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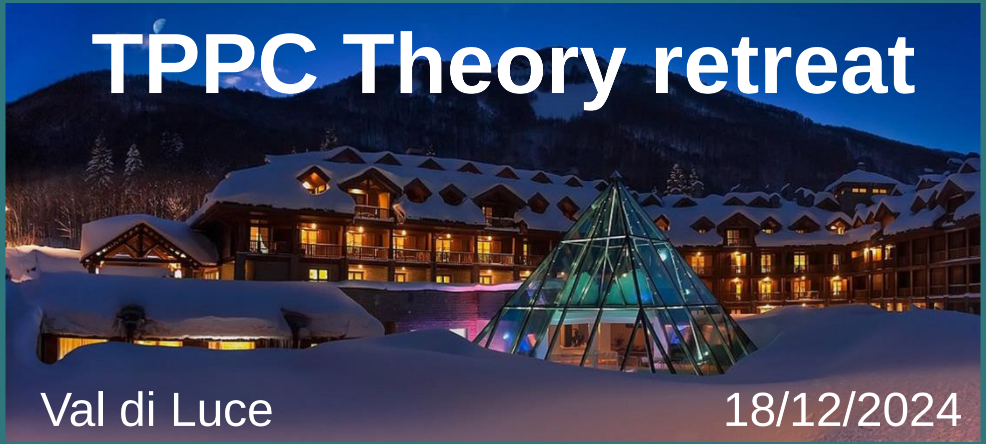

INFN
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AN ITERATIVE METHOD TO BUILD NON-LINEAR RELATIONS BETWEEN COSMOLOGICAL OBSERVABLES

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CONTENTS

NON-LINEARITIES IN THE LATE UNIVERSE

- In the study of the large-scale structure (LSS) of the Universe, the overall matter content is considered as a fluid on scales larger than the typical size of galaxies.
- Cosmological perturbation theory is a powerful tool to study the LSS.
  Background level: homogeneous and isotropic fluid; deviations: small perturbations.

- Non-linear corrections and higher-order perturbation theory in the LSS become increasingly important at late times, since matter tends to cluster under the action of gravity.
- Loop corrections in the matter power spectra and bispectra.

- Large amount of data from current and next generation surveys (Euclid, DESI, LSST, SKA, etc.).

- Relativistic N-body simulations provide an interesting arena to test non-perturbative inhomogeneities and handle fully non-linear processes.

PERTURBATIONS OF THE LUMINOSITY DISTANCE

Second-order corrections of the luminosity distance due to gravitational lensing:

$$d_L\left(z\right) \simeq d_L^{(0)}\left(z\right)\left(1+\sigma^{(1)}+\sigma^{(2)}\right)\,,$$

$$\sigma^{(1)} = \int_0^{r_s} dr \frac{r-r_s}{r r_s}\Delta_2\psi\left(r\right)\,,$$

$$\sigma^{(2)} = \frac{1}{2}\sigma^{(1)\,2} + \Sigma^{(2)} + \sigma^{(2)}_{LSS}\,,$$

ψ , ϕ: Bardeen gravitational potentials

r_s: comoving distance of the source

Δ_2: dimensionless angular Laplacian on the 2-sphere

$$\bar{\gamma}_0^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^{-2}\theta \end{pmatrix} \qquad \gamma_0^{ab} = r^{-2}\bar{\gamma}_0^{ab}$$

$$\sigma^{(2)}_{LSS} \equiv \frac{1}{2}\int_0^{r_s} dr \frac{r-r_s}{r r_s}\Delta_2\left[\psi^{(2)}+\phi^{(2)}\right](r)\,,$$

$$\begin{aligned}\Sigma^{(2)} \equiv\ & 2\int_0^{r_s} dr \frac{r-r_s}{r r_s}\partial_b\left[\Delta_2\psi(r)\right]\int_0^{r_s} dr \frac{r-r_s}{r r_s}\bar{\gamma}_0^{ab}\partial_a\psi(r) \\ & +2\int_0^{r_s} dr\left\{\gamma_0^{ab}\partial_b\left[\int_0^{r} dr'\,\psi(r')\right]\int_0^{r} dr' \frac{r'-r}{r r'}\partial_a\Delta_2\psi(r')\right\} \\ & +\int_0^{r_s} dr \frac{r-r_s}{r r_s}\Delta_2\left[\gamma_0^{ab}\partial_a\left(\int_0^{r} dr'\,\psi(r')\right)\partial_b\left(\int_0^{r} dr'\,\psi(r')\right)\right]\end{aligned}$$

BEN-DAYAN *et al* JCAP11(2012)045

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UMEH *et al* 2014 *Class. Quantum Grav.* 31 205001

FANIZZA *et al* JCAP08(2015)020

A GENERAL METHOD IN PERTURBATION THEORY

- Identifying a set of 4 conditions/constraints: (4 observables)

$$f^A(x^\mu) = 0$$

x^μ: set of coordinates

$A = 0, \ldots, 3$

- Perturbative approach
  Expanding the 4 constraints
  and the coordinate system

$$\begin{cases} f^A(x^\mu) = \bar{f}^A(x^\mu) + \delta f^A(x^\mu) \\ x^\mu = \bar{x}^\mu + \delta x^\mu \end{cases}$$

$\bar{\ }$: background quantity

$\delta(\ldots)$: all the perturbative series up to a given order n

$$f^A(x^\mu) = \bar{f}^A(\bar{x}^\mu) + \delta f^A(\bar{x}^\mu) + \delta x^\nu\, \partial_\nu \bar{f}^A(\bar{x}^\mu) + Q^A(\bar{x}^\mu) + h.o.$$

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$$

- $Q^A(\bar{x}^\mu)$ contains all the terms in the expansion that depend on $\delta x^\mu(\bar{x}^\mu)$ and derivatives of $\delta f^A(\bar{x}^\mu)$.
  $Q^A_{(n)}(\bar{x}^\mu)$ involves only low-order perturbations. For $n = 1,2,3$:

$$Q^A_{(1)} = 0$$

$$Q^A_{(2)} = \delta x^\rho_{(1)} \partial_\rho \delta f^A_{(1)} + \frac{1}{2} \delta x^\rho_{(1)} \delta x^\sigma_{(1)} \partial^2_{\rho\sigma} \bar{f}^A$$

$$Q^A_{(3)} = \delta x^\rho_{(2)} \partial_\rho \delta f^A_{(1)} + \delta x^\rho_{(1)} \partial_\rho \delta f^A_{(2)} + \delta x^\rho_{(2)} \delta x^\sigma_{(1)} \partial^2_{\rho\sigma} \bar{f}^A + \frac{1}{2} \delta x^\rho_{(1)} \delta x^\sigma_{(1)} \partial^2_{\rho\sigma} \delta f^A_{(1)} + \frac{1}{6} \delta x^\rho_{(1)} \delta x^\sigma_{(1)} \delta x^\tau_{(1)} \partial^3_{\rho\sigma\tau} \bar{f}^A$$

A GENERAL METHOD IN PERTURBATION THEORY

- Imposing 4 conditions/constraints order by order

$$f^A_{(n)}(x^\mu) = 0$$

Background condition

$$\bar{f}^A(\bar{x}^\mu) = 0$$

$$A = 0, \dots, 3$$

- n-th order:

$$f^A_{(n)}(x^\mu) = \delta f^A_{(n)}(\bar{x}^\mu) + \delta x^\nu_{(n)}\, \partial_\nu \bar{f}^A(\bar{x}^\mu) + Q^A_{(n)}(\bar{x}^\mu) = 0$$

- We obtain an algebraic system for $\delta x^\mu_{(n)}$:

$$\delta x^\mu_{(n)}\left(\partial_\mu \bar{f}^A\right)_{x^\nu = \bar{x}^\nu} = -\delta f^A_{(n)}\,(\bar{x}^\nu) - Q^A_{(n)}\,(\bar{x}^\nu)$$

$$\begin{array}{l} A = 0 \\ A = 1 \\ A = 2 \\ A = 3 \end{array} \left\{ \begin{array}{l}
\delta x^0_{(n)}\partial_0 \bar{f}^0(\bar{x}^\nu) + \delta x^1_{(n)}\partial_1 \bar{f}^0(\bar{x}^\nu) + \delta x^2_{(n)}\partial_2 \bar{f}^0(\bar{x}^\nu) + \delta x^3_{(n)}\partial_3 \bar{f}^0(\bar{x}^\nu) = -\delta f^0_{(n)}\,(\bar{x}^\nu) - Q^0_{(n)}\,(\bar{x}^\nu) \\
\delta x^0_{(n)}\partial_0 \bar{f}^1(\bar{x}^\nu) + \delta x^1_{(n)}\partial_1 \bar{f}^1(\bar{x}^\nu) + \delta x^2_{(n)}\partial_2 \bar{f}^1(\bar{x}^\nu) + \delta x^3_{(n)}\partial_3 \bar{f}^1(\bar{x}^\nu) = -\delta f^1_{(n)}\,(\bar{x}^\nu) - Q^1_{(n)}\,(\bar{x}^\nu) \\
\delta x^0_{(n)}\partial_0 \bar{f}^2(\bar{x}^\nu) + \delta x^1_{(n)}\partial_1 \bar{f}^2(\bar{x}^\nu) + \delta x^2_{(n)}\partial_2 \bar{f}^2(\bar{x}^\nu) + \delta x^3_{(n)}\partial_3 \bar{f}^2(\bar{x}^\nu) = -\delta f^2_{(n)}\,(\bar{x}^\nu) - Q^2_{(n)}\,(\bar{x}^\nu) \\
\delta x^0_{(n)}\partial_0 \bar{f}^3(\bar{x}^\nu) + \delta x^1_{(n)}\partial_1 \bar{f}^3(\bar{x}^\nu) + \delta x^2_{(n)}\partial_2 \bar{f}^3(\bar{x}^\nu) + \delta x^3_{(n)}\partial_3 \bar{f}^3(\bar{x}^\nu) = -\delta f^3_{(n)}\,(\bar{x}^\nu) - Q^3_{(n)}\,(\bar{x}^\nu)
\end{array} \right.$$

A GENERAL METHOD IN PERTURBATION THEORY

- Imposing 4 conditions/constraints order by order $\quad f^A_{(n)}(x^\mu) = 0$

Background condition

$$\bar{f}^A(\bar{x}^\mu) = 0 \qquad A = 0, \dots, 3$$

- n-th order: $\quad f^A_{(n)}(x^\mu) = \delta f^A_{(n)}(\bar{x}^\mu) + \delta x^\nu_{(n)}\, \partial_\nu \bar{f}^A(\bar{x}^\mu) + Q^A_{(n)}(\bar{x}^\mu) = 0$

- We obtain an algebraic system for $\delta x^\mu_{(n)}$: $\quad \delta x^\mu_{(n)}\left(\partial_\mu \bar{f}^A\right)_{x^\nu = \bar{x}^\nu} = -\delta f^A_{(n)}\,(\bar{x}^\nu) - Q^A_{(n)}\,(\bar{x}^\nu)$

- An iterative solution for $\delta x^\mu_{(n)}$ at any perturbative order is obtained by using the Kramer method:

$$\boxed{\delta x^\mu_{(n)} = \frac{\det \mathcal{A}^{[\mu]}_{(n)}}{\det C}}$$

The matrix $\mathcal{A}^{[\mu]}_{(n)}$ is given by replacing the μ-th column in C with $-\delta f^A_{(n)}\,(\bar{x}^\nu) - Q^A_{(n)}\,(\bar{x}^\nu)$

Coefficient matrix

$$C = \left(\partial_\mu \bar{f}^A(\bar{x}^\nu)\right) = \begin{array}{c} \xrightarrow{\partial_\mu} \\ \begin{pmatrix} \partial_0 \bar{f}^0 & \partial_1 \bar{f}^0 & \partial_2 \bar{f}^0 & \partial_3 \bar{f}^0 \\ \partial_0 \bar{f}^1 & \partial_1 \bar{f}^1 & \partial_2 \bar{f}^1 & \partial_3 \bar{f}^1 \\ \partial_0 \bar{f}^2 & \partial_1 \bar{f}^2 & \partial_2 \bar{f}^2 & \partial_3 \bar{f}^2 \\ \partial_0 \bar{f}^3 & \partial_1 \bar{f}^3 & \partial_2 \bar{f}^3 & \partial_3 \bar{f}^3 \end{pmatrix}_{x^\nu = \bar{x}^\nu} \end{array} \quad A \downarrow$$

$$\det C \neq 0$$

ITERATIVE METHOD
FOR LATE-TIME COSMOLOGICAL PERTURBATIONS

- Now we will apply this iterative method to late-time cosmology, focusing on a cosmological observable given in terms of the observed redshift and the incoming direction of photons in the observer's frame to obtain solutions for the time delay, radial shift, and deflection angles up to the third order in perturbation theory.

FIDUCIAL BACKGROUND COSMOLOGY

Background metric: spatially flat FLRW line element.

Observer's frame.

Fiducial coordinates: $\bar{x}_z^{\mu} = (\bar{\eta}_z, \bar{r}_z, \bar{\theta}_z^a)$

Observed redshift: $1 + \mathrm{z}_{\mathrm{obs}} = a^{-1}(\bar{\eta}_z)$

Observed light-cone: $\bar{w}_z = \bar{\eta}_z + \bar{r}_z$

Observed directions: $\bar{\theta}_z^a \quad a = 1,2$

INHOMOEGENEOUS COSMOLOGY

Perturbations wrt background: inhomogeneities are projected along a fictitious past light-cone.

Coordinates: $\mathrm{x}^{\mu} = (\eta, r, \theta^a)$

Perturbed redshift: $1 + \mathrm{z}$

Perturbed light-cone: w

Deflection angles: $\Theta^a \quad a = 1,2$

ITERATIVE METHOD FOR LATE-TIME COSMOLOGICAL PERTURBATIONS

- Constraints:

$$f^A(x^\mu) = \begin{pmatrix} (1+z)(x^\mu) - 1 + z_{obs} \\ w(x^\mu) - \bar{w}_z \\ \Theta^a(x^\mu) - \bar{\theta}^a_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vec{0} \end{pmatrix}$$

$$A = \eta, r, 1{,}2$$
$$a = 1{,}2$$

Background

$$a^{-1}(\eta) = 1 + z_{obs}$$
$$\eta + r = \bar{\eta}_z + \bar{r}_z$$
$$\theta^a(x^\mu) = \bar{\theta}^a_z$$

- Expanding light-cone perturbations:

$$1 + z = \frac{1}{a(\eta)}\left(1 + \delta z_{(1)} + \delta z_{(2)} + \ldots\right)$$
$$w = \eta + r + \delta w_{(1)} + \delta w_{(2)} + \cdots$$
$$\Theta^a = \theta^a + \delta\Theta^a{}_{(1)} + \delta\Theta^a{}_{(2)} + \cdots$$

$$\longrightarrow \quad \delta f^A_{(n)}(x^\mu) \equiv \begin{pmatrix} \dfrac{\delta z_{(n)}}{a(\eta)} \\ \delta w_{(n)} \\ \delta\Theta^a_{(n)} \end{pmatrix}$$

Shifts at the n-th order

- Algebraic system:

$$\delta x^\mu_{(n)}\left(\partial_\mu \bar{f}^A\right)_{x^\nu = \bar{x}^\nu} = -\delta f^A_{(n)}(\bar{x}^\nu) - Q^A_{(n)}(\bar{x}^\nu)$$

Coefficient matrix

$$C = \begin{pmatrix} -\dfrac{\mathcal{H}}{a}(\bar{\eta}_z) & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\det C \neq 0$$
$$\mathcal{H} \equiv \frac{a'}{a}$$
$$(\ldots)' \equiv \partial_\eta$$

ITERATIVE METHOD FOR LATE-TIME COSMOLOGICAL PERTURBATIONS

$$\boxed{\delta x^{\mu}_{(n)} = \frac{\det \mathcal{A}^{[\mu]}_{(n)}}{\det C}}$$

- Using our iterative method, we obtain the following solutions at the n-th order:

TIME SHIFTS

$$\boxed{\delta \eta_z^{(n)}(\bar{x}_z^{\mu}) = \frac{\delta z_{(n)}}{\mathcal{H}(z_{obs})} + \frac{Q^{\eta}_{(n)}}{(1+z_{obs})\mathcal{H}(z_{obs})}}$$

DEFLECTION ANGLES

$$\boxed{\delta \theta_z^{a\,(n)}(\bar{x}_z^{\mu}) = -\delta\Theta^a_{(n)} - Q^a_{(n)}}$$

RADIAL DISPLACEMENT

$$\boxed{\delta r_z^{(n)}(\bar{x}_z^{\mu}) = -\delta w_{(n)} - \frac{\delta z_{(n)}}{\mathcal{H}(z_{obs})} - Q^r_{(n)} - \frac{Q^{\eta}_{(n)}}{(1+z_{obs})\mathcal{H}(z_{obs})}}$$

Perturbations $\delta z_{(n)}$, $\delta w_{(n)}$, $\delta\Theta^a_{(n)}$ can be evaluated apart, solving the light-like geodesic equations (use the Geodesic Light-Cone (GLC) gauge)

ITERATIVE METHOD FOR LATE-TIME COSMOLOGICAL PERTURBATIONS

LINEAR TERMS

(n=1) $Q^A_{(1)} = 0$

Background constraints

$\eta = \bar{\eta}_z$

$r = \bar{r}_z$

$\theta^a = \bar{\theta}^a_z$

TIME SHIFTS

$$\boxed{\delta\eta_z^{(1)} = \frac{\delta z_{(1)}}{\mathcal{H}}}$$

The shift in the time component is entirely sourced at linear order by redshift perturbations

RADIAL DISPLACEMENT

$$\boxed{\delta r_z^{(1)} = -\delta w_{(1)} - \frac{\delta z_{(1)}}{\mathcal{H}}}$$

The radial displacement is sourced not only by redshift perturbations but also by light-cone distortions.

DEFLECTION ANGLES

$$\boxed{\delta\theta_z^{a\,(1)} = -\delta\Theta^a_{(1)}}$$

ITERATIVE METHOD
SECOND-ORDER TERMS

(n=2) $Q^A_{(2)} = \delta x^\rho_{(1)} \partial_\rho \delta f^A_{(1)} + \frac{1}{2} \delta x^\rho_{(1)} \delta x^\sigma_{(1)} \partial^2_{\rho\sigma} \bar{f}^A$

Background constraints

$\eta = \bar{\eta}_z$
$r = \bar{r}_z$
$\theta^a = \bar{\theta}^a_z$

TIME SHIFTS

$$\delta\eta_z^{(2)} = \frac{1}{\mathcal{H}}\left[\delta z^{(2)} + \frac{\delta z^{(1)}\partial_\eta \delta z^{(1)}}{\mathcal{H}} - \delta w^{(1)}\partial_r \delta z^{(1)} - \frac{\delta z^{(1)}\partial_r \delta z^{(1)}}{\mathcal{H}} - \delta\Theta^a_{(1)}\partial_a \delta z^{(1)} - \frac{1}{2}\left(1 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right)\left(\delta z^{(1)}\right)^2\right],$$

RADIAL DISPLACEMENT

$$\delta r_z^{(2)} = -\delta w^{(2)} - \frac{\delta z^{(1)}}{\mathcal{H}}\partial_\eta \delta w^{(1)} + \delta w^{(1)}\partial_r \delta w^{(1)} + \frac{\delta z^{(1)}}{\mathcal{H}}\partial_r \delta w^{(1)} + \delta\Theta^a_{(1)}\partial_a \delta w^{(1)} - \frac{1}{\mathcal{H}}\left[\delta z^{(2)} + \frac{\delta z^{(1)}\partial_\eta \delta z^{(1)}}{\mathcal{H}} - \delta w^{(1)}\partial_r \delta z^{(1)} - \frac{\delta z^{(1)}\partial_r \delta z^{(1)}}{\mathcal{H}} - \delta\Theta^a_{(1)}\partial_a \delta z^{(1)} - \frac{1}{2}\left(1 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right)\left(\delta z^{(1)}\right)^2\right],$$

DEFLECTION ANGLES

$$\delta\theta_z^{a(2)} = -\delta\Theta^a_{(2)} - \frac{\delta z^{(1)}}{\mathcal{H}}\partial_\eta \delta\Theta^a_{(1)} + \delta w^{(1)}\partial_r \delta\Theta^a_{(1)} + \frac{\delta z^{(1)}}{\mathcal{H}}\partial_r \delta\Theta^a_{(1)} + \delta\Theta^b_{(1)}\partial_b \delta\Theta^a_{(1)}.$$

ITERATIVE METHOD

SECOND-ORDER TERMS

(n=2) $$Q^A_{(2)} = \delta x^\rho_{(1)} \partial_\rho \delta f^A_{(1)} + \frac{1}{2} \delta x^\rho_{(1)} \delta x^\sigma_{(1)} \partial^2_{\rho\sigma} \bar{f}^A$$

Background constraints

$$\eta = \bar{\eta}_z$$
$$r = \bar{r}_z$$
$$\theta^a = \bar{\theta}^a_z$$

TIME SHIFTS

$$\delta\eta_z^{(2)} = \frac{1}{\mathcal{H}}\left[\delta z^{(2)} + \frac{\delta z^{(1)}\partial_\eta \delta z^{(1)}}{\mathcal{H}} - \delta w^{(1)}\partial_r \delta z^{(1)} - \frac{\delta z^{(1)}\partial_r \delta z^{(1)}}{\mathcal{H}}\right.$$
$$\left. - \delta\Theta^a_{(1)}\partial_a \delta z^{(1)} - \frac{1}{2}\left(1 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right)\left(\delta z^{(1)}\right)^2\right],$$

RADIAL DISPLACEMENT

$$\delta r_z^{(2)} = -\delta w^{(2)} - \frac{\delta z^{(1)}}{\mathcal{H}}\partial_\eta \delta w^{(1)} + \delta w^{(1)}\partial_r \delta w^{(1)} + \ldots$$
$$-\frac{1}{\mathcal{H}}\left[\delta z^{(2)} + \frac{\delta z^{(1)}\partial_\eta \delta z^{(1)}}{\mathcal{H}} - \delta w^{(1)}\partial_r \delta z^{(1)} - \ldots\right.$$
$$\left. - \delta\Theta^a_{(1)}\partial_a \delta z^{(1)} - \frac{1}{2}\left(1 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right)\left(\delta z^{(1)}\right)^2\right],$$

DEFLECTION ANGLES

$$\delta\theta_z^{a(2)} = -\delta\Theta^a_{(2)} - \frac{\delta z^{(1)}}{\mathcal{H}}\partial_\eta \delta\Theta^a_{(1)} + \delta w^{(1)}\partial_r \delta\Theta^a_{(1)} + \ldots$$

Agreement with the calculation of:

- Second-order distance-redshift relation
- Second-order expansion of the galaxy number count
  [arXiv:1407.0376, arXiv:1510.04202]
- Next-to-leading order corrections to CMB spectra (only for the angular part)
  [arXiv:1605.08761, arXiv:1612.07263, arXiv:1612.07650]

[arXiv:gr-qc/9609040, arXiv:1805.05959, arXiv:1705.05839]

ITERATIVE METHOD
THIRD-ORDER TERMS

Background constraints

$$\eta = \bar{\eta}_z \quad r = \bar{r}_z \quad \theta^a = \bar{\theta}^a_z$$

(n=3)

$$Q^A_{(3)} = \delta x^\rho_{(2)} \partial_\rho \delta f^A_{(1)} + \delta x^\rho_{(1)} \partial_\rho \delta f^A_{(2)} + \delta x^\rho_{(2)} \delta x^\sigma_{(1)} \partial^2_{\rho\sigma} \bar{f}^A + \frac{1}{2} \delta x^\rho_{(1)} \delta x^\sigma_{(1)} \partial^2_{\rho\sigma} \delta f^A_{(1)} + \frac{1}{6} \delta x^\rho_{(1)} \delta x^\sigma_{(1)} \delta x^\tau_{(1)} \partial^3_{\rho\sigma\tau} \bar{f}^A$$

TIME SHIFTS

$$\begin{aligned}
\delta\eta_z^{(3)} = \frac{\delta z^{(3)}}{\mathcal{H}} + \frac{1}{\mathcal{H}} \Bigg\{ & \delta x^\rho_{(2)} \partial_\rho \delta z^{(1)} + \delta x^\rho_{(1)} \partial_\rho \delta z^{(2)} - \left(1 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right) \delta z^{(2)} \delta z^{(1)} \\
& - \delta z^{(1)} \frac{\mathcal{H}'}{\mathcal{H}^2} \left[\frac{\delta z^{(1)} \partial_\eta \delta z^{(1)}}{\mathcal{H}} - \delta w^{(1)} \partial_r \delta z^{(1)} - \frac{\delta z^{(1)} \partial_r \delta z^{(1)}}{\mathcal{H}} - \delta \Theta^a_{(1)} \partial_a \delta z^{(1)} \right] \\
& + \frac{1}{2} \delta x^\rho_{(1)} \delta x^\sigma_{(1)} \partial^2_{\rho\sigma} \left(\delta z^{(1)}\right) - \delta z^{(1)} \delta x^\rho_{(1)} \partial_\rho \left(\delta z^{(1)}\right) \\
& + \left(\delta z^{(1)}\right)^3 \left[\frac{1}{3} + \frac{\mathcal{H}'}{2\mathcal{H}^2} - \frac{\mathcal{H}''}{6\mathcal{H}^3} + \frac{1}{2} \left(\frac{\mathcal{H}'}{\mathcal{H}^2}\right)^2 \right] \Bigg\} ,
\end{aligned}$$

ITERATIVE METHOD
THIRD-ORDER TERMS

Background constraints

$$\eta = \bar{\eta}_z$$
$$r = \bar{r}_z$$
$$\theta^a = \bar{\theta}^a_z$$

(n=3)

$$Q^A_{(3)} = \delta x^\rho_{(2)} \partial_\rho \delta f^A_{(1)} + \delta x^\rho_{(1)} \partial_\rho \delta f^A_{(2)} + \delta x^\rho_{(2)} \delta x^\sigma_{(1)} \partial^2_{\rho\sigma} \bar{f}^A + \frac{1}{2} \delta x^\rho_{(1)} \delta x^\sigma_{(1)} \partial^2_{\rho\sigma} \delta f^A_{(1)} + \frac{1}{6} \delta x^\rho_{(1)} \delta x^\sigma_{(1)} \delta x^\tau_{(1)} \partial^3_{\rho\sigma\tau} \bar{f}^A$$

RADIAL DISPLACEMENT

$$\begin{aligned}
\delta r_z^{(3)} = & -\delta w^{(3)} - \frac{\delta z^{(3)}}{\mathcal{H}} - \delta x^\rho_{(2)} \partial_\rho \delta w^{(1)} - \delta x^\rho_{(1)} \partial_\rho \delta w^{(2)} - \frac{1}{2} \delta x^\rho_{(1)} \delta x^\sigma_{(1)} \partial^2_{\rho\sigma} \delta w^{(1)} \\
& - \frac{1}{\mathcal{H}} \Bigg\{ \delta x^\rho_{(2)} \partial_\rho \delta z^{(1)} + \delta x^\rho_{(1)} \partial_\rho \delta z^{(2)} - \left(1 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right) \delta z^{(2)} \delta z^{(1)} \\
& - \delta z^{(1)} \frac{\mathcal{H}'}{\mathcal{H}^2} \left[\frac{\delta z^{(1)} \partial_\eta \delta z^{(1)}}{\mathcal{H}} - \delta w^{(1)} \partial_r \delta z^{(1)} - \frac{\delta z^{(1)} \partial_r \delta z^{(1)}}{\mathcal{H}} - \delta \Theta^a_{(1)} \partial_a \delta z^{(1)} \right] \\
& + \frac{1}{2} \delta x^\rho_{(1)} \delta x^\sigma_{(1)} \partial^2_{\rho\sigma} \left(\delta z^{(1)}\right) - \delta z^{(1)} \delta x^\rho_{(1)} \partial_\rho \left(\delta z^{(1)}\right) \\
& + \left(\delta z^{(1)}\right)^3 \left[\frac{1}{3} + \frac{\mathcal{H}'}{2\mathcal{H}^2} - \frac{\mathcal{H}''}{6\mathcal{H}^3} + \frac{1}{2} \left(\frac{\mathcal{H}'}{\mathcal{H}^2}\right)^2 \right] \Bigg\} ,
\end{aligned}$$

ITERATIVE METHOD

THIRD-ORDER TERMS

Background constraints

$$\eta = \bar{\eta}_z$$
$$r = \bar{r}_z$$
$$\theta^a = \bar{\theta}^a_z$$

(n=3)

$$Q^A_{(3)} = \delta x^\rho_{(2)} \partial_\rho \delta f^A_{(1)} + \delta x^\rho_{(1)} \partial_\rho \delta f^A_{(2)} + \delta x^\rho_{(2)} \delta x^\sigma_{(1)} \partial^2_{\rho\sigma} \bar{f}^A + \frac{1}{2} \delta x^\rho_{(1)} \delta x^\sigma_{(1)} \partial^2_{\rho\sigma} \delta f^A_{(1)} + \frac{1}{6} \delta x^\rho_{(1)} \delta x^\sigma_{(1)} \delta x^\tau_{(1)} \partial^3_{\rho\sigma\tau} \bar{f}^A$$

DEFLECTION ANGLES

$$\delta\theta^{a(3)}_z = -\delta\Theta^a_{(3)} - \delta x^\rho_{(2)} \partial_\rho \delta\Theta^a_{(1)} - \delta x^\rho_{(1)} \partial_\rho \delta\Theta^a_{(2)} - \frac{1}{2} \delta x^\rho_{(1)} \delta x^\sigma_{(1)} \partial^2_{\rho\sigma} \delta\Theta^a_{(1)} \,.$$

ITERATIVE METHOD
LUMINOSITY DISTANCE SPACE

- As a further relevant application of our procedure, we will now implement our method to express a cosmological observables in the luminosity space rather than in the redshift one.
- LSST is predicted to observe $\sim 10^6$ SNe [arXiv:2010.05926]; third generation GW observatories will see a similar number of GW events: the observed luminosity distance will be an essential probe of LSS.
- We express a cosmological observable in terms of the observed luminosity distance along the observed past light cone.

FIDUCIAL BACKGROUND COSMOLOGY

Background metric: spatially flat FLRW line element.

Observer's frame.

Fiducial coordinates: $\bar{x}^{\mu}_d = (\bar{\eta}_d, \bar{r}_d, \bar{\theta}^a_d)$

Observed luminosity distance: $\mathrm{d}^{obs}_{\mathrm{L}} = \frac{\bar{r}_d}{a(\bar{\eta}_d)}$

Observed light-cone: $\bar{w}_d = \bar{\eta}_d + \bar{r}_d$

Observed directions: $\bar{\theta}^a_d \quad a = 1,2$

INHOMOEGENEOUS COSMOLOGY

Perturbations wrt background: inhomogeneities are projected along a fictitious past light-cone.

Coordinates: $\mathrm{x}^{\mu} = (\eta, r, \theta^a)$

Perturbed luminosity distance: $\mathrm{d_L}$

Perturbed light-cone: w

Deflection angles: $\Theta^a \quad a = 1,2$

ITERATIVE METHOD
LUMINOSITY DISTANCE SPACE

- Constraints:

$$f^A(x^\mu) = \begin{pmatrix} d_L(x^\mu) - d_L^{obs} \\ w(x^\mu) - \bar{w}_d \\ \Theta^a(x^\mu) - \bar{\theta}_d^a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vec{0} \end{pmatrix}$$

$$A = \eta, r, 1,2$$
$$a = 1,2$$

Background

$$\mathrm{d}_\mathrm{L}^{obs} = \frac{\bar{r}_d}{a(\bar{\eta}_d)}$$
$$\eta + r = \bar{\eta}_d + \bar{r}_d$$
$$\theta^a(x^\mu) = \bar{\theta}_d^a$$

- Expanding perturbations in the luminosity distance space:

$$d_L = \frac{r}{a(\eta)}\left(1 + \delta d_L^{(1)} + \delta d_L^{(2)} + \dots\right)$$
$$w = \eta + r + \delta w_{(1)} + \delta w_{(2)} + \cdots$$
$$\Theta^a = \theta^a + \delta\Theta^a{}_{(1)} + \delta\Theta^a{}_{(2)} + \cdots$$

$$\Rightarrow \quad \delta f_{(n)}^A(x^\mu) \equiv \begin{pmatrix} \frac{r}{a(\eta)}\delta d_L^{(n)} \\ \delta w_{(n)} \\ \delta\Theta_{(n)}^a \end{pmatrix}$$

Shifts at the n-th order

- Algebraic system:

$$\delta x_{(n)}^\mu \left(\partial_\mu \bar{f}^A\right)_{x^\nu = \bar{x}^\nu} = -\delta f_{(n)}^A(\bar{x}^\nu) - Q_{(n)}^A(\bar{x}^\nu)$$

Coefficient matrix

$$C = \begin{pmatrix} -\frac{\bar{r}_d}{a(\bar{\eta}_d)}\mathcal{H}_d & \frac{1}{a(\bar{\eta}_d)} & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\det C \neq 0$$

$$\mathcal{H}_d \equiv \mathcal{H}(\bar{\eta}_d)$$

ITERATIVE METHOD
LUMINOSITY DISTANCE SPACE

$$\boxed{\delta x^{\mu}_{(n)} = \frac{\det \mathcal{A}^{[\mu]}_{(n)}}{\det C}}$$

- Using our iterative method, we obtain the following solutions at the n-th order:

TIME SHIFTS

$$\boxed{\delta\eta_d^{(n)}(\bar{x}_d^{\mu}) = \frac{1}{(1+\bar{r}_d\mathcal{H}_d)}\left[\bar{r}_d\,\delta d_L^{(n)} + a(\bar{\eta}_d)\,Q^{\eta}_{(n)} - \delta w_{(n)} - Q^{r}_{(n)}\right]}$$

RADIAL DISPLACEMENT

$$\boxed{\delta r_d^{(n)}(\bar{x}_d^{\mu}) = -\frac{1}{(1+\bar{r}_d\mathcal{H}_d)}\left[\bar{r}_d\mathcal{H}_d\left(\delta w_{(n)} + Q^{r}_{(n)}\right) + \bar{r}_d\,\delta d_L^{(n)} + a(\bar{\eta}_d)\,Q^{\eta}_{(n)}\right]}$$

DEFLECTION ANGLES

$$\boxed{\delta\theta_d^{a\,(n)}(\bar{x}_d^{\mu}) = -\delta\Theta^{a}_{(n)} - Q^{a}_{(n)}}$$

ITERATIVE METHOD
LUMINOSITY DISTANCE SPACE

LINEAR TERMS

(n=1) $Q^A_{(1)} = 0$

TIME SHIFTS

$$\boxed{\delta\eta_d^{(1)} = \frac{1}{(1+\bar{r}_d\mathcal{H}_d)}\left[\bar{r}_d\, \delta d_L^{(1)} - \delta w_{(1)}\right]}$$

RADIAL DISPLACEMENT

$$\boxed{\delta r_d^{(1)} = -\frac{1}{(1+\bar{r}_d\mathcal{H}_d)}\left[\bar{r}_d\mathcal{H}_d\, \delta w_{(1)} + \bar{r}_d\, \delta d_L^{(1)}\right]}$$

DEFLECTION ANGLES

$$\boxed{\delta\theta_d^{a\,(1)} = -\delta\Theta^a_{(1)}}$$

The shift in the time component is sourced both by luminosity distance perturbations and light-cone distortions (differently from $\delta\eta_z^{(1)}$).

Time and radial shifts are suppressed when the source approaches the observer $\bar{r}_d \to 0$ due to the presence of a background factor (differently from $\delta r_z^{(1)}$).

No differences in $\delta\theta_d^{a\,(1)}$ between redshift and luminosity spaces.

ITERATIVE METHOD
LUMINOSITY DISTANCE SPACE

SECOND-ORDER TERMS

(n=2) $Q^{A}_{(2)} = \delta x^{\rho}_{(1)} \partial_\rho \delta f^{A}_{(1)} + \frac{1}{2} \delta x^{\rho}_{(1)} \delta x^{\sigma}_{(1)} \partial^2_{\rho\sigma} \bar{f}^{A}$

TIME SHIFTS

$$\delta\eta_d^{(2)} = \frac{1}{(1+\bar{r}_d\mathcal{H}_d)} \Bigg\{ \bar{r}_d \delta d_L^{(2)} - \delta w^{(2)} + \frac{1}{(1+\bar{r}_d\mathcal{H}_d)} \Bigg[\left(\bar{r}_d \delta d_L^{(1)} - \delta w^{(1)}\right) \partial_\eta \left(\bar{r}_d \delta d_L^{(1)} - \delta w^{(1)}\right)$$
$$- \left(\bar{r}_d \mathcal{H}_d \delta w^{(1)} + \bar{r}_d \delta d_L^{(1)}\right) \left(\bar{r}_d \partial_r \delta d_L^{(1)} - \partial_r \delta w^{(1)}\right)\Bigg] - \delta\Theta^{b(1)} \partial_b \left(\bar{r}_d \delta d_L^{(1)} - \delta w^{(1)}\right)$$
$$- \bar{r}_d \left(\delta d_L^{(1)}\right)^2 - \frac{1}{2} \frac{r_d}{\left(1+\bar{r}_d\mathcal{H}_d\right)^2} \Bigg[\left(\mathcal{H}'_d - \mathcal{H}^2_d\right) \left(\bar{r}_d \delta d_L^{(1)} - \delta w^{(1)}\right)^2$$
$$-2\frac{\mathcal{H}_d}{r_d} \left(\bar{r}_d \delta d_L^{(1)} - \delta w^{(1)}\right) \left(\bar{r}_d \mathcal{H}_d \delta w^{(1)} + \bar{r}_d \delta d_L^{(1)}\right)\Bigg]\Bigg\} ,$$

ITERATIVE METHOD

LUMINOSITY DISTANCE SPACE

SECOND-ORDER TERMS

(n=2) $Q^A_{(2)} = \delta x^\rho_{(1)} \partial_\rho \delta f^A_{(1)} + \frac{1}{2} \delta x^\rho_{(1)} \delta x^\sigma_{(1)} \partial^2_{\rho\sigma} \bar{f}^A$

RADIAL DISPLACEMENT

$$
\begin{aligned}
\delta r_d^{(2)} = & -\frac{1}{(1+\bar{r}_d\mathcal{H}_d)}\Big\{\bar{r}_d\mathcal{H}_d\delta w^{(2)} + \bar{r}_d\delta d_L^{(2)} \\
& + \frac{1}{(1+\bar{r}_d\mathcal{H}_d)}\Big[\left(\bar{r}_d\delta d_L^{(1)} - \delta w^{(1)}\right)\left(\bar{r}_d\partial_\eta\delta d_L^{(1)} + \bar{r}_d\mathcal{H}_d\partial_\eta\delta w^{(1)}\right) \\
& - \left(\bar{r}_d\mathcal{H}_d\delta w^{(1)} + \bar{r}_d\delta d_L^{(1)}\right)\left(\bar{r}_d\partial_r\delta d_L^{(1)} + \bar{r}_d\mathcal{H}_d\partial_r\delta w^{(1)}\right)\Big] - \delta\Theta^{b(1)}\partial_b\left(\bar{r}_d\delta d_L^{(1)} + \bar{r}_d\mathcal{H}_d\delta w^{(1)}\right) \\
& - \bar{r}_d\left(\delta d_L^{(1)}\right)^2 - \frac{1}{2}\bar{r}_d\frac{1}{(1+\bar{r}_d\mathcal{H}_d)^2}\Big[\left(\mathcal{H}'_d - \mathcal{H}_d^2\right)\left(\bar{r}_d\delta d_L^{(1)} - \delta w^{(1)}\right)^2 \\
& - 2\frac{\mathcal{H}_d}{r\ \ d}\left(\bar{r}_d\delta d_L^{(1)} - \delta w^{(1)}\right)\left(\bar{r}_d\mathcal{H}_d\delta w^{(1)} + \bar{r}_d\delta d_L^{(1)}\right)\Big]\Big\} ,
\end{aligned}
$$

ITERATIVE METHOD
LUMINOSITY DISTANCE SPACE

SECOND-ORDER TERMS

(n=2) $$Q^A_{(2)} = \delta x^\rho_{(1)} \partial_\rho \delta f^A_{(1)} + \frac{1}{2} \delta x^\rho_{(1)} \delta x^\sigma_{(1)} \partial^2_{\rho\sigma} \bar{f}^A$$

DEFLECTION ANGLES

$$\delta\theta_d^{a(2)} = -\delta\Theta^{a(2)} - \frac{1}{(1+\bar{r}_d\mathcal{H}_d)} \left\{ \left[\bar{r}_d \delta d_L^{(1)} - \delta w^{(1)} \right] \partial_\eta \delta\Theta^{a(1)} - \left[\bar{r}_d \mathcal{H}_d \delta w^{(1)} + \bar{r}_d \delta d_L^{(1)} \right] \partial_r \delta\Theta^{a(1)} \right\} + \delta\Theta^{b(1)} \partial_b \delta\Theta^{a(1)} \,.$$

NUMBER COUNT IN THE LUMINOSITY DISTANCE SPACE AT THE FIRST ORDER IN PERTURBATION THEORY

- We compute the number count fluctuations at first order (n=1) in perturbation theory in terms of luminosity distance corrections.
- Number of objects N observed at a given distance in a given direction.

Number count

$$\Delta = \frac{N-\bar{N}}{\bar{N}} = \frac{\rho(x^\mu)\,V(x^\mu) - \bar{\rho}(x^\mu)\,\bar{V}(x^\mu)}{\bar{\rho}(x^\mu)\,\bar{V}(x^\mu)}$$

$\rho = \frac{N}{V}$ number density

V: volume

- Linear expansion

$$\left.\begin{aligned} \rho(x^\mu) &= \bar{\rho}\,(x^\mu) + \delta\rho^{(1)}(x^\mu) \\ V(x^\mu) &= \bar{V}\,(x^\mu) + \delta V^{(1)}(x^\mu) \\ x^\mu &= \bar{x}_d^\mu + \delta x_d^{\mu\,(1)} \end{aligned}\right] \Longrightarrow \Delta \approx \frac{\delta\rho^{(1)}(x^\mu)}{\bar{\rho}(\bar{x}^\mu)} + \frac{\delta V^{(1)}(x^\mu)}{\bar{V}(\bar{x}^\mu)}$$

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$$\rho(x^\mu) = \bar{\rho}\,(x^\mu) + \delta\rho^{(1)}(x^\mu)$$

$$V(x^\mu) = \bar{V}\,(x^\mu) + \delta V^{(1)}(x^\mu)$$

$$x^\mu = \bar{x}_d^\mu + \delta x_d^{\mu\,(1)}$$

$$\Longrightarrow \quad \Delta \approx \frac{\delta\rho^{(1)}(x^\mu)}{\bar{\rho}(\bar{x}^\mu)} + \frac{\delta V^{(1)}(x^\mu)}{\bar{V}(\bar{x}^\mu)}$$

$$\frac{\delta\rho^{(1)}(x^\mu)}{\bar{\rho}(\bar{x}^\mu)} = \delta_n - \frac{\bar{\rho}'(\bar{\eta}_d)}{\bar{\rho}(\bar{\eta}_d)}\,\delta\eta_d^{(1)}$$

number density contrast $\quad \delta_n = \dfrac{\rho(x^\mu) - \bar{\rho}(\bar{x}^\mu)}{\bar{\rho}(\bar{x}^\mu)}$

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$$\rho(x^\mu) = \bar{\rho}\,(x^\mu) + \delta\rho^{(1)}(x^\mu)$$
$$V(x^\mu) = \bar{V}\,(x^\mu) + \delta V^{(1)}(x^\mu)$$
$$x^\mu = \bar{x}^\mu_d + \delta x^{\mu\,(1)}_d$$

$$\Longrightarrow \quad \Delta \approx \frac{\delta\rho^{(1)}(x^\mu)}{\bar{\rho}(\bar{x}^\mu)} + \frac{\delta V^{(1)}(x^\mu)}{\bar{V}(\bar{x}^\mu)}$$

$$\frac{\delta\rho^{(1)}(x^\mu)}{\bar{\rho}(\bar{x}^\mu)} = \delta_n - \frac{\bar{\rho}'(\bar{\eta}_d)}{\bar{\rho}(\bar{\eta}_d)}\,\delta\eta^{(1)}_d$$

number density contrast
$$\delta_n = \frac{\rho(x^\mu) - \bar{\rho}(\bar{x}^\mu)}{\bar{\rho}(\bar{x}^\mu)}$$

$$\boxed{\delta\eta^{(1)}_d = \frac{1}{(1+\bar{r}_d\mathcal{H}_d)}\left[\bar{r}_d\,\delta d^{(1)}_L - \delta w_{(1)}\right]}$$

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$$\rho(x^\mu) = \bar{\rho}\,(x^\mu) + \delta\rho^{(1)}(x^\mu)$$

$$V(x^\mu) = \bar{V}\,(x^\mu) + \delta V^{(1)}(x^\mu)$$

$$x^\mu = \bar{x}_d^\mu + \delta x_d^{\mu\,(1)}$$

$$\boxed{\Delta = \delta_n - \gamma\left(\delta d_L^{(1)} - \frac{\delta w^{(1)}}{\bar{r}_d}\right)\left(\frac{\partial \ln \bar{n}}{\partial \ln a} - 3\right) + \frac{\delta V^{(1)}(x^\mu)}{\bar{V}(\bar{x}^\mu)}}$$

$$\boxed{\delta\eta_d^{(1)} = \frac{1}{(1+\bar{r}_d\mathcal{H}_d)}\left[\bar{r}_d\,\delta d_L^{(1)} - \delta w_{(1)}\right]}$$

with $\gamma \equiv \frac{\bar{r}_d\,\mathcal{H}_d}{1+\bar{r}_d\,\mathcal{H}_d}$ and $n \equiv \rho\, a^3$ comoving number density

See the comparison with arXiv:2304.14253

CONCLUSIONS

- Analytical techniques in the non-linear regime via a perturbation approach.
- Relativistic numerical simulations of cosmic structure formation.
- Tremendous amount of data from forthcoming LSS surveys (Euclid, LSST) and GWs detectors.

▪ Need for higher-order perturbation theory for predicting the statistics of the LSS of the Universe.

▪ General method in perturbation theory with simple mathematical tools.
▪ Iterative method to compute shifts in redshift and luminosity distance spaces at the desired order.
▪ It can be implemented in Mathematica to obtain analytical expressions for time delays, radial displacements, and deflection angles.

▪ Number count fluctuations, distance-redshift relation, and angular corrections to CMB spectra in higher-order perturbation theory.

Back-up slides

MATTER BISPECTRUM

the 3-point function and the matter bispectrum

δ: density contrast
δ_D: Dirac delta

$$\overline{\delta(\boldsymbol{k_1}, r_1)\delta(\boldsymbol{k_2}, r_2)\delta^{(2)}(\boldsymbol{k_3}, r_3)} = (2\pi)^3\,\delta_D(\boldsymbol{k_1} + \boldsymbol{k_2} + \boldsymbol{k_3})\,B(k_1, k_2, k_3, r_1, r_2, r_3)$$

Non-symmetrized matter bispectrum in Fourier space:

$$B(k_1, k_2, k_3, r_1, r_2, r_3) = 2\,D_1(r_1)\,D_1(r_2)\,D_1^2(r_3)\,F_2(k_1, k_2, k_3)\,P(k_1)\,P(k_2)$$

with $F_2\,(k_1, k_2, k_3) = \frac{5}{7} + \frac{1}{4}\,\frac{k_3^2 - k_1^2 - k_2^2}{k_1\,k_2}\left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right) + \frac{1}{14}\left(\frac{k_3^2 - k_1^2 - k_2^2}{k_1\,k_2}\right)^2$

D_1: growth function

Symmetrized bispectrum:

$$B_{sym}\,(k_1, k_2, k_3, r_1, r_2, r_3) = B(k_1, k_2, k_3, r_1, r_2, r_3) + B(k_2, k_3, k_1, r_2, r_3, r_1) + B(k_3, k_1, k_2, r_3, r_1, r_2)$$

AVERAGING PROBLEM IN COSMOLOGY

Describe the large-scale geometry and dynamics, averaging cosmological quantities on scales below roughly 100 Mpc

- Photons as probes of the matter distribution in the large-scale structure (LSS)
- Mathematical average prescription, covariant and gauge-invariant formalism
- Powerful tool to evaluate the effect of (local) inhomogeneities on (global) cosmological measurements

Average of a generic scalar observable S over a region of space-time (i.e. past light-cone):

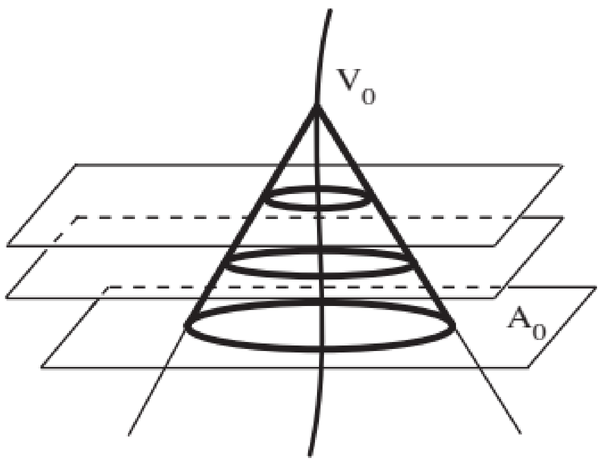
$$\langle S(x)\rangle \equiv \frac{\int d\mu\, S(x)}{\int d\mu} \equiv \frac{I(S,W_\Omega)}{I(1,W_\Omega)} \qquad \text{with} \qquad I(S,W_\Omega) = \int_{M_4} d^4x \sqrt{-g(x)}\; W_\Omega(x)\, S(x)$$

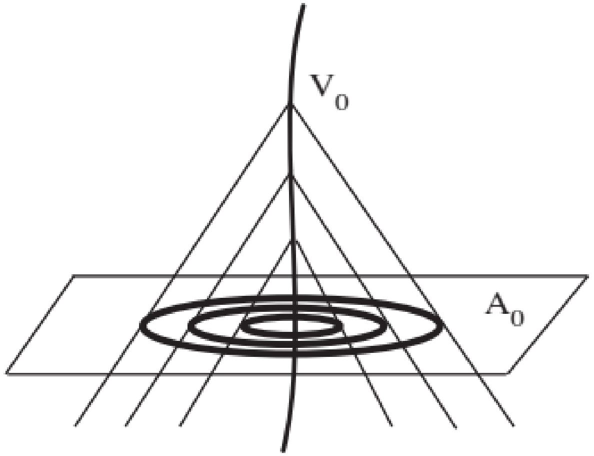
μ: measure weighting the averaging procedure

$W_\Omega(x)$: window function that selects only a region Ω of the space-time

BUCHERT (2000), Gen. Rel. Grav. 32, 105
GASPERINI, MAROZZI, & VENEZIANO (2009), JCAP 03, 011
GASPERINI et al. (2011), JCAP 07, 008
FANIZZA et al. (2020), JCAP 02, 017

AVERAGING PRESCRIPTIONS ON THE PAST LIGHT-CONE





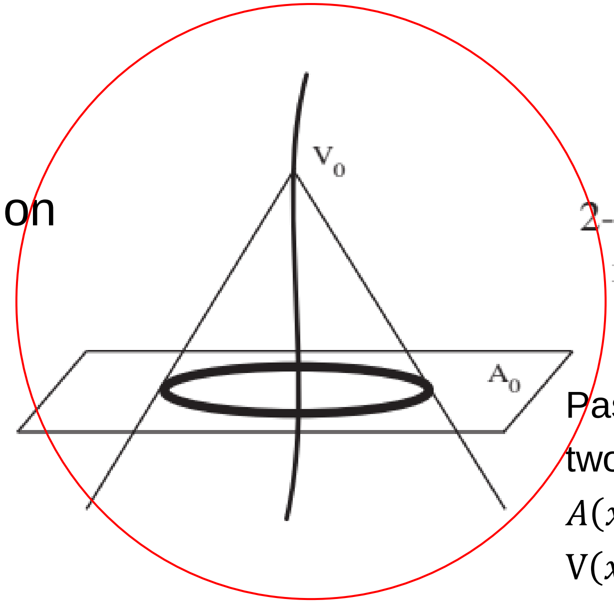
- Foliation of the space-time M_4
- Space-like hypersurfaces $\Sigma(A)$:
  scalar function $A(x) \ \rightarrow \ \partial_\mu A\, \partial^\mu A < 0$
- Normal vector of $\Sigma(A)$:

$$n_\mu = -\frac{\partial_\mu A}{\sqrt{-\partial_\nu A\, \partial^\nu A}} \text{ with } n_\mu n^\mu = -1$$

- Null hypersurfaces:
  scalar function $V(x) \ \rightarrow \ \partial_\mu V\, \partial^\mu V = 0$
- Average of a scalar quantity $S(x)$, i.e. $d_L(z)$, over the 2-D region embedded in the light-cone:

$$W_\Omega(x) = -n^\mu \nabla_\mu \Theta[A(x) - A_0]\, n^\nu \nabla_\nu\, \Theta[V_0 - V(x)]$$
$$= \left|\partial_\mu V(x) \partial^\mu A(x)\right| \delta_D[A(x) - A_0]\, \delta_D|V_0 - V(x)|$$

Θ: Heaviside step function; δ_D: Dirac delta distribution



Past light-cone bounded by two hypersurfaces:
$A(x) = A_0$ const. and
$V(x) = V_0$ const.

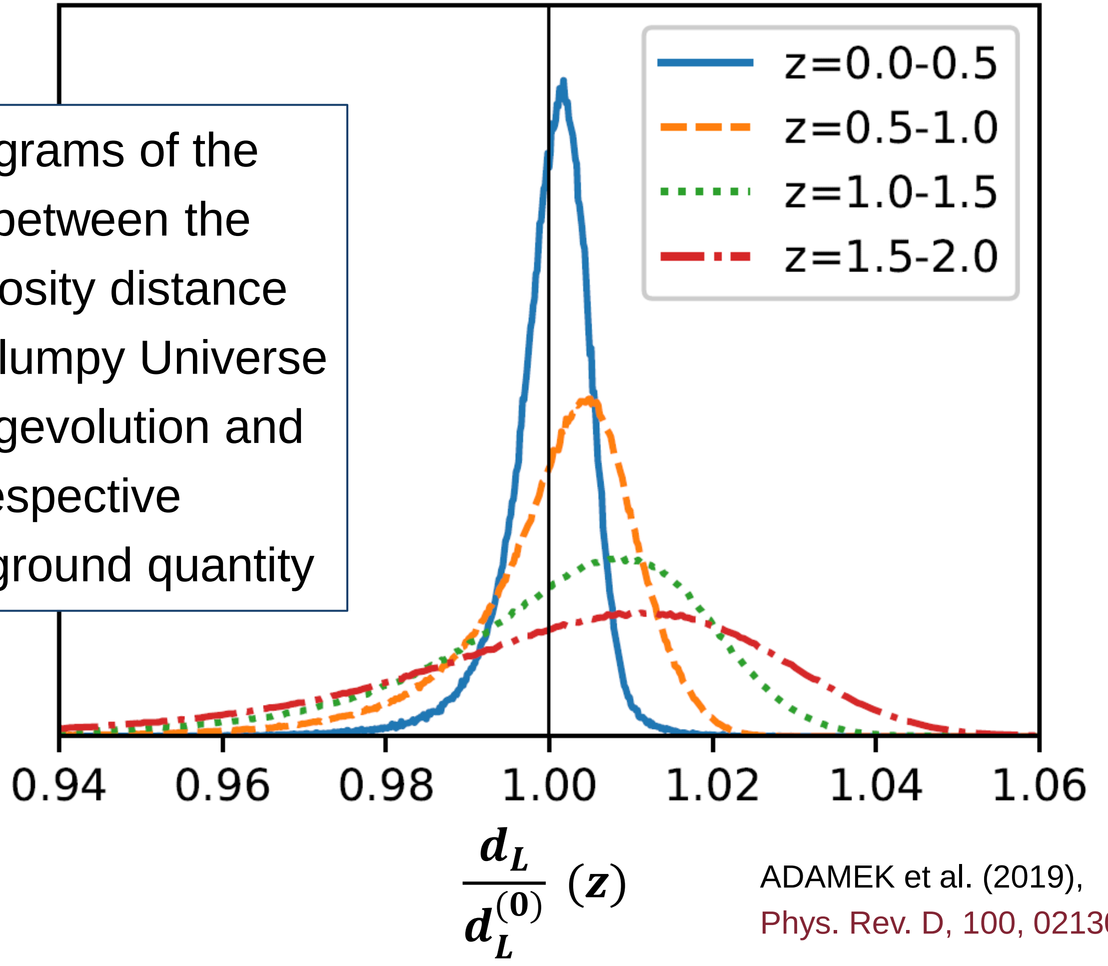
GASPERINI et al. (2011), JCAP 07, 008

Non-Gaussianities in the matter distribution → Non-Gaussianities in the Hubble diagram

<u>Results from numerical simulations</u>

- *gevolution* code: ray-tracing of light-like geodesic from local overdensities (halos)
- Evolution of structures
- Box size 2.4 Gpc/h, grid space 312.5 kpc/h
- 4 redshift bins
- Evidence of non-Gaussianities: non-null skewness in the distribution of $d_L(z)$

Histograms of the ratio between the luminosity distance in a clumpy Universe from gevolution and the respective background quantity

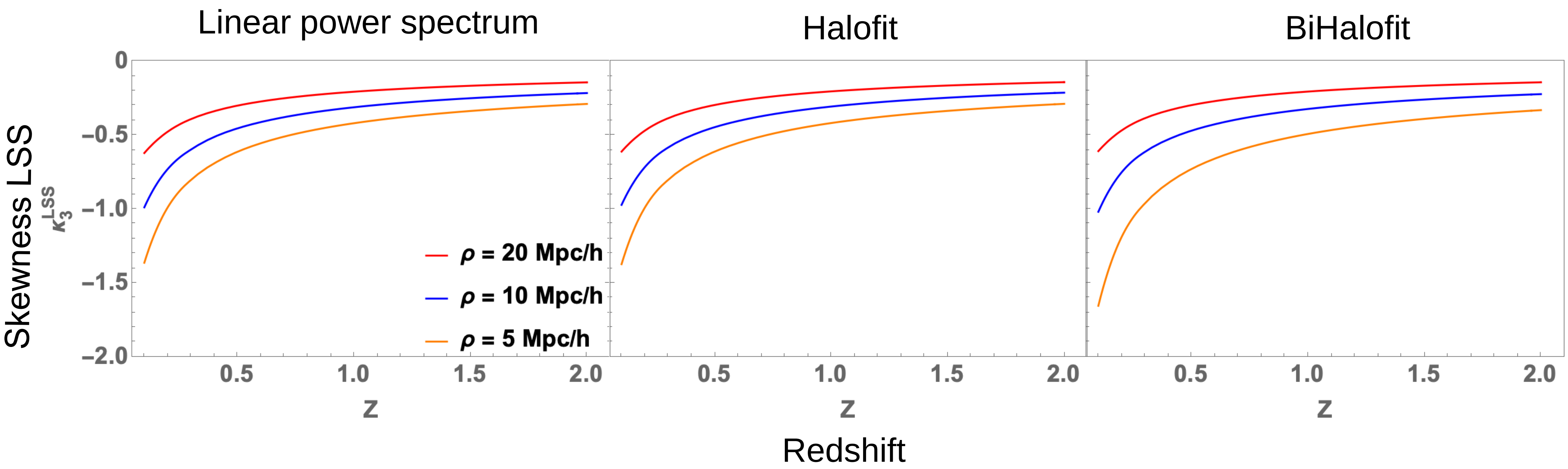


ADAMEK et al. (2016), Nature Phys. 12, 346 (2016), arXiv:1509.01699
https://github.com/gevolution-code/gevolution-1.0

ADAMEK et al. (2019), Phys. Rev. D, 100, 021301

SKEWNESS OF THE DISTANCE-REDSHIFT RELATION

Standardized third moment – skewness k_3



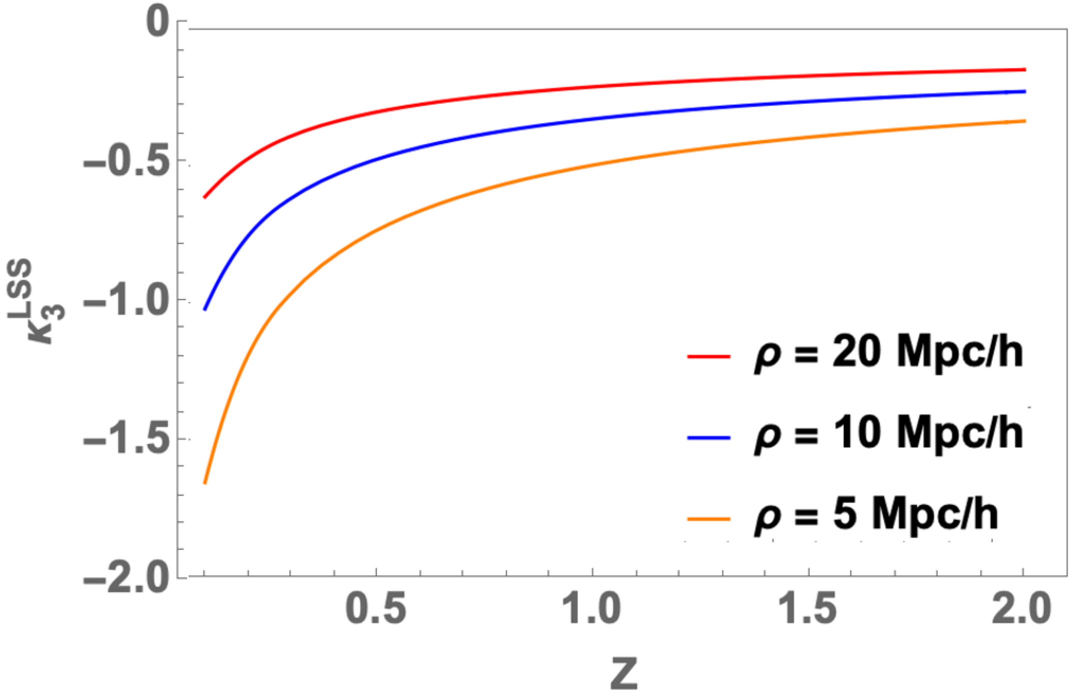
- The dominant term is related to the matter bispectrum driven by gravitational lensing. It contains information about non-linearities in the LSS and non-Gaussianities in the Hubble-Lemaître diagram

SCHIAVONE, DI DIO, & FANIZZA, arXiv:2307.13455, JCAP 02 (2024) 050

COMPARISON WITH NUMERICAL SIMULATIONS

Bin	Redshift Range	Skewness κ_3 from *gevolution*
1	0 - 0.5	-2.27
2	0.5 - 1	-1.44
3	1 - 1.5	-0.72
4	1.5 - 2	-0.44

Our results with BiHalofit



- Negative values of k_3 and same order of magnitude [differences $\sim O(1)$]

However:

- Finite redshift bins - VS - infinitesimal bin formalism
- Grid space of *gevolution* as unique smoothing scale - VS - Limit of our perturbative scheme with smoothing scales in the non-linear regime