Jet Bundle Geometry of Scalar Field Theories

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SMEFT vs. HEFT

- In the standard model effective field theory (SMEFT) the Higgs boson is embedded in an SU(2) doublet
- In the Higgs effective field theory (HEFT) the Higgs boson is introduced as a singlet
- In terms of expansions we typically say:
  - SMEFT: Expansion around the EW-symmetric point
  - HEFT: Expansion around the EW-vacuum

R. Alonso, E. Jenkins, A. Manohar., arXiv:1605.03602

SMEFT vs. HEFT: Motivation for Geometry

- A smooth map from SMEFT to HEFT can always be found.

- Any map from HEFT to SMEFT will have singularities

R. Gomez-Ambrosio, et al., arXiv:2204.01763

- The singularities are not necessarily physical.

R. Alonso, E. Jenkins, A. Manohar., arXiv:1511.00724

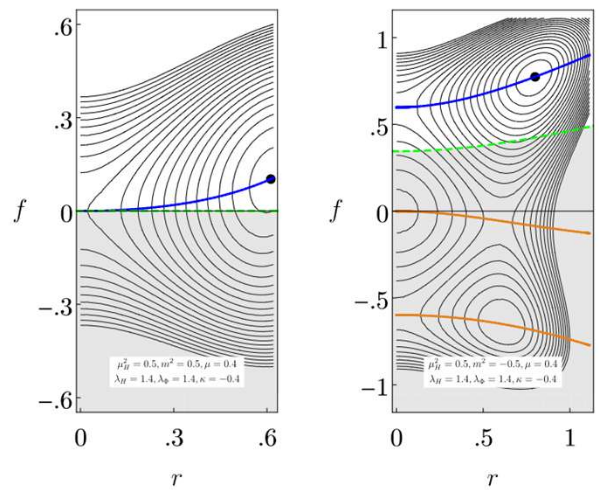
- Geometry allows us to reveal the nature of the singularities in a systematic manner

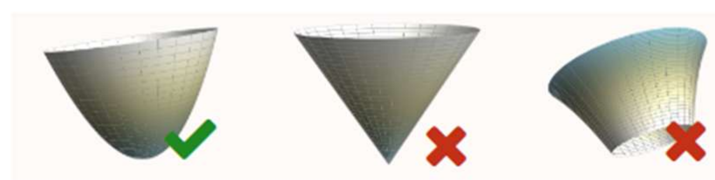
T. Cohen, et al, arXiv:2008.08597

SMEFT vs HEFT: Physical Singularities

- In the case of physical singularities, the theory cannot be matched onto SMEFT

- Particles that induce additional electroweak symmetry breaking as well as particles that obtain more than half their mass from electroweak symmetry breaking are examples of such theories





T.Cohen, et al., arXiv:2108.03240

Geometric formalism of EFTs

- Treat fields as if they are coordinates on a manifold M equipped with a Riemannian metric g

- Two derivative term in the Lagrangian comes from the metric

$$L = \frac{1}{2} g_{ij} \partial_\mu \phi^i \partial^\mu \phi^j - V(\phi) + O(\partial^4)$$

- Physical amplitudes related to tensor on the manifold M

$$\overline{V}_{;(\alpha_1\alpha_2\alpha_3\alpha_4)} + \frac{2}{3}\left(s_{12}\overline{R}_{\alpha_1(\alpha_3\alpha_4)\alpha_2} + s_{13}\overline{R}_{\alpha_1(\alpha_2\alpha_4)\alpha_3} + s_{14}\overline{R}_{\alpha_1(\alpha_2\alpha_3)\alpha_4}\right)$$

Two-to-two scattering. T.Cohen, et al., arXiv:2108.03240

Standard Geometric Formalism: Mathematical Details

Fields as coordinates on a manifold M

Introduce a (pseudo-)Riemannian metric on M

$$u^i \in M$$

$$g = g_{ij}(u) du^i \otimes du^j : TM \times TM \to \mathbb{R}$$

Pullback to space-time along a map

Coordinates pullback to fields

One-forms pullback to derivatives

Two derivative term provided by geometry

$$\phi : \Sigma \to M$$

$$(\phi^* u^i)(x) = u^i(\phi(x)) = \phi^i(x)$$

$$(\phi^* du^i)(x) = d(u^i(\phi(x)) = d\phi^i(x) = \partial_\mu \phi^i(x) dx^\mu$$

$$\phi^* g = g_{ij}(\phi(x)) \partial_\mu \phi^i \partial_\nu \phi^j dx^\mu \otimes dx^\nu$$

Potential must be added in by hand

Formalism limited to two derivative terms

$$V(\phi(x)) \notin \phi^* g$$

$$ddu = 0 \implies \partial_\mu \partial_\nu \phi^i \notin \phi^* g$$

Fibre Bundles

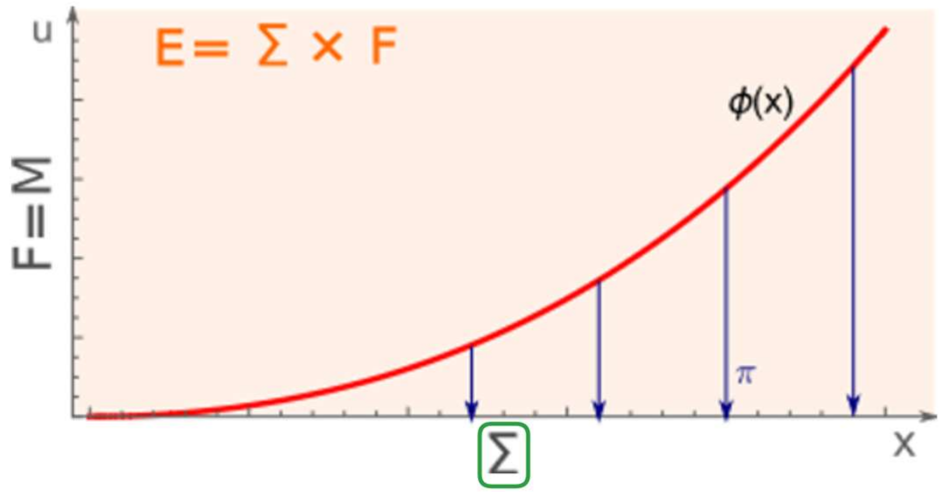
A bundle consists of a:

Total space

Base Space

Surjective Map

Locally the bundle looks like a direct product of the base space and the fibres



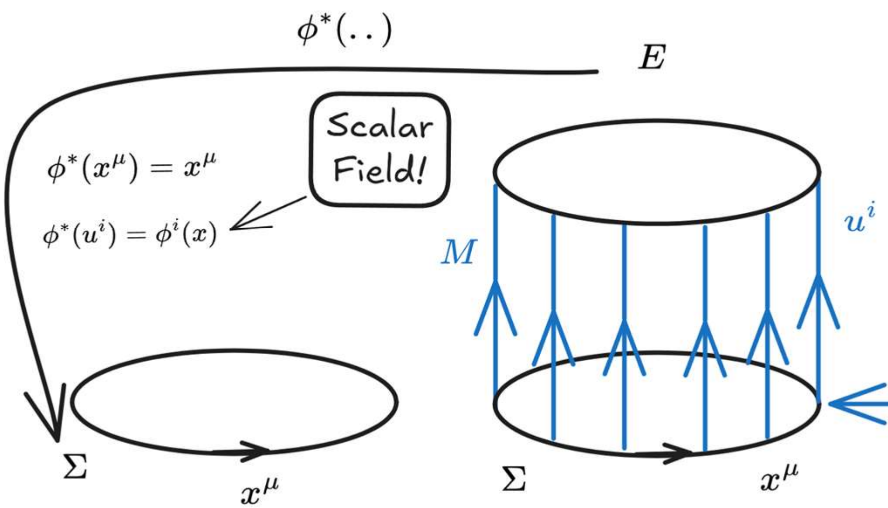
At every point in the bundle local inverses to the surjective map exist. The maps are called sections

Fibre Bundles Geometry

Locally a fibre bundle admits fibred coordinates ⟷ $\forall p \in E, \quad \exists E_p \quad s.t \quad (x^\mu, u^i) \in E_p$

The coordinates naturally split into fibre and base coordinates

$$E_p = (M \times \Sigma)_p$$
$$u_i \in M, x^\mu \in \Sigma$$



A metric on E consists of four blocks

$$\begin{pmatrix} g_{\mu\nu} & g_{\mu i} \\ g_{\mu i} & g_{ij} \end{pmatrix}$$

Poincaré invariance implies the off-diagonal blocks vanish →

$$\begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & g_{ij} \end{pmatrix}$$

Lagrangian from Metric

- After Poincare invariance

$$g = g_{\mu\nu} dx^{\mu} \otimes dx^{\nu} + g_{ij} du^{i} \otimes du^{j}$$

$\phi^{*}(\dots)$ $\phi^{*}(\dots)$

$$g_{\mu\nu}\eta^{\mu\nu} = -V(\phi) \subset L \qquad g_{ij}\eta^{\mu\nu}\partial_{\mu}\phi^{i}\partial_{\nu}\phi^{j} \subset L$$

- Scalar potential is now geometric!

Amplitudes on Fibre Bundle

- Individual Feynman rules obtained from taking partial derivatives of the two blocks of the metric $g_{\mu\nu}$ and g_{ij}

- Using the compatibility of the Levi Civita connection we can rewrite derivatives in terms of Christoffel symbols

$$\nabla g = 0 \leftrightarrow \partial_k g_{ij} = \Gamma^q_{ki} g_{qj} + \Gamma^q_{kj} g_{qi}$$

Amplitudes on Fibre Bundles

$$\Gamma^i_{j\mu} = \Gamma^\mu_{ij} = \Gamma^\mu_{\nu\rho} = 0$$

$$\Gamma^i_{\mu\nu} = -\frac{1}{2} g^{il} g_{\mu\nu,l}$$

$$\Gamma^\mu_{i\nu} = \frac{1}{2} g^{\mu\rho} g_{\rho\nu,i}$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{im} (g_{jm,k} + g_{km,j} - g_{jk,m})$$

$\longrightarrow$

$$R^i_{\mu\nu\rho} = R^\mu_{i\nu\rho} = R^\mu_{\nu i\rho} = R^\mu_{\nu\rho i} = 0$$

$$R^i_{jk\mu} = R^i_{j\mu\mathrm{k}} = R^i_{\mu jk} = R^\mu_{ijk} = 0$$

Non-zero Riemann tensors:

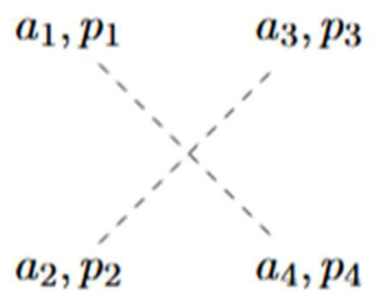
$$R^i_{j\mu\nu}, R^i_{\mu j\nu}, R^\mu_{ij\nu}, R^\mu_{\nu ij}, R^\mu_{\nu\rho\sigma}, R^i_{jkl}$$

Amplitudes on Fibre Bundles

- The three-point amplitude is given as*:

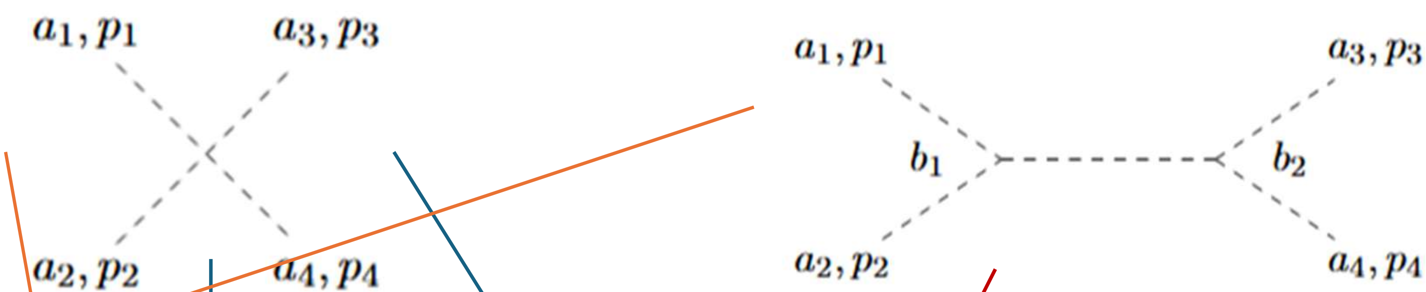
$$-\frac{i}{2}\nabla_{a_3}R^{\mu}_{a_1\mu a_2}+\Gamma_{a_1a_2a_3}\left(p^2_{a_1}-m^2_{a_1}\right)+\Gamma_{a_2a_1a_3}\left(p^2_{a_2}-m^2_{a_2}\right)+\Gamma_{a_3a_2a_1}(p^2_{a_3}-m^2_{a_3})$$

Amplitudes on Fibre Bundles



$$
\begin{aligned}
& i\Big(- R_{a_1 a_3 a_4 a_2} s_{a_1 a_2} + 2 R^{\mu}{}_{a_1 \mu a_2} R^{\mu}{}_{a_3 \mu a_4} \\
& - \frac{1}{12} \nabla_{a_1} \nabla_{a_3} R^{\mu}{}_{a_2 \mu a_4} - \frac{3}{4} \nabla_{a_2} R^{\mu b_1}{}_{\nu a_1} \nabla_{a_4} R^{\nu}{}_{b_1 \mu a_3} (p_{b_1}^2 - m_{b_1}^2)^{-1} \\
& + 6 \nabla_{a_4} R^{\mu b_1}{}_{\mu a_3} \Gamma_{a_1 a_2 b_1} (p_{a_1}^2 - m_{a_1}^2)(p_{b_1}^2 - m_{b_1}^2)^{-1} + 4 \partial_{a_2} \Gamma_{a_1 a_3 a_4} (p_{a_1}^2 - m_{a_1}^2) \\
& - 12 g^{b_1 b_2} \Gamma_{a_3 a_4 b_2} \Gamma_{a_1 a_2 b_1} (p_{a_1}^2 - m_{a_1}^2)(p_{b_1}^2 - m_{b_1}^2)^{-1} (p_{a_3}^2 - m_{a_3}^2) - 12 \Gamma^{b_1}{}_{a_2 b_1} \Gamma_{a_1 a_3 a_4} (p_{a_1}^2 - m_{a_1}^2) \\
& - 4 \Gamma^{b_1}{}_{a_2 a_3} \Gamma_{b_1 a_1 a_4} (p_{a_1}^2 - m_{a_1}^2) \Big) + \text{perms}(a_1 a_2 a_3 a_4)
\end{aligned}
$$

Amplitudes on Fibre Bundles



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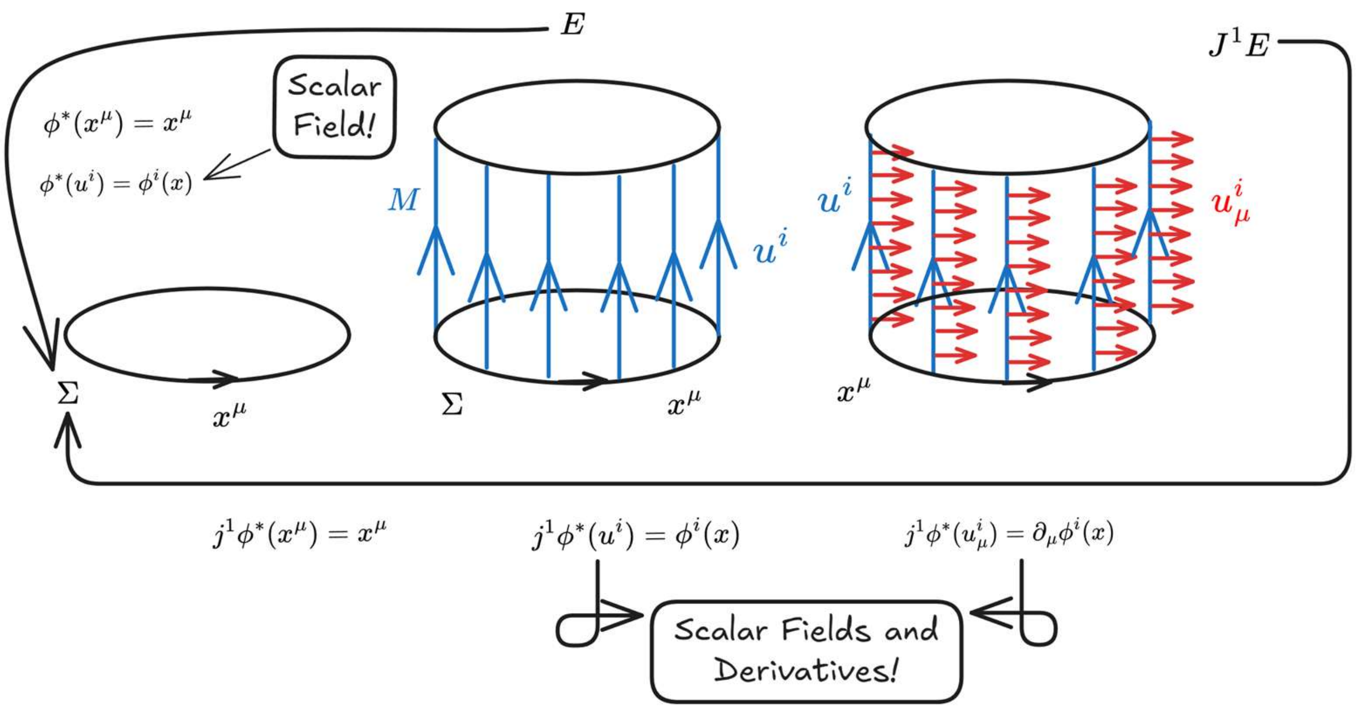
\begin{aligned}
i\Big(& -R_{a_1 a_3 a_4 a_2} s_{a_1 a_2} + 2R^{\mu}{}_{a_1 \mu a_2} R^{\mu}{}_{a_3 \mu a_4} \\
& - \frac{1}{12} \nabla_{a_1} \nabla_{a_3} R^{\mu}{}_{a_2 \mu a_4} - \frac{3}{4} \nabla_{a_2} R^{\mu b_1}{}_{\nu a_1} \nabla_{a_4} R^{\nu}{}_{b_1 \mu a_3} (p_{b_1}^2 - m_{b_1}^2)^{-1} \\
& + 6 \nabla_{a_4} R^{\mu b_1}{}_{\mu a_3} \Gamma_{a_1 a_2 b_1} (p_{a_1}^2 - m_{a_1}^2)(p_{b_1}^2 - m_{b_1}^2)^{-1} + 4 \partial_{a_2} \Gamma_{a_1 a_3 a_4} (p_{a_1}^2 - m_{a_1}^2) \\
& - 12 g^{b_1 b_2} \Gamma_{a_3 a_4 b_2} \Gamma_{a_1 a_2 b_1} (p_{a_1}^2 - m_{a_1}^2)(p_{b_1}^2 - m_{b_1}^2)^{-1}(p_{a_3}^2 - m_{a_3}^2) - 12 \Gamma^{b_1}{}_{a_2 b_1} \Gamma_{a_1 a_3 a_4} (p_{a_1}^2 - m_{a_1}^2) \\
& - 4 \Gamma^{b_1}{}_{a_2 a_3} \Gamma_{b_1 a_1 a_4} (p_{a_1}^2 - m_{a_1}^2) \Big) + \text{perms}(a_1 a_2 a_3 a_4)
\end{aligned}
$$

Jet Bundles

- Given a smooth fibre bundle (E, Σ, π) we can extend it to create a jet bundle
- At each point $p \in E$ we define equivalence classes $j_p^1\phi$. The set of all equivalence classes is a smooth manifold J^1E
- The manifolds Σ, J^1E create the first jet bundle (Σ, J^1E, π_1)

Jet Bundle Geometry



Equivalence Classes

$$\phi, \psi \in j^1_p \phi$$

1. $\phi(p) = \psi(p)$

2. $\left. \dfrac{\partial \phi \circ u^i}{\partial x^\mu} \right|_p = \left. \dfrac{\partial \psi \circ u^i}{\partial x^\mu} \right|_p$

Geometry on Jet Bundle

- The first jet bundle introduces an additional coordinate u_μ^i thus the metric becomes

$$g = \begin{pmatrix} dx^\mu & du^i & du_\mu^i \end{pmatrix} \begin{pmatrix} g_{\mu\nu} & g_{\mu i} & g_{\mu i}^{\ \ \nu} \\ g_{\mu i} & g_{ij} & g_{ij}^{\ \ \nu} \\ g_{\mu i}^{\ \ \nu} & g_{ij}^{\ \ \nu} & g_{ij}^{\mu\nu} \end{pmatrix} \begin{pmatrix} dx^\nu \\ du^j \\ du_\nu^j \end{pmatrix}$$

- The coordinates are independent on the manifold J^1E

Lagrangian from Jet Bundle Metric

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu + g_{ij} du^i \otimes du^j + g_{ij}^{\mu\nu} du_\mu^i \otimes du_\nu^j$$

$$+ g_{i\mu} du^i \otimes dx^\mu + g_{ij}^{\mu} du_\mu^i \otimes du^j + g_{i\nu}^{\mu} du_\mu^i \otimes dx^\nu$$

$$g_{ij}\eta^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j \subset L$$

$$g_{\mu\nu}\eta^{\mu\nu} = -V(\phi) + \cdots \subset L$$

$$g_{ij}^{\mu\nu}\eta^{\rho\sigma}\partial_\rho\partial_\mu\phi^i\partial_\sigma\partial_\nu\phi^j \subset L$$

Conclusions

- Fibre bundles allow us to describe all terms in a geometric manner
- On-shell amplitudes correspond to geometric tensors
- Invariance of S-Matrix given by the transformation properties of tensors

Outlook

- Computation of amplitudes in higher jet bundle orders
- Renormalization of scalar theories
- Introduction of gauge fields into the formalism
- Automation of the computation up to higher point functions

Back up

- The propagator is defined by

$$\tilde{\Delta}^{d_1 d_2}(p_3 + p_4) = (\frac{1}{2}\eta^{\rho\sigma}\tilde{g}_{\rho\sigma, d_1 d_2} + (p_3 + p_4)^2 \tilde{g}_{d_1 d_2})^{-1}$$

- If the masses are diagonal and the kinetic term is canonical then it is possible to rewrite as

$$\Delta(p) = \frac{g^{ij}}{p^2 - m^2}$$