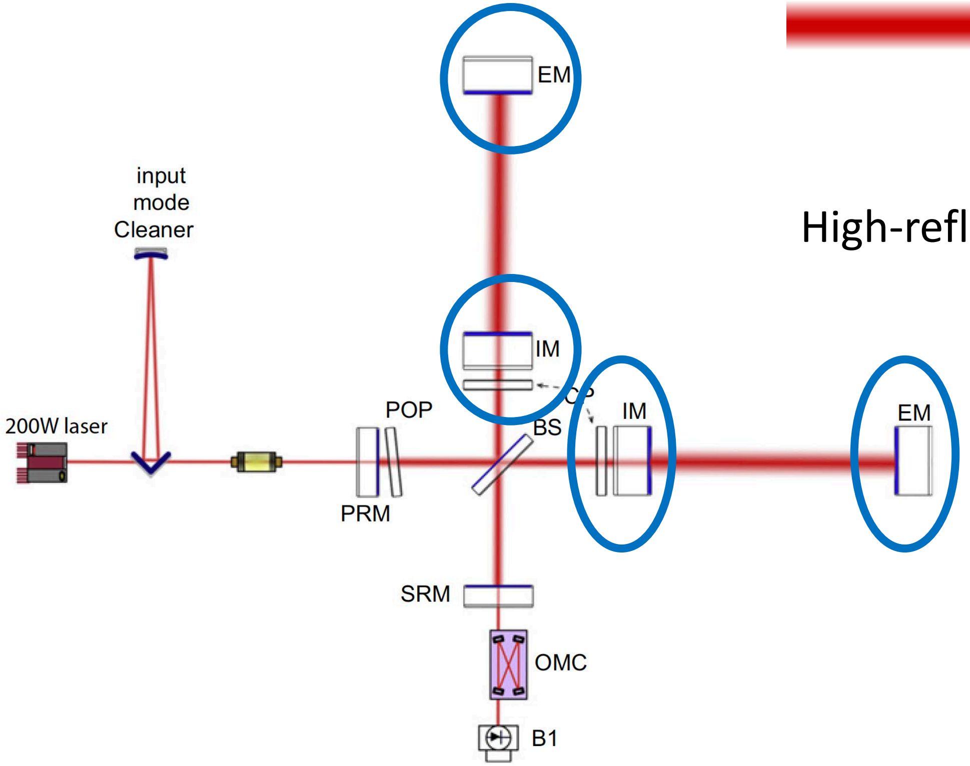
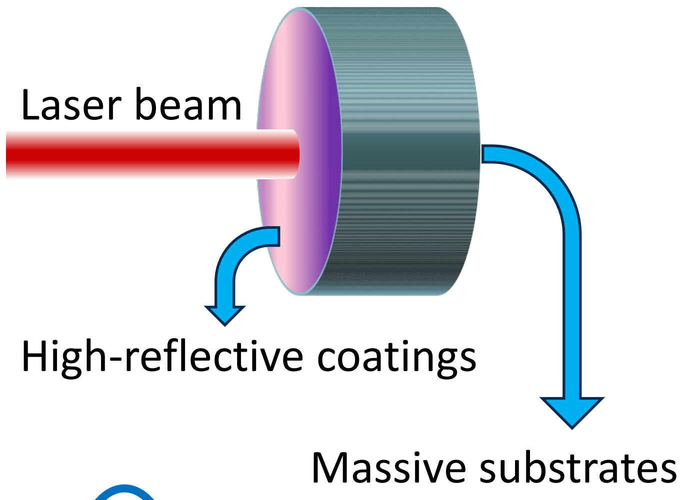


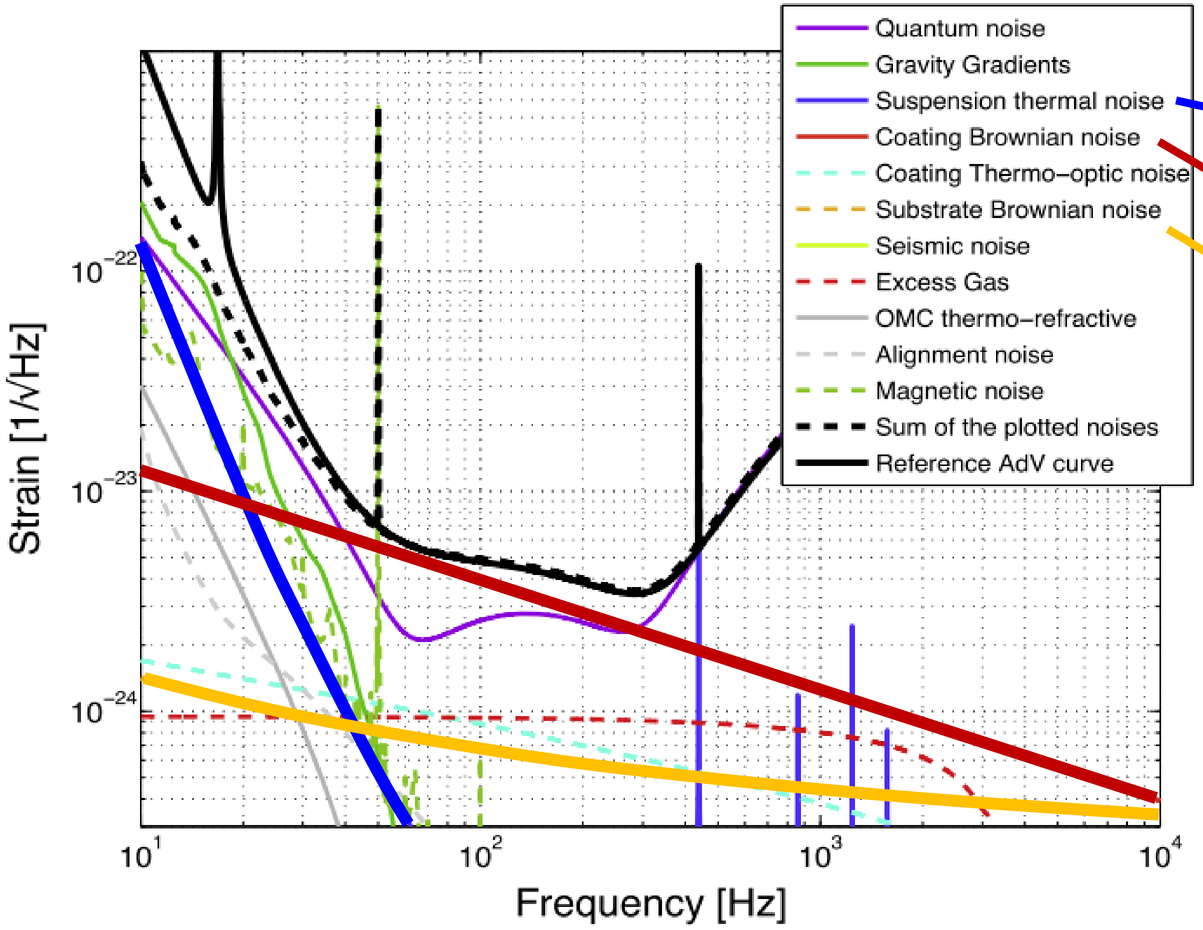
Figure 3. Simplified optical layout of the AdV ITF. Each 3 km long arm cavity is formed by an IM and an EM. The recycling cavities at the center of the ITF are 12 meters long and are formed by the PRM, the SRM and the two IM.

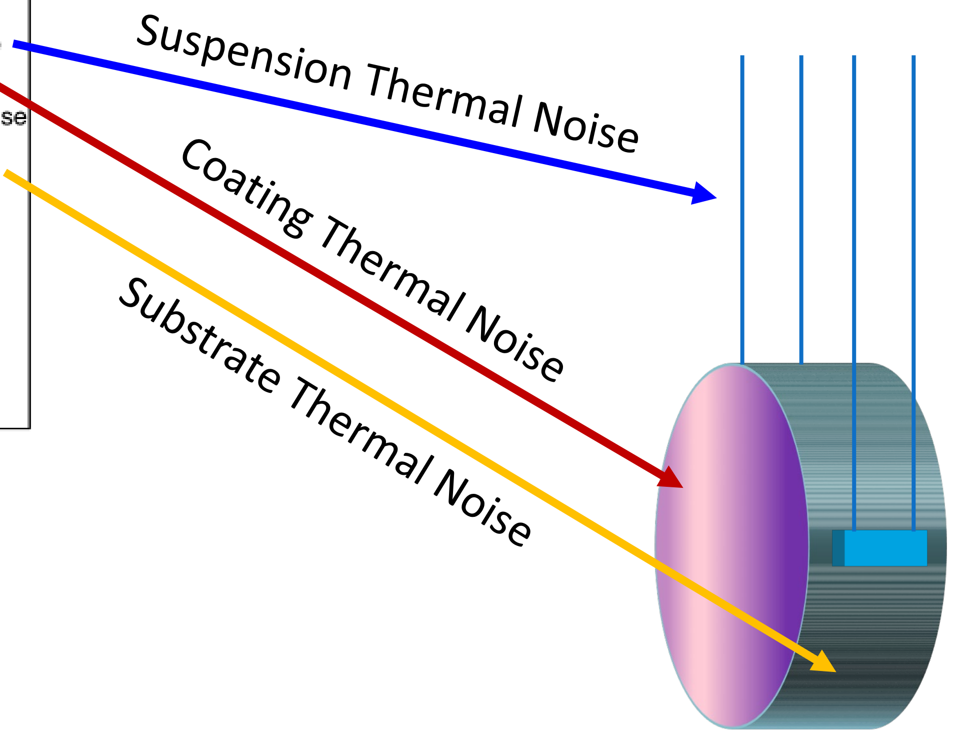
Class. Quantum Grav. 32 (2015) 024001 (52pp)



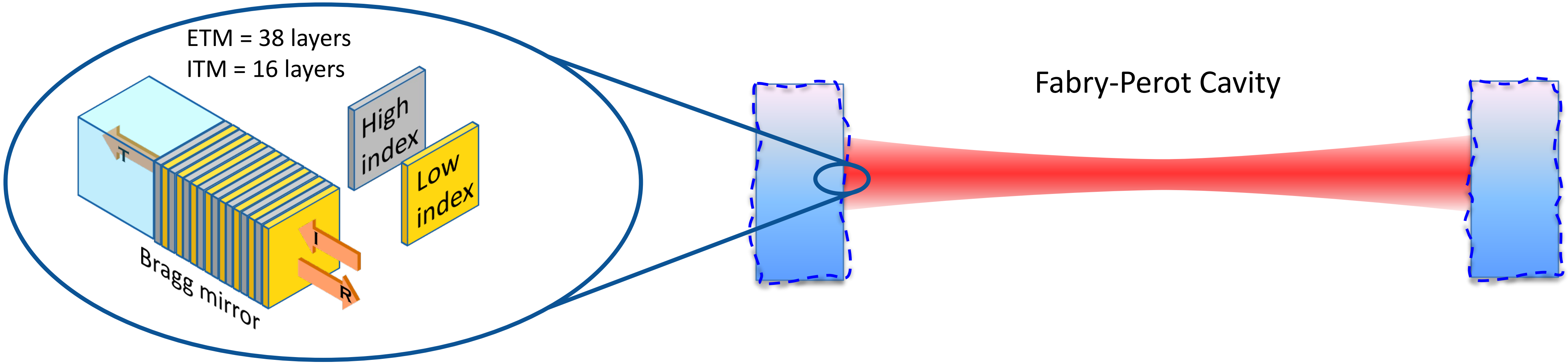
Noise in Gravitational-Wave Detectors (GWDs)

Thermal noise is one of the fundamental noises limiting the detection of gravitational waves.





Thermal Noise in GWDs



- The laser beam is reflected by a Bragg mirror (the coating!) deposited on the internal surfaces of the cavity
  - The reflection is caused by interference of multiple reflections

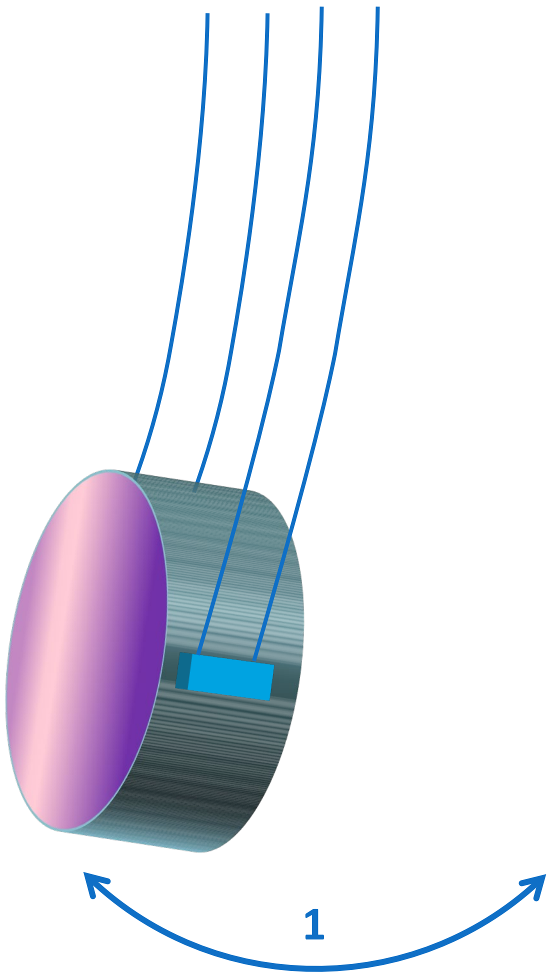
- One can imagine the existence of a virtual surface located inside the coating stacks where the beam is bounced off
  - Any fluctuation of the position of this surface is regarded as noise
  - The fluctuations that are driven by the temperature are called Thermal Noise

Thermal Noise in GWDs

There are 3 types of thermal noises in GWDs.

If the reflecting surface moves because of a:

1. rigid body motion of the entire mirror:
   SUSPENSION NOISE



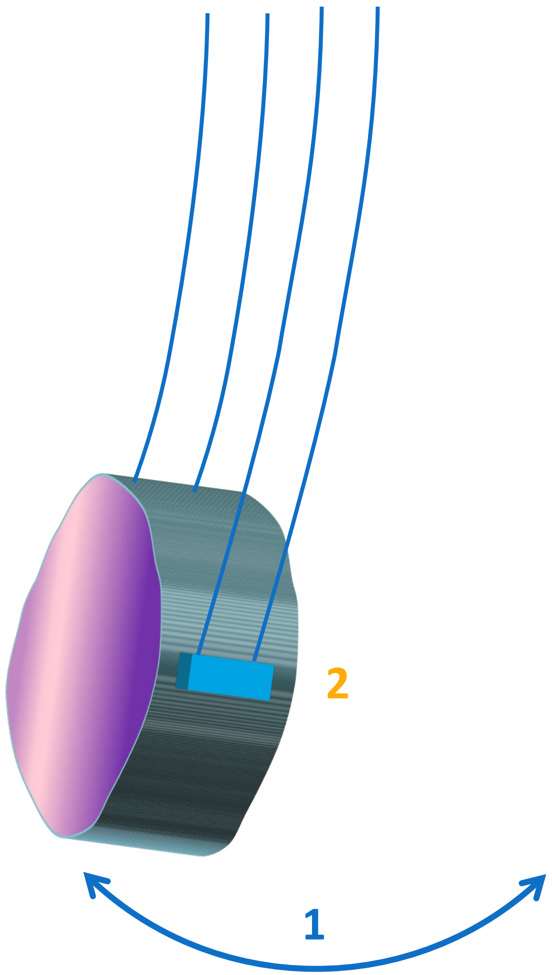
Thermal Noise in GWDs

There are 3 types of thermal noises in GWDs.

If the reflecting surface moves because of a:

1. rigid body motion of the entire mirror:
   SUSPENSION NOISE

2. deformation of the mirror substrate
   SUBSTRATE NOISE



Thermal Noise in GWDs

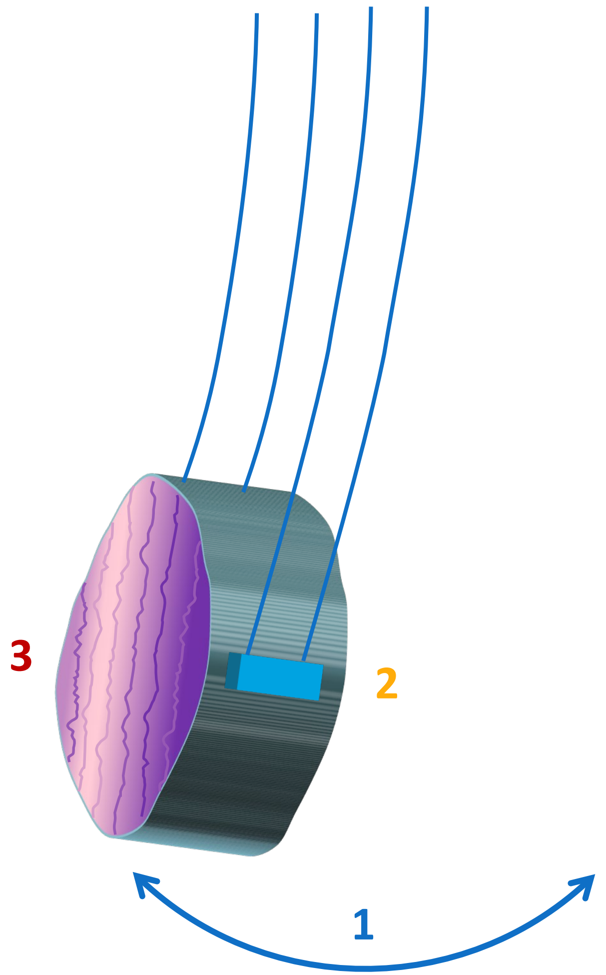
There are 3 types of thermal noises in GWDs.

If the reflecting surface moves because of a:

1. rigid body motion of the entire mirror:
   SUSPENSION NOISE

2. deformation of the mirror substrate
   SUBSTRATE NOISE

3. mechanical or optical fluctuation inside the Bragg mirror
   COATING NOISE



Thermal Noise in GWDs

Thermal noise arises from fluctuations of state functions (like volume) driven by temperature.

If we look at the reflecting surface as a thermodynamic system, and we consider volume and temperature as independent variables, the probability of finding the system out of equilibrium is given by:

$$w \propto \exp\left(-\frac{C_v}{2T^2}\Delta T^2 + \frac{1}{2\mathrm{T}}\left(\frac{\partial P}{\partial V}\right)_{\mathrm{T}}\Delta V^2\right), \qquad (112.4)$$

L.D. Landau, E.M. Lifshitz
Statistical Physics – Part 1

Mean squared fluctuations of volume

$$\langle(\Delta V)^2\rangle = -k_b T\left(\frac{\partial V}{\partial p}\right)_T \qquad (112.7)$$

- limited by the compressibility (rigidity)
- directly coupled to the interferometer

Mean squared fluctuations of temperature

$$\langle(\Delta T)^2\rangle = \frac{k_b T^2}{C_v} \qquad (112.6)$$

- limited by the heat capacity
- induces volume changes (through the thermal expansion) and refractive index changes.

Note that there is no fluctuations at $T = 0$

Thermal Noise in GWDs

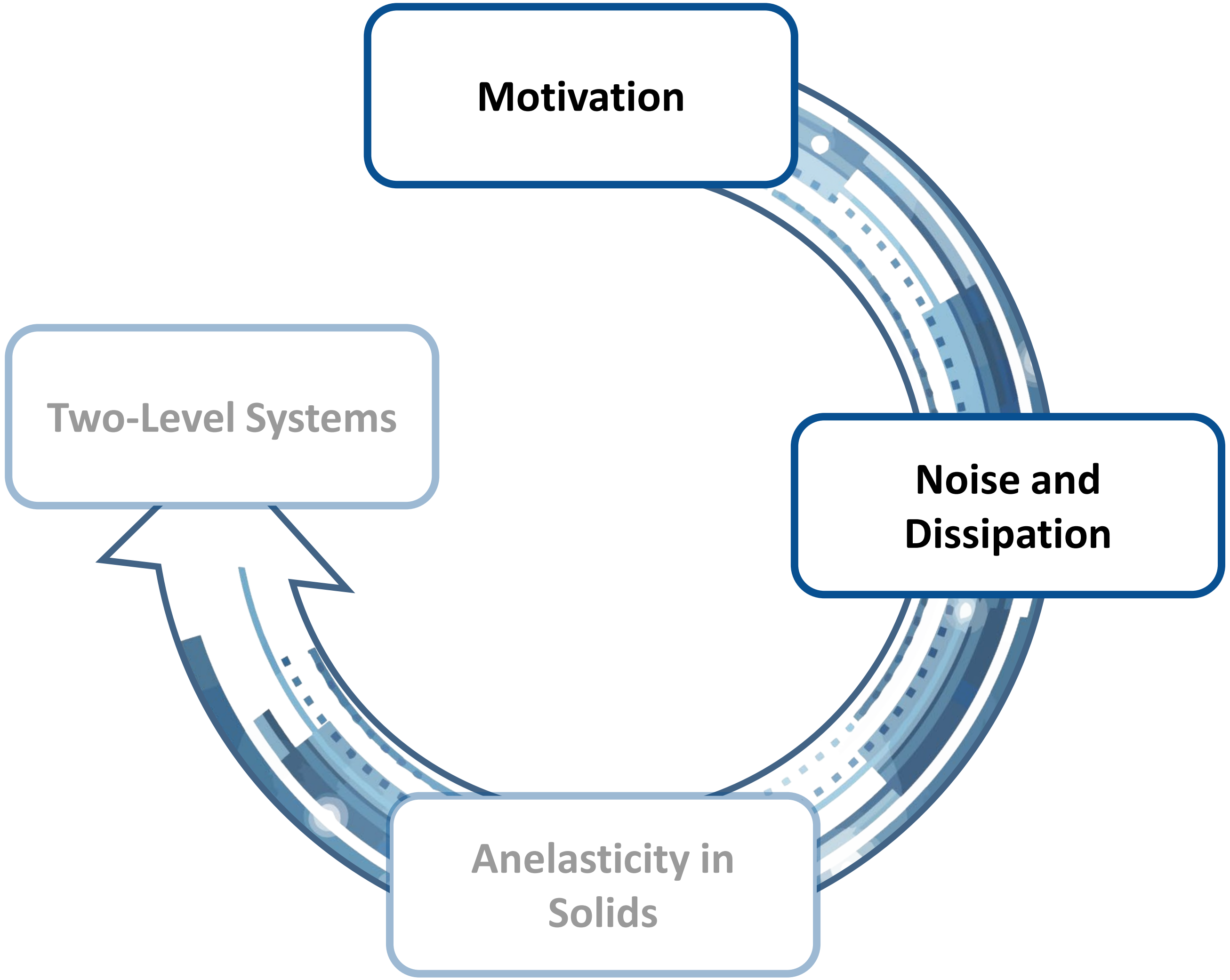
Is it enough to know basic notions of thermodynamics?

We are missing:

- ❑ Spectral distribution of fluctuations and not just the variance
- ❑ The complexity of the system → suspension, substrate and coating

We need a theory!

Noise and Dissipation

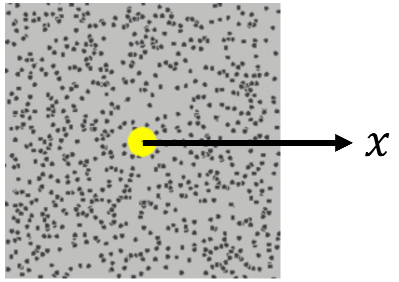


The Brownian Motion

The theory comes from the Fluctuation-Dissipation Theorem established in 1951 by Callen and Welton.
This theorem is also called generalized Johnson-Nyquist theorem (1927-1928).

However, before Johnson and Nyquist, in 1827 Robert Brown described a phenomenon that was modeled by Einstein in 1905 and that relates the fluctuations of a quantity to a finite temperature.

Grain of pollen in water



Mean squared displacement

$$\langle (\Delta x)^2 \rangle = 2Dt$$

Depends on the diffusivity

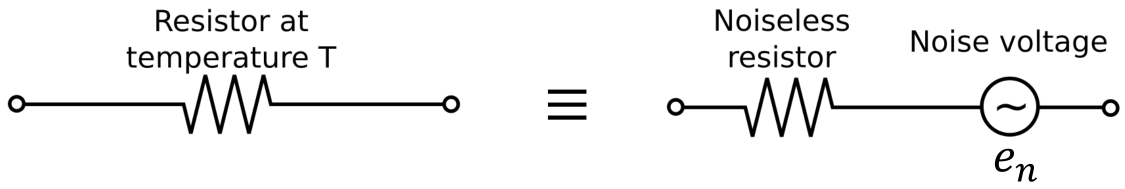
$$D = \mu\, k_B T = \frac{k_\mathrm{B}\mathrm{T}}{6\pi r \eta}$$

μ = mobility of the pollen in water
r = pollen grain radius
η = viscosity

The Johnson-Nyquist Noise

The theory comes from the Fluctuation-Dissipation Theorem established in 1951 by Callen and Welton.
This theorem is also called generalized Johnson-Nyquist theorem.

- In 1926 Johnson discovered that the newly born electronic amplifiers produced a noise whose power is proportional to the resistance.
- In 1928 Nyquist produced the equation:



$$e_n = \sqrt{4\, k_B\, T R}$$

  - The noise voltage power is proportional to the resistance and the temperature
  - The model provided a solution for the thermal noise calculation in any circuit

Noise and dissipation are related!

The Fluctuation-Dissipation Theorem

Power Spectral Density

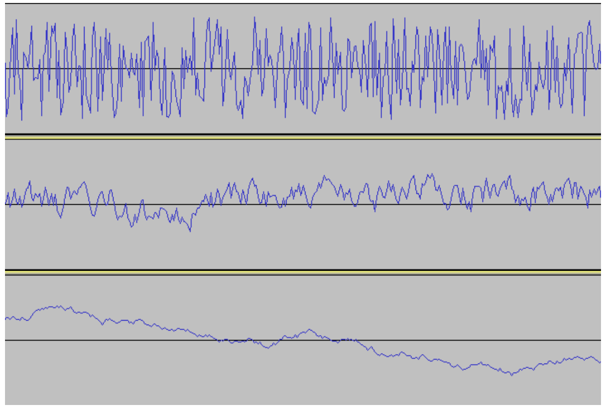
- Why PSD has been invented :
  To characterize the spectral content of noise
- Why FT is not good :
  - It is complex
  - Its modulus is proportional to the square root of the integration time
  - A noise $n(t)$ has infinite energy, whereas its power $\overline{n^2}$ is finite
- What the PSD is :
  It gives the distribution of power over frequencies, i.e.

$$\langle n^2 \rangle = \int_0^{\infty} PSD(f)\, df$$

One definition :

$$PSD(f) = \lim_{T\to\infty} \frac{1}{2T} \left| \int_{-T}^{T} n(t) \cdot e^{-i\,2\pi f t}\; dt \right|^2 \quad ; \quad [PSD] = \frac{[n^2]}{\mathrm{Hz}}$$

3 noises with equivalent rms



Voltage noise (Nyquist)

$$[PSD_{e_n}] = \frac{\mathrm{V}^2}{Hz}$$

The Fluctuation-Dissipation Theorem

Callen and Welton in 1951 and 1952 generalized the Nyquist formula to any linear system. The PSD is associated to the dissipating quantity.

For an observable x, the power spectral density is given by

$$S_{xx}(\omega) = -4\hbar \cdot \left\{\frac{1}{2} + \frac{1}{e^{\frac{\hbar\omega}{k_B T}} - 1}\right\} \cdot \chi''(\omega)$$

Response function or susceptibility

That can be expressed in the classical limit ($\hbar\omega \ll k_b T$) as

$$S_{xx}(\omega) = -\frac{4k_B T}{\omega} \cdot \chi''(\omega)$$

$k_B T = 25\ \mathrm{meV}\ (\text{at } 300\mathrm{K})$

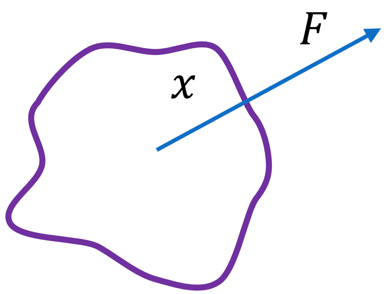
$\hbar\omega = hf$

$h = 4.136 \times 10^{-15} \mathrm{eV} \cdot \mathrm{Hz}^{-1}$

if $f \sim 10\mathrm{kHz}$ the limit is valid

Response Function and Dissipation

In order to evaluate the response function $\chi(\omega)$, one may imagine to expose the observable x to an external solicitation $F(t)$. If the external solicitation is weak, from the linear response theory we obtain



$$x(t) = \int_{-\infty}^{+\infty} \chi(t-\zeta)F(\zeta)d\zeta$$

This is the linear response theory of Kubo and it has general validity!

Polarization

$$P(t) = \int_{-\infty}^{t} \chi_e(t-\zeta)E(\zeta)d\zeta$$

$\chi_e(t-\zeta)$ = electric susceptibility

Response Function and Dissipation

In order to evaluate the response function $\chi(\omega)$, one may imagine to expose the observable x to an external solicitation $F(t)$. If the external solicitation is weak, from the linear response theory we obtain

$$x(t) = \int_{-\infty}^{+\infty} \chi(t-\zeta)F(\zeta)d\zeta$$

In frequency domain, the convolution is

$$x(\omega) = \chi(\omega)F(\omega) \quad \text{with} \quad \chi(\omega) = \chi'(\omega) + i\chi''(\omega)$$

$$\chi(\omega) = \frac{x(\omega)}{F(\omega)}$$

One can define the **admittance** as

$$Y(\omega) = i\omega\frac{x(\omega)}{F(\omega)} = i\omega\,\chi(\omega) \quad \text{with} \quad -\chi''(\omega) = \frac{Y'(\omega)}{\omega}$$

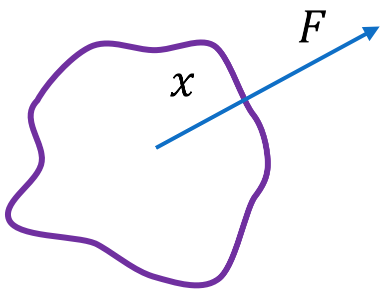
So that

$$S_{xx}(\omega) = \frac{4k_BT}{\omega^2}\cdot Y'(\omega)$$

Response Function and Dissipation

Why $\chi''(\omega)$ is related to the dissipation?

imagine to expose the observable x to a periodic external solicitation $F(t)$. To obtain the dissipated energy, one must evaluate what is the energy after one cycle of excitation (we will use the power):



$$E_{1T} = \int_0^T P(t)dt = \int_0^T F(t)v(t)dt$$

$F(t) = F_0\,\Re\{e^{i\omega t}\}$ and $v(t) = \dot{x}(t)$

$$x(t) = \Re\{\chi(\omega)F_0 e^{i\omega t}\} = \Re\{|\chi(\omega)|F_0 e^{i(\omega t+\phi)}\}$$

Note that $\chi(\omega) = |\chi(\omega)|e^{i\phi}$

$$v(t) = \dot{x}(t) = \Re\{i\omega|\chi(\omega)|F_0 e^{i(\omega t+\phi)}\}$$

$$E_{1T} = \int_0^T F_0\,\Re\{e^{i\omega t}\}\,\Re\{i\omega\chi(\omega)F_0 e^{i(\omega t+\phi)}\}dt = \frac{F_0^2}{4}(i\omega\chi - i\omega\chi^*)T = \frac{F_0^2}{4}2\pi\,(-2\chi'') = -F_0^2\pi\chi''$$

$$E_{1T} = -F_0^2\pi\chi''$$

The total work done by the force after one cycle is negative, which means the energy has been dissipated

The Levin's Equation

The dissipation is related to the imaginary part of the response function

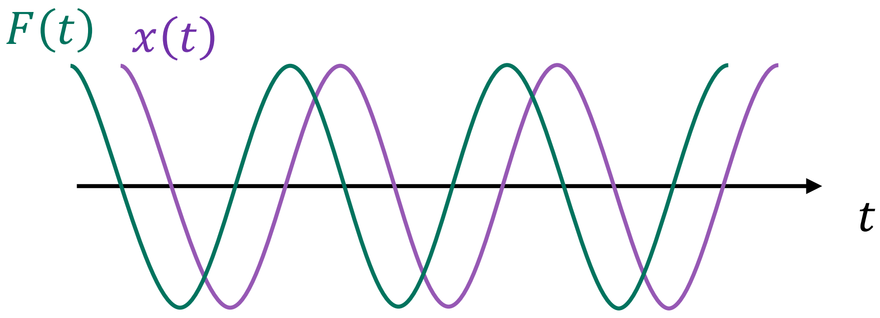
$$W = |E_{1T}| = F_0^2 \pi \chi''$$

Levin replaced χ'' in the fluctuation-dissipation theorem by the work done in one cycle (period), W:

$$S_{xx}(\omega) = -\frac{4k_B T}{\omega} \cdot \chi''(\omega) \longrightarrow \boldsymbol{S_{xx}(\omega) = -\frac{4k_B T}{\omega} \cdot \frac{W}{\pi F_0^2}}$$

The Loss Angle

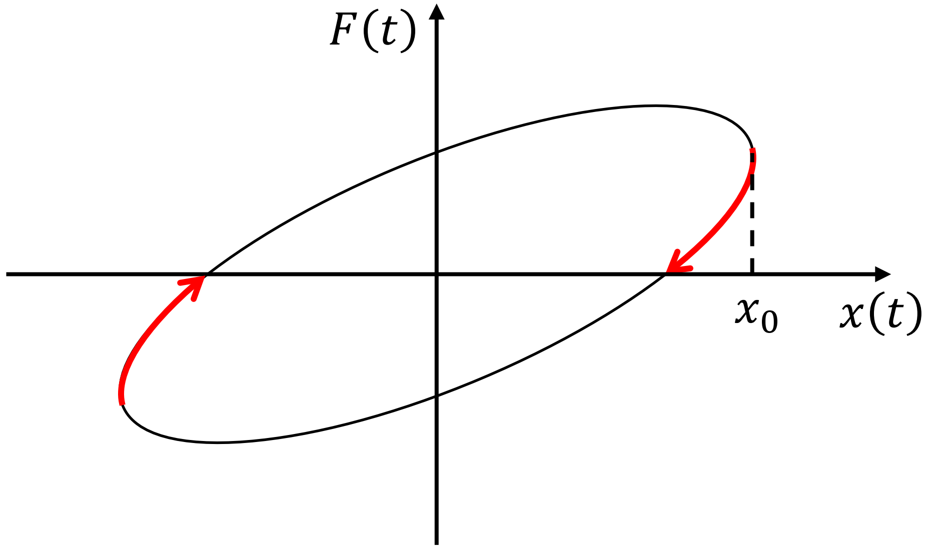
The response function is delayed, and this process is called relaxation. This delay introduce a phase lag between the stress and the strain.

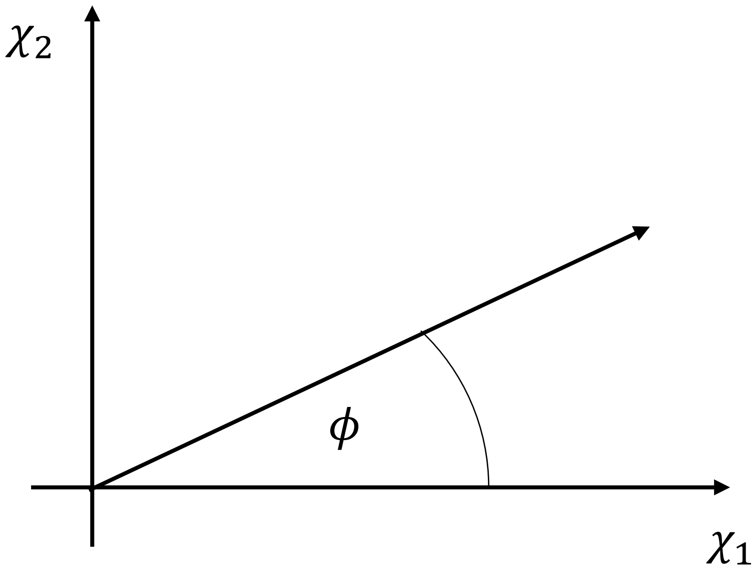


$$x(t) = \int_{-\infty}^{+\infty} \chi(t-\zeta)F(\zeta)d\zeta \quad \text{with} \quad F(t) = F_0 e^{i\omega t}$$

$$\chi(\omega) = |\chi(\omega)|e^{i\phi} \;\rightarrow\; \chi = \chi_1 + i\chi_2$$

$$x(t) = x_0 e^{i(\omega t+\phi)} \quad \text{where} \quad |\chi(\omega)| = \frac{x_0}{F_0}$$

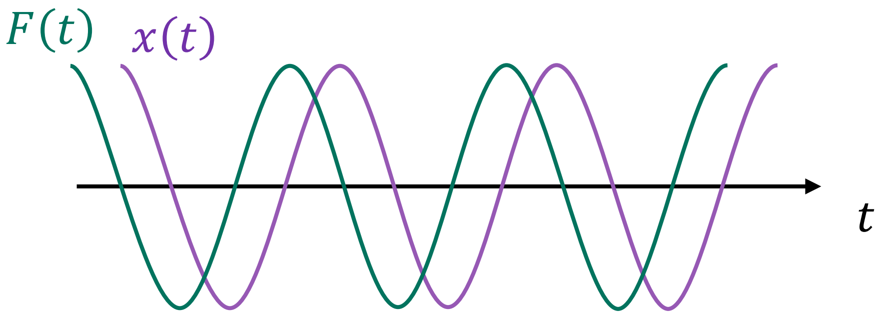




$$\phi = \frac{\chi_2}{\chi_1}$$

The Loss Angle

The response function is delayed, and this process is called relaxation. This delay introduce a phase lag between the stress and the strain.



$$x(t) = \int_{-\infty}^{+\infty} \chi(t-\zeta)F(\zeta)d\zeta \quad \text{with} \quad F(t) = F_0 e^{i\omega t}$$

$$\chi(\omega) = |\chi(\omega)|e^{i\phi} \;\rightarrow\; \chi = \chi_1 + i\chi_2$$

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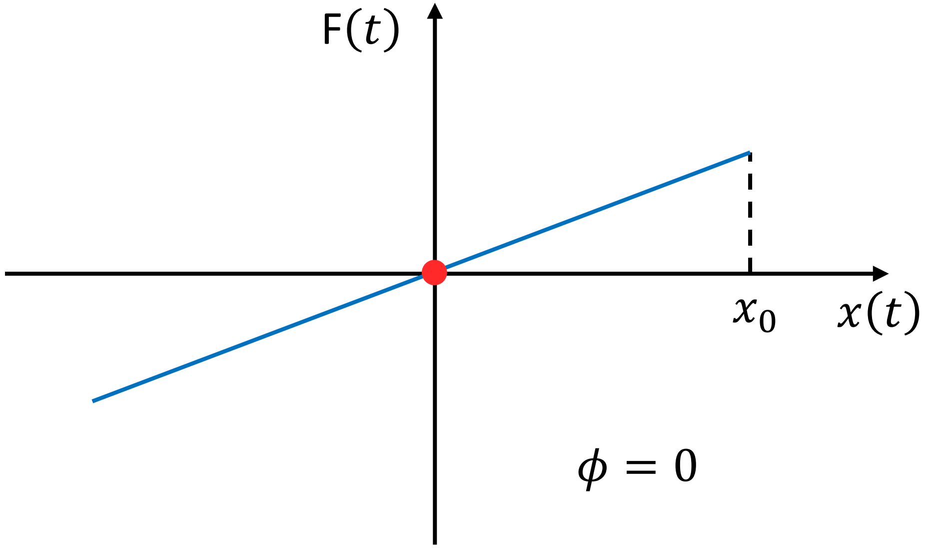
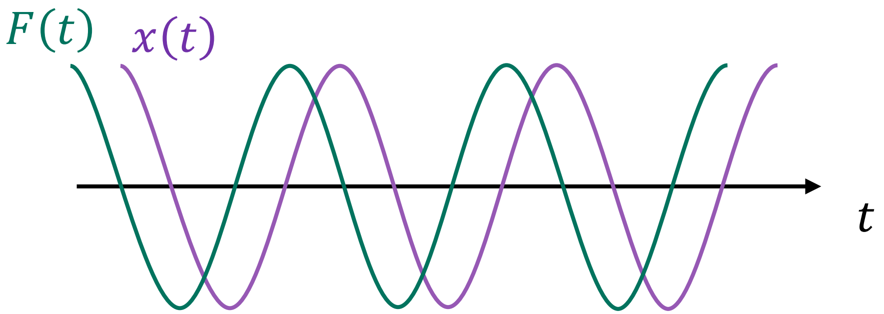


In case of an ideal response, the loss angle is zero, the response is a real quantity and there is no delay

$$\chi_2 = 0 \quad \rightarrow \quad \phi = 0$$

The Loss Angle

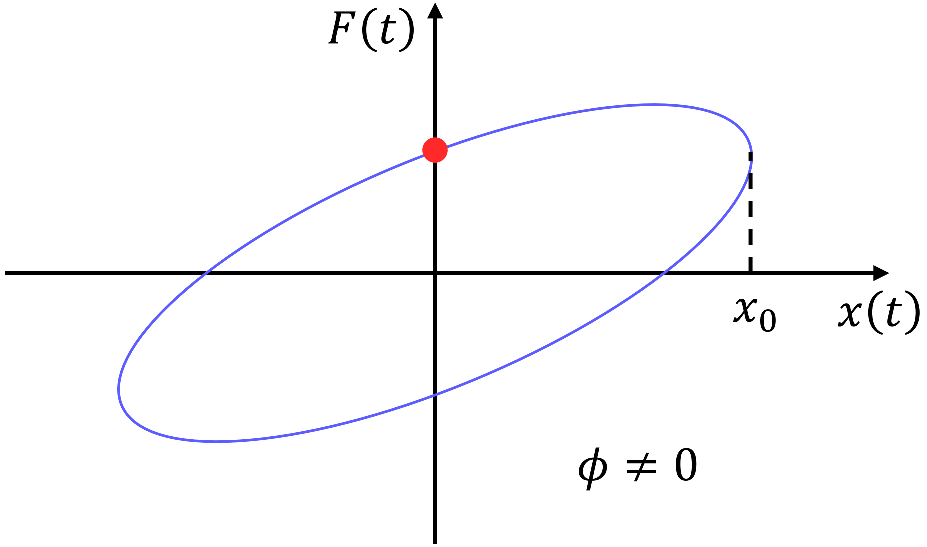
The response function is delayed, and this process is called relaxation. This delay introduce a phase lag between the stress and the strain.



$$x(t) = \int_{-\infty}^{+\infty} \chi(t-\zeta)F(\zeta)d\zeta \quad \text{with} \quad F(t) = F_0 e^{i\omega t}$$

$$\chi(\omega) = |\chi(\omega)|e^{i\phi} \;\rightarrow\; \chi = \chi_1 + i\chi_2$$

$$x(t) = x_0 e^{i(\omega t+\phi)} \quad \text{where} \quad |\chi(\omega)| = \frac{x_0}{F_0}$$

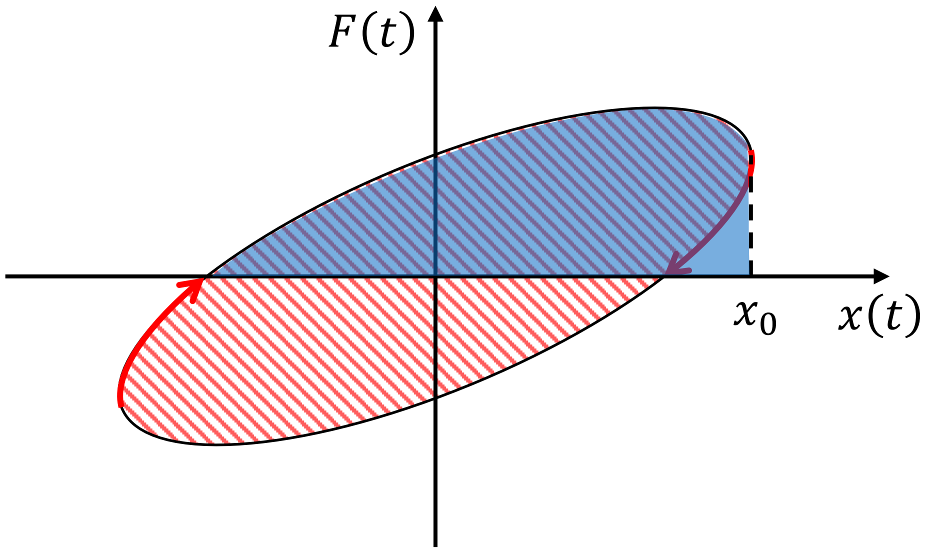


In case of dissipation, the loss angle is not zero, the response function is a complex quantity and there is a delay

$$\chi_2 \neq 0 \;\rightarrow\; \phi \neq 0$$

The Loss Angle

Why is it called **loss** angle?



$$x(t) = x_0 e^{i\omega t+\phi} \qquad x = (\chi_1 + i\chi_2)u$$

In order to evaluate the energy loss, we can evaluate the dissipated power over a cycle with respect to the maximum energy.

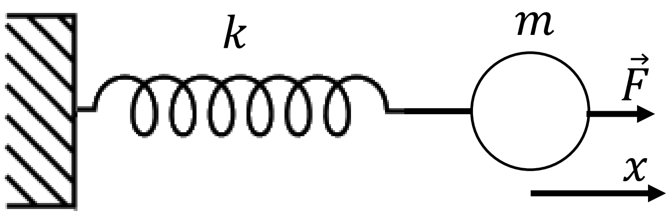
$$P(t) = F(t)\dot{x}(t)$$

$$E_{max} = \int_{-\frac{\pi}{2\omega}}^{0} P(t)dt = \frac{1}{2}x_0^2(\chi_1 + \chi_2\backslash 2) \xrightarrow{\chi_2 \to 0} \frac{1}{2}x_0^2\chi_1$$

$$E_{diss} = \int_0^{\frac{2\pi}{\omega}} P(t)dt = x_0^2\pi\chi_2$$

$$E_{diss} = 2\pi\frac{\chi_2}{\chi_1}E_{max} = 2\pi\boldsymbol{\phi}E_{max} \qquad \boldsymbol{\phi \propto \chi_2}$$

Thermal Noise of a Harmonic Oscillator



What is the thermal noise related to a perfect elastic, harmonic, system?

Let's consider for simplicity a spring-mass system and calculate the PSD using the response function

Time domain

$$m\ddot{x} = -kx + F$$

Frequency domain

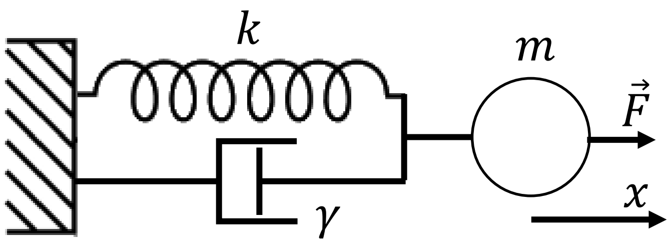
$$-m\omega^2 x = -kx + F$$

$$\chi(\omega) = \frac{x(\omega)}{F(\omega)} = \frac{1}{k - m\omega^2}$$

$$\chi''(\omega) = 0$$

There is no thermal noise

Thermal Noise of a Damped Harmonic Oscillator



What happen if we add a dashpot and introduce a delay in the response?

$$\Gamma = \frac{\gamma}{m^2}$$

$$\chi''(\omega) = -\frac{\omega\Gamma}{(\omega_0^2 - \omega^2)^2 + (\omega\Gamma)^2}$$

$$\omega_0^2 = \frac{k}{m}$$

Time domain

$$m\ddot{x} = -kx - \gamma\dot{x} + F$$

Frequency domain

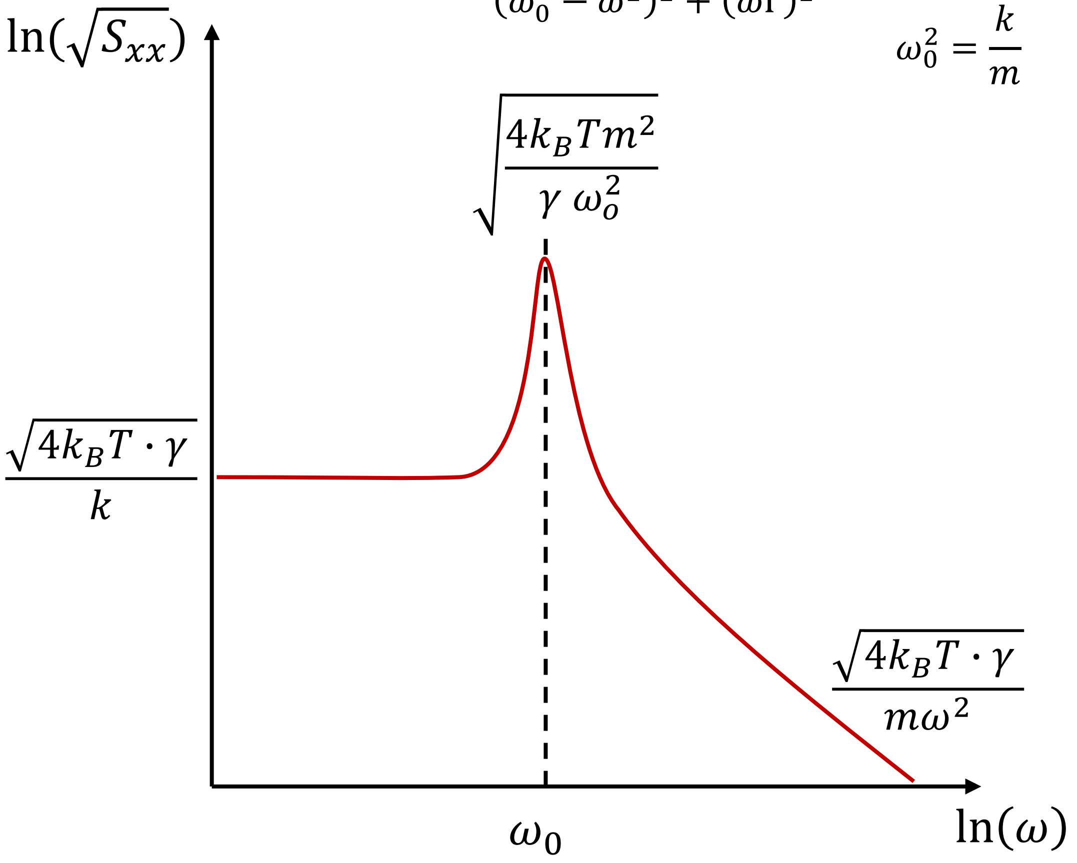
$$-m\omega^2 x = -kx - i\omega\gamma x + F$$

$$\chi(\omega) = \frac{x(\omega)}{F(\omega)} = \frac{1}{k - m\omega^2 + i\omega\gamma}$$

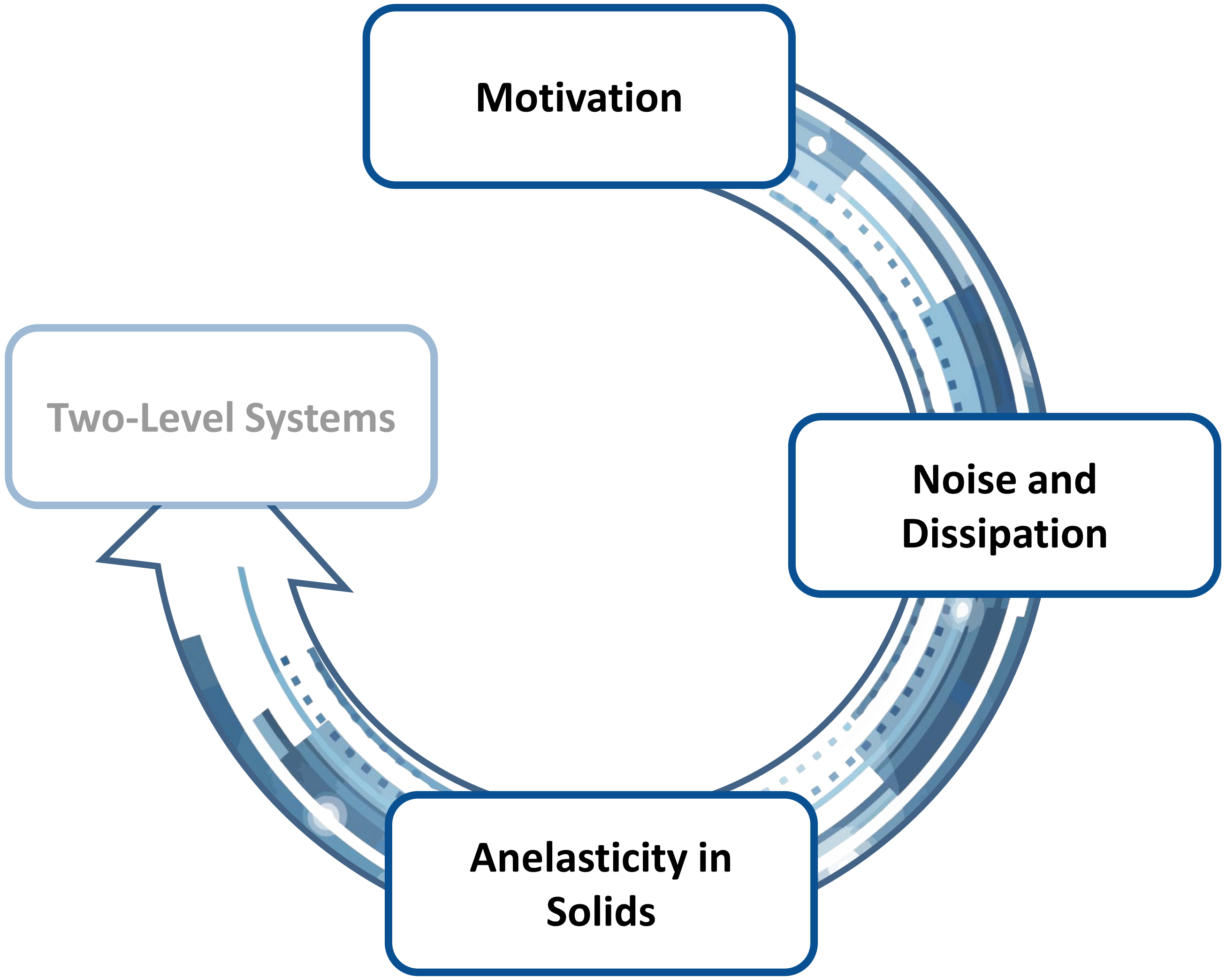
$$\chi''(\omega) = -\frac{\omega\gamma}{(k - m\omega^2)^2 + (\omega\gamma)^2} = -\omega\gamma|\chi(\omega)|^2$$

The Power Spectral Density

$$S_{xx}(\omega) = -\frac{4k_BT}{\omega} \cdot \chi''(\omega) = 4k_BT\gamma|\chi(\omega)|^2$$



Anelasticity in Solids



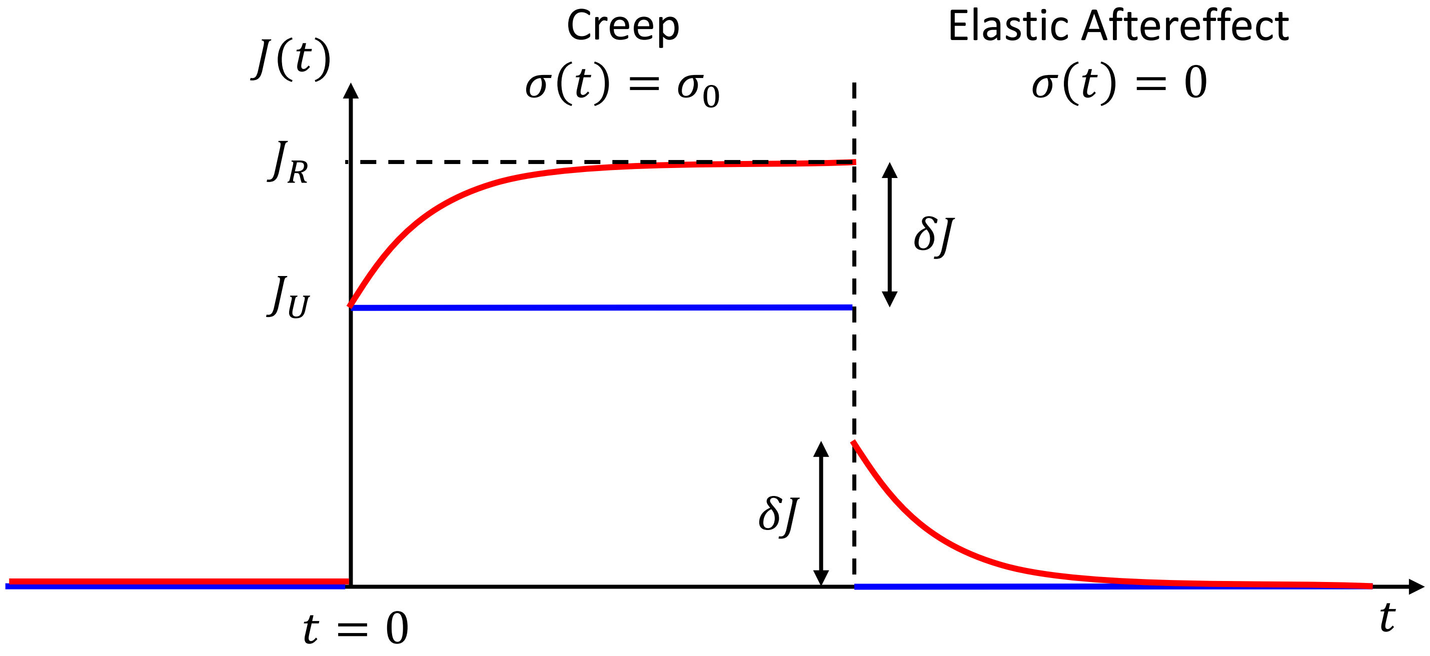
Delayed Response

In order to understand the behavior of anelastics solids is useful to look at the differences from ideal elastic materials. If we apply the stress $\sigma(t)$ to the solid, the strain $u(t)$ is modeled following the linear-response theory

$$u(t) = \int_{-\infty}^{+\infty} j(t-\zeta)\sigma(\zeta)d\zeta$$

A.S. Nowick – Anelastic Relaxation In Crystalline Solids

Let's apply a constant stress at $t = 0$.



Compliance function

$$J(t) = \frac{u(t)}{\sigma_0}$$

Strain

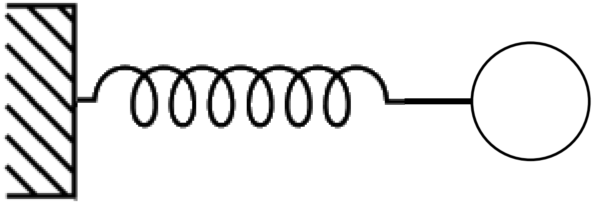
Stress

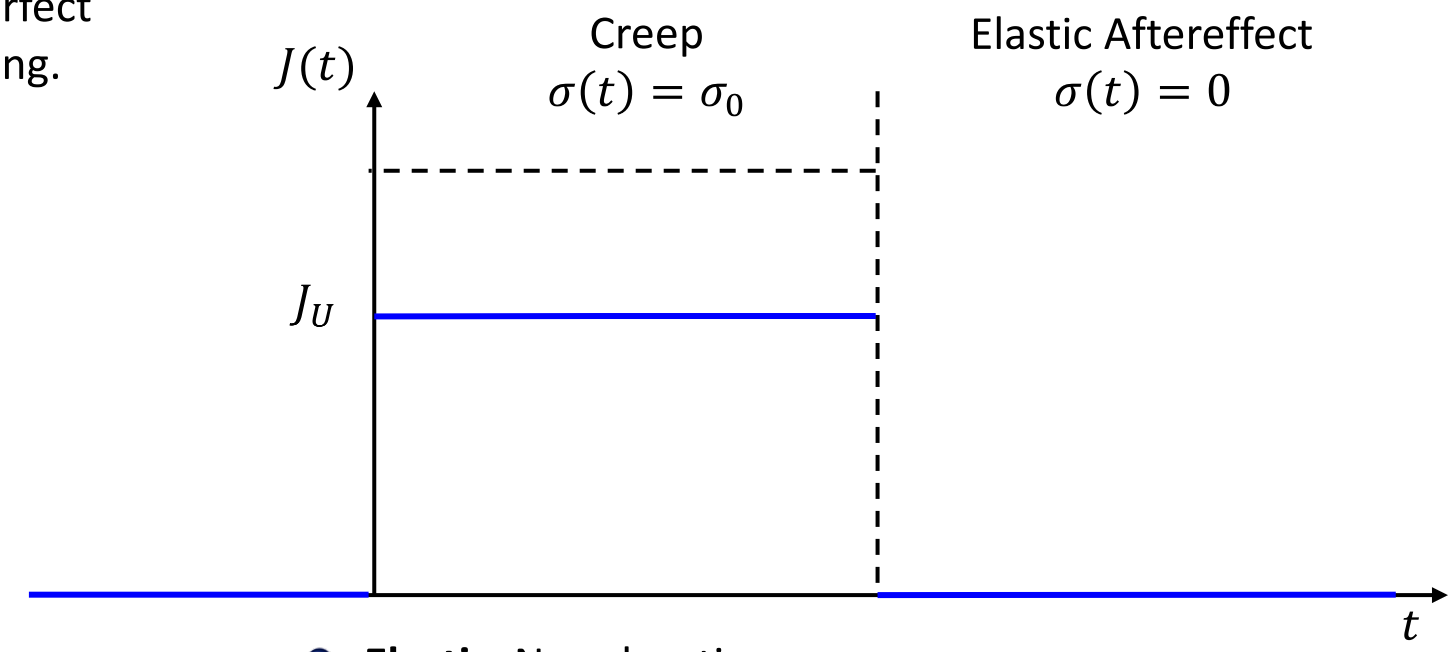
- **Elastic:** No relaxation processes
- **Anelastic:** Relaxation processes occur

Delayed Response

The instantaneous response is typical of perfect elastic materials and it is modeled by a spring.

The Hooke's model



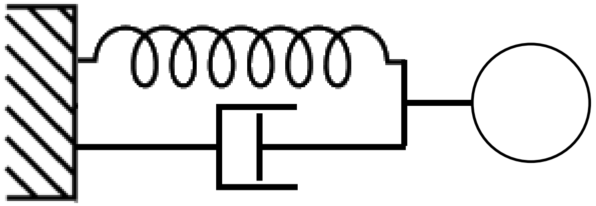


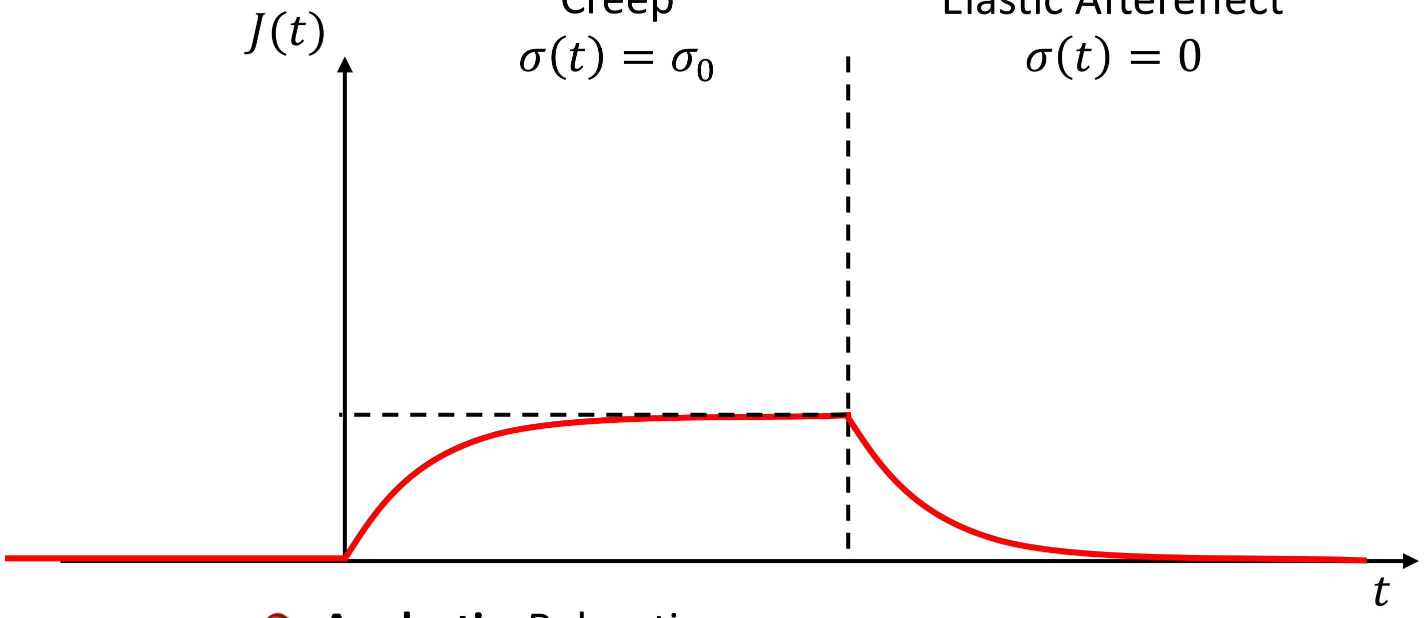
- **Elastic:** No relaxation processes

Delayed Response

In order to have a delayed response, we need to add a dashpot to our model.

The Voigt-Kelvin model



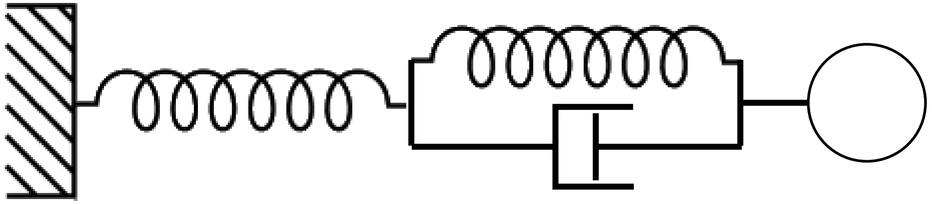


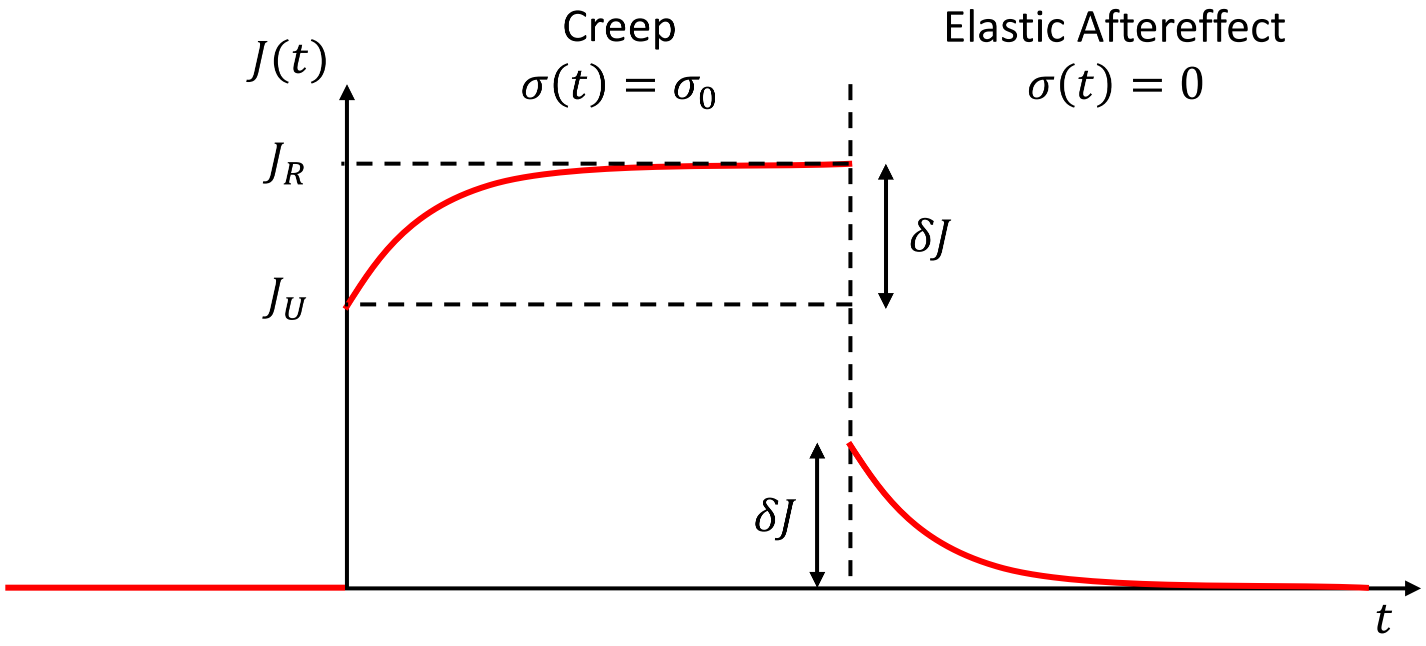
- **Anelastic:** Relaxation processes occur

Delayed Response

To fully capture the behavior of anelastic material we need to add another spring.

The three-phase model





The Elastic Response Function

It is easy to think about an applied stress $\sigma(t)$ to the solid and the creep experiments. However, a very useful quantity is obtained when we consider an applied strain $u(t)$ and evaluate the stress

$$u(t) = \int_{-\infty}^{+\infty} j(t-\zeta)\sigma(\zeta)d\zeta \quad \rightarrow \quad \sigma(t) = \int_{-\infty}^{+\infty} m(t-\zeta)u(\zeta)d\zeta$$

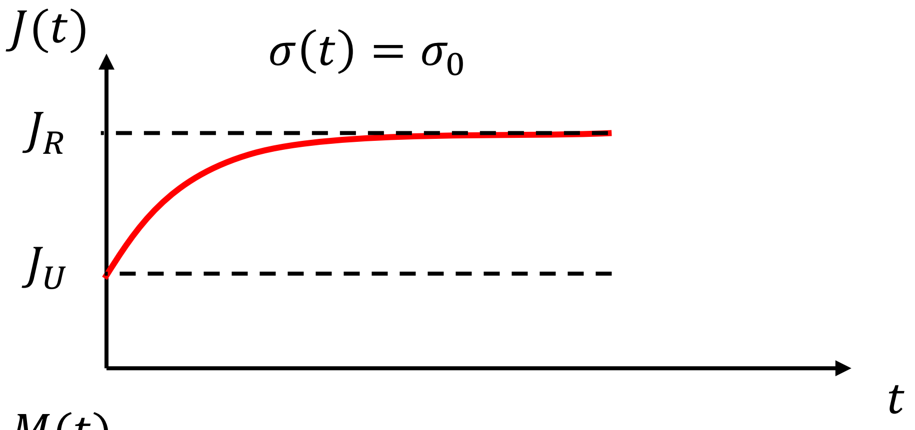
The Modulus function

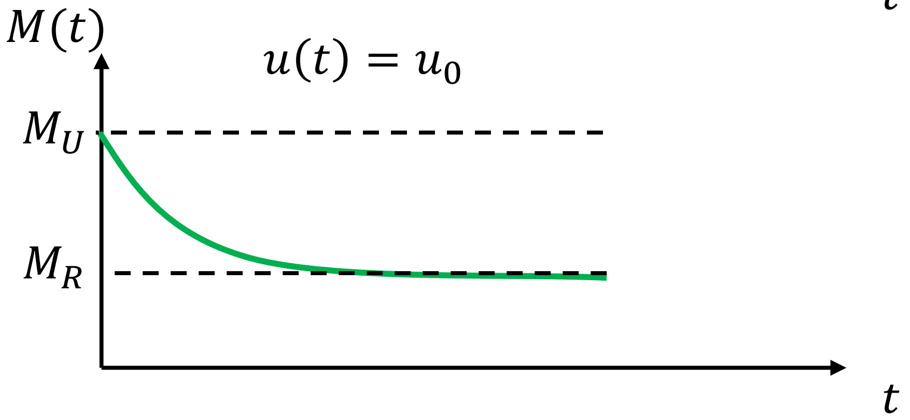
$$M(t) = \frac{\sigma(t)}{u_0}$$

For bulk deformations along one axis, the modulus function is called Young's modulus.

$$M_R = \frac{1}{J_R}$$

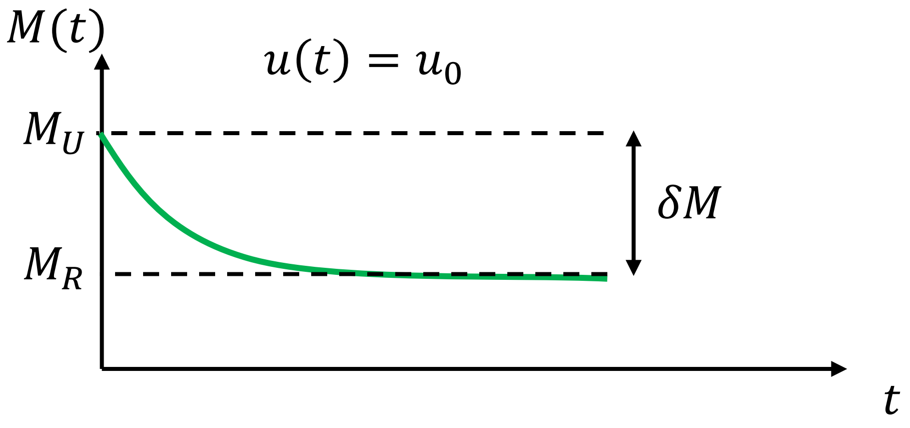
$$M_U = \frac{1}{J_U}$$

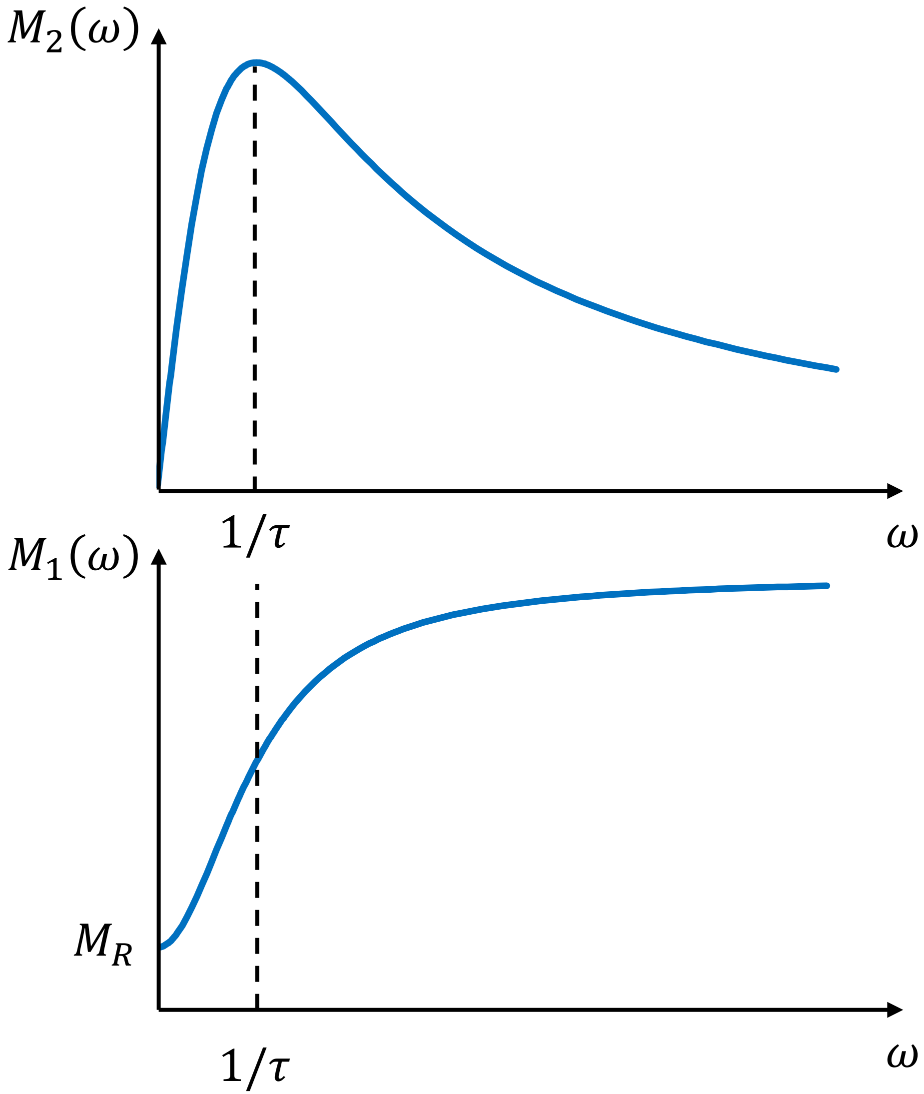




Debye Equation

The response function is delayed, and this process is called relaxation. The relaxation follows an exponential with a single relaxation time τ





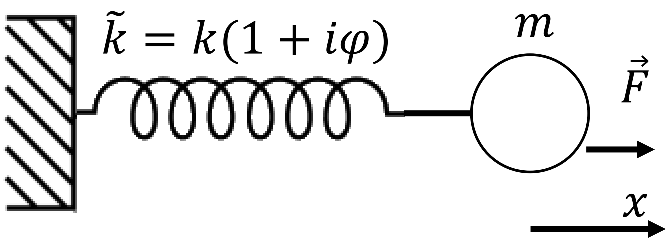
It is possible to obtain M_1 and M_2 as a function of $\omega\tau$.

Debye Equation

$$M_1 = M_R + \delta M \frac{\omega^2\tau^2}{1+\omega^2\tau^2}$$

$$M_2 = \delta M \frac{\omega\tau}{1+\omega^2\tau^2}$$

Thermal Noise for an Anelastic Harmonic Oscillator



We know that in case of anelasticity, the elastic modulus of our system is a complex quantity. Let's use this knowledge in a spring-mass system, where the spring constant is now a complex quantity.

Time domain → **Frequency domain**

$$m\ddot{x} = -k(1 + i\varphi)x + F$$

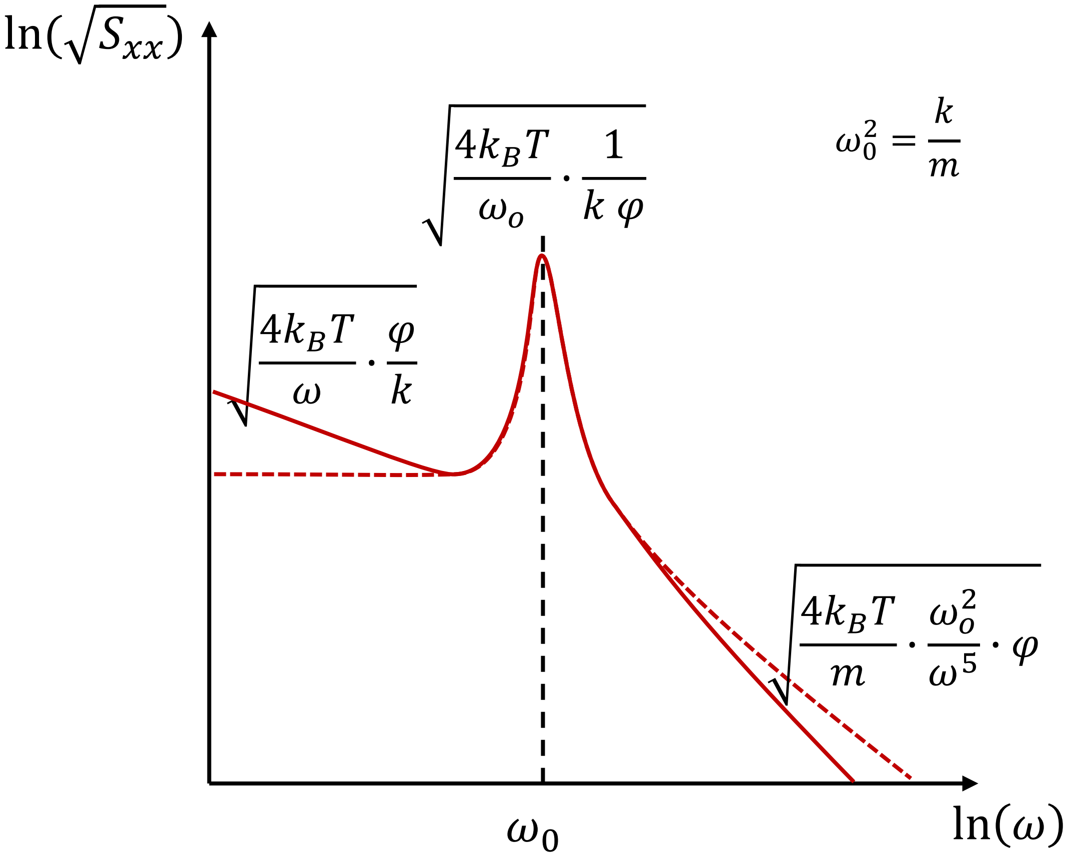
$$-m\omega^2 x = -k(1 + i\varphi)x + F$$

$$\chi(\omega) = \frac{x(\omega)}{F(\omega)} = \frac{1}{k - m\omega^2 + ik\varphi}$$

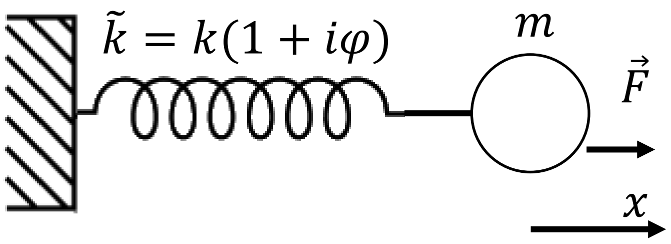
$$\chi''(\omega) = -\frac{k\varphi}{(k - m\omega^2)^2 + (k\varphi)^2} = -k\varphi|\chi(\omega)|^2$$

The Power Spectral Density

$$S_{xx}(\omega) = -\frac{4k_BT}{\omega} \cdot \chi''(\omega) = \frac{4k_BT}{\omega} k\varphi|\chi(\omega)|^2$$



Thermal Noise for an Anelastic Harmonic Oscillator

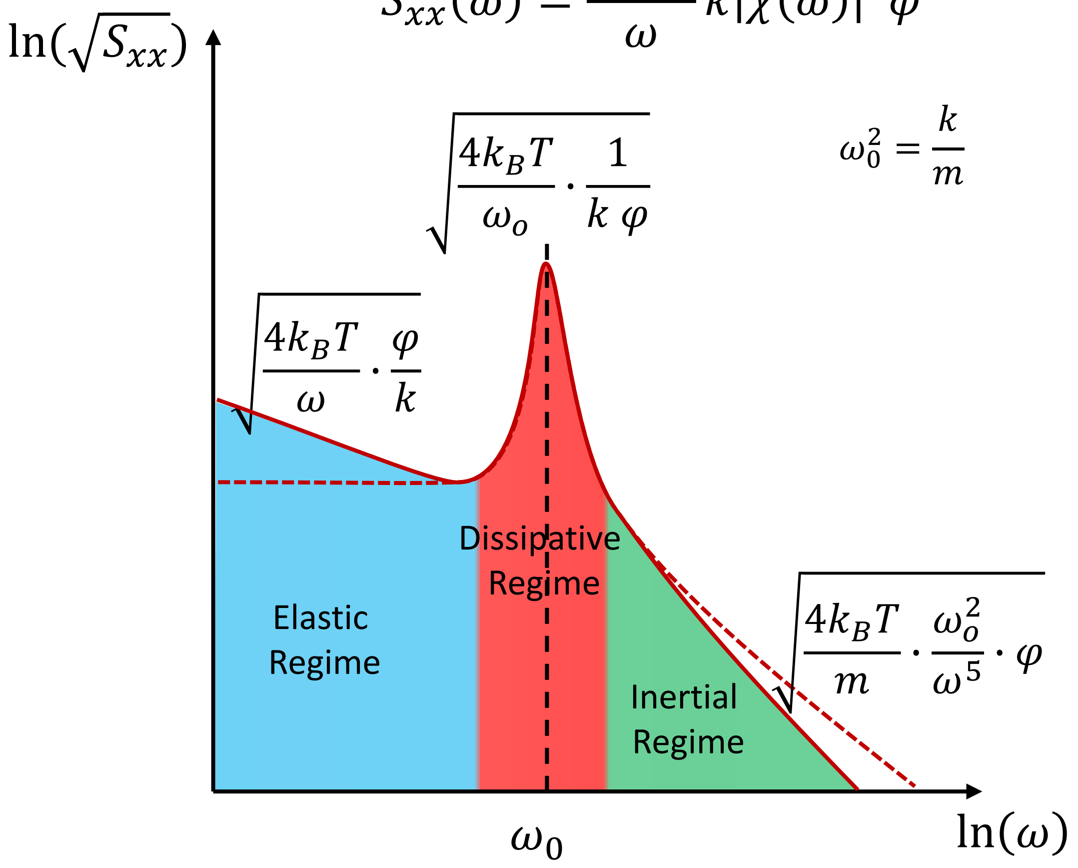


We know that in case of anelasticity, the elastic modulus of our system is a complex quantity. Let's use this knowledge in a spring-mass system, where the spring constant is now a complex quantity.

$$S_{xx}(\omega) = \frac{4k_BT}{\omega} k|\chi(\omega)|^2\varphi$$

Three possible regimes

- **Elastic:** the elasticity k is dominant ($\omega \ll \omega_0$)
- **Dissipative:** the loss $k\varphi$ is dominant ($\omega \sim \omega_0$)
- **Inertial:** the inertia $m\omega$ is dominant ($\omega \gg \omega_0$)



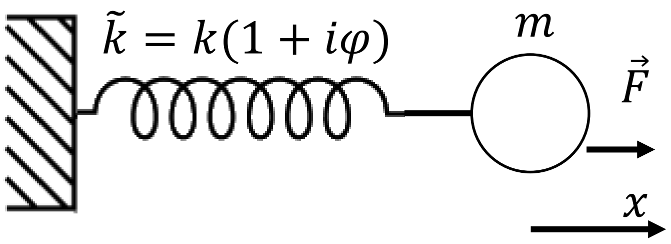
General expression of the PSD

$$S_{xx}(\omega) = \frac{4k_BT}{\omega} \cdot \frac{\text{Dilution}}{\text{Rigidity}} \cdot \varphi$$

Dilution = Dissipated Energy / Total Energy

Rigidity = $1/|\chi(\omega)|$

The Quality Factor



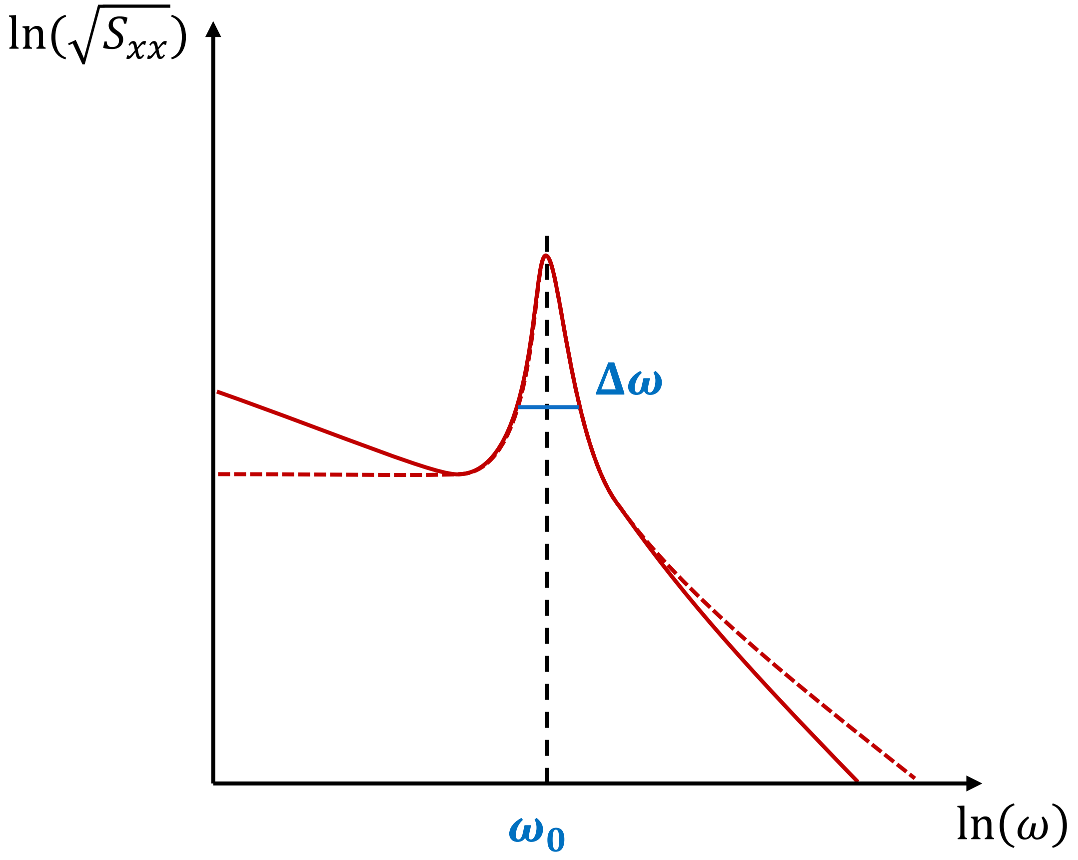
The quality factor is a dimensionless quantity that describe how much an oscillator is damped.

- High Q $\rightarrow$ low damping
- Low Q $\rightarrow$ high damping

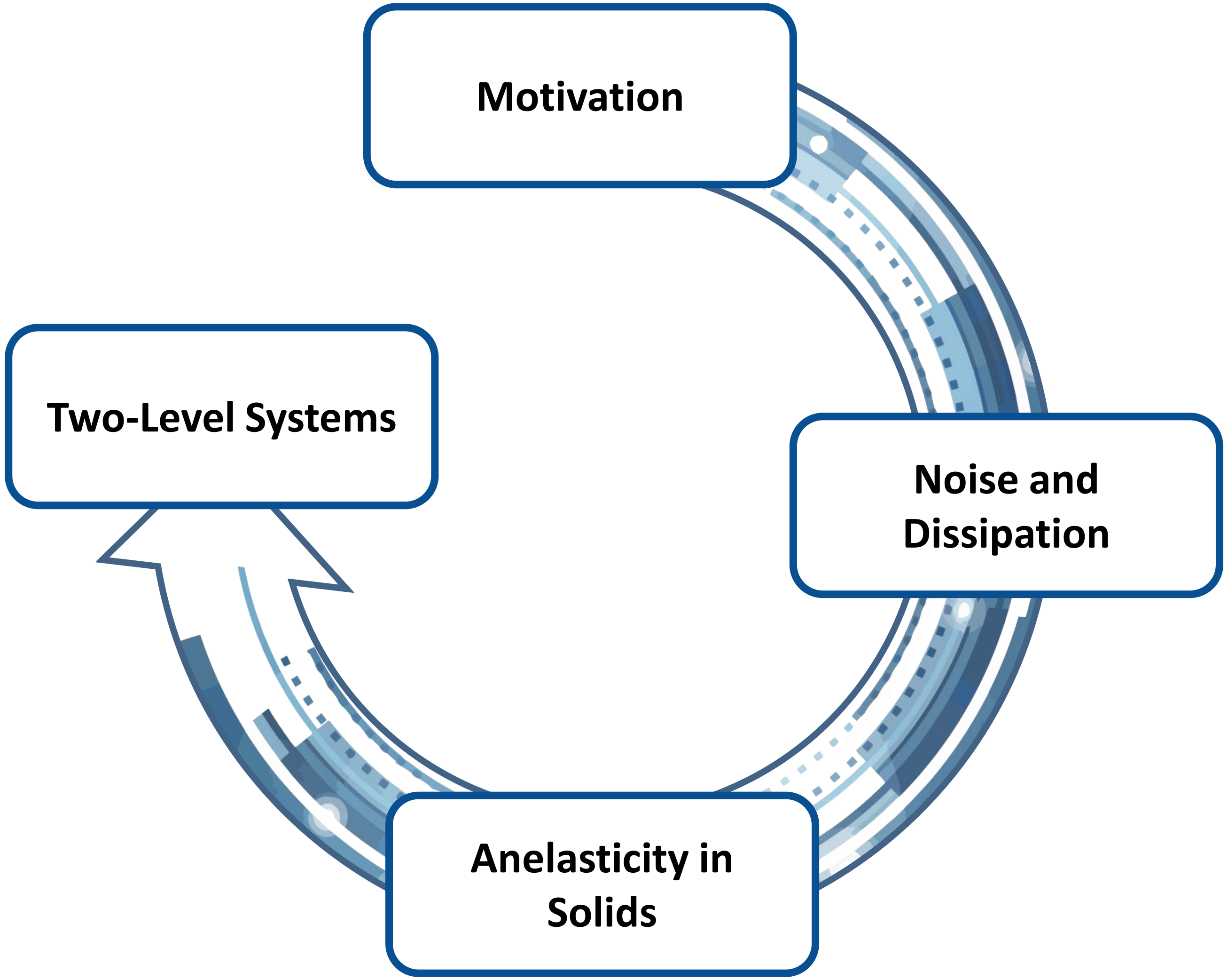
Two definitions

$$Q = \frac{\omega_0}{\Delta\omega}$$

$$Q = 2\pi \frac{\text{stored energy}}{\text{dissipated energy}} = \frac{1}{\varphi(\omega_0)}$$

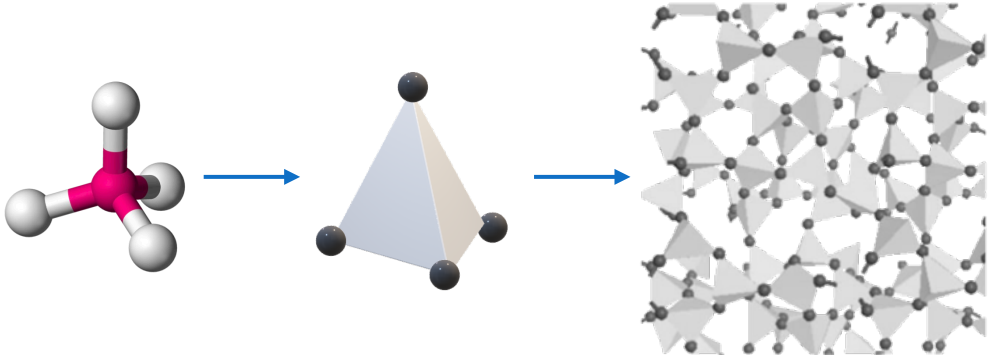


Types of Thermal Noise

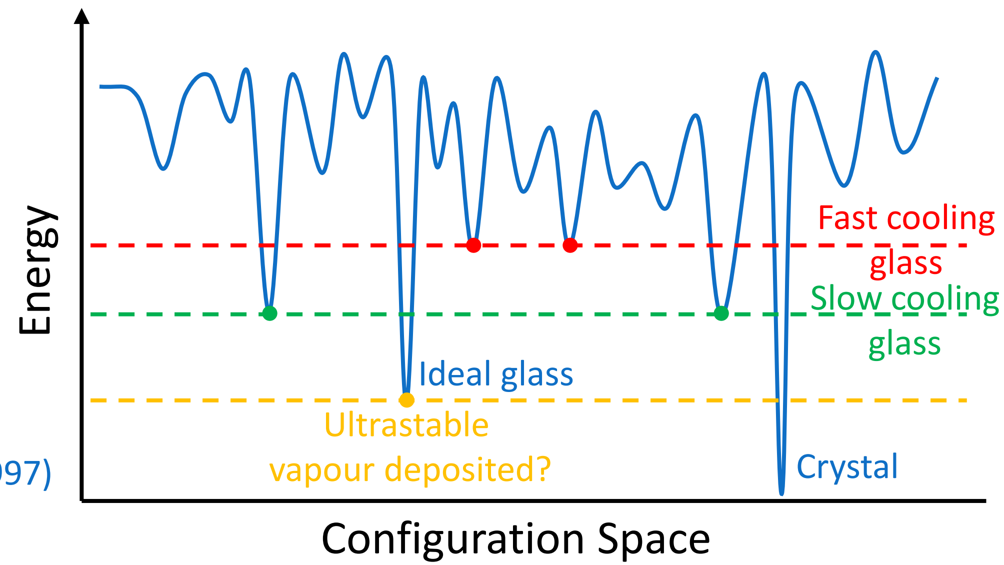


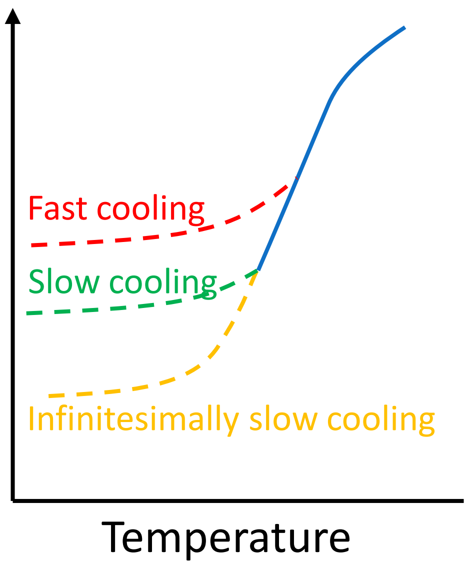
Two-Level System

In current detectors the coatings are made of amorphous materials. The SiO_2 is found in both coating and substrate.



C.A. Angell,
Physica D 107, 122 (1997)

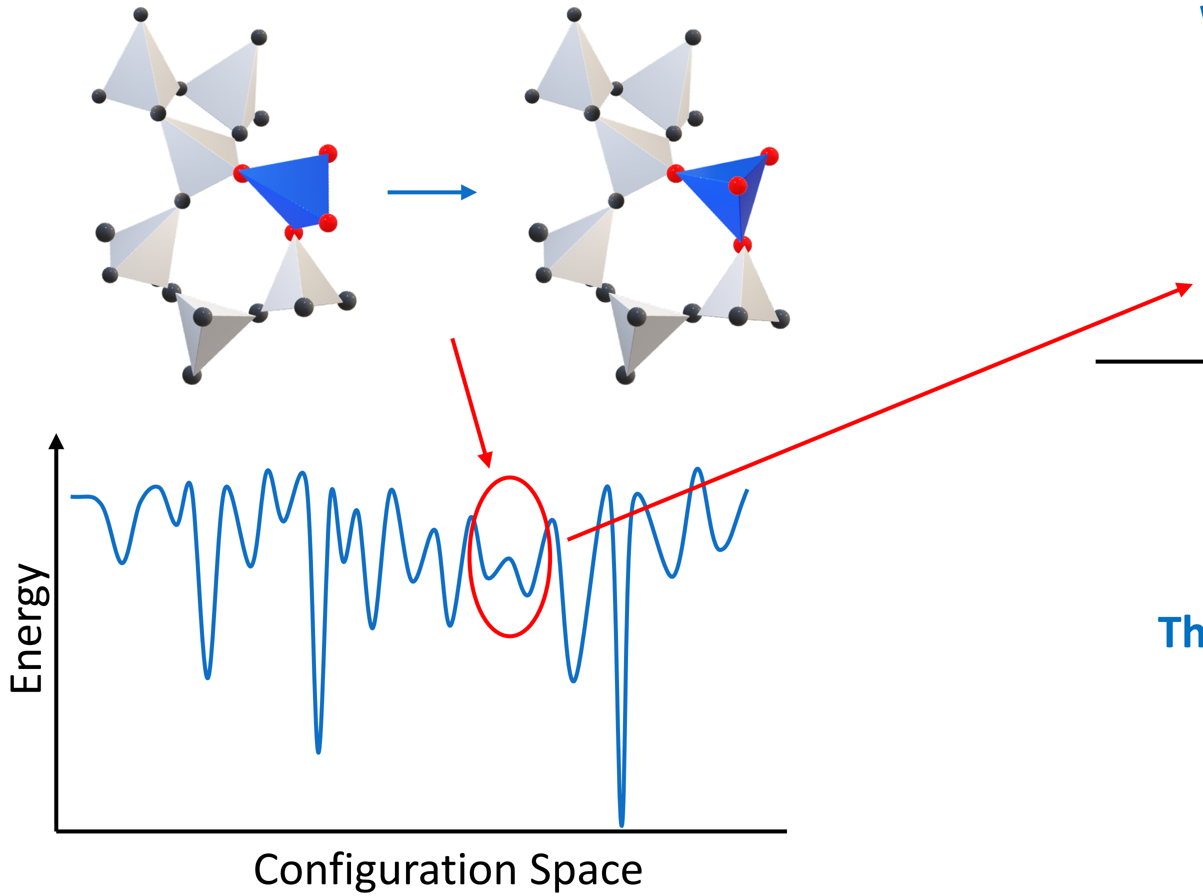


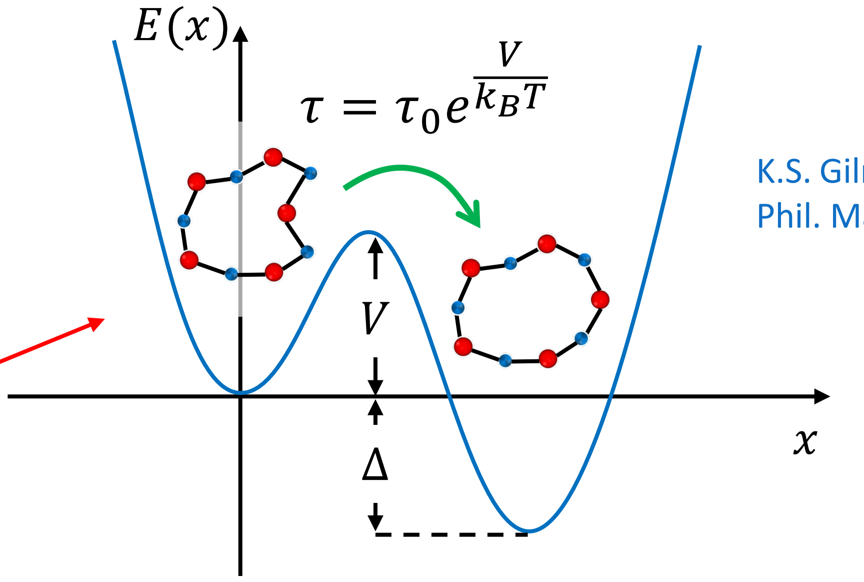


Kearns, Swallow, Ediger,
J. Chem. Phys. 127,
154702 (2007)

Two-Level System

O-Si-O bonds are fixed angles, but Si-O-Si angle is quite floppy → different energy scales





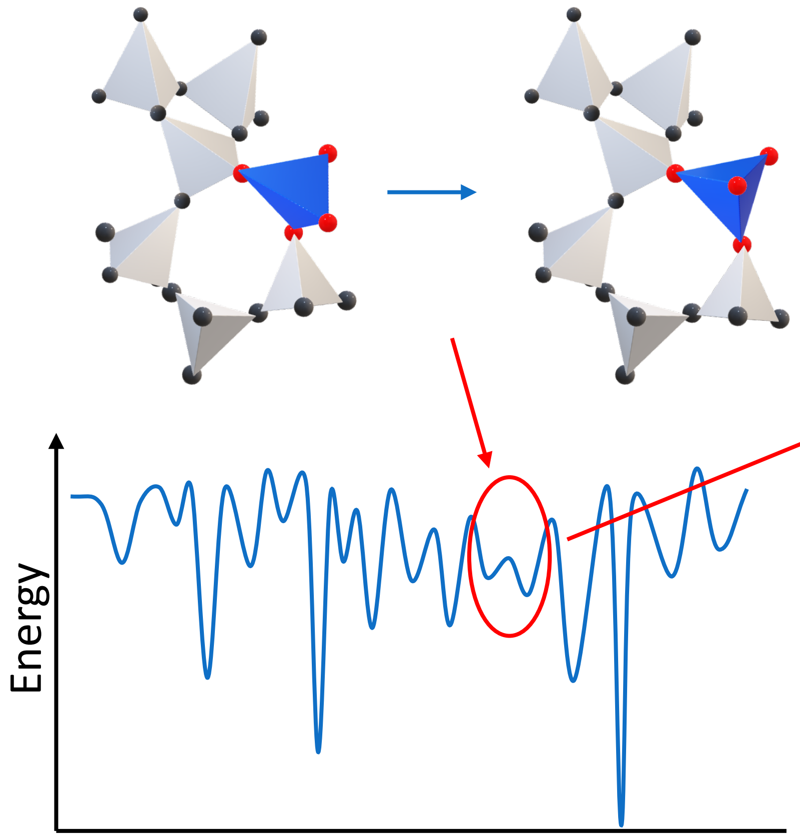
K.S. Gilroy and W.A. Phillips, 1981
Phil. Mag. B, 43 (5) 735-746

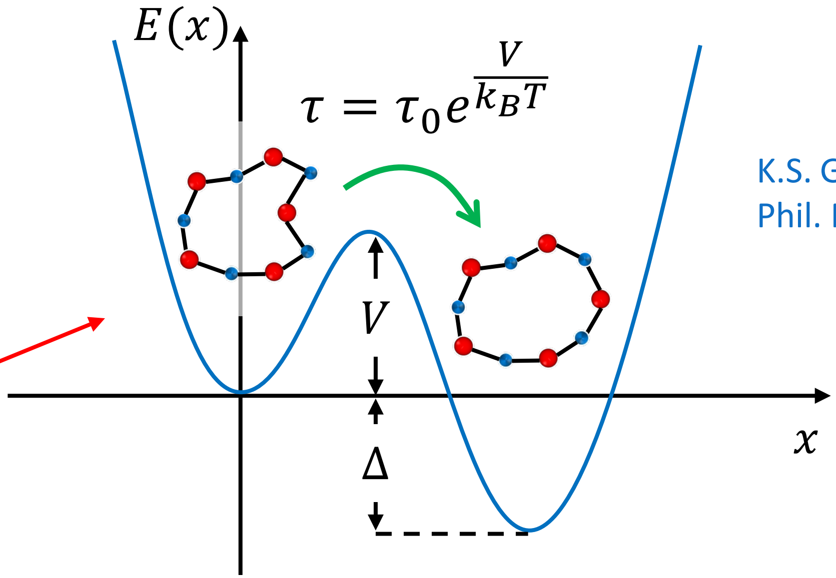
The elastic modulus is locally changing

$$\phi(\omega) \propto M_2 = \delta M \frac{\omega\tau}{1+\omega^2\tau^2}$$

Two-Level System

O-Si-O bonds are fixed angles, but Si-O-Si angle is quite floppy → different energy scales





K.S. Gilroy and W.A. Phillips, 1981
Phil. Mag. B, 43 (5) 735-746

Superposition of single relaxation time processes

$$\phi(\omega) \propto M_2 = \int_0^\infty \int_0^\infty \delta M \frac{\omega\tau}{1+\omega^2\tau^2} g(\Delta) f(V)\, d\Delta dV$$

The two distributions g and f depends on the material.
g is assumed flat within a specific range

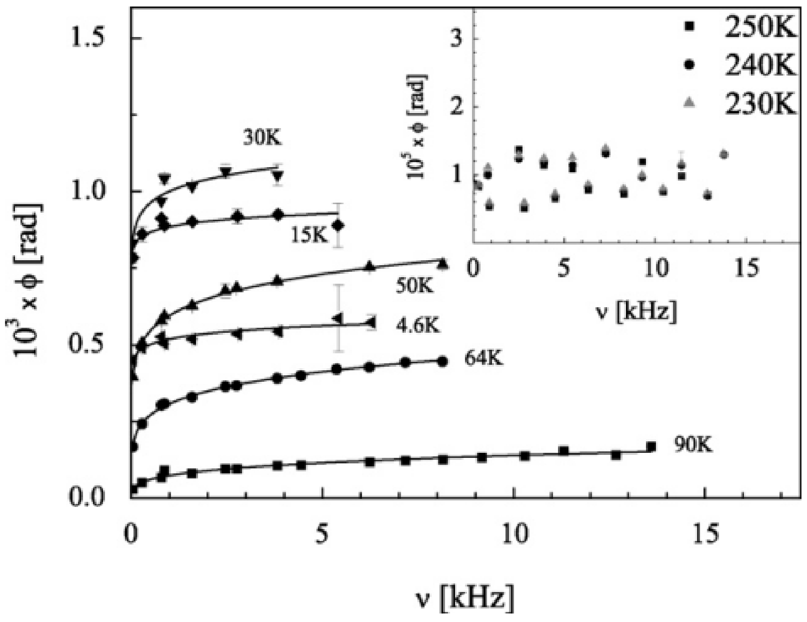
Two-Level System: Fused Silica

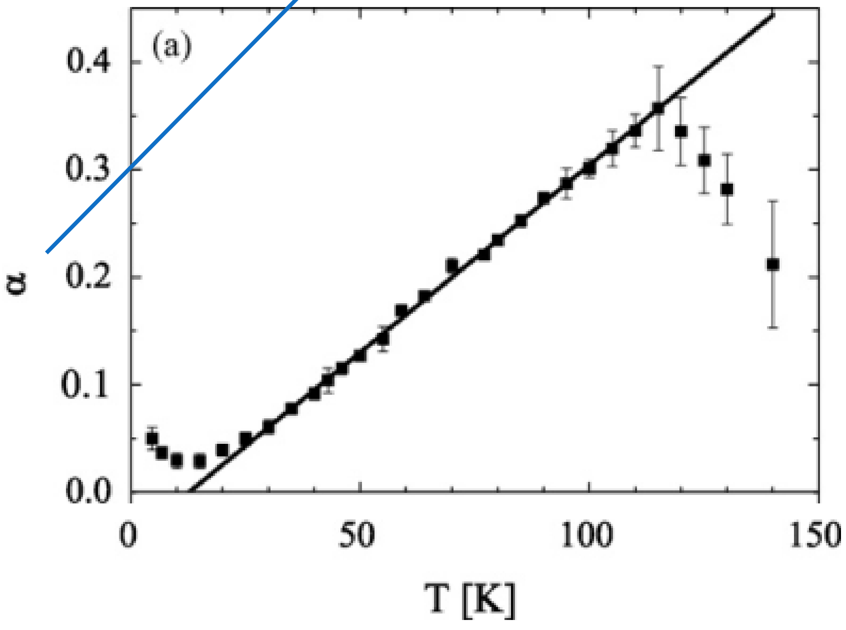
Between 30K and 110K fused silica seems to have an exponential barrier height distribution :

$$f(V) = \frac{1}{V_0} e^{-V/V_0}$$

After some calculations

$$\phi \text{ or } M_2 \propto k_B T (\omega \tau_0)^{-\frac{k_B T}{V_0}}$$

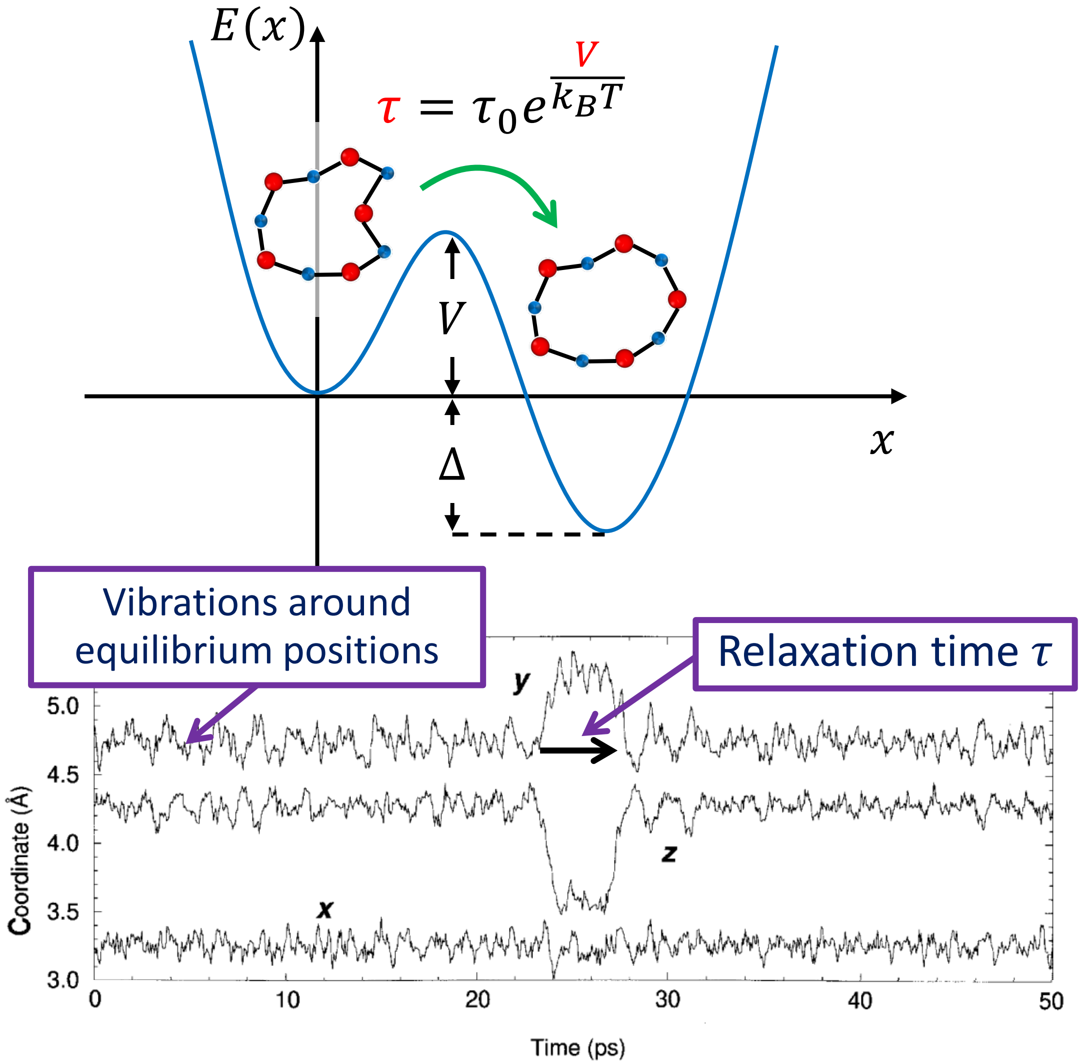




F. Travasso et al.,
Materials Science and Engineering A
521–522, 268–271 (2009)

Two-Level System

Not all the relaxation times introduce noise to the detector!



M.T. Dove at al., Mineralogical Magazine, 64(3), 377-388 (2000)