Quadratic Quasi-Normal Modes

of a Schwarzschild Black Hole

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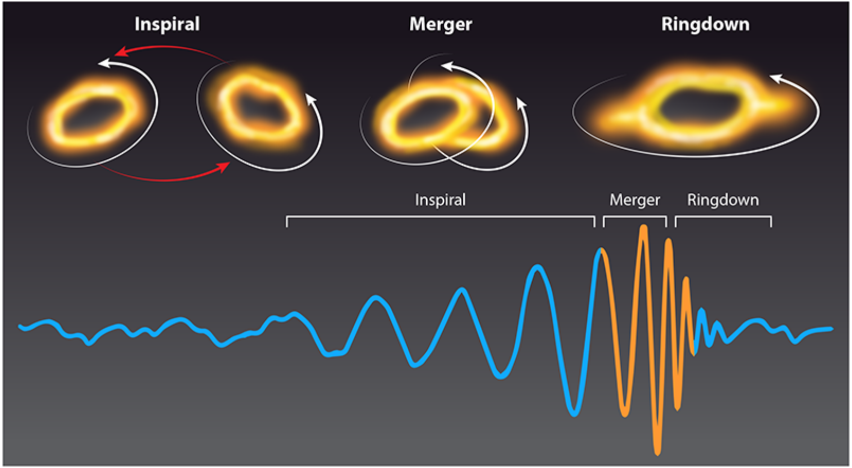
SCUOLA
NORMALE
SUPERIORE

[1] with Leonardo Juliano, Adrien Kuntz and Enrico Trincherini

Inspire-Merger-Ringdown



(Top) Kip Thorne; (Bottom) B. P. Abbott et al.; adapted by APS/Carin Cain

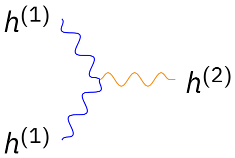
Linear Perturbations

and Beyond

Linear Quasi-Normal Modes

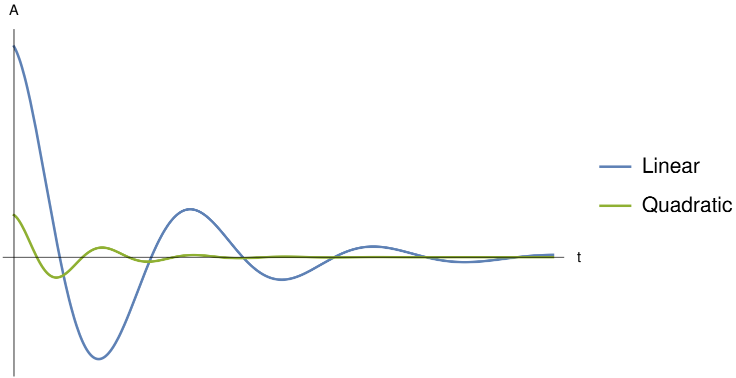
- Small Amplitude Perturbations of the final BH metric
- Obey Linearized Einstein Equations
- Characterized by discrete set of frequencies
- Describe the Ringdown Signal very accurately

Where do the non-linearities go?



Quadratic Perturbations

Detected in Numerical GR by [Cheung et al. '23]



GOAL:

Compute Frequency and Amplitude of Quadratic Modes

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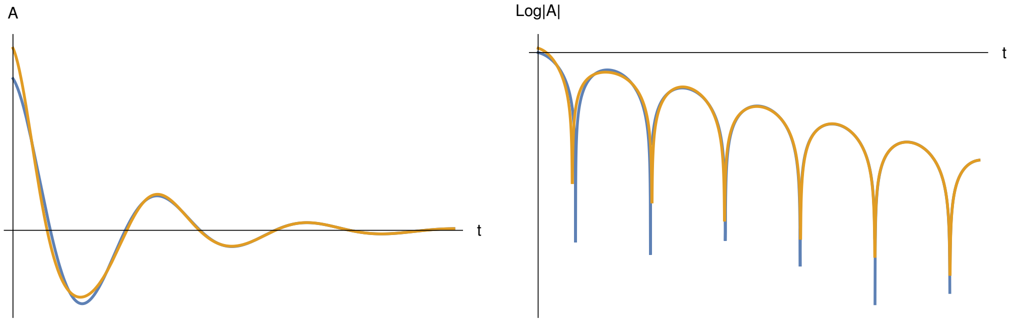

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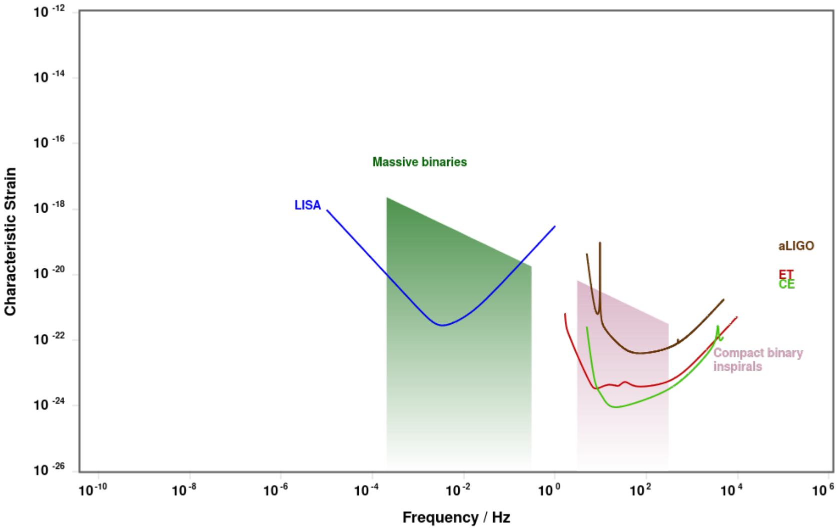


Detectability by LISA claimed by [Berti et al. ’24]

Black Hole Perturbation Theory

Metric Ansatz

Metric Perturbations $h^{(1)}_{\mu\nu}$, $h^{(2)}_{\mu\nu}$

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon\, h^{(1)}_{\mu\nu} + \epsilon^2 h^{(2)}_{\mu\nu}$$

$$h_{\mu\nu} = \begin{pmatrix} h_{tt} & h_{tr} & [\quad] \\ & h_{rr} & [\quad] \\ & & \square \end{pmatrix}$$

- → 2 vectors h_{t+}, h_{t-}
- → 2 vectors h_{r+}, h_{r-}
- → 3 tensors $h_+, h_-, h_\circ$

$$\gamma_{AB} = \begin{pmatrix} 1 & \\ & \sin^2\theta \end{pmatrix}, \quad Y_{\ell,m},\; D_A Y_{\ell,m},\; \epsilon_{AB} D^A Y_{\ell,m}, \ldots$$

$$e^{-i\omega t}, \quad \Re[\omega] > 0,\; \Re[\omega] < 0$$

In Regge-Wheeler gauge

$$h_{t+} = h_{r+} = h_+ = h_- = 0$$

Even | Odd sector

$7 + 3$

$\downarrow$ Gauge fixing

$4 + 2$

$\downarrow$ Constraint eq.'s

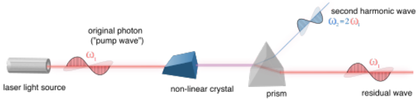
$1 + 1$

Black Hole Perturbation Theory

Einstein Equations

Einstein Equations in vacuum

$$\epsilon^0\, G^{(0)}_{\mu\nu}\left[\bar{g}\right] = 0 \text{ (trivial)}$$
$$\epsilon^1\, G^{(1)}_{\mu\nu}\left[h^{(1)}\right] = 0$$



$$\epsilon^2\, G^{(1)}_{\mu\nu}\left[h^{(2)}\right] = -G^{(2)}_{\mu\nu}\left[\epsilon\, h^{(1)}, \epsilon\, h^{(1)}\right] \equiv \epsilon^2 S_{\mu\nu}\left[h^{(1)}, h^{(1)}\right]$$

Symmetries of Background

$\Rightarrow$ Can take eigenstates of angular momentum, frequency, and parity, and dynamics will not mix them.

Black Hole Perturbation Theory

Master Scalars, Linear Order

The two physical d.o.f. of the graviton are captured by **master scalars**

$$\psi_+ = \frac{2r}{\lambda_1^2}\left[r^{-2}\tilde{h}_\circ + \frac{2}{\Lambda(r)}\left(f^2\tilde{h}_{rr} - rf(r^{-2}\tilde{h}_\circ)'\right)\right]$$

$$\psi_- = \frac{2r}{\mu^2}\left[\partial_r\tilde{h}_{t-} + \frac{M}{r^2 f(r)}(\tilde{h}_{t-} - \tilde{h}_{r-}) - \partial_t\tilde{h}_{r-} - \frac{2}{r}\tilde{h}_{t-}\right]$$

Einstein equations reduce to the Regge-Wheeler and Zerilli equations

$$\frac{\mathrm{d}\psi_\pm}{\mathrm{d}r_*^2} + \omega^2\psi_\pm - V_\pm(r)\psi_\pm = 0, \quad r_* = r + \ln\left(\frac{r}{2M} - 1\right)$$

Knowing $\psi_\pm$ we can fully reconstruct the metric

$$h_{\mu\nu} \longleftrightarrow \psi_\pm$$

Linear Quasi-Normal Modes

Green function decomposes into [Leaver '86]

- Prompt Response (high frequency/free propagation)
- Quasi-Normal Modes (poles)
- Late time tail (cut)

Numerical results by [Mitman et al.'24]

QNMs are found by imposing boundary conditions

$$\psi \underset{r_* \to +\infty}{\sim} \mathcal{A}\, e^{i\omega r_*}, \quad \psi \underset{r_* \to -\infty}{\propto} e^{-i\omega r_*}$$

where $\mathcal{A}$ is the QNM amplitude.

Many techniques available: Leaver method (fully numerical), (high order) WKB, Uniform approximations, Liouville Theory, ...

Linear Spectrum

ℓ	n	Uniform (2-nd order)	6-th order WKB
2	0	$0.3854 + 0.0909i$ (3.1%)	$0.37371 + 0.08892i$ (0.014%)
	1	$0.3590 + 0.2796i$ (3.1%)	$0.34672 + 0.27388i$ (0.0089%)
	2	$0.3146 + 0.4868i$ (2.8%)	$0.30005 + 0.47883i$ (0.2%)
	3	$0.2670 + 0.7146i$ (2.4%)	$0.24551 + 0.71159i$ (1.2%)
	1000	$0.000 + 249.771i$ (0.06%)	–
3	0	$0.6075 + 0.0935i$ (1.3%)	$0.59944 + 0.09270i$ (0.000049%)
	1	$0.5909 + 0.2837i$ (1.3%)	$0.58264 + 0.28129i$ (0.00088%)
	2	$0.5605 + 0.4830i$ (1.3%)	$0.55160 + 0.47906i$ (0.013%)
	3	$0.5215 + 0.6956i$ (1.3%)	$0.51111 + 0.69049i$ (0.1%)
	4	$0.4807 + 0.9219i$ (1.2%)	$0.46688 + 0.91799i$ (0.39%)
	5	$0.4428 + 1.1591i$ (1.1%)	$0.42437 + 1.16253i$ (1.%)

Table: The quasi-normal frequencies of a Schwarzschild black hole in units where $GM = 1$.

[BB, Adrien Kuntz, Francesco Serra, Enrico Trincherini ’23]

Selection Rules

for Quadratic Order Modes

Before doing any computation, let's exploit symmetry.

Couple two linear modes of given

frequencies $\omega_{1,2}$, angular momenta $(\ell_{1,2}, m_{1,2})$, parity $P_{1,2} = 0, 1$

- $\cos \omega_1 t\, e^{-\gamma_1 t} \cos \omega_2 t\, e^{-\gamma_2 t} \propto (\cos(\omega_1 + \omega_2)t + \cos(\omega_1 - \omega_2)t)e^{-(\gamma_1+\gamma_2)t}$
- $\ell = |\ell_1 - \ell_2|, \ldots, \ell_1 + \ell_2; \quad m_1 + m_2 = m$ (Clebsh-Gordan coefficient)
- $(-1)^{\ell_1+P_1}(-1)^{\ell_2+P_2} = (-1)^{\ell+P}$

But what are the **amplitudes** of the quadratic modes?

Black Hole Perturbation Theory

Master Scalars, Quadratic Order

Similarly, defining the master scalars $\psi_\pm^{(2)}$ using $h_{\mu\nu}^{(2)}$, they obey

$$\frac{\mathrm{d}\psi_\pm^{(2)}}{\mathrm{d}r_*^2} + \omega^2 \psi_\pm^{(2)} - V_\pm(r)\psi_\pm^{(2)} = S[\psi_\pm^{(1)}, \psi_\pm^{(1)}] \leftarrow \text{ Source term}$$

[Hui et al. '22; Spiers,Pound,Wardell '23]

But $\psi_\pm^{(2)}$ diverge at large r as $\psi_\pm^{(2)} \propto r^2 e^{i\omega r_*}$, so for them

- QNM boundary conditions cannot be imposed
- Cannot extract a finite amplitude $\mathcal{A}$

Resolution

The Good Master Scalars

Divergences due to *poor choice* of master scalars

[Ioka, Nakano '07; Brizuela et al. '09]

$$h^{(2)}_{\mu\nu} = \mathcal{D}_{\mu\nu}[\psi^{(2)}_{\pm}] + [\text{divergent terms}]_{\mu\nu}$$

Resolution

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$$h^{(2)}_{\mu\nu} = \mathcal{D}_{\mu\nu}[\psi^{(2)}_{\pm}] + [\text{divergent terms}]_{\mu\nu}$$

For each Parity sector, we redefine $\psi^{(2)}_{\pm} \to \Psi^{(2)}_{\pm}$

$$\Psi^{(2)}_{\pm} = \psi^{(2)}_{\pm} + \Delta(r)\psi^{(1)}_{\pm}\psi^{(1)}_{\pm}, \quad \Delta(r) = c_2 r^2 + c_1 r$$

choosing c_1, c_2 to reabsorb divergences.

$\Psi^{(2)}_{\pm}$ obeys RWZ equation with new source term.

$\Psi^{(2)}_{\pm} \underset{r_* \to +\infty}{\sim} \mathcal{A}^{(2)} e^{i\omega r_*}$. Can impose QNM b.c. and extract $\mathcal{A}^{(2)}$.

Physical Waveform

Transverse-Traceless gauge

We can now reconstruct the metric $h^{(2)}_{\mu\nu}$ in Regge-Wheeler gauge.

To extract the physical waveform, we go to *asymptotically transverse-traceless gauge* and extract $+, \times$ polarizations

$$h^{TT}_{ab} = \mathcal{O}(r^{-2}),\; h^{TT}_{a\pm} = \mathcal{O}(r^{-1}),\; h^{TT}_{\circ} = \mathcal{O}(r^{0}),\; h^{TT}_{\pm} = \mathcal{O}(r)$$

$$x^\mu \to x^\mu + \epsilon\,\xi^{(1)\mu}(x) + \epsilon^2\,\xi^{(2)\mu}(x)$$

$$h_{AB} = \begin{pmatrix} h^{TT}_{\circ} + h^{TT}_{+} & h^{TT}_{-} \\ h^{TT}_{-} & h^{TT}_{\circ} - h^{TT}_{+} \end{pmatrix}$$

$$\Delta h^{(1)}_{\mu\nu} = \mathcal{L}_{\xi^{(1)}}\bar{g}_{\mu\nu}, \qquad \Delta h^{(2)}_{\mu\nu} = \mathcal{L}_{\xi^{(2)}}\bar{g}_{\mu\nu} + \tfrac{1}{2}\mathcal{L}^2_{\xi^{(1)}}\bar{g}_{\mu\nu} + \mathcal{L}_{\xi^{(1)}}h^{(1)}_{\mu\nu}$$

Physical Waveform

Quantifying outgoing radiation

Finally, a convenient parametrization is the Newman Penrose scalar Ψ_4

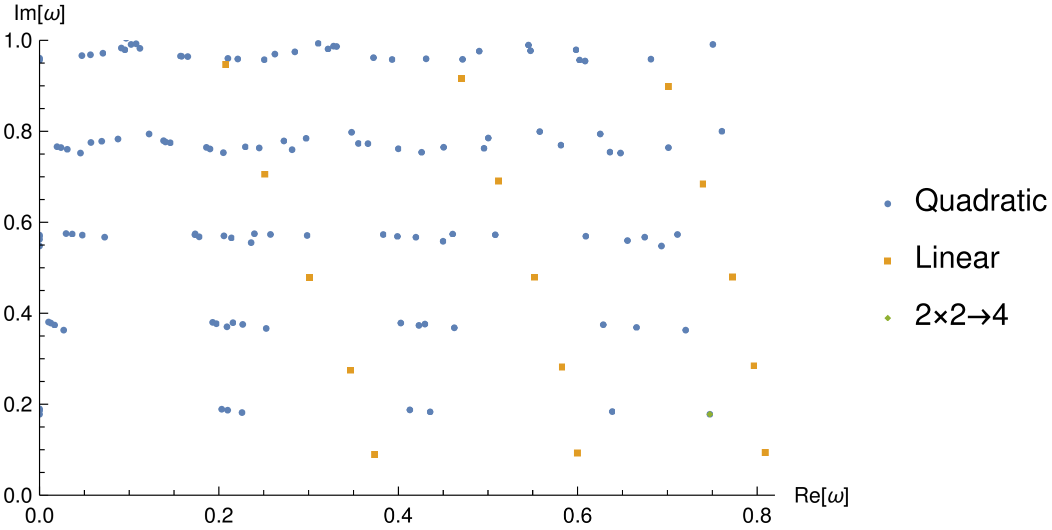
$$\Psi_4 = \mathfrak{h}_+ - i\mathfrak{h}_\times = \frac{M}{r}\sum \mathcal{A}_{\ell m \mathcal{N}} e^{-i\omega_{\ell\mathcal{N}}(r_* - t)} \; {}_{-2}Y^{\ell m}(\theta,\phi), \quad \mathcal{N} = (n, \pm)$$

$\mathcal{R} \equiv \dfrac{\mathcal{A}^{(2)}}{\mathcal{A}_1^{(1)}\mathcal{A}_2^{(1)}}$ are universal predictions of GR, like the spectrum of linear ω.

The peeling theorem supports this ansatz, which excludes $\ell = 0, 1$ even at quadratic order ($\ell \geq |s|$) [Lagos&Hui'22], [Geiller,Laddha,Zwikel'24].

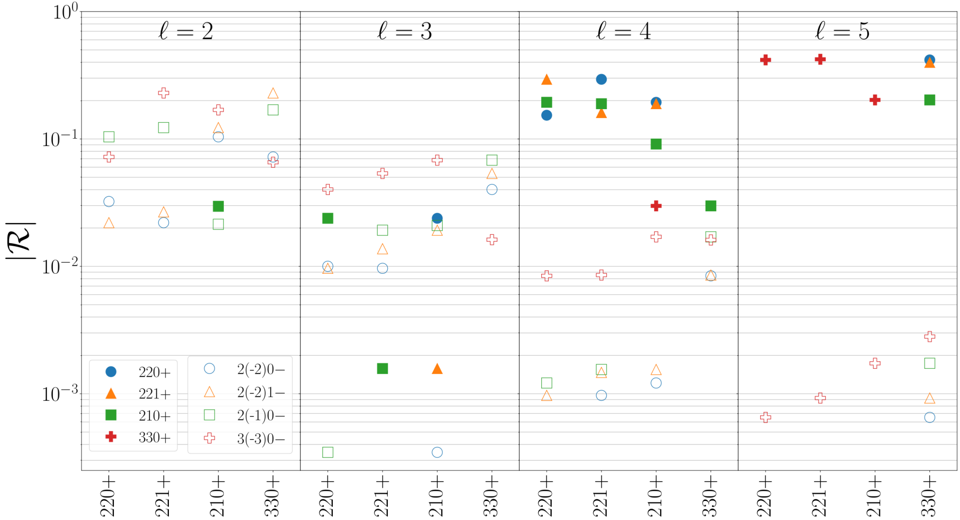
Conclusions

Quadratic Frequencies



Conclusions

Quadratic Amplitudes



$\mathcal{A}^{(1)} \sim 10\%$

$\mathcal{A}^{(2)} \sim 1\%$

Conclusions

- We confirmed $\mathcal{R}$ for (ℓ, m) $2,2 \times 2,2 \to 4,4$: $|\mathcal{R}| \simeq 0.15$ and $2,2 \times 3,3 \to 5,5$: $|\mathcal{R}| \simeq 0.4$ against existing NR simulations [Cheung et al.; Mitman et al.; Zhu et al.], but we also found new modes
- Trusting GR, we reduce overfitting because of the more detailed ringdown model (no new parameters!)
- Detection prospects of Quadratic QNMs [Berti et al. '24]: ground detectors should see $\mathcal{O}(10)/y$, while LISA $\mathcal{O}(100)/y$
- Can we study deviations from GR?

Thank you

Why is Kerr Hard?

$$\text{Spherical Symmetry} \longrightarrow \text{Axial Symmetry}$$

Can separate r, θ dependence at linear order, using Spheroidal Harmonics.

Source term is problematic. When breaking a symmetry,

$$S \sim \quad \not{f}(t) e^{-i\omega_1 t} e^{-i\omega_2 t}, \qquad g(\cos\theta)\mathcal{Y}_1(\theta,\phi)\mathcal{Y}_2(\theta,\phi)$$

To isolate the source of a given $\mathcal{Y}$, we need

$$\int \mathcal{Y}^*(\theta,\phi) g(\cos\theta)\mathcal{Y}_1(\theta,\phi)\mathcal{Y}_2(\theta,\phi)\, \mathrm{d}\Omega$$

[Ma&Yang'24] [Khera,Ma,Yang'24] [Spiers et al.'23] [Spiers '24]