Ring laser gyros: skew-Hermitian coupling from system-bath interaction

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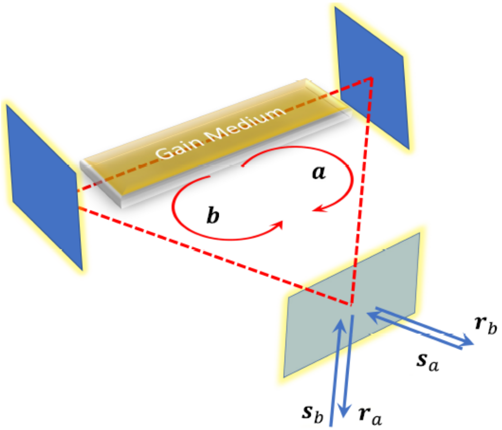
RLG as an open quantum system

Inspired by Mecozzi's article [1], we want to describe a RLG as an open quantum system via a quantum Langevin equation approach.



RLG as an open quantum system (OQS)

We want to build a model for a ring laser gyroscope (RLG) that includes (at least) the following properties:

- the optical cavity supports two counter-propagating modes of frequencies $\omega_0 \pm \frac{\Omega_0}{2}$; Ω_0 is the Sagnac frequency;
- EM modes interact with the environment (power leaks through the mirrors);
- back-scattering is taken into account;
- the modes interact with a gain medium represented by N two-level atoms with transition frequency Ω;
- the atoms are subject to stimulated emission and are driven into an inverted state by an appropriate pumping.

RLG as an OQS: features

We choose to work in the following setting:

- we adopt the Heisenberg picture approach;
- bath operators associated to different noise ports commute;
- we assume that the rotating-wave approximations and the Markovian assumptions are legit;
- the N atoms are identical (however, they can be distinguished by their position in the gain medium);
- the interaction between each atom and the incoming EM field does not depend on the polarization or direction of the incoming field (spherical symmetry) so the atom-field coupling constants depend only on the mode;

RLG as an OQS: features

We introduce the following (total) Hamiltonian:

$$H = H_S + H_B + H_{SB}, \tag{1}$$

where:

$$H_S = H_S^{\mathrm{EM}} + H_S^{\mathrm{Atoms}} + H_S^{\mathrm{EM+Atoms}} + H_S^{b.s.}, \tag{2}$$

$$H_B = H_B^{\mathrm{EM}} + H_B^{\mathrm{s.e.}} + H_B^{\mathrm{p.}}, \tag{3}$$

$$H_{SB} = H_{SB}^{\mathrm{EM}} + H_{SB}^{\mathrm{s.e.}} + H_{SB}^{\mathrm{p.}}. \tag{4}$$

Note that both the EM-modes and the atoms constitute the open subsystem of our-interest.

RLG as an OQS: the Hamiltonian

EM-terms:

$$H_S^{\mathrm{EM}} = \left(\omega_0 + \frac{\Omega_0}{2}\right) a^\dagger a + \left(\omega_0 - \frac{\Omega_0}{2}\right) b^\dagger b, \tag{5}$$

$$H_S^{b.s.} = ??? \tag{6}$$

$$H_B^{\mathrm{EM}} = \int_0^{+\infty} \mathrm{d}\omega\, \omega d_a^\dagger(\omega) d_a(\omega) + (a \leftrightarrow b), \tag{7}$$

$$H_{SB}^{\mathrm{EM}} \approx i \int_{-\infty}^{+\infty} \mathrm{d}\omega \kappa_a(\omega) \left(d_a^\dagger(\omega) a - a^\dagger d_a(\omega)\right) + (a \leftrightarrow b). \tag{8}$$

with $\kappa_a(\omega) \approx \sqrt{\frac{\gamma_a}{2\pi}}$, $\kappa_b(\omega) \approx \sqrt{\frac{\gamma_b}{2\pi}}$, $\gamma_a, \gamma_b > 0$.

RLG as an OQS: the Hamiltonian

Atomic terms:

$$H_S^{\text{Atoms}} = -\sum_{i=1}^{N} \frac{\Omega}{2} \sigma_{z,i}, \tag{9}$$

$$H_S^{\text{EM+Atoms}} = \sum_{i=1}^{N} \left\{ g_{a,i} \left(f_a^*(\vec{r}_i)\sigma_i a^\dagger + f_a(\vec{r}_i)\sigma_i^\dagger a \right) + (a \leftrightarrow b) \right\}, \tag{10}$$

$$H_B^{\text{s.e.}} = \sum_{i=1}^{N} \int_0^{+\infty} \mathrm{d}\omega\, \omega l_i^\dagger(\omega) l_i(\omega), \quad H_B^{\text{p.}} = \sum_{i=1}^{N} \int_0^{+\infty} \mathrm{d}\omega\, \omega h^\dagger(\omega) h(\omega), \tag{11}$$

$$H_{SB}^{\text{s.e.}} \approx i \int_{-\infty}^{+\infty} \mathrm{d}\omega \sum_{i=1}^{N} m_i(\omega) \left(l_i^\dagger(\omega)\sigma_i - \sigma_i^\dagger l_i(\omega) \right), \; m_i(\omega) \approx \sqrt{\frac{\Gamma_l}{\pi}} \tag{12}$$

$$H_{SB}^{\text{p.}} \approx i \int_{-\infty}^{+\infty} \mathrm{d}\omega \sum_{i=1}^{N} p_i(\omega) \left(h_i(\omega)\sigma_i - \sigma_i^\dagger h_i^\dagger(\omega) \right), \; p_i(\omega) \approx \sqrt{\frac{\Gamma_p}{\pi}} \tag{13}$$

RLG as an OQS: quantum Langevin equations

Neglecting the back-scattering, we arrive at the following QLEs:

$$\frac{\mathrm{d}}{\mathrm{d}t}a = -i\left(\omega_0 + \frac{\Omega_0}{2}\right)a - \frac{\gamma_a}{2}a - ig_a\sum_{i=1}^{N} f_a^*\left(\boldsymbol{r}_i\right)\sigma_i + \sqrt{\gamma_a}d_{in,a}, \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}b = -i\left(\omega_0 - \frac{\Omega_0}{2}\right)b - \frac{\gamma_b}{2}b - ig_b\sum_{i=1}^{N} f_b^*\left(\boldsymbol{r}_i\right)\sigma_i + \sqrt{\gamma_b}d_{in,b}, \tag{15}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_i = -i\Omega\sigma_i - \left(\Gamma_l + \Gamma_p\right)\sigma_i - i\left[g_a f_a\left(\boldsymbol{r}_i\right)a + g_b f_b\left(\boldsymbol{r}_i\right)b\right]\sigma_{z,i} + B_{in,i}^{-}, \tag{16}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sigma_{z,i} =& 2\left(\Gamma_l - \Gamma_p\right)\mathbb{I} - 2\left(\Gamma_l + \Gamma_p\right)\sigma_{z,i} + B_{in,i}^{z} \\ &+ 2i\left[g_a\left(f_a\left(\boldsymbol{r}_i\right)a\sigma_i^\dagger - f_a^*\left(\boldsymbol{r}_i\right)\sigma_i a^\dagger\right) + g_b\left(f_b\left(\boldsymbol{r}_i\right)b\sigma_i^\dagger - f_b^*\left(\boldsymbol{r}_i\right)\sigma_i b^\dagger\right)\right]\end{aligned} \tag{17}$$

Introducing the back-scattering

In order to describe the back-scattering from corner mirrors, we want to introduce another contribution to the Hamiltonian. We assume that it is given by a linear combination of the following operators of the following form:

$$a^\dagger b, \quad b^\dagger a \tag{18}$$

The ansatz is motivated by the fact that these operators describe the annihilation of a photon and the creation of another in the opposite direction. Moreover, they conserve the total number of photons in the cavity:

$$\left[a^\dagger b, a^\dagger a + b^\dagger b\right] = \left[b^\dagger a, a^\dagger a + b^\dagger b\right] = 0 \tag{19}$$

Introducing the back-scattering: hermitian Hamiltonian

The only Hermitian linear combination of the previous operators is given by:

$$H_S^{b.s.} = i\left(\kappa_m a^\dagger b - \kappa_m^* b^\dagger a\right), \tag{20}$$

where $\kappa_m = |\kappa_m| e^{i\varphi_m}$ is an arbitrary complex coupling parameter. Neglecting all other interaction terms and noises for simplicity, the back-scattering Hamiltonian modifies the QLEs for the mode operators:

$$\frac{\mathrm{d}}{\mathrm{d}t} a = -i\left(\omega_0 + \frac{\Omega_0}{2}\right) a + \kappa_m b, \tag{21}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} b = -i\left(\omega_0 - \frac{\Omega_0}{2}\right) b - \kappa_m^* a, \tag{22}$$

Introducing the back-scattering: hermitian Hamiltonian

Defining:

$$\alpha = \langle a \rangle = |\alpha| e^{i\varphi_a}, \quad \beta = \langle b \rangle = |\beta| e^{i\varphi_b}, \tag{23}$$

$$\Delta\varphi = \varphi_b - \varphi_a + \varphi_m, \tag{24}$$

we see that:

$$\frac{\mathrm{d}\Delta\varphi}{\mathrm{d}t} = \Omega_0 - |\kappa_m| \left(\frac{|\beta|}{|\alpha|} - \frac{|\alpha|}{|\beta|} \right) \sin \Delta\varphi \tag{25}$$

This equation is not in accord with the phenomenology!
In particular, if $|\alpha| = |\beta|$, it is impossible to recover the lock-in!

Introducing the back-scattering: skew-hermitian Hamiltonian

In order to recover the expected behaviour of the system, we "just" need to replace κ^* with $-\kappa^*$ in eq. (86), hence also in (6):

$$H_S^{b.s.} = i\left(\kappa_m a^\dagger b + \kappa_m^* b^\dagger a\right). \tag{26}$$

The associated evolution equation for the frequency difference is of the form expected from the (semi-)classical theory:

$$\frac{\mathrm{d}\Delta\varphi}{\mathrm{d}t} = \Omega_0 - |\kappa_m|\left(\frac{|\beta|}{|\alpha|} + \frac{|\alpha|}{|\beta|}\right)\sin\Delta\varphi \tag{27}$$

Note that the new Hamiltonian is skew-Hermitian!

$$H_S^{b.s.\,\dagger} = -H_S^{b.s.} \tag{28}$$

However this is not surprising if we look at classical equations...

Wilkinson's classical equations for a RLG

In Wilkinson's review [2] the classical equations of motion of a RLG are derived. The coupling of the modes is due to their interaction with the same "intracavity medium", so it is described in terms of the Fourier components of the susceptibility. For single-mode operation:

$$\frac{\mathrm{d}E_{+n}}{\mathrm{d}t} = -i\frac{\Omega_0}{2}E_{+n} + \frac{i}{2}\chi_0^+\omega_n E_{+n} + \frac{i}{2}\chi_{2n}^-\omega_n E_{-n}, \tag{29}$$

$$\frac{\mathrm{d}E_{-n}}{\mathrm{d}t} = +i\frac{\Omega_0}{2}E_{-n} + \frac{i}{2}\chi_0^-\omega_n E_{-n} + \frac{i}{2}\chi_{-2n}^+\omega_n E_{+n}, \tag{30}$$

In this approach the susceptibilities describe all the system components that interact with the modes, e.g. the active medium, the mirrors, ecc.

Wilkinson's classical equations for a RLG

The derivation of eqs. (29) and (30) require the following assumptions:

1. all the high-order Fourier expansion terms of the rotation parameter are neglected (i.e. the form of the RLG is not relevant);
2. slow rotation, hence all quadratic terms in the rotation parameter are neglected;
3. slowly evolving envelop for the electromagnetic field, so we neglect the second order time derivatives of the electromagnetic field (in the rotated frame) and all first order time derivative of the electromagnetic field multiplied to small expansion parameters;
4. single-mode operation, i.e. we assume that only two modes are relevant, in particular those associated to the same "unperturbed" frequency (e.g. the main resonant frequency of the cavity).

Wilkinson's classical equations for a RLG: couplings

In [2] the time evolution of the phase difference θ is given by:

$$\frac{d\theta}{dt} = \Omega + XT - |K_s| \cos(\theta - \mu_s) + X|K_h| \cos(\theta - \mu_h)$$

where X is the log difference of the mode amplitudes and T, K_s, K_h are constant coupling parameters. In particular:

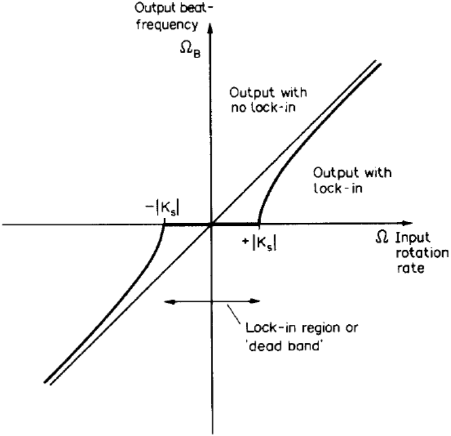
$$K_h = |K_h| \exp(i\mu_h) = \frac{\omega_n}{2}(\chi^{-}_{+2n} + \chi^{+*}_{-2n})$$

$$K_s = |K_s| \exp(i\mu_s) = \frac{\omega_n}{2}(\chi^{-}_{+2n} - \chi^{+*}_{-2n})$$

They are called hermitian and skew-Hermitian couplings, respectively.

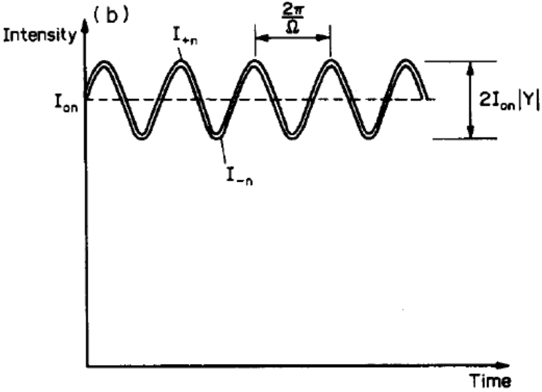
Wilkinson's classical equations for a RLG: $K_h = 0$

Note that for $X = 0$, $K_h = 0$, the evolution equation for θ is completely determined by K_s and is equivalent in form to eq. (27). The associated time-averaged beat-frequency has the expected behavior:



Wilkinson's classical equations for a RLG: $K_h = 0$

For $X = 0$, $K_h = 0$, the intensities of the modes are completely in phase, hence energy is not conserved:



The modes exchange energy with the medium at the same rate and this leading to their (at least partial) synchronization.

Physical meaning of the couplings

For a reciprocal medium, i.e. $\chi^{-}_{+2n} = \chi^{+}_{-2n} = \chi$, we see that:

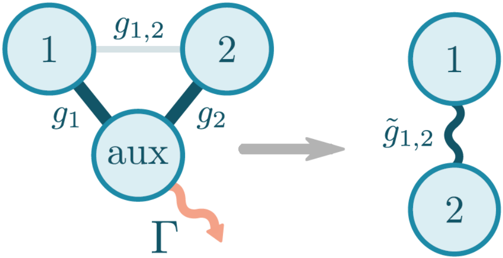
$$K_h \propto \mathrm{Re}\{\chi\} \quad \longleftrightarrow \quad \text{dispersion, e.g. pure dielectric material}$$
$$K_s \propto \mathrm{Im}\{\chi\} \quad \longleftrightarrow \quad \text{dissipation, e.g. pure conducting material}$$

Even if the hermitian coupling contributes to the beat-frequency in general, the lock-in is dictated by the skew-hermitian one.
So, dissipation plays an important role in the output response of a RLG!
Hence, if we want to retrieve the correct phenomenology of a RLG, we need to include in our model non-Hermitian (dissipative) phenomena that couple the modes.

Reduced non-Hermitian dynamics from Hermitian one

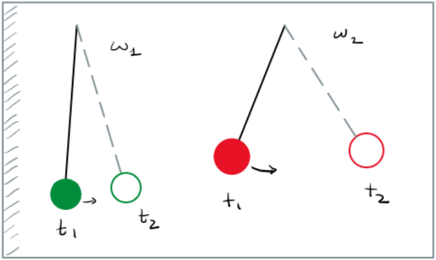
We want to understand how to induce a coupled non-Hermitian dynamics of the counter-propagating modes in a RLG starting from first principles in a quantum framework.

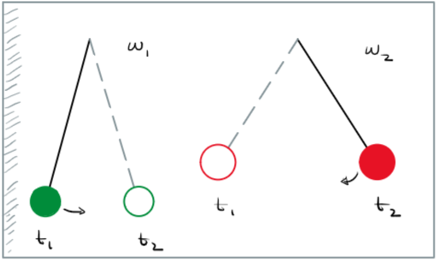
An hint comes from recent works [3, 4], where the link between non-hermitian Hamiltonians and phase-synchronization is investigated. In particular, the authors show that is possible to recover an effective non-hermitian Hamiltonian that couples two modes starting from an Hermitian Hamiltonian that couples them to an high-dissipative auxiliary mode.



Phase synchronization... of pendulum clocks?

It is instructive to think to a mechanical analogue.
Two pendulum clocks hanging from a wall synchronize because of the interaction with sound solitons propagating in the wall [5].





Skew-Hermitian coupling from system-bath interaction

Here we follow another procedure to retrieve a skew-Hermitian coupling between the modes. In our approach, the role of the auxiliary mode in [3] is taken by the environment. In particular the modes are coupled (*a la* Caldeira-Leggett) to the same two independent external baths. The system-bath interaction Hamiltonians are given by:

$$H_{SB,a} \approx \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}\omega \left[\sqrt{\gamma_a^{(a)}} \left(\alpha_a d_a^\dagger(\omega) a - \alpha_a^* a^\dagger d_a(\omega) \right) + \sqrt{\gamma_a^{(b)}} \left(\beta_a d_b^\dagger(\omega) a - \beta_a^* a^\dagger d_b(\omega) \right) \right],$$

$$H_{SB,b} \approx \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}\omega \left[\sqrt{\gamma_b^{(a)}} \left(\alpha_b d_a^\dagger(\omega) b - \alpha_b^* b^\dagger d_a(\omega) \right) + \sqrt{\gamma_b^{(b)}} \left(\beta_b d_b^\dagger(\omega) b - \beta_b^* b^\dagger d_b(\omega) \right) \right],$$

where $\alpha_i = e^{i\phi_i^{(a)}}$, $\beta_i = e^{i\phi_i^{(b)}}$. Moreover, we introduce a direct (Hermitian) interaction between the modes:

$$H_{S,ab} = i \left[\tilde{\kappa}_h a^\dagger b - \tilde{\kappa}_h^* b^\dagger a \right] \tag{31}$$

Skew-Hermitian coupling from system-bath interaction

Neglecting the interaction with the atoms, the QLEs are the following:

$$\frac{d}{dt}a(t) = -i\omega_a a(t) - \frac{1}{2}\left(\gamma_a^{(a)} + \gamma_a^{(b)}\right)a(t) + \left(\tilde{\kappa}_s + \tilde{\kappa}_h\right)b(t) + \sqrt{\gamma_a^{(a)}}e^{-i\phi_a^{(a)}}d_{in}^{(a)}(t) + \sqrt{\gamma_a^{(b)}}e^{-i\phi_a^{(b)}}d_{in}^{(b)}(t), \tag{32}$$

$$\frac{d}{dt}b(t) = -i\omega_b b(t) - \frac{1}{2}\left(\gamma_b^{(a)} + \gamma_b^{(b)}\right)b(t) + \left(\tilde{\kappa}_s^* - \tilde{\kappa}_h^*\right)a(t) + \sqrt{\gamma_b^{(a)}}e^{-i\phi_b^{(a)}}d_{in}^{(a)}(t) + \sqrt{\gamma_b^{(b)}}e^{-i\phi_b^{(b)}}d_{in}^{(b)}(t), \tag{33}$$

where:

$$\tilde{\kappa}_s = -\frac{1}{2}\left(\sqrt{\gamma_a^{(a)}\gamma_b^{(a)}}e^{i\left(\phi_b^{(a)}-\phi_a^{(a)}\right)} + \sqrt{\gamma_a^{(b)}\gamma_b^{(b)}}e^{i\left(\phi_b^{(b)}-\phi_a^{(b)}\right)}\right). \tag{34}$$

Classical equations from QLEs

Taking the expected values of eqs. (32) and (33) written in the rotating frame, we obtain the following equations:

$$\frac{d}{dt}\alpha'(t) = -i\frac{\Omega_0}{2}\alpha'(t) - \frac{1}{2}\left(\gamma_a^{(a)} + \gamma_a^{(b)}\right)\alpha'(t) + (\tilde{\kappa}_s + \tilde{\kappa}_h)\,\beta'(t), \quad (35)$$

$$\frac{d}{dt}\beta'(t) = +i\frac{\Omega_0}{2}\beta'(t) - \frac{1}{2}\left(\gamma_b^{(a)} + \gamma_b^{(b)}\right)\beta'(t) + (\tilde{\kappa}_s^* - \tilde{\kappa}_h^*)\,\alpha'(t). \quad (36)$$

A comparison with eqs. (29) and (30) shows that:

$$\tilde{\kappa}_s \propto K_s, \quad \tilde{\kappa}_h \propto K_h \quad (37)$$

Hence, we see that the direct Hermitian interaction between the modes completely determine the Hermitian coupling of the classical description of the system, while the system-bath interaction of our quantum model reproduce the wanted skew-Hermitian coupling.

Classical equations from QLEs: phase difference evolution

Concerning the instantaneous beat-frequency, eqs. (35) and (36) lead the following equation (at the first order in X):

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\Omega_0 - 2|\tilde{\kappa}_s| \sin(\theta(t) - \phi_s) + 2|\tilde{\kappa}_h| X(t) \sin(\theta(t) - \phi_s) \tag{38}$$

In the case $\tilde{\kappa}_h = 0$, defining $\theta' = \theta + \phi_s$, we obtain:

$$\frac{\mathrm{d}\theta'}{\mathrm{d}t} = +\Omega_0 - 2|\tilde{\kappa}_s| \sin\theta'(t) \tag{39}$$

that admits explicit analytical solutions.

Classical equations from QLEs: beat-frequency

In particular, if the Sagnac frequency is below the lock-in frequency:

$$\Omega_L = 2|\tilde{\kappa}_s| \tag{40}$$

the equation admits a stationary solution, i.e. perfect synchronization in phase is reached and the beat note frequency vanish.
When $|\Omega_0| > \Omega_L$, eq. (39) admits periodic solutions for $\theta'(t)$. The inverse of the period gives the time averaged beat-frequency. It's absolute value is:

$$\Omega_B = \sqrt{\Omega_0^2 - \Omega_L^2} \tag{41}$$

This is precisely the expected classical behavior we have shown before.

"Complete" QLEs for a RLG

In [2] the classical equations take into account the presence of the active medium from the start via the explicit expression of the susceptibility parameters, derived assuming a semi-classical interaction between the field and the atoms.
In our case we separately introduce the interaction in a full quantum framework via a Jaynes-Cummings model. The resulting QLEs are more involved, however their physical interpretation remain simple.
In the following we only display the QLE for the annihilation operator of a single EM mode interacting with an ensemble of N atoms evenly distributed between two different species.

"Complete" QLEs for a RLG

$$\frac{d}{dt}a(t) = -i\left(\omega_0 + \frac{\Omega_0}{2}\right)a(t) - \frac{1}{2}\left(\gamma_a^{(a)} + \gamma_a^{(b)}\right)a(t) + (\tilde{\kappa}_s + \tilde{\kappa}_h)\,b(t)$$
$$- ig_{aa}\sum_{i=1}^{N/2} f_a^*(\boldsymbol{r}_i)\,\sigma_i - ig_{ab}\sum_{i=N/2+1}^{N} f_a^*(\boldsymbol{r}_i)\,\sigma_i$$
$$+ \sqrt{\gamma_a^{(a)}}e^{-i\phi_a^{(a)}}d_{in}^{(a)}(t) + \sqrt{\gamma_a^{(b)}}e^{-i\phi_a^{(b)}}d_{in}^{(b)}(t)$$

Via the adiabatic elimination of the atomic variables, already in the case of identical "close by" atoms, it is possible to see that the interaction with the active medium induces further skew-Hermitian coupling terms between the modes and additional decay [1].

Conclusions

- We have seen that the the back-scattering terms in the QLE necessarily come from an effective non-Hermitian coupling between the counter-propagating modes;
- in particular, the skew-Hermitian coupling is responsible for the lock-in effect; this could been seen as an example of synchronization in non-hermitianly coupled systems;
- skew-hermitian coupling between two modes can be implemented via a standard coupling between the modes and the same two thermal baths;
- the expected values of the QLEs obtained in this way are in agreement with the standard classical equations (in particular those in [6]).

Next steps

- We need to study the classical equations associated to the QLEs of the complete model (comprising the interaction with the active medium);
- the noise contribution to the measured beat-frequency needs to be investigated;
- a possible approach is the one detailed in *Gardiner and Zoller (2020)* [7] (asymptotic expansion, stationary solutions, Schrodinger picture equivalent...).


WORK IN PROGRESS

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Back-up slides

Quantum Langevin equation: an overview

Inspired by Mecozzi's work, we want to propose a RLG model describing it as an open quantum system.
We will adopt a formalism analogous to that of *C. Gardiner, and P. Zoller (2020)* [7], starting from the introduction of the Hamiltonian of the system and then working out the associated quantum Langevin equations for the system operators.

In order to clarify the physical motivation behind the final equations, we will briefly review the quantum Langevin equation approach, then we will separately introduce the dynamics of various sub-systems constituting the RLG in our model (e.g. EM modes, the atoms ...)

Quantum Langevin equation: an overview

Open quantum system $\Longrightarrow$ $\mathcal{H} = \mathcal{H}_{\mathrm{S}} \otimes \mathcal{H}_B$

- total Hilbert space
- (open sub-)system Hilbert space
- environment Hilbert space

We adopt the Heisenberg picture. We are (primarily) interested in time evolution of system operators $a(t)$ when $\dim(\mathcal{H}_B) >> \dim(\mathcal{H}_S)$:

$$a(t) \in \mathscr{L}(\mathcal{H}) \ : \quad a(t_0) \equiv a \otimes \mathbb{I}_B \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} a(t) = ???$$

- $a \in \mathscr{L}(\mathcal{H}_S)$
- $\mathbb{I}_B \in \mathscr{L}(\mathcal{H}_B)$ identity operator on $\mathcal{H}_B$

Quantum Langevin equation: an overview

We start from the Liouville equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} a(t) = i[H, a] \tag{42}$$

where $H = H^\dagger \in \mathscr{L}(\mathcal{H})$ is the Hamiltonian.
We assume the following decomposition:

$$H = H_S + H_B + H_{SB}, \tag{43}$$

- H_S contains system operators only
- H_B contains bath operators only
- H_{SB} contains products of system and bath operators

Quantum Langevin equation: an overview

In particular, we describe the system-bath interaction through the Caldeira-Leggett model in the rotating-wave approximation (single port):

$$H_B = \int_0^{+\infty} \mathrm{d}\omega\, \omega d^\dagger(\omega) d(\omega), \tag{44}$$

$$H_{SB} = i \int_{-\infty}^{+\infty} \mathrm{d}\omega\, \kappa(\omega) \left[d^\dagger(\omega) c - c^\dagger d(\omega) \right] \tag{45}$$

$c(t)$: a case-dependent system operator

$\{d(\omega)\}$: a set of frequency-dependent bath operators

$$\left[d(\omega), d^\dagger(\omega') \right] = \delta(\omega - \omega'), \quad \left[d(\omega), d(\omega') \right] = 0 \tag{46}$$

Quantum Langevin equation: an overview

The Liouville equation (42) for bath operators is:

$$\frac{\mathrm{d}}{\mathrm{d}t} d(\omega, t) = i[H, d(\omega, t)] = -i\omega d(\omega, t) + \kappa(\omega) c(t) \tag{47}$$

whose formal solution is:

$$d(\omega, t) = d(\omega, t_0) e^{-i\omega(t-t_0)} + \kappa(\omega) \int_{t_0}^{t} dt' e^{-i\omega(t-t')} c(t') \tag{48}$$

to be substituted in the Liouville equation for a general system operator a:

$$\frac{\mathrm{d}}{\mathrm{d}t} a(t) = i[H, a(t)] \tag{49}$$

Quantum Langevin equation: an overview

Under the Markovian assumption:

$$\kappa(\omega) \approx \sqrt{\frac{\gamma}{2\pi}}, \quad \gamma > 0 \tag{50}$$

and defining the input noise operator:

$$d_{in}(t) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}\omega \, e^{-i\omega(t-t_0)} d(\omega, t_0) \tag{51}$$

we obtain the following quantum Langevin equation (QLE):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} a(t) =& i\,[H_S, a] + \left(\frac{\gamma}{2} c^\dagger(t) - \sqrt{\gamma} d_{in}^\dagger(t)\right)[a, c] \\ & - [a, c^\dagger]\left(\frac{\gamma}{2} c(t) - \sqrt{\gamma} d_{in}(t)\right) \end{aligned} \tag{52}$$

This equation is easily generalizable to the case of a system coupled to the environment through multiple independent noise ports.

QLE for a harmonic oscillator (1)

A single harmonic oscillator (e.g. an EM mode of a cavity) is described by the following Hamiltonian:

$$H_{S,HO} = \omega_0 a^\dagger a \tag{53}$$

where the creation and annihilation operators are defined by the following commutation relations:

$$[a, a^\dagger] = 1, \quad [a, a] = 0 \tag{54}$$

In particular, choosing $c = a$, we obtain the following QLE:

$$\frac{\mathrm{d}}{\mathrm{d}t} a(t) = -i\omega_0 a(t) - \frac{\gamma}{2} a(t) + \sqrt{\gamma} d_{in}(t) \tag{55}$$

"free" evolution term → $-i\omega_0 a(t)$

damping term → $\frac{\gamma}{2} a(t)$

noise term → $\sqrt{\gamma} d_{in}(t)$

QLE for a system coupled to an amplifier

If the environment behaves as an amplifier, i.e. it pumps energy into the system, we can describe it as an inverted heat bath [7]:

$$H_{B,inv} = -\int_0^{+\infty} \mathrm{d}\omega\, h(\omega)h^\dagger(\omega), \tag{56}$$

$$H_{SB,inv} = i\int_0^{+\infty} \mathrm{d}\omega\, \omega p(\omega) \left[h^\dagger(\omega,t)c(t) - c^\dagger(t)h(\omega,t)\right] \tag{57}$$

$$\left[h^\dagger(\omega), h(\omega')\right] = \delta(\omega - \omega') \tag{58}$$

Under Markov approximation:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}a(t) =& i\,[H_S, a] + \left(-\frac{\gamma_p}{2}c^\dagger(t) - \sqrt{\gamma_p}h^\dagger_{in}(t)\right)[a,c] \\ & - \left[a, c^\dagger\right]\left(-\frac{\gamma_p}{2}c(t) - \sqrt{\gamma_p}h_{in}(t)\right) \end{aligned} \tag{59}$$

Note that other amplifier models (e. g. phase conjugating amplifier) differ only for the noise terms.

QLE for a harmonic oscillator (2)

For a harmonic oscillator coupled to both a regular (dissipative) and an inverted (pumping) heat bath, choosing $c = a$ for both the noise ports, the QLE for the annihilation operator a are given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}a(t) = -i\omega_0 a(t) + \frac{\gamma_p - \gamma_l}{2}a(t) + n_{in}(t) \tag{60}$$

where:

$$n_{in}(t) = \sqrt{\gamma_l}d_{in}(t) + \sqrt{\gamma_p}h_{in}(t) \tag{61}$$

Equation (60) can be formally solved:

$$a(t) = a(t_0)e^{(-i\omega_0 + \frac{\gamma_p - \gamma_l}{2})(t-t_0)} + \int_{t_0}^{t} dt' e^{(-i\omega_0 + \frac{\gamma_p - \gamma_l}{2})(t-t')} n_{in}(t') \tag{62}$$

The mode is amplified or damped depending on the sign of $\gamma_p - \gamma_l$.

QLE for two cavity EM modes (1)

We consider two electromagnetic (EM) modes in a cavity, with associated annihilation operators a and b ($[a, b] = \left[a, b^\dagger\right] = 0$). We assume that they not interact with each other and that they are coupled to independent (bosonic) dissipative reservoirs ($c_a = a$, $c_b = b$). The QLEs are given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}a(t) = -i\omega_a a(t) - \frac{\gamma_a}{2}a(t) + \sqrt{\gamma_a}d_{in,a}(t), \tag{63}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}b(t) = -i\omega_b b(t) - \frac{\gamma_b}{2}b(t) + \sqrt{\gamma_b}d_{in,b}(t) \tag{64}$$

QLE for two cavity EM modes (2)

For a closed two-mode optical cavity, in general there is a non-zero probability that the corner mirrors scatter the light of a mode in the direction of propagation of the other, inducing an indirect interaction between the modes (back-scattering). The simplest way to implement such an interaction is through the following Hamiltonian:

$$H_{S,ab} = i \left[\kappa_m a^\dagger b - \kappa_m^* b^\dagger a \right] \tag{65}$$

that modifies the eqs. (63) and (64) in the following way:

$$\frac{\mathrm{d}}{\mathrm{d}t} a(t) = -i\omega_a a(t) - \frac{\gamma_a}{2} a(t) + \kappa_m b + \sqrt{\gamma_a} d_{in,a}(t), \tag{66}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} b(t) = -i\omega_b b(t) - \frac{\gamma_b}{2} b(t) - \kappa_m^* a + \sqrt{\gamma_b} d_{in,b}(t) \tag{67}$$

Note: $\left[H_{S,ab}, a^\dagger a + b^\dagger b\right] = 0$ (the total number of photons is conserved).

QLE for an atom (1)

A simple model of an atom is that of a two level system.
The corresponding system-Hamiltonian is given by:

$$H_{S,atom} = -\frac{\Omega}{2}\sigma_z \tag{68}$$

where:

$$\sigma_z(t_0) = |0\rangle\langle 0| - |1\rangle\langle 1| \tag{69}$$

We also define the lowering operator and its conjugate:

$$\sigma(t_0) = |0\rangle\langle 1|, \quad \sigma^\dagger(t_0) = |1\rangle\langle 0| \tag{70}$$

QLE for an atom (1)

The atomic spontaneous emission can be introduced through an appropriate bath-system interaction Hamiltonian, where we set $c = \sigma$.

The resulting QLE equations for σ and σ_z are given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma = -i\Omega\sigma \boxed{-\Gamma\sigma} + \sqrt{2\Gamma}\sigma_z l_{in}, \tag{71}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_z = 2\Gamma\mathbb{I} \boxed{-2\Gamma\sigma_z} - 2\sqrt{2\Gamma}\left(l_{in}^\dagger\sigma + \sigma^\dagger l_{in}\right) \tag{72}$$

where the damping terms have been evidenced.

Note that multiplicative noise terms appear in the equation and the input noise operator does not commute with system operators in general.

QLE for an atom (2)

A pumped atom can be modeled as a two-level system interacting with an inverted bath, choosing again $c = \sigma$. Including both the effect of the amplifier and of the spontaneous emission, we obtain the following QLEs:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma = -i\Omega\sigma - (\Gamma_l + \Gamma_p)\,\sigma + B_{in}^{-}, \tag{73}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_z = 2\,(\Gamma_l - \Gamma_p)\,\mathbb{I} - 2\,(\Gamma_l + \Gamma_p)\,\sigma_z + B_{in}^{z} \tag{74}$$

where $\Gamma_l > 0$ and $\Gamma_p > 0$ are the dissipation and amplification constants, and:

$$B_{in}^{-}(t) = \sqrt{2\Gamma_l}\sigma_z l_{in} + \sqrt{2\Gamma_p} h_{in}\sigma_z, \tag{75}$$

$$B_{in}^{z}(t) = -2\sqrt{2\Gamma_l}\left(l_{in}^{\dagger}\sigma + \sigma^{\dagger} l_{in}\right) - 2\sqrt{2\Gamma_p}\left(h_{in}\sigma^{\dagger} + \sigma h_{in}^{\dagger}\right). \tag{76}$$

Atom-EM mode interaction

To describe the interaction between an EM-mode (e.g. a) and a single atom we choose the following Hamiltonian (Jaynes-Cummings model):

$$H^{a}_{S,JC} = g_a \left[f_a(\boldsymbol{r})\, a\sigma^\dagger + f^*_a(\boldsymbol{r})\, \sigma a^\dagger \right] \tag{77}$$

where $g_a \in \mathbb{R}$ is a coupling constant and $f_a(\boldsymbol{r})$ is the normalized spatial mode profile of the mode a evaluated at the position of the atom $\boldsymbol{r}$ inside the optical cavity [8, 7]. The resulting QLEs are:

$$\frac{\mathrm{d}}{\mathrm{d}t} a(t) = -i\omega_a a(t) - \frac{\gamma_a}{2} a(t) - i g_a f^*_a(\boldsymbol{r})\, \sigma + \sqrt{\gamma_a} d_{in,a}(t), \tag{78}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \sigma = -i\Omega\sigma - (\Gamma_l + \Gamma_p)\, \sigma - i g_a f_a(\boldsymbol{r})\, a\sigma_z + B^-_{in}, \tag{79}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \sigma_z = 2(\Gamma_l - \Gamma_p)\, \mathbb{I} - 2(\Gamma_l + \Gamma_p)\, \sigma_z + 2 i g_a \left[f_a(\boldsymbol{r})\, a\sigma^\dagger - f^*_a(\boldsymbol{r})\, \sigma a^\dagger \right] + B^z_{in} \tag{80}$$

Frequency pulling and lock-in

...

A model for the RLG: two atomic species

In real He-Ne RLGs two isotopes of Neon are present in a 50:50 mixture, in order to stabilize both the modes in the cavity.
Accordingly, we can modify the previous equations letting the atomic parameters to be atom dependent. In particular:

$$\Omega \to \Omega_i = \begin{cases} \Omega_a & i = 1, ..., \frac{N}{2} \\ \Omega_b & i = \frac{N}{2} + 1, ..., \frac{N}{2} \end{cases} \tag{81}$$

and similar substitutions for the other parameters.

However, if all atoms interact with both the EM modes, the detuning frequencies cannot all vanish simultaneously. Moreover, extra time-exponentials appear and they cannot be absorbed by rotating-frame transformations.

A model for the RLG: quantum Langevin equations (1)

A form of the QLEs similar to the ones in Mecozzi's paper is achieved with the following assumptions:

- atomic resonance frequency is exactly given by the mean resonance frequency of the cavity, i.e. $\Delta = \omega_0 - \Omega = 0$ (the detuning frequency vanishes);
- we perform rotating-frame transformations on the operators a, b and σ substituting:

$$a(t) \to a(t)e^{-i\omega_0 t},\ b(t) \to b(t)e^{-i\omega_0 t}, \quad \sigma(t) \to \sigma(t)e^{-i\omega_0 t};$$

- we assume: $\gamma_a = \gamma_b = \gamma$, $g_a = g_b = g$;
- we define: $R_p = 2N(\Gamma_p - \Gamma_l)$, $\Gamma = \frac{1}{2\tau} = \Gamma_l + \Gamma_p$;
- (cont.)

RLG as an OQS: more assumptions...

- we define the collective variables:

$$\sigma_- = \frac{1}{N}\sum_{i=1}^{N}\sigma_i\,, \quad n = N\sigma_3 = -\sum_{i=1}^{N}\sigma_{z,i} \tag{82}$$

- we re-define the noise terms:

$$s_a(t) = d_{in,a}(t), \quad s_b(t) = d_{in,b}(t), \tag{83}$$

$$\sqrt{2N\Gamma}s_-(t) = \sum_{i=1}^{N} B^-_{in,i}(t), \quad \sqrt{2N\Gamma}s_n(t) = -\sum_{i=1}^{N} B^z_{in,i}(t); \tag{84}$$

- the spatial profiles of the two-modes are those of counter-propagating plane waves;
- the atoms are "not too far apart from each other", so we can factorize the terms depending the position of the atoms.

RLG as an OQS: quantum Langevin equations (2)

The resulting equations are the following:

$$\frac{\mathrm{d}}{\mathrm{d}t}a = -i\frac{\Omega_0}{2}a - \frac{\gamma}{2}a - ige^{-i\vec{k}_a\cdot\vec{r}}N\sigma_- + \sqrt{\gamma}s_a(t), \quad (85)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}b = +i\frac{\Omega_0}{2}b - \frac{\gamma}{2}b - ige^{+i\vec{k}_b\cdot\vec{r}}N\sigma_- + \sqrt{\gamma}s_b(t), \quad (86)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_- = -\Gamma\sigma_- + ig(e^{+i\vec{k}_a\cdot\vec{r}}a + e^{-i\vec{k}_b\cdot\vec{r}}b)\sigma_3 + \sqrt{\frac{2\Gamma}{N}}s_-(t), \quad (87)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}n = R_p - \frac{n}{\tau} + \sqrt{2\Gamma N}s_n(t) + i2gN\left[(e^{-i\vec{k}_a\cdot\vec{r}}a^\dagger + e^{+i\vec{k}_b\cdot\vec{r}}b^\dagger)\sigma_- - h.c.\right] \quad (88)$$

The equations are *similar* to those in Mecozzi's paper (there is an extra noise term in the last equation and different parameters).

Non-Hermitian Hamiltonians: when?

We want to justify the introduction of a non-hermitian Hamiltonian from first principles. We list here some considerations:

- in literature, approaches to open quantum systems can be found that make use of non-hermitian Hamiltonians, in particular for PT-symmetric systems, quantum scattering [6, 9]) ...;
- ... Moreover the standard literature on RLGs seems suggesting this approach [2];
- if we assume that both under time inversion and parity transformations a and b go one into the other, the back-scattering Hamiltonian is PT-symmetric for $\varphi_m = \pm\frac{\pi}{2}$; it is never T-symmetric;
- this Hamiltonian hide an interaction between mirrors and EM modes, that in principle is dissipative...;
- ... however we are implicitly supposing that a photon is either not affected or is scattered exactly into the opposite-travelling mode (total photon number is conserved, no-photons scattered in "unwanted" directions).

Phase synchronization in non-hermitian systems

A recent paper [4] suggests that there is a deep link between non-hermitian Hamiltonians and phase-synchronization, in general.

Note that both the lock-in and the frequency pulling (above threshold) are essentially phase-synchronization phenomenona induced from non-hermitian couplings [2]. So, could be interesting to study in more detail the correlation between the beams in a RLG from the point of view of non-Hermitian couplers.

In particular, it is interesting that it is possible to recover an effective non-hermitian Hamiltonian of the form we are interested in from a standard hermitian system.

Non-Hermitian Hamiltonians: how?

The usual QLEs for a set of N annihilation operators:

$$\frac{\mathrm{d}}{\mathrm{d}t} a_i(t) = i[H_S, a_i(t)] - \gamma_i a(t) + n_i^{(in)}(t) \tag{89}$$

where $H_S = H_S^\dagger$ and are equivalent to the following ones:

$$\frac{\mathrm{d}}{\mathrm{d}t} a_i(t) = i[H_{NH}, a_i(t)] + n_i^{(in)}(t) \tag{90}$$

where:

$$H_{NH} = H_S - i \sum_i \gamma_i a_i^\dagger a_i \tag{91}$$

i.e., we are introducing imaginary frequency components to take into account the dissipation.
Note that $H_{NH} \neq H_{NH}^\dagger$, however it has a physical meaning.

Non-Hermitian Hamiltonians: how?

The Hamiltonians H_{NH} we are interested in are of the following form:

$$H_{NH} = \Psi^\dagger h \Psi, \tag{92}$$

where:

$$\Psi = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_N \end{pmatrix}, \quad h = \begin{pmatrix} \omega_1 - i\gamma_1 & g_{12} & \dots \\ g_{21} & \omega_2 - i\gamma_2 & \dots \\ \dots & \dots & \dots \end{pmatrix}, \quad g_{ij}^* = g_{ji}. \tag{93}$$

The QLEs are given by:

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = -ih\Psi(t) - \beta_{in}(t) \tag{94}$$

where $\beta_{in}(t)$ is an appropriately defined noise term.

Non-Hermitian Hamiltonians: how?

It is possible to show [3] that every unitary operator $S = S^\dagger$ acting on the system Hilbert space $\mathcal{H}_S$ in the following way:

$$S\Psi S^\dagger = U\Psi, \quad U = U^\dagger \in M_N(\mathbb{C}) \tag{95}$$

define an evolution equation equivalent to the previous one. It is given by:

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = -i\tilde{h}\Psi(t) - U^\dagger\beta_{in}(t), \quad \tilde{h} = U^\dagger h U \tag{96}$$

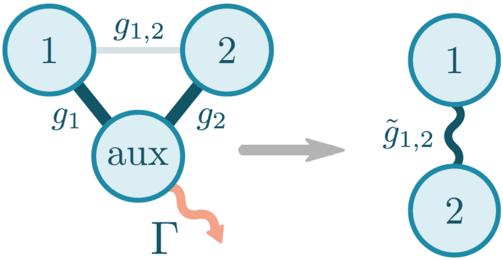
It is in the form of a QLE with the following system Hamiltonian:

$$\tilde{H}_{NH} = \Psi^\dagger \tilde{h}\Psi \tag{97}$$

Note that $\tilde{g}_{ij}^* \neq \tilde{g}_{ji}$, in general. Therefore, we can still make sense of some Hamiltonians with non-hermitian couplings.

Phase synchronization in non-hermitian system

The idea of [3, 4] is the following.
Consider an open system S made of two bosonic modes, namely 1 and 2. The modes are hermitianly coupled to each other and to the same auxiliary mode 0. If the auxiliary mode is high-dissipative, the subsystem S can be equivalently described as a non-hermitian system in which modes 1 and 2 are non-hermitianly coupled and possibly synchronize.



Non-Hermitian systems from hermitian ones

In [3] the author shows two alternative ways to formally reduce a three-mode hermitian system to a two-mode non-hermitian one:

- performing a Schrieffer-Wolff transformation that effectively decouple the auxiliary mode from the others;
- performing an adiabatic elimination of the auxiliary mode.

In both cases the underlying assumption is that the auxiliary mode is high-dissipative.

Schrieffer-Wolff transformation from three to two modes

Consider the following Hamiltonian:

$$H = \sum_{i=0}^{2} (\omega_i - i\gamma_i) a_i^\dagger a_i + \sum_{i<j} g_{ij} \left(a_i^\dagger a_j + a_j^\dagger a_i \right) \tag{98}$$

where $g_{ij} \in \mathbb{R}$. Assuming $g_{i0} >> g_{12}$ and $\gamma_0 >> \gamma_i, |\omega_i - \omega_0| \; \forall i$, it is possible to show that up to second order in g_{i0}, the Hamiltonian H is unitary equivalent to the following one:

$$\tilde{H} = (\tilde{\omega}_0 - i\tilde{\gamma}_0) a_0^\dagger a_0 + \sum_{i=1,2} (\tilde{\omega}_i - i\tilde{\gamma}_i) a_i^\dagger a_i + \tilde{g}_{12} \left(a_1^\dagger a_2 + a_2^\dagger a_1 \right) \tag{99}$$

where $\tilde{g}_{12}^* \neq \tilde{g}_{21}$, in general:

$$\tilde{g}_{12} = g_{12} + \frac{g_{01} g_{02}}{2} \sum_{i=1,2} \frac{1}{(\omega_i - \omega_0) - i(\gamma_i - \gamma_0)} \tag{100}$$

Schrieffer-Wolff transformation from three to two modes

In summary:

- a transformation can be found from a three-mode hermitian system to an effective non-hermitian two-mode system;
- the transformation form H to $\tilde{H}$ can be identified with a Schrieffer-Wolff unitary transformation up to some order in a perturbative expansion of the parameters;
- strong coupling between the auxiliary high-dissipating mode and the other ones is required.

However:

- a careful inspection of the equivalence of the two description is needed since the Schrieffer-Wolff transformation *is not exactly unitary*;
- moreover, the effect of the transformation on other interaction terms is needed (e.g. EM field- atoms);
- an analysis of the noise terms is needed.

Adiabatic elimination of the auxiliary mode

Consider again the following Hamiltonian:

$$H = \sum_{i=0}^{2} (\omega_i - i\gamma_i) a_i^\dagger a_i + \sum_{i<j} g_{ij} \left(a_i^\dagger a_j + a_j^\dagger a_i \right), \quad g_{ij} \in \mathbb{R}. \tag{101}$$

Assuming vanishing expectation values for the noise terms, the associated classical Langevin equations are given by:

$$\frac{\mathrm{d}\alpha_i}{\mathrm{d}t} = -\gamma_i \alpha_i - i g_i \lambda e^{-i\Delta_{i0} t}, \tag{102}$$

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = -\gamma_0 \lambda - i(g_1^* \alpha_1 e^{i\Delta_{10} t} + g_2^* \alpha_2 e^{i\Delta_{20} t}), \tag{103}$$

where $g_{0i} = g_i$, $\Delta_{i0} = \omega_0 - \omega_i$, $\langle a_i \rangle = \alpha_i e^{-i\omega_i t}$, $\langle a_0 \rangle = \lambda e^{-i\omega_0 t}$. In the following $i+1 = 2$ for $i = 1$ and $i+1 = 1$ for $i = 2$.

Adiabatic elimination of the auxiliary mode

Equation (103) can be formally solved. Assuming that $\gamma_0 >> \gamma_i$, we obtain:

$$\lambda(t) = -\frac{ig_1^*}{i\Delta_{10} + \gamma_0}\alpha_1(t) - \frac{ig_2^*}{i\Delta_{20} + \gamma_0}\alpha_2(t) \tag{104}$$

and substituting in eq. (102) we get:

$$\frac{\mathrm{d}\alpha_i}{\mathrm{d}t} = -\left(\gamma_i + \frac{|g_i|^2}{\gamma_0 + i\Delta_{i0}}\right)\alpha_i(t) - \frac{g_i g_{i+1}^*}{\gamma_0 + i\Delta_{(i+1)0}}\alpha_{i+1}(t) \tag{105}$$

The above equations obtained as the classical equivalent of the QLEs associated to an effective two-mode Hamiltonian $\tilde{H}$ with new parameters.

Adiabatic elimination of the auxiliary mode

In particular, if $\gamma_0 >> \Delta_{i0}$:

$$\tilde{g}_{12} = -i\frac{g_1 g_2^*}{\gamma_0} = -\tilde{g}_{21}^* \tag{106}$$

i.e., we obtain an effective skew-Hermitian coupling between the modes.

It is interesting to note that the coupling constants found via the Schrieffer-Wolff transformation eq. (100) and adiabatic elimination eq. (106) coincide in the limit $\gamma_0 >> \gamma_i$ and $g_1, g_2 \in \mathbb{R}$.

Adiabatic elimination of the auxiliary mode

In summary:

- we have seen that adiabatic elimination of a high-dissipative auxiliary mode lead to an effective non-hermitian interaction between the modes in the remaining subsystem;
- additional assumptions lead to the same type of skew-Hermitian of the Schrieffer-Wolff transformation;
- this technique is standard and is the same used in [1] to derive the coupling between the modes mediated by the gain medium;

However:

- in the previous analysis the role of the noise terms is completely neglected (but it's easy to restore it, in principle).