Correlated D decays at the $\Psi(3770)$

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To calculate correlated D decay rates at the J/Ψ(37700 we calculate the correlated amplitude for the D and the $\overline{D}$ to decay to the states α and β at times t_1 and t_2 respectively, where the times are measured in the center-of-mass (CM) system and $t = 0$ is the time of the $e^+e^- \to c\bar{c}$ production. Because the $\Psi(3770)$ is $J^{PC} = 1^{--}$ state, we antisymmetrize the amplitude with respect to charge conjugation.

$$\mathcal{M} = \frac{1}{\sqrt{2}}\left[\langle\alpha|\mathcal{H}|D^0(t_1)\rangle\langle\beta|\mathcal{H}|\overline{D}^0(t_2)\rangle - \langle\beta|\mathcal{H}|D^0(t_2)\rangle\langle\alpha|\mathcal{H}|\overline{D}^0(t_1)\rangle\right] \tag{1}$$

The time evolution of the D^0–$\overline{D}^0$ system is described by the Schrödinger equation

$$i\frac{\partial}{\partial t}\begin{pmatrix} D^0(t) \\ \overline{D}^0(t)\end{pmatrix} = \left(\mathrm{M} - \frac{i}{2}\Gamma\right)\begin{pmatrix} D^0(t) \\ \overline{D}^0(t)\end{pmatrix}, \tag{2}$$

where the M and Γ matrices are Hermitian, and CPT invariance requires $M_{11} = M_{22} \equiv M$ and $\Gamma_{11} = \Gamma_{22} \equiv \Gamma$.

Some Notation

The two eigenstates D_1 and D_2 of the effective Hamiltonian are

$$|D_{1,2}\rangle = p|D^0\rangle \pm q|\overline{D}^0\rangle \,, \qquad |p|^2 + |q|^2 = 1\,. \tag{3}$$

The corresponding eigenvalues are

$$\lambda_{1,2} \equiv m_{1,2} - \frac{i}{2}\Gamma_{1,2} = \left(M - \frac{i}{2}\Gamma\right) \pm \frac{q}{p}\left(M_{12} - \frac{i}{2}\Gamma_{12}\right), \tag{4}$$

where $m_{1,2}$, $\Gamma_{1,2}$ are the masses and decay widths and

$$\frac{q}{p} = \sqrt{\frac{M_{12}^* - \frac{i}{2}\Gamma_{12}^*}{M_{12} - \frac{i}{2}\Gamma_{12}}}\,. \tag{5}$$

The proper time evolution of the eigenstates of Eq. 2 is

$$|D_{1,2}(t)\rangle = e_{1,2}(t)|D_{1,2}\rangle,\ e_{1,2}(t) = e^{[-i(m_{1,2} - \frac{i\Gamma_{1,2}}{2})t]}. \tag{6}$$

A state that is prepared as a flavor eigenstate $|D^0\rangle$ or $|\overline{D}^0\rangle$ at $t = 0$ will evolve according to

$$|D^0(t)\rangle = \frac{1}{2p}\Big[p(e_1(t) + e_2(t))|D^0\rangle + q(e_1(t) - e_2(t))|\overline{D}^0\rangle\Big] \tag{7}$$

$$|\overline{D}^0(t)\rangle = \frac{1}{2q}\Big[p(e_1(t) - e_2(t))|D^0\rangle + q(e_1(t) + e_2(t))|\overline{D}^0\rangle\Big]\,. \tag{8}$$

We adopt a version of the standard notation

$$\Gamma = \frac{\Gamma_1 + \Gamma_2}{2}\,, \qquad x = \frac{m_1 - m_2}{\Gamma}\,, \qquad y = \frac{\Gamma_1 - \Gamma_2}{2\Gamma}\,. \tag{9}$$

Forms of $\mathcal{M}$ and $|\mathcal{M}|^2$

After a bit of algebra we can write the matrix element as

$$
\begin{aligned}
2\sqrt{2}\,\mathcal{M} \;&=\; \left(\frac{q}{p}\,\overline{\mathcal{A}}_\alpha \overline{\mathcal{A}}_\beta - \frac{p}{q}\,\mathcal{A}_\alpha \mathcal{A}_\beta\right) [e_1(t_1)e_2(t_2) - e_1(t_2)e_2(t_1)] \\
&+\; (\mathcal{A}_\alpha \overline{\mathcal{A}}_\beta - \overline{\mathcal{A}}_\alpha \mathcal{A}_\beta)\,[e_1(t_1)e_2(t_2) + e_1(t_2)e_2(t_1)]
\end{aligned} \tag{10}
$$

which has the form

$$
2\sqrt{2}\,\mathcal{M} = X(e_{11}e_{22} - e_{12}e_{21}) \;+\; Y(e_{11}e_{22} + e_{12}e_{21})\,. \tag{11}
$$

From this one calculates

$$
\begin{aligned}
8|\mathcal{M}|^2 \;&=\; e^{-\Gamma(t_1+t_2)} \times \{ \quad XX^*\,(\cosh y\Gamma\Delta t - \cos x\Gamma\Delta t) \\
&-\quad 2\,\Re(XY^*)\,\sinh y\Gamma\Delta t + \;\; 2\,\Im(XY^*)\,\sin x\Gamma\Delta t \\
&+\quad\quad YY^*\,(\cosh y\Gamma\Delta t + \cos x\Gamma\Delta t\,\}
\end{aligned} \tag{12}
$$

For $x\Gamma\Delta t$, $y\Gamma\Delta t \ll 1$ this can be approximated by

$$
\begin{aligned}
4|\mathcal{M}|^2 \;&=\; e^{-\Gamma(t_1+t_2)} \times \left\{ \quad XX^* \left[\frac{(x^2+y^2)}{4}(\Gamma\Delta t)^2\right] \right. \\
&-\quad \Re(XY^*)\,y\Gamma\Delta t + \Im(XY^*)\,x\Gamma\Delta t \\
&+\quad\quad \left. YY^* \left[1 + \frac{(y^2-x^2)}{4}(\Gamma\Delta t)^2\right]\right\}
\end{aligned} \tag{13}
$$

- Y is the unmixed amplitude
- X is the mixing amplitude
- XY^* controls the interference terms in the mixing rate

see also Zhi-zhong Xing, Phys.Rev. D55 (1997) 196-218

Correlated $(K^- K^+,\ K^- K^+)$ decays
CP even versus CP even

For two CP-even eigenstates α and β,

$$\begin{aligned} Y &= 0 \\ X &= \left(\frac{q}{p} - \frac{p}{q}\right) \mathcal{A}_\alpha \mathcal{A}_\beta \,. \end{aligned} \tag{14}$$

so the rate is

$$|\mathcal{M}|^2 = e^{-\Gamma(t_1+t_2)} \times \left|\frac{q}{p} - \frac{p}{q}\right|^2 |A_\alpha|^2 \, |A_\beta|^2 \left(\frac{x^2+y^2}{4}\right) (\Gamma \Delta t)^2 \,. \tag{15}$$

In the limit that CP is a good symmetry, this rate goes to zero. To estimate what might be possible at Super$B$, we take the numbers of $K^\mp \pi^\pm$ versus CP even events observed by CLEO-c (605), scale by the approximate ratio of $K^- K^+$ plus $\pi^- \pi^+$ events observed (≈ 0.13) [to account for the value of $|A_\alpha|^2 \, |A_\beta|^2$], and scale by the nominal relative luminosity. This procedure gives approximately 120K as the coefficient of $(x^2+y^2)\,(\Gamma\Delta t)^2/4$. Using $(x^2+y^2)\,(\Gamma\Delta t)^2/2$ as an estimate of the time integral, and taking $x^2+y^2 = 10^{-4}$, the integrated signal will be about

$$\left|\frac{q}{p} - \frac{p}{q}\right|^2 \times 6 \text{ events} \,. \tag{16}$$

Correlated $(K^-\pi^+\ ,\ \ K^-\pi)$ decays

A similar result obtains for common final states such as $K^-\pi^+$. If $\alpha = \beta$ then $\mathcal{A}_\beta = \mathcal{A}_\alpha$ and $\overline{\mathcal{A}}_\beta = \overline{\mathcal{A}}_\alpha$. Again, the unmixed amplitude goes to zero. However, the pure mixing term does not require CP violation to be non-zero.

$$
\begin{aligned}
Y &= 0 \qquad (17)\\
X &= \left(\frac{q}{p}\overline{\mathcal{A}}_\alpha\overline{\mathcal{A}}_\alpha - \frac{p}{q}\mathcal{A}_\alpha\mathcal{A}_\alpha\right).
\end{aligned}
$$

In this case, $\mathcal{A}_\alpha$ corresponds to the Cabibbo-favored amplitude and $\overline{\mathcal{A}}_\alpha$ to the doubly Cabibbo-suppressed amplitude. With $\overline{\mathcal{A}}_\alpha = ke^{i\delta}\mathcal{A}_\alpha$ the rate can be written

$$
|\mathcal{M}|^2 = e^{-\Gamma(t_1+t_2)} \times \left|\frac{q}{p}k^2e^{i2\delta} - \frac{p}{q}\right|^2 |A_\alpha|^2\,|A_\alpha|^2 \left(\frac{x^2+y^2}{4}\right)(\Gamma\Delta t)^2\,. \qquad (18)
$$

As a first approximation, we can ignore both the doubly Cabibbo-suppressed amplitude and CP violation. In this case

$$
|\mathcal{M}|^2 \approx e^{-\Gamma(t_1+t_2)} \times |A_\alpha|^2\,|A_\alpha|^2 \left(\frac{x^2+y^2}{4}\right)(\Gamma\Delta t)^2\,. \qquad (19)
$$

CLEO-c observes 600 $K^-\pi^+$, $K^+\pi^-$ events, which corresponds to $2\,|A_\alpha|^2\,|A_\alpha|^2$. Scaling by relative luminosities, and again using 10^{-4} for (x^2+y^2), we can project a mixing signal of 23 events in this channel and a similar number in $K^+\pi^-$ versus $K^+\pi^-$. While differences nominally can be due to direct CP violation, indirect CP violation, or statistical fluctuation, given the existing HFAG bounds on direct and indirect CP violation, any variation we observe in this channel will be predominantly due to statistical fluctuations.

Correlated $(K\ell\nu,\ K\ell\nu)$ decays

For opposite-sign semileptonic decays we can choose $\alpha = K^-\ell^+\nu$ and $\beta = K^+\ell^-\overline{\nu}$ for which

$$Y \ = \ \mathcal{A}_\alpha \overline{\mathcal{A}}_\beta; \qquad X = 0 \tag{20}$$

The rate is proportional to

$$|\mathcal{M}|^2 = e^{-\Gamma(t_1+t_2)} \times |\mathcal{A}_\alpha|^2\,|\mathcal{A}_\beta|^2 \left[1 + \frac{(y^2 - x^2)}{4}(\Gamma\Delta t)^2\right]. \tag{21}$$

The only signature of mixing in this final state is the quadratic departure from purely exponential decay which is proportional to $(y^2 - x^2)$. This is less than one part in 10^4, significantly less than the rate of statistical fluctuations. This final state has no sensitivity to CP violation in mixing $(q/p \neq 1)$.

For same-sign semileptonic decays we can choose $\alpha = \beta = K^-\ell^+\nu$. In this case

$$Y \ = \ 0 \qquad X = -\frac{p}{q}\Big(\mathcal{A}(D^0 \to K^- e^+ \overline{\nu}_e)\Big)\,. \tag{22}$$

The corresponding rate is

$$|\mathcal{M}|^2 = e^{-i\Gamma(t_1+t_2)} \left|\left(\frac{p}{q}\right)\mathcal{A}_\alpha\mathcal{A}_\beta\right|^2 \left(\frac{x^2+y^2}{4}\right)(\Gamma\Delta t)^2\,. \tag{23}$$

Extrapolating from CLEO-c's opposite-sign $K\pi$ rate, we estimate 23 mixing events in each of $K^-e^+\nu_e$ versus $K^-e^+\nu_e$ and $K^+e^-\overline{\nu}_e$ versus $K^+e^-\overline{\nu}_e$.

Correlated $(K\ell\nu,\ K^-K^+)$ decays

The correlated decays of $D^0\overline{D}^0$ into a CP eigenstate and and semileptonic final state are also (relatively) easy to understand. Consider $\mathcal{A}_\alpha = \mathcal{A}(D^0 \to K^- e^+ \nu_e)$ and $\mathcal{A}_\beta = \mathcal{A}(D^0 \to K^- K^+)$ as an example such a final state. In this case

$$Y = \mathcal{A}_\alpha \mathcal{A}_\beta\,; \qquad X = -\frac{p}{q}\mathcal{A}_\alpha \mathcal{A}_\beta \tag{24}$$

The (small $y\Gamma\Delta t$, small $x\Gamma\Delta t$) limit for $(K^-\ell^+X,\ K^-K^+)$ is

$$\begin{aligned} |\mathcal{M}|^2 \ &= \ e^{-\Gamma(t_1+t_2)}\,|\mathcal{A}_\alpha|^2\,|\mathcal{A}_\beta|^2 \ \times \\ &\left\{ 1 \ \mp \ \Re(\frac{p}{q})\,y\Gamma\Delta t \ \pm \ \Im(\frac{p}{q})\,x\Gamma\Delta t + \frac{y^2}{2}(\Gamma\Delta t)^2 \right\}. \end{aligned} \tag{25}$$

For $\overline{D}^0 \to K^+\ell^- X$ detected in conjunction with a CP even final state, $(-p/q)$ in XY^* becomes $(+q/p)$ and $\mathcal{A}_\alpha = \mathcal{A}(\overline{D}^0 \to K^+\ell^- X)$. As a first approximation, the difference between positive and negative decay time distributions will be proportional to

$$\left(\Re(\frac{p}{q})\,y - \Im(\frac{p}{q})\,x\right) \times \Gamma\,|\Delta t| \ = \ y'\Gamma\,|\Delta t| \tag{26}$$

for $D^0 \to K^-\ell^+X$ and to

$$\left(\Re(\frac{q}{p})\,y - \Im(\frac{q}{p})\,x\right) \times \Gamma\,|\Delta t| \ = \ y''\Gamma\,|\Delta t| \tag{27}$$

for $\overline{D}^0 \to K^+\ell^-X$. For $q/p \approx 1$, the difference between y' and y'' measures $|q/p|$.

Correlated $(K^-\ell^+\nu,\ K^-\pi^+)$ decays

The correlated decays to a semileptonic final state and a hadronic non-CP eigenstate are somewhat more complicated. For the final state $(K^-\pi^+, K^-e^+\nu_e)$ we can write

$$\mathcal{A}_\alpha = \mathcal{A}(D^0 \to K^-\pi^+) \qquad \overline{\mathcal{A}}_\alpha = k e^{i\delta_{K\pi}}\mathcal{A}_\alpha$$
$$\mathcal{A}_\beta = \mathcal{A}(D^0 \to K^-e^+\nu_e) \qquad \overline{\mathcal{A}}_\beta = 0$$

where a, δ, ϕ, k and $\delta_{K\pi}$ are real numbers. Writing The factor $k \approx \tan^2\theta_C$ is the ratio of the magnitudes of the doubly Cabibbo-suppressed (DCS) and Cabibbo-favored (CF) amplitudes. The angle $\delta_{K\pi}$ is the relative strong phase between the CF and DCS amplitudes to the same final state. The mixing and direct amplitudes for $(K^-\pi^+, K^-e^+\nu_e)$ are

$$X = -\frac{p}{q}\mathcal{A}_{\alpha\mathcal{A}\beta} \qquad Y = k e^{i\delta_{k\pi}}\mathcal{A}_\alpha\mathcal{A}_\beta$$

The (small $y\Gamma\Delta t$, small $x\Gamma\Delta t$) limit for the $(K^-\ell^+X, K^-\pi^+)$ decay rate is

$$\begin{aligned} |\mathcal{M}|^2 \;=\; \frac{1}{4}e^{-\Gamma(t_1+t_2)}\,|\mathcal{A}_\alpha|^2\,|\mathcal{A}_\beta|^2 \;\times & \left\{\left|\frac{p}{q}\right|^2\left(\frac{x^2+y^2}{4}\right)(\Gamma\Delta t)^2\right. \\ & - \left(\Re(\frac{p}{q})\cos\delta_{K\pi} + \Im(\frac{p}{q})\sin\delta_{K\pi}\right) k\, y\Gamma\Delta t \\ & + \left(\Im(\frac{p}{q})\cos\delta_{K\pi} - \Re(\frac{p}{q})\sin\delta_{K\pi}\right) k\, x\Gamma\Delta t \\ & \left. + k^2\left[1 + \left(\frac{y^2-x^2}{4}\right)(\Gamma\Delta t)^2\right]\right\}. \end{aligned} \tag{28}$$

Correlated $(K^-\pi^+,\ K^-K^+)$ decays

The correlated decays to a CP eigenstate and a hadronic non-CP eigenstate are somewhat more complicated. Consider, as a first example, the final state $(K^-\pi^+, K^-K^+)$. We can write

$$\mathcal{A}_\alpha = \mathcal{A}(D^0 \to K^-\pi^+) \qquad \overline{\mathcal{A}}_\alpha = k e^{i\delta_{K\pi}} \mathcal{A}_\alpha$$
$$\mathcal{A}_\beta = \mathcal{A}(D^0 \to K^-K^+) \qquad \overline{\mathcal{A}}_\beta = \mathcal{A}_\beta$$

The mixing and direct amplitudes for $(K^-\pi^+, K^-K^+)$ are

$$X = \left(\frac{q}{p} k e^{i\delta_{K\pi}} - \frac{p}{q}\right) \mathcal{A}_\alpha \mathcal{A}_\beta \qquad Y = (1 - k e^{i\delta_{K\pi}}) \mathcal{A}_\alpha \mathcal{A}_\beta$$

As is well-known, the time-integrated rate is dominated by the term

$$YY^* = (1 - 2k\cos\delta_{K\pi} + k^2) \mathcal{A}_\alpha \overline{\mathcal{A}}^*_\alpha \mathcal{A}_\beta \overline{\mathcal{A}}^*_\beta \tag{29}$$

which depends linearly on $\cos\delta_{K\pi}$.

The real and imaginary parts of the interference term are

$$\begin{aligned} \Re(XY^*) &= k\left(1 + \left|\frac{q}{p}\right|^2\right)\left[\Re\left(\frac{p}{q}\right)\cos\delta - \Im\left(\frac{p}{q}\right)\sin\delta\right] - \Re\left(\frac{p}{q}\right)(1+k^2) \\ \Im(XY^*) &= k\left(1 - \left|\frac{q}{p}\right|^2\right)\left[\Im\left(\frac{p}{q}\right)\cos\delta + \Re\left(\frac{p}{q}\right)\sin\delta\right] - \Im\left(\frac{p}{q}\right)(1+k^2) \end{aligned} \tag{30}$$

Correlated $(K^-\ell^+X, K^-\pi^+\pi^0)$ decays

With the notation

$$\begin{aligned}
\mathcal{A}(D^0 \to K^-\pi^+\pi^0) &= A_r\zeta(s_{12}, s_{13}) \\
\overline{\mathcal{A}}(\overline{D}^0 \to K^-\pi^+\pi^0) &= \overline{A}_r\overline{\zeta}(s_{12}, s_{13}) = k e^{i\delta_{K\pi\pi^0}} A_r\overline{\zeta}(s_{12}, s_{13})
\end{aligned} \tag{31}$$

The (small $y\Gamma\Delta t$, small $x\Gamma\Delta t$) limit for the $(K^-\ell^+X, K^-\pi^+\pi^0)$ decay rate is

$$\begin{aligned}
|\mathcal{M}|^2 \quad &= \frac{1}{4} e^{-\Gamma(t_1+t_2)} |A_r|^2 |\mathcal{A}_\beta|^2 \times \\
&\left\{ \left|\frac{p}{q}\right|^2 \zeta(s_{12}, s_{13})\, \zeta^*(s_{12}, s_{13}) \left(\frac{x^2+y^2}{4}\right) (\Gamma\Delta t)^2 \right. \\
&- \left[\Re\left(\frac{p}{q}\zeta(s_{12}, s_{13})\overline{\zeta}^*(s_{12}, s_{13})\right) \cos\delta_{K\pi\pi^0} \right. \\
&\left. + \Im\left(\frac{p}{q}\zeta(s_{12}, s_{13})\overline{\zeta}^*(s_{12}, s_{13}))\right) \sin\delta_{K\pi\pi^0} \right] k\, y\Gamma\Delta t \\
&+ \left[\Im\left(\frac{p}{q}\zeta(s_{12}, s_{13})\overline{\zeta}^*(s_{12}, s_{13})\right) \cos\delta_{K\pi\pi^0} \right. \\
&\left. - \Re\left(\frac{p}{q}\zeta(s_{12}, s_{13})\overline{\zeta}^*(s_{12}, s_{13})\right) \sin\delta_{K\pi\pi^0} \right] k\, x\Gamma\Delta t \\
&\left. + k^2\, \overline{\zeta}(s_{12}, s_{13})\, \overline{\zeta}^*(s_{12}, s_{13}) \left[1 + \left(\frac{y^2-x^2}{4}\right) (\Gamma\Delta t)^2\right] \right\} .
\end{aligned} \tag{32}$$

As a first approximation, the time-integrated rate is dominated by the doubly-Cabibbo suppressed rate associated with YY^*. To a lesser degree, the pure mixing rate propotional to the Cabibbo favored rate, XX^*, also contributes.

Correlated $(K^-K^+, K^-\pi^+\pi^0)$ decays

In this case, the time-integrated rate will be dominated by

$$YY^* \; = \zeta\zeta^* + k^2\overline{\zeta\zeta}^* - 2k\left[\Re(\zeta\overline{\zeta}^*)\,\cos\delta_{K\pi\pi^0} + \Im(\zeta\overline{\zeta}^*)\,\sin\delta_{K\pi\pi^0}\right]. \tag{33}$$

The time-odd rate will depend on the real and imaginary parts of

$$\begin{aligned} XY^* \; &= \; -\frac{q}{p}k^2\overline{\zeta\zeta}^* - \frac{p}{q}\zeta\zeta^* \\ &+ \; k\left[\frac{q}{p}e^{i\delta_{K\pi\pi^0}}\overline{\zeta}\zeta^* + \frac{p}{q}e^{-i\delta_{K\pi\pi^0}}\zeta\overline{\zeta}^*\right] \end{aligned} \tag{34}$$

In the limit $p/q = 1$,

$$XY^* \rightarrow -k^2\overline{\zeta\zeta}^* - \zeta\zeta^* + 2k\left[\Re(\zeta\overline{\zeta}^*)\cos\delta_{K\pi\pi^0} + \Im(\zeta\overline{\zeta}^*)\sin\delta_{K\pi\pi^0}\right] \tag{35}$$

which is purely real and equal in magnitude to YY^*. In this limit, the time-odd part of the rate is proportional only to $y\Gamma\Delta t$ and is independent of x.

Correlated $(K^-\pi^+, K^-\pi^+\pi^0)$ decays

Here we will write

$$
\begin{aligned}
\mathcal{A}_\alpha &= \mathcal{A}(D^0 \to K^-\pi^+) \\
\overline{\mathcal{A}}_\alpha &= k_1 e^{i\delta_1} \mathcal{A}_\alpha \\
\mathcal{A}_\beta &= \mathcal{A}(D^0 \to K^-\pi^+\pi^0) = A_r \zeta \\
\overline{\mathcal{A}}_\beta &= k_2 e^{i\delta_2}\, A_r\, \overline{\zeta}
\end{aligned}
\tag{36}
$$

so that

$$
\begin{aligned}
X &= \left(\frac{q}{p}\, k_1 k_2\, e^{i(\delta_1+\delta_2)}\, \overline{\zeta} \;-\; \frac{p}{q}\, \zeta\right)\; A_r \mathcal{A}_\beta \\
Y &= \left(k_2\, e^{i\delta_2}\, \overline{\zeta} - k_1\, e^{i\delta_1}\, \zeta\right)\; A_r \mathcal{A}_\beta \,.
\end{aligned}
\tag{37}
$$

It follows that

$$
\begin{aligned}
YY^* &= k_2^2\, \overline{\zeta}\overline{\zeta}^* + k_1^2 \zeta\zeta^* \\
&\quad -2\, k_1 k_2 \left[\Re(\zeta\overline{\zeta}^*)\, \cos(\delta_1-\delta_2) - \Im(\zeta\overline{\zeta}^*)\, \sin(\delta_1-\delta_2)\right]\, |\mathcal{A}_\alpha|^2\, |A_r|^2 \,,
\end{aligned}
\tag{38}
$$

and as a good first approximation,

$$
\begin{aligned}
XY^* \approx -\frac{p}{q}\, &\Big\{\Big[k_2\, \cos\delta_2\, \Re(\zeta\, \overline{\zeta}^*) + k_2\, \sin\delta_2\, \Im(\zeta\, \overline{\zeta}^*) - k_1\, \cos\delta_1\, \zeta\zeta^*\Big] \\
&+ i\, \Big[k_2\, \cos\delta_2\, \Im(\zeta\, \overline{\zeta}^*) - k_2\, \sin\delta_2\, \Re(\zeta\, \overline{\zeta}^*) + k_1 \sin\delta_1\, \zeta\, \zeta^*\Big]\Big\}\; |\mathcal{A}_\alpha|^2\, |A_r|^2 \,.
\end{aligned}
\tag{39}
$$

Some Notation for $D^0 \to K_S^0\pi^-\pi^+$

At first sight, $K_S^0\pi^-\pi^+$, appears to be similar to $K^-\pi^+\pi^0$ as both are three-body decays whose amplitudes are often described using isobar models. However, in the limit of no direct CP violation in D decay, and ignoring the known CP violation in K_S^0 decay, we can exploit the relationship

$$\mathcal{A}(\overline{D}^0 \to K_S^0\pi^-\pi^+)(s_{12},\, s_{13}) = \mathcal{A}(D^0 \to K_S^0\pi^-\pi^+)(s_{13},\, s_{12}) \tag{40}$$

Using the notation

$$\mathcal{A}(D^0 \to K_S^0\pi^-\pi^+)(s_{13},\, s_{12}) = A_r\zeta(s_{12}, s_{13}) \tag{41}$$

and assuming no direct CP violation, we have

$$\mathcal{A}_\alpha = A_r\zeta(s_{12}, s_{13})\,; \quad \overline{\mathcal{A}}_\alpha = A_r\zeta(s_{13}, s_{12}) \tag{42}$$

It is sometimes useful to re-write $\zeta(s_{12}, s_{13})$ and $\zeta(s_{13}, s_{12})$ in terms of symmetric and antisymmetric functions

$$\begin{aligned} \zeta_S(s_{13}, s_{12}) &= \frac{1}{2}\left[\zeta(s_{12}, s_{13}) + \zeta(s_{13}, s_{12})\right] \\ \zeta_A(s_{13}, s_{12}) &= \frac{1}{2}\left[\zeta(s_{12}, s_{13}) - \zeta(s_{13}, s_{12})\right] \end{aligned} \tag{43}$$

so that

$$\begin{aligned} \zeta(s_{12}, s_{13}) &= \zeta_S(s_{13}, s_{12}) + \zeta_A(s_{13}, s_{12}) \\ \zeta(s_{13}, s_{12}) &= \zeta_S(s_{13}, s_{12}) - \zeta_A(s_{13}, s_{12})\,. \end{aligned} \tag{44}$$

Note that we can use the same notation for $D^0 \to \pi^0\pi^-\pi^+$.

Correlated $(K^-\ell^+\nu, K_S^0\pi^-\pi^+)$ decays

With the notation introduced for $\mathcal{A}_\alpha = \mathcal{A}(D^0 \to K_S^0\pi^-\pi^+)$,

$$\begin{aligned} X &= -\frac{p}{q}(\zeta_S + \zeta_A) A_r \mathcal{A}_\beta \\ Y &= -(\zeta_S - \zeta_A) A_r \mathcal{A}_\beta \end{aligned} \tag{45}$$

which gives

$$\begin{aligned} YY^* &= (2\,\zeta_S\,\zeta_S^* + 2\,\zeta_A\,\zeta_A^* - \zeta\,\zeta^*)\,|A_r|^2\,|\mathcal{A}_\beta|^2 \\ XY^* &= \frac{p}{q}\left[\zeta_S\,\zeta_S^* - \zeta_A\,\zeta_A^* - 2\,i\,\Im(\zeta_S\,\zeta_A^*)\,\right]\,|A_r|^2\,|\mathcal{A}_\beta|^2 \end{aligned} \tag{46}$$

so that

$$\begin{aligned} \Re(XY^*) &= \left[\Re\left(\frac{p}{q}\right)(\zeta_S\,\zeta_S^* - \zeta_A\,\zeta_A^*) + 2\,\Im\left(\frac{p}{q}\right)\Im(\zeta_S\,\zeta_A^*)\right]\,|A_r|^2\,|\mathcal{A}_\beta|^2 \\ \Im(XY^*) &= \left[-2\,\Re\left(\frac{p}{q}\right)\,\Im(\zeta_S\,\zeta_A^*) + \Im\left(\frac{p}{q}\right)\,(\zeta_S\,\zeta_S^* - \zeta_A\,\zeta_A^*)\right]\,|A_r|^2\,|\mathcal{A}_\beta|^2 \end{aligned} \tag{47}$$

Correlated $(K^-K^+, K_S^0\pi^-\pi^+)$ decays

As usual, the time integrated rate is dominated by

$$YY^* = 4\,\zeta_A(s_{12}, s_{13})\zeta_A^*(s_{12}, s_{13})\,|A_r|^2\,|\mathcal{A}|^2 \tag{48}$$

which we can identify as the antisymmetric rate. Were we to consider $K_S^0\pi^-\pi^+$ produced in conjunction with a pure CP odd eigenstate rather than CP even, YY^* would be the symmetric rate instead. The time-odd rates are proportional to the real and imaginary parts of XY^* which is

$$XY^* = 2\left[\zeta_S\,\zeta_A^*\left(\frac{p}{q} - \frac{q}{p}\right) + \zeta_A\,\zeta_A^*\left(\frac{p}{q} + \frac{q}{p}\right)\right]|A_r|^2\,|\mathcal{A}|^2\,. \tag{49}$$

In the limit $p = q$, $XY^* \to YY^*$. Were we to consider $K_S^0\pi^-\pi^+$ produced in conjunction with a pure CP odd eigenstate rather than CP even, XY^* becomes

$$XY^* = 2\left[(\zeta_S\,\zeta_A^*)^*\left(\frac{p}{q} - \frac{q}{p}\right) + \zeta_S\,\zeta_S^*\left(\frac{p}{q} + \frac{q}{p}\right)\right]|A_r|^2\,|\mathcal{A}|^2\,. \tag{50}$$

The roles of ζ_S and ζ_A are interchanged. If $p \neq q$ the $\Re(\zeta_S\,\zeta_A^*) = \Re(\zeta_S\,\zeta_A^*)^*$ but the $\Im(\zeta_S\,\zeta_A^*) = -\Im(\zeta_S\,\zeta_A^*)^*$ so the time-odd asymmetries will differ and the difference of the two as a function of position in the Dalitz plot will provide additional sensitivity to the real and imaginary parts of p/q.

Correlated $(K^-\pi^+, K^0_S\pi^-\pi^+)$ decays

With the same type of notation as used earlier,

$$\begin{aligned} YY^* &= \zeta'\,\zeta'^* + k^2\,\zeta\,\zeta^* - 2\,k\,\Re(e^{i\delta}\,\zeta\,\zeta'^*) \\ &= \zeta'\,\zeta'^* + k^2\,\zeta\,\zeta^* - 2k\,\cos\delta\,\Re(\zeta\,\zeta'^*) + 2k\,\sin\delta\,\Im(\zeta\,\zeta'^*)\,. \end{aligned} \tag{51}$$

We can identify the real and imaginary parts of $\zeta\,\zeta'^*$ with ζ_S and ζ_A writing

$$\zeta\,\zeta'^* = \zeta_S\,\zeta_S^* - \zeta_A\,\zeta_A^* - 2i\Im(\zeta_S\,\zeta_A^*) \tag{52}$$

from which we find

$$\Re(\zeta\,\zeta'^*) = \zeta_S\,\zeta_S^* - \zeta_A\,\zeta_A^*\,; \quad \Im(\zeta\,\zeta'^*) = -2\Im(\zeta_S\,\zeta_A^*)\,. \tag{53}$$

This gives

$$YY^* = \zeta'\,\zeta'^* + k^2\,\zeta\,\zeta^* - 2k\,\cos\delta\,(\zeta_S\,\zeta_S^* - \zeta_A\,\zeta_A^*) - 4k\,\sin\delta\,\Im(\zeta_S\,\zeta_A^*)\,. \tag{54}$$

The time-odd rate is proportional to the real and imaginary parts of

$$\begin{aligned} XY^* &= \frac{q}{p}\,k^2\,(\zeta\,\zeta'^*)^* + \frac{p}{q}(\zeta\,\zeta'^*) \\ &+ k\left[\frac{q}{p}\,e^{i\delta}\,(\zeta'\,\zeta'^*) - \frac{p}{q}\,e^{-i\delta}\,(\zeta\,\zeta^*)\right] \end{aligned} \tag{55}$$

In the limit $p/q \to 1$, these become

$$\begin{aligned} \Re(XY^*) &= (1+k^2)\,[\zeta_S\zeta_S^* - \zeta_A\,\zeta_A^*] + \cos\delta\,[\zeta\zeta^* - k\,\zeta'\zeta'^*] \\ \Im(XY^*) &= \sin\delta\,[\zeta\,\zeta^* + k\,\zeta'\zeta'^*] - (1-k^2)\,\Im(\zeta_S\zeta_A^*) \end{aligned} \tag{56}$$

Correlated $(K^-\pi^+\pi^0, K^-\pi^+\pi^0)$ decays

For this correlated final state we will use the notation

$$\begin{aligned}
\mathcal{A}_\alpha(s_{12}, s_{13}) &= A_r\zeta(s_{12}, s_{13}) \\
\overline{\mathcal{A}}_\alpha(s_{12}, s_{13}) &= A_r\overline{\zeta}(s_{12}, s_{13}) = \kappa(s_{12}, s_{13})\, e^{i\epsilon(s_{12},s_{13})}\, A_r\zeta(s_{12}, s_{13}) \\
\mathcal{A}_\beta(s'_{12}, s'_{13}) &= A_r\zeta'(s'_{12}, s'_{13}) \\
\overline{\mathcal{A}}_\beta(s'_{12}, s'_{13}) &= A_r\overline{\zeta}'(s'_{12}, s'_{13}) = \kappa'(s'_{12}, s'_{13})\, e^{i\epsilon(s'_{12},s'_{13})}\, A_r\zeta'(s'_{12}, s'_{13})\,.
\end{aligned} \tag{57}$$

The real functions κ, κ', ϵ, and ϵ' are chosen so that κ and κ' are positive definite.

$$YY^* = \zeta\,\zeta^*\,\overline{\zeta}'\,\overline{\zeta}'^* + \overline{\zeta}\,\overline{\zeta}^*\,\zeta'\,\zeta'^* + 2\Re\left[(\zeta\,\overline{\zeta}^*)\,(\overline{\zeta}'\,\zeta'^*)\right]|A_r|^4\,. \tag{58}$$

The first two terms are the products of the Cabibbo favored rate for one decay and the doubly-Cabibbo suppressed rate for the other. The last term is the product of two Cabibbo favored, doubly-Cabibbo suppressed interference rates. As a good approximation, we can calculate the interference term ignoring the doubly-Cabibbo suppressed term in X:

$$XY^* \approx \frac{p}{q}\left[(\zeta\,\zeta^*)\,(\zeta'\,\overline{\zeta}'^*) - (\zeta\,\overline{\zeta}^*)\,(\zeta'\,\zeta'^*)\right]|A_r|^4\,. \tag{59}$$

Here, each term is the product of a Cabibbo favored rate for one decay and the interference of amplitudes for the other. Events will populate a four-dimensional phase space corresponding to the two Dalitz plot positions (s_{12}, s_{13}) and (s'_{12}, s'_{13}). Furthermore, this interference term is antisymmetric under the interchange of the ζ and ζ'. This is evident algebraically from the form of Eqn. (59). Physically, it corresponds to identifying one or the other Dalitz plot position as that of the first D to decay. As the interference rate is time-odd, XY^* must be antisymmetric when the two Dalitz plot positions are interchanged.

Correlated $K_S^0\pi^-\pi^+$, $K_S^0\pi^-\pi^+$ decays

Here, we use notation here more similar to that used for $K^-\pi^+\pi^0$, $K^-\pi^+\pi^0$ than for $K_S^0\pi^-\pi^+$, $K^-\pi^+$:

$$\begin{aligned}
\mathcal{A}_\alpha(s_{12}, s_{13}) &= \zeta(s_{12}, s_{13})\, A_r &= (\zeta_S + \zeta_A)\, A_r \\
\overline{\mathcal{A}}_\alpha(s_{12}, s_{13}) &= \overline{\zeta}(s_{13}, s_{12})\, A_r &= (\zeta_S - \zeta_A)\, A_r \\
\mathcal{A}_\beta(s'_{12}, s'_{13}) &= \zeta(s'_{12}, s'_{13})\, A_r &= (\zeta'_S + \zeta'_A)\, A_r \\
\overline{\mathcal{A}}_\beta(s'_{12}, s'_{13}) &= \overline{\zeta}(s'_{13}, s'_{12})\, A_r &= (\zeta'_S - \zeta'_A)\, A_r
\end{aligned} \tag{60}$$

The prime superscript ($'$) distinguishes the amplitudes associated with the two Dalitz plot positions of the $K_S^0\pi^-\pi^+$ decays rather than the amplitudes associated with direct D^0 and $\overline{D}^0$ decay. With this notation

$$\begin{aligned}
YY^* = 4\Big\{&\zeta_A\zeta_A^*\,\zeta'_S\zeta'^*_S + \zeta_S\zeta_S^*\,\zeta'_A\zeta'^*_A \\
&-2\left[\Re(\zeta_S\zeta_A^*)\,\Re(\zeta'_S\zeta'^*_A) + \Im(\zeta_S\zeta_A^*)\,\Im(\zeta'_S\zeta'^*_A)\right]\Big\}
\end{aligned} \tag{61}$$

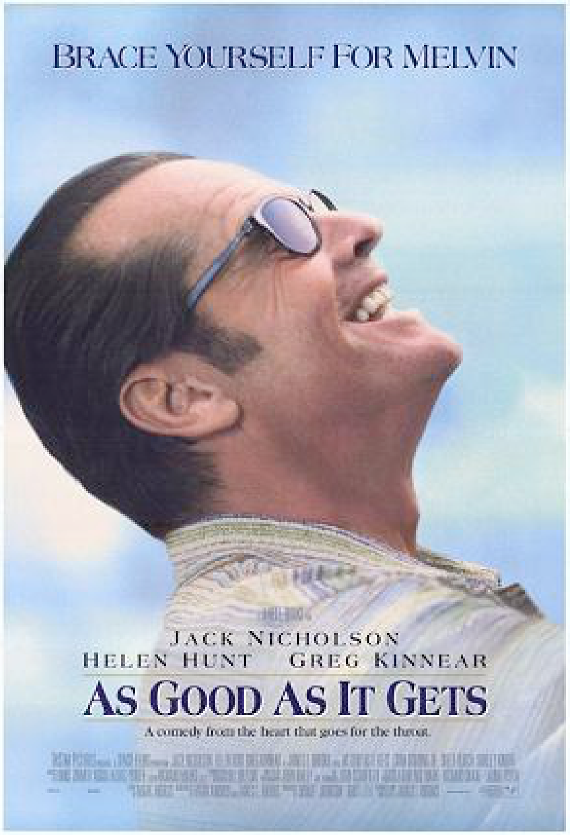
In the limit $p = q$, the mixing amplitude becomes

$$\begin{aligned}
X &= (\zeta_S - \zeta_A)\,(\zeta'_S - \zeta'_A) - (\zeta_S + \zeta_A)\,(\zeta'_S + \zeta'_A) \\
&= -2\,[\zeta_A\,\zeta'_S + \zeta_S\,\zeta'_A]
\end{aligned} \tag{62}$$

in which case

$$\begin{aligned}
XY^* &= -4\,(\zeta_A\,\zeta'_S + \zeta_S\,\zeta'_A)\,(\zeta_A^*\,\zeta'^*_S - \zeta_S\zeta'^*_A) \\
&= -4\Big[\zeta_A\,\zeta_A^*\,\zeta'_S\,\zeta'^*_S - \zeta_S\,\zeta_S^*\,\zeta'_A\,\zeta'^*_A + 2\,i\,\Im(\zeta_S\zeta_A^*\zeta'_A\zeta'^*_S)\Big] \\
&= -4\Big[\zeta_A\,\zeta_A^*\,\zeta'_S\,\zeta'^*_S - \zeta_S\,\zeta_S^*\,\zeta'_A\,\zeta'^*_A \\
&\qquad +2\,i\,\Big(-\Re(\zeta_S\,\zeta_A^*)\,\Im(\zeta'_S\,\zeta'^*_A) + \Re(\zeta'_S\,\zeta'^*_A)\,\Im(\zeta_S\,\zeta_A^*)\Big)\Big]
\end{aligned} \tag{63}$$

Sensitivities - As Good As It Gets



Use a Toy Monte Carlo procedure to generate and fit datasets

- extrapolate (roughly) from numbers of events seen by CLEO-c to 500 fb^{-1}
- assume we know all amplitudes and related terms exactly: XX^*, YY^*, $\Re(XY^*)$, $\Im(XY^*)$
- assume the quadratic expansion for time-dependence
- calculate expected numbers of events in each (Dalitz plot bin) $\times$ (time bin) . Use 100×100 or 400×400 Dalitz plots. Use 18 postive time and 18 negative time bins, starting with 0.25 lifetime width.

Additional Comments

- We assume CP symmetry
- Results are generally insensitive to Dalitz plot binning
- Results are generally insensitive to time binning
- XX^* and YY^* can usually be extracted from time-integrated data with minimal assumptions
- $\Re(XY^*)$ and $\Im(XY^*)$ can probably be extracted from time-integrated data, but it will be more difficult.

Sensitivities

channel	# of events	δx	δy	Comments
$K_S^0\pi^-\pi^+,\ K^-e^+\nu_e$	720K	0.15%	0.09%	BaBar $K_S^0\pi^-\pi^+$ amplitudes
$K_S^0\pi^-\pi^+,\ K^-\pi^+$	865K	0.18%	0.05%	$\cos\delta_{K\pi} = 0.95$
$K_S^0\pi^-\pi^+,\ h^-h^+$	110K	–	0.21%	
$K_S^0\pi^-\pi^+,\ K_S^0\pi^-\pi^+$	285K	0.24%	0.16%	
$K^-\pi^+\pi^0,\ K^-e^+\nu_e$	4500	0.06%	0.06%	$\cos\delta_{K\pi\pi^0} = 0.95,\ R_D = 0.16\%$ BaBar $K\pi\pi^0$ amplitudes
$K^-\pi^+\pi^0,\ K^-\pi^+$	5000	0.06%	0.05%	
$K^-\pi^+\pi^0,\ K^-\pi^+\pi^0$	7200	0.07%	0.06%	
$K^-\pi^+\pi^0,\ h^-h^+$	460K	–	0.10%	
$K^-\pi^+,\ K^-e^+\nu_e$	10,600	0.27%	0.08%	$\cos\delta_{K\pi} = 0.95$
$K^-\pi^+,\ h^-h^+$	187K	–	0.16%	$\cos\delta_{K\pi} = 0.95$
$h^-h^+,\ K^-e^+\nu_e$	345K	–	0.12%	
$\pi^-\pi^+\pi^0,\ K^-e^+\nu_e$	120K	0.28%	0.22%	BaBar $\pi^-\pi^+\pi^0$ amplitudes
$\pi^-\pi^+\pi^0,\ K^-\pi^+$	120K	0.56%	0.15%	
$\pi^-\pi^+\pi^0,\ h^-h^+$	20K	–	0.5%	

Mixing Summary

In the limit of no CP violation in mixing ($p/q = 1$),

- The formulas presented here allow one to write correlated events generators for the modes studied. These almost certainly include most of the important modes for studying mixing and CP violation in mixing (*i.e.*, $p/q \neq 1$). This allows one to determine efficiency and Δt resolutions for possible machine and detector configurations ($\beta\gamma$, B-field strength, etc.).
- The rates associated with direct decays can be extracted from data independently of models using time-integrated measurements. Mixing perturbs these determinations at the 10^{-4} level if one integrates over all decay times, and even less if one integrates over limited ranges of Δt.
- Relative (strong interaction) phases between Cabibbo favored and doubly-Cabibbo suppressed decays to the same final states can be extracted from time-integrated measurements.
- The interference rate terms can generally be extracted from data independently of models using a multiplicity of time-integrated measurements, if relative strong phases are kind to us. If not, a multiplicity of time-dependent measurements allows these rates as well as the mixing parameters to be extracted with no need for a model.
- The formalism for $K_L^0\pi^-\pi^+$ decays is the same as that for $K_S^0\pi^-\pi^+$, and the statistics will be greater for most correlated channels.
- The formalism for $K^+\pi^+\pi^-\pi^-$ is the same as that for $K^+\pi^-\pi^0$ *except* that the phase space is 5-dimensional rather than 2-dimensional. This makes extrapolating the amplitudes more difficult.

CPV in Time Asymmetric Rates

The time asymmetric part of the correlated rate depends on

$$X = \frac{q}{p}\,\overline{\mathcal{A}}_\alpha \overline{\mathcal{A}}_\beta - \frac{p}{q}\,\mathcal{A}_\alpha \mathcal{A}_\beta\ ; \qquad Y = \mathcal{A}_\alpha \overline{\mathcal{A}}_\beta - \overline{\mathcal{A}}_\alpha \mathcal{A}_\beta \tag{64}$$

through

$$\begin{aligned} 4|\mathcal{M}|^2 &= e^{-\Gamma(t_1+t_2)} \times \Big\{ \ X X^* \left[\frac{(x^2+y^2)}{4}(\Gamma\Delta t)^2\right] \\ &- \ \Re(XY^*)\, y\Gamma\Delta t + \Im(XY^*)\, x\Gamma\Delta t \\ &+ \ \ Y Y^* \left[1 + \frac{(y^2-x^2)}{4}(\Gamma\Delta t)^2\right]\Big\} \end{aligned} \tag{65}$$

In the limit of no direct CPV, $\mathcal{A}_\alpha \to \overline{\mathcal{A}}_{\overline{\alpha}}$; $\mathcal{A}_\beta \to \overline{\mathcal{A}}_{\overline{\beta}}$ and $q/p \to p/q$ when we interchange D^0 and $\overline{D}^0$. This leads to CPV in the time-odd rate proportional to the real and imaginary parts of XY^*. As an example, recall that for $(K^-\pi^+, K^-\pi^+\pi^0)$,

$$\begin{aligned} XY^* \approx -\frac{p}{q} \Big\{ &\left[k_2 \cos\delta_2 \,\Re(\zeta\,\overline{\zeta}^*) + k_2 \sin\delta_2\,\Im(\zeta\,\overline{\zeta}^*) - k_1 \cos\delta_1\,\zeta\zeta^*\right] \\ &+ i\,\left[k_2 \cos\delta_2\,\Im(\zeta\,\overline{\zeta}^*) - k_2 \sin\delta_2\,\Re(\zeta\,\overline{\zeta}^*) + k_1 \sin\delta_1\,\zeta\,\zeta^*\right]\Big\}\ |\mathcal{A}_\alpha|^2\,|A_r|^2\,. \end{aligned} \tag{66}$$

Time-Odd CPV Sensitivities

channel	$(x = 0.3\%,\ y = 0.8\%)$				$(x = 1.2\%,\ y = 3.2\%))$			
	δx	δy	$\delta(\|q/p\|)$	$\delta\phi$	δx	δy	$\delta(\|q/p\|)$	$\delta\phi$
$K_S^0\pi^0\pi^+$, $K^-e^+\nu$	0.11 %	0.11%	0.11	0.12	0.13%	0.11 %	0.028	0.036
$K_S^0\pi^-\pi^+$, h^-h^+	–	0.21%	0.20	0.14	–	0.22%	0.065	0.100
$K_S^0\pi^-\pi^+$, $K^-\pi^+$	0.11%	0.10%	0.11	0.11	0.13 %	0.10%	0.031	0.030
$\pi^-\pi^+\pi^0$, $K^-e^+\nu$	0.34%	0.26%	0.20	0.13	0.37%	0.24%	0.069	0.100
$\pi^-\pi^+\pi^0$, h^+h^-	–	0.42%	0.27	0.11	–	0.47%	0.159	0.099
$\pi^-\pi^+\pi^-$, $K^-\pi^+$	0.35%	0.26%	0.19	0.14	0.32%	0.23%	0.063	0.091

Some observations:

- δx and δy sensitivities somewhat worse when allowing for CPV;
- δx and δy sensitivities independent of central values;
- δx and δy sensitivities scale like $1/\sqrt{n}$;
- $\delta|q/p|$ and $\delta\phi$ sensitivities depend on central values of x and y;
- $\delta|q/p|$ and $\delta\phi$ sensitivities scale like $1/\sqrt{n}$.

Final Thoughts

Some final thoughts on running at threshold:

- XX^* and YY^* can be extracted from time-integrated measurements with no model dependence.
- $\Re(XY^*)$ and $\Im(XY^*)$ can be extractd from time-integrated measuerments with little (or no) model dependence (if nature is kind.
- time-integrated measurements of XX^*, YY^*, $\Re(XY^*)$, and $\Im(XY^*)$ can be useful for tagged mixing and CPV studies [can eliminate, or strongly reduce, model dependencies]. This is especially true for WS channels like $D^0 \to K^+\pi^-\pi^0$.
- $K_L^0\pi^-\pi^+$ and $K^+\pi^+\pi^-\pi^-$ are potentially power channels in correlated decays at threshold.
- 500 fb^{-1} at threshold has less physics reach for charm mixing and CPV than does 75 ab^{-1} at the $\Upsilon(4S)$.
- 5 ab^{-1} at threshold has more physics reach for charm mixing and CPV than does 75 ab^{-1} at the $\Upsilon(4S)$.