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On-shell renormalization of effective field theories

Luigi C. Bresciani([1])([2])

([1]) *Dipartimento di Fisica e Astronomia "G. Galilei", Università di Padova - Via Marzolo 8, 35131, Padova, Italy*
([2]) *INFN, Sezione di Padova - Via Marzolo 8, 35131, Padova, Italy*

Summary. — We apply on-shell and unitarity-based methods to compute renormalization group coefficients in effective field theories. We extend this framework to include leading mass effects via the Higgs low-energy theorem and to capture the most general operator-mixing contributions. We demonstrate that these techniques significantly reduce computational complexity compared to standard diagrammatic approaches. We evaluate phase-space cut integrals using different integration methods, showing that the double-cut integration via Stokes' theorem is technically the most efficient. Our results provide a powerful tool for interpreting experimental measurements of low-energy observables, such as flavor-violating processes or electric and magnetic dipole moments, as induced by new physics emerging above the electroweak scale.

1. – Introduction

The Standard Model (SM) of particle physics has passed several experimental tests in all its sectors. The lack of heavy new physics (NP) at the LHC has firmly established the SM as a very successful theory describing the fundamental interactions of nature up to the TeV scale. However, it is a common belief that the SM has to be regarded as the low-energy description of a more fundamental theory emerging at a large, yet unknown, energy scale Λ. New interactions can be then described by an effective field theory (EFT) containing non-renormalizable operators that are invariant under the SM gauge group. EFTs provide a very powerful and model-independent approach to NP which does not rely on the details of the underlying (unknown) high-energy theory but just on its symmetries.

Predictions for physical processes are obtained by evaluating matrix elements of the EFT Lagrangian at energy scales accessible by collider experiments. Therefore, the high-scale Lagrangian needs to be evolved from the scale Λ down to the experimental scale $E \ll \Lambda$. Such a program can be carried out by computing the anomalous dimension matrix of the higher-dimensional operators, which is a crucial ingredient for interpreting experimental results.

Recently, anomalous dimensions have been calculated using on-shell and unitarity-based techniques for scattering amplitudes. In ref. [1] it was remarkably observed that anomalous dimensions can be directly related to unitarity cuts. Specifically, the discontinuities of form factors of EFT operators can be calculated via phase-space integrals and are related to the corresponding anomalous dimensions. One of the most intriguing features of this approach is to make manifest hidden structures with the appearance of non-trivial zeros in the anomalous dimension matrix [2, 3, 4, 5].

So far, the method proposed in ref. [1] has been applied to derive the anomalous dimensions of non-renormalizable massless theories including only mixing effects among operators of the same dimension. However, the non-trivial inclusion of mixings among operators with different dimensions and leading mass effects, which play a key role in many popular EFT extensions of the SM, has not yet been discussed in this framework. The main goal of ref. [6] is to fill this gap. In particular, we derive a master formula for the most general operator-mixing contributions and present a method to include leading mass effects, still working in the massless limit, by exploiting the Higgs low-energy theorem [7, 8].

2. – The on-shell method

The on-shell method for computing anomalous dimensions is based on the non-perturbative relation [1]

$$e^{-i\pi D} F_i^* = S F_i^* \tag{1}$$

that involves the S-matrix $S = \mathbb{1} + i\mathcal{M}$, the dilatation operator $D = \sum_k p_k \cdot \partial/\partial p_k$, and form factors associated with local and gauge-invariant operators $\mathcal{O}_i$ of the Lagrangian $\mathcal{L}_{\mathrm{EFT}} = \sum_i c_i \mathcal{O}_i / \Lambda^{[\mathcal{O}_i]-4}$:

$$F_i(\vec{n}; q) = \frac{1}{\Lambda^{[\mathcal{O}_i]-4}} \langle \vec{n} | \mathcal{O}_i(q) | 0 \rangle \,, \tag{2}$$

where $\langle \vec{n}|$ is an outgoing on-shell state and q is the off-shell momentum injected by the operator. Eq. (1) is a consequence of unitarity, CPT theorem, and analyticity of form factors [9]. Additionally, form factors, in dimensional regularization, satisfy the Callan-Symanzik equation

$$\left(\delta_{ij} \frac{\partial}{\partial \ln \mu} + \frac{\partial \beta_i}{\partial c_j} - \delta_{ij} \gamma_{i,\mathrm{IR}} + \delta_{ij} \beta_g \frac{\partial}{\partial g} \right) F_i = 0 \,, \tag{3}$$

where g collectively denotes the couplings of the renormalizable Lagrangian, $\gamma_{i,\mathrm{IR}}$ is the infrared anomalous dimension, which results from soft and collinear particle emissions, and $\beta_i = dc_i/d\ln\mu$ is the renormalization group equation (RGE) for the Wilson coefficient c_i. Since D reduces to $-\partial/\partial \ln \mu$ in dimensional regularization and in the massless limit, eqs. (1) and (3) can be connected, allowing to relate the anomalous dimensions to the S-matrix phase when applied to a form factor. In particular, at one-loop order, one finds

$$\left(\frac{\partial \beta_i^{(1)}}{\partial c_j} - \delta_{ij} \gamma_{i,\mathrm{IR}}^{(1)} + \delta_{ij} \beta_g^{(1)} \frac{\partial}{\partial g} \right) F_i^{(0)} = -\frac{1}{\pi} (\mathcal{M} F_j)^{(1)} \,, \tag{4}$$

where the right-hand side corresponds to a sum over all one-loop two-particle unitarity cuts

$$(\mathcal{M}F_j)^{(1)}(1,\ldots,n) = \sum_{k=2}^{n} \sum_{\{x^{h_1},y^{h_2}\}} \int d\text{LIPS}_2 F_j^{(0)}(x^{h_1},y^{h_2},\ldots,n)\mathcal{M}^{(0)}(1,\ldots,k;y^{h_2},x^{h_1}) \tag{5}$$
$$+\,\text{permutations}\,,$$

where $\mathcal{M}(\vec{n};\vec{m}) = \langle\vec{n}|\mathcal{M}|\vec{m}\rangle$ and $d\text{LIPS}_2$ is the two-particle Lorentz-invariant phase-space measure. The corresponding cut integral can be evaluated by employing different parameterizations: *e.g.*, via angular integration [10] or via Stokes' integration method [11]. The latter has proven to be technically more efficient as it makes manifest the rational terms, which are the only ones that can contribute to the ultraviolet anomalous dimensions; see, *e.g.*, refs. [12, 13].

2˙1. *General operator mixing*. – Near the Gaussian fixed point (denoted by "$*$"), where $c_i = 0$ for every i, each β_i can be Taylor expanded as

$$\beta_i = \sum_{n>0} \frac{1}{n!}\gamma_{i\leftarrow j_1,\ldots,j_n} c_{j_1}\ldots c_{j_n}\,, \qquad \text{where} \qquad \gamma_{i\leftarrow j_1,\ldots,j_n} = \left.\frac{\partial^n\beta_i}{\partial c_{j_1}\ldots\partial c_{j_n}}\right|_* \tag{6}$$

is the most general anomalous dimension tensor. At one-loop order, one then finds the following master formulae for the most relevant single- and double-operator insertions [6]:

$$\left(\gamma^{(1)}_{i\leftarrow j} - \delta_{ij}\gamma^{(1)}_{i,\text{IR}}\right) F_i|^{(0)}_* = -\frac{1}{\pi}(\mathcal{M}F_j)|^{(1)}_*\,, \quad \gamma^{(1)}_{i\leftarrow j,k}F_i|^{(0)}_* = -\frac{1}{\pi}\left.\frac{\partial}{\partial c_k}\right|_* (\mathcal{M}F_j)^{(1)}\,. \tag{7}$$

The extension for multiple operator insertions is straightforward.

2˙2. *Leading mass effects*. – As a natural consequence of operator mixing, chirality-violating and -preserving operators do generally mix under the renormalization flow. In case of EFTs defined below the electroweak scale, the required chirality flip proceeds through a fermion mass insertion. However, such a mass dependence cannot be directly implemented in the massless method of ref. [1].

The key observation to circumvent this issue is that the fermionic Higgs interactions in the SM can be written in the form

$$\mathcal{L}_h^{\text{int}} = -\left(1+\frac{h}{v}\right)\sum_f m_f\bar{f}f\,. \tag{8}$$

In the limit where the Higgs field h has a vanishing momentum, $p_h \to 0$, h becomes a constant field and its effect is equivalent to redefining all mass parameters as $m_f \to m_f(1+h/v)$. This implies the Higgs low-energy theorem [7, 8]

$$\lim_{\{p_h\}\to 0}\mathcal{M}(A\to B+Nh) = \sum_f \frac{m_f^N}{v^N}\frac{\partial^N}{\partial m_f^N}\mathcal{M}(A\to B) \tag{9}$$

relating the amplitudes of two processes differing by N insertions of zero momentum Higgs bosons. In practice, whenever an amplitude requires N fermion mass insertions not to vanish, we consider an equivalent amplitude entailing N extra massless Higgs fields; see fig. 1.

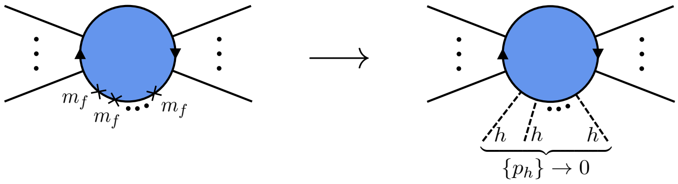

Fig. 1. – Diagrammatic representation of the massification procedure via Higgs low-energy theorem. Taken from ref. [6].

The number N can be determined as follows. By dimensional analysis we argue that whenever the anomalous dimension $\gamma_{i\leftarrow j_1,\dots,j_n}$ vanishes in the limit of massless fermions, possible mass effects must be of order $(m_f/\Lambda)^N$, where

$$N = 4 - [\mathcal{O}_i] + \sum_{k=1}^{n}([\mathcal{O}_{j_k}] - 4) \tag{10}$$

is the superficial degree of divergence associated with the loop amplitude under consideration [6]. The number of needed Higgs insertions coincides with the superficial degree of divergence because scaleless integrals vanish in dimensional regularization. For $N < 0$, $\gamma_{i\leftarrow j_1,\dots,j_n}$ is trivially zero. For $N \geq 0$, the anomalous dimension is obtained by renormalizing the operator $\mathcal{O}_i^{Nh} = (h/v)^N \mathcal{O}_i/N!$ instead of $\mathcal{O}_i$ [6]. Remarkably, the approximation of setting the Higgs mass to zero is justified since we are interested in the evaluation of anomalous dimensions that are related to the ultraviolet properties of a theory.

3. – Phenomenological applications

In this section, we review some phenomenological applications of the above techniques.

3˙1. *Effective field theory for axion-like particles.* – A first application concerns the calculation of one-loop anomalous dimensions in a general EFT for axion-like particles (ALPs), with both CP-odd and CP-even interactions, reproducing and extending previous results [12]. In this scenario, we extensively applied the above techniques to compute the RGEs of the Wilson coefficients of the dimension-five ALP-SM effective operators and to evaluate the anomalous dimensions of the SM effective operators, both above and below the electroweak scale, induced by double insertions of ALP-SM operators. A close comparison between the standard Feynman diagrammatic approach and the on-shell method enabled us to explicitly verify the reduction of complexity in the latter case, along with a more direct and elegant way to establish a connection between anomalous dimensions of operators that are dual under the CP symmetry [12].

To illustrate this with an example, we can consider the ALP-SM interactions [14, 15, 16]

$$\mathcal{L}_\phi \supset \frac{\tilde{\mathcal{C}}_\gamma}{\Lambda}\,\phi\, F_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{\mathcal{C}_\gamma}{\Lambda}\,\phi\, F_{\mu\nu}F^{\mu\nu} + \mathcal{Y}_P^{ij}\,\phi\,\bar{f}_i i\gamma_5 f_j + \mathcal{Y}_S^{ij}\,\phi\,\bar{f}_i f_j\,, \tag{11}$$

where ϕ is the ALP field, $\Lambda \gg v \approx 246$ GeV is the EFT cutoff scale, $f \in \{e, u, d\}$, and $\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}/2$ (with $\epsilon^{0123} = 1$). The anomalous dimension for the operator $\mathcal{O}_S^{ij} = \phi \bar{f}_i f_j$ induced by $\mathcal{O}_\gamma = \phi F_{\mu\nu} F^{\mu\nu}$ needs a fermionic mass insertion due to a chirality mismatch. Following the procedure outlined in sect. **2**˙2, the situation can be handled by renormalizing the operator $\mathcal{O}_S^{h,ij} = (h/v)\phi \bar{f}_i f_j$ instead of $\mathcal{O}_S^{ij}$. The master formula reads as

$$\gamma_{S\leftarrow\gamma}^{ij(1)} F_S^{h,ij}|_*^{(0)} = -\frac{1}{\pi} (\mathcal{M} F_\gamma)|_*^{(1)} \tag{12}$$

and is diagrammatically represented in fig. 2.

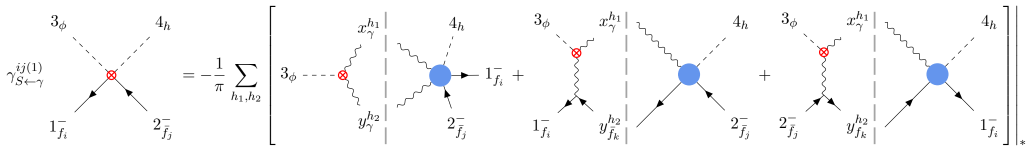

Fig. 2. – Unitarity cuts contributing to the master formula in eq. (12). Red and blue blobs refer to form factors and amplitudes, respectively. Taken from ref. [6].

Using [6, 12]

(13)
$$F_S^h|_*^{(0)}(1_{f_i}^-, 2_{\bar{f}_j}^-, 3_\phi, 4_h) = \frac{\langle 12 \rangle}{v} \quad \text{and} \quad (\mathcal{M} F_\gamma)|_*^{(1)}(1_{f_i}^-, 2_{\bar{f}_j}^-, 3_\phi, 4_h) = -\frac{3 m_i}{2\pi \Lambda v} \delta^{ij} e^2 Q_f^2 \langle 12 \rangle$$

one can then find the result

$$\gamma_{S\leftarrow\gamma}^{ij(1)} = \frac{3}{2\pi^2} \frac{m_i}{\Lambda} \delta^{ij} e^2 Q_f^2 \,. \tag{14}$$

Finally, the anomalous dimension $\gamma_{P\leftarrow\tilde{\gamma}}^{ij(1)}$ for the corresponding CP-dual operators $\mathcal{O}_P^{ij} = \phi \bar{f}_i i\gamma_5 f_j$ and $\mathcal{O}_{\tilde{\gamma}} = \phi F_{\mu\nu} \tilde{F}^{\mu\nu}$ follows from $\gamma_{S\leftarrow\gamma}^{ij(1)}$ via CP symmetry arguments [12]:

$$\gamma_{P\leftarrow\tilde{\gamma}}^{ij(1)} = -\gamma_{S\leftarrow\gamma}^{ij(1)} = -\frac{3}{2\pi^2} \frac{m_i}{\Lambda} \delta^{ij} e^2 Q_f^2 \,. \tag{15}$$

3˙2. *General effective gauge theory*. – In ref. [13], instead of focusing on a particular EFT, we adopted a general approach. We first classified the physical operators of the most general bosonic effective gauge theory up to dimension six, involving an arbitrary (possibly non-compact) gauge group and any number of scalar fields in arbitrary representations. Based on this classification, we then computed the complete one-loop anomalous dimensions by employing the above techniques (see also refs. [17, 18] for similar results achieved via diagrammatic and functional approaches). These results can be used to derive the one-loop RGEs for specific models just by performing group-theoretic manipulations, as we explicitly demonstrated by reproducing the known anomalous dimensions for the SM EFT [19, 20, 21] and other theories.

4. – Conclusions

We have generalized the application of on-shell and unitarity-based methods for evaluating renormalization group coefficients, to account for the most general operator-mixing contributions and leading mass effects [6], which are fundamental to the renormalization of several effective field theories. We first validated our approach by reproducing established results in well-studied effective field theories. We then applied it to derive new anomalous dimensions for axion-like particle effective field theories [12] and for a general effective gauge theory with arbitrary bosonic degrees of freedom [13].

Our results can be applied to a number of new physics scenarios, defined above the TeV scale, in order to analyze their impact on low-energy observables (such as flavor violating processes, electric and magnetic dipole moments, etc.) occurring at or below the GeV scale. Such a large separation of scales requires the inclusion of running effects at two-loop order to obtain sensible predictions. While this is a very challenging task when approached with standard techniques, on-shell and unitarity-based methods offer a simpler, more efficient, and elegant way to reach this goal.

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