Status of the $(g-2)_\mu$ puzzle

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FOR FUNDAMENTAL PHYSICS

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Outline

Introduction: $(g-2)_\mu$ in the Standard Model

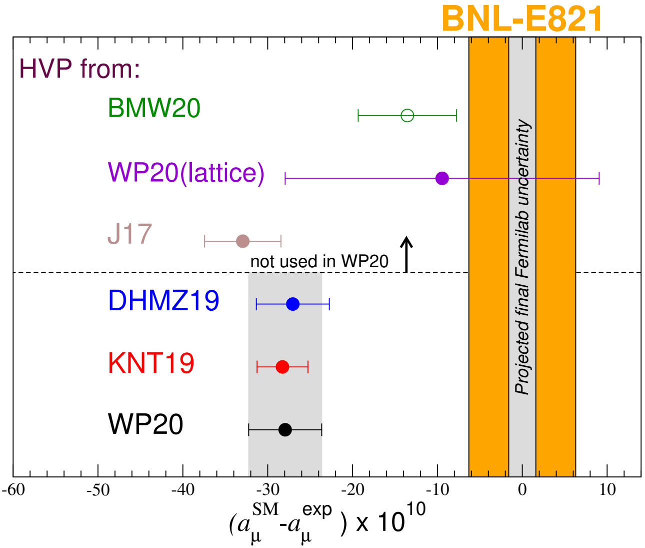
Hadronic light-by-light contribution
- Dispersive
- Lattice

Hadronic Vacuum Polarization contribution
- Dispersive
- Lattice

Conclusions and Outlook

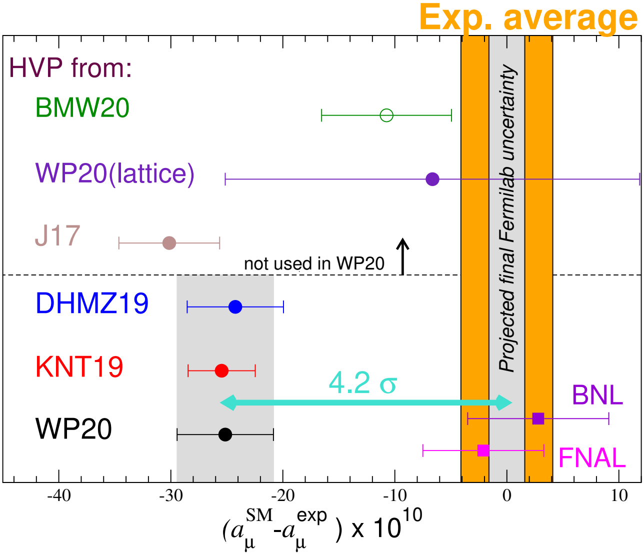
Present status of $(g-2)_\mu$: experiment vs SM

Before



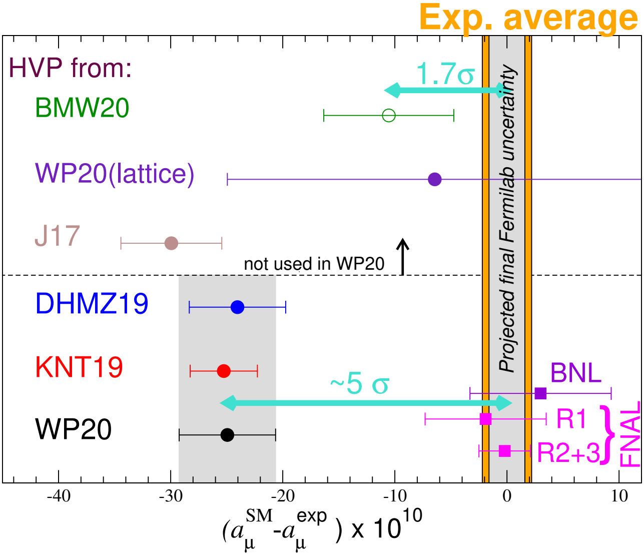
Present status of $(g-2)_\mu$: experiment vs SM

After the 2021 Fermilab result



Present status of $(g-2)_\mu$: experiment vs SM

After the 2023 Fermilab result



White Paper (2020): $(g-2)_\mu$, experiment vs SM

Contribution	Value $\times 10^{11}$
HVP LO (e^+e^-)	6931(40)
HVP NLO (e^+e^-)	$-98.3(7)$
HVP NNLO (e^+e^-)	12.4(1)
HVP LO (lattice, *udsc*)	7116(184)
HLbL (phenomenology)	92(19)
HLbL NLO (phenomenology)	2(1)
HLbL (lattice, *uds*)	79(35)
HLbL (phenomenology + lattice)	90(17)
QED	116 584 718.931(104)
Electroweak	153.6(1.0)
HVP (e^+e^-, LO + NLO + NNLO)	6845(40)
HLbL (phenomenology + lattice + NLO)	92(18)
Total SM Value	116 591 810(43)
Experiment	116 592 059(22)
Difference: $\Delta a_\mu := a_\mu^{\text{exp}} - a_\mu^{\text{SM}}$	249(48)

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White Paper (2020): $(g-2)_\mu$, experiment vs SM

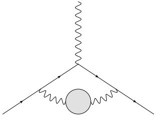
White Paper:
T. Aoyama et al. Phys. Rep. 887 (2020) = WP(20)

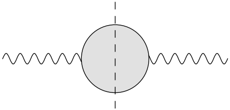
Muon $g-2$ Theory Initiative
Steering Committee:
GC
Michel Davier (vice-chair)
Aida El-Khadra (chair)
Martin Hoferichter
Laurent Lellouch
Christoph Lehner (vice-chair)
Tsutomu Mibe (J-PARC E34 experiment)
Lee Roberts (Fermilab E989 experiment)
Thomas Teubner
Hartmut Wittig

White Paper 2: to appear soon ($\sim$ April 2025)

Theory uncertainty comes from hadronic physics

- Hadronic contributions responsible for most of the theory uncertainty
- Hadronic vacuum polarization (HVP) is $\mathcal{O}(\alpha^2)$, dominates the total uncertainty, despite being known to $< 1\%$



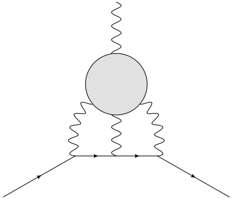


  - unitarity and analyticity $\Rightarrow$ dispersive approach
  - $\Rightarrow$ direct relation to experiment: $\sigma_{\rm tot}(e^+e^- \to \rm hadrons)$
  - e^+e^- Exps: BaBar, Belle, BESIII, CMD2/3, KLOE2, SND
  - alternative approach: lattice, now competitive

(BMW, ETMC, Fermilab, HPQCD, Mainz, MILC, RBC/UKQCD)

Theory uncertainty comes from hadronic physics

- Hadronic contributions responsible for most of the theory uncertainty
- Hadronic vacuum polarization (HVP) is $\mathcal{O}(\alpha^2)$, dominates the total uncertainty, despite being known to $< 1\%$
- Hadronic light-by-light (HLbL) is $\mathcal{O}(\alpha^3)$, known to $\sim 20\%$, second largest uncertainty (now subdominant)



- earlier: model-based—uncertainties difficult to quantify
- recently: dispersive approach $\Rightarrow$ data-driven, systematic treatment
- more recently: lattice QCD also competitive (Mainz, RBC/UKQCD, BMW)

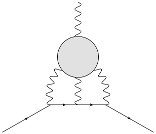
The 2 $\times$ 2 matrix of Hadronic Contributions

	dispersive	lattice
HLbL	??	??
HVP	??	??

The 2 $\times$ 2 matrix of Hadronic Contributions

	dispersive	lattice
HLbL	??	??
HVP	??	??

HLbL contribution: Master Formula



$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{48\pi^2} \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1-\tau^2} \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau)$$

Q_i^μ are the Wick-rotated four-momenta and τ the four-dimensional angle between Euclidean momenta: $Q_1 \cdot Q_2 = |Q_1||Q_2|\tau$
The integration variables $Q_1 := |Q_1|$, $Q_2 := |Q_2|$.

GC, Hoferichter, Procura, Stoffer (15)

- T_i: known kernel functions
- $\bar{\Pi}_i$ are amenable to a dispersive treatment: imaginary parts are related to measurable subprocesses

Improvements obtained with the dispersive approach

Contribution	PdRV(09) *Glasgow cons.*	N/JN(09)	J(17)	WP(20)	HSZ (25)
π^0, η, η'-poles	114(13)	99(16)	95.45(12.40)	93.8(4.0)	$91.2\ ^{+2.9}_{-2.4}$
π, K-loops/boxes	−19(19)	−19(13)	−20(5)	−16.4(2)	−16.4(2)
S-wave $\pi\pi$ rescattering	−7(7)	−7(2)	−5.98(1.20)	−8(1)	−9.1(1.0)
subtotal	88(24)	73(21)	69.5(13.4)	69.4(4.1)	$65.7\ ^{+3.1}_{-2.6}$
scalars	–	–	–	} −1(3) (scalars + tensors)	} 33.2(7.2) (scalars, tensors, axial vectors, u, d, s-loops / short-distance)
tensors	–	–	1.1(1)		
axial vectors	15(10)	22(5)	7.55(2.71)	6(6)	
u, d, s-loops / short-distance	–	21(3)	20(4)	15(10)	
c-loop	2.3	–	2.3(2)	3(1)	3(1)
total	105(26)	116(39)	100.4(28.2)	92(19)	102(8)

- significant reduction of uncertainties in the first three rows

  CHPS (17), Masjuan, Sánchez-Puertas (17) Hoferichter, Hoid et al. (18), Gerardin, Meyer, Nyffeler (19)

- $\eta^{(\prime)}$ contributions, resonances and short-distance constraints have recently been improved

  Lüdtke, Procura, Stoffer (23), Bijnens et al. (23,24), Hoferichter, Stoffer, Zillinger (25) (=HSZ (25)), Mager, (Cappiello), Leutgeb, Rebhan (23-25)

Recent progress on HLbL

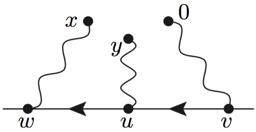
- Pseudoscalars:
  dispersive analysis for $\eta^{(\prime)}$ just completed — Hoferichter, Hoid, Holz, Kubis, (24)
- Axials:
  - TFF analyzed in terms of VMD — Hoferichter, Kubis, Zanke (23)
  - Optimized basis — Hoferichter, Stoffer, Zillinger (24)
- Tensors: $\Rightarrow$ dispersion relation for $g-2$ kinematics ($q_4 = 0$)
  Lüdtke, Procura, Stoffer (23-24)
- SDC:
  - complete analysis in QCD at NLO in all regimes
    Bijnens, Hermansson-Truedsson, Rodríguez-Sánchez, (23-24)
  - hQCD models further refined — Mager, (Cappiello), Leutgeb, Rebhan (23-25)
- Total:
  - Dispersive — Hoferichter, Stoffer, Zillinger (25)
  - hQCD — Mager, Cappiello, Leutgeb, Rebhan (25)

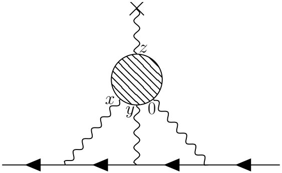
$\Rightarrow$ WP2 (work in progress)

The 2 $\times$ 2 matrix of Hadronic Contributions

	dispersive	lattice
HLbL	✓	??
HVP	??	??

Master formula for HLbL lattice calculations





$$a_\mu^{\text{HLbL}} = \frac{me^6}{3} \int d^4x\, d^4y\, \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(p,x,y)\; i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y),$$

$$i\widehat{\Pi}_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int d^4z\, z_\rho \left\langle j_\mu(x)\, j_\nu(y)\, j_\sigma(z)\, j_\lambda(0) \right\rangle_{\text{QCD}} .$$

with $\mathcal{L}_{\ldots}(p,x,y)$ the analytically calculable QED kernel:

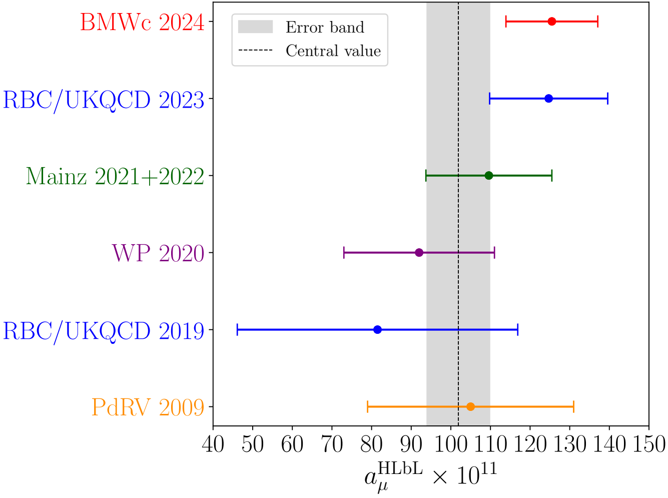
$$\mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(p,x,y) = \frac{1}{16m^2} \int d^4u d^4v d^4w G(w-x)\, G(u-y)\, G(v)\, e^{-ip\cdot(w-v)}$$
$$\times \operatorname{Tr}\left\{[\gamma_\rho,\gamma_\sigma]\, (-i\not{p}+m)\gamma_\mu S(w-u)\gamma_\nu S(u-v)\gamma_\lambda(-i\not{p}+m)\right\}$$

HLbL Lattice results

Collab.	$10^{11}\, a_\mu^{\mathrm{HLbL},\ell}$	$10^{11}\, a_\mu^{\mathrm{HLbL},s}$	$10^{11}\, a_\mu^{\mathrm{HLbL},c}$
Mainz/CLS	107.4(11.3)(9.2)(6.0)	−0.6(2.0)	2.8(5)
RBC/UKQCD	122.0(10.1)(9.5)	−0.0(2.2)(0.3)	–
BMW	122.6(11.6)	−1.7(8)(3)	2.73(27)

HLbL: comparison dispersive/lattice

Figure from Hoferichter, Stoffer, Zillinger (25)

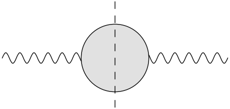


The 2 × 2 matrix of Hadronic Contributions

	dispersive	lattice
HLbL	✓	✓
HVP	??	??

HVP contribution: Master Formula

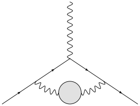
Unitarity relation: simple, same for all intermediate states



$$\mathrm{Im}\bar{\Pi}(q^2) \propto \sigma(e^+e^- \to \mathrm{hadrons}) = \sigma(e^+e^- \to \mu^+\mu^-)R(q^2)$$

Analyticity $\left[\bar{\Pi}(q^2) = \frac{q^2}{\pi}\int ds \frac{\mathrm{Im}\bar{\Pi}(s)}{s(s-q^2)}\right] \Rightarrow$ Master formula for HVP

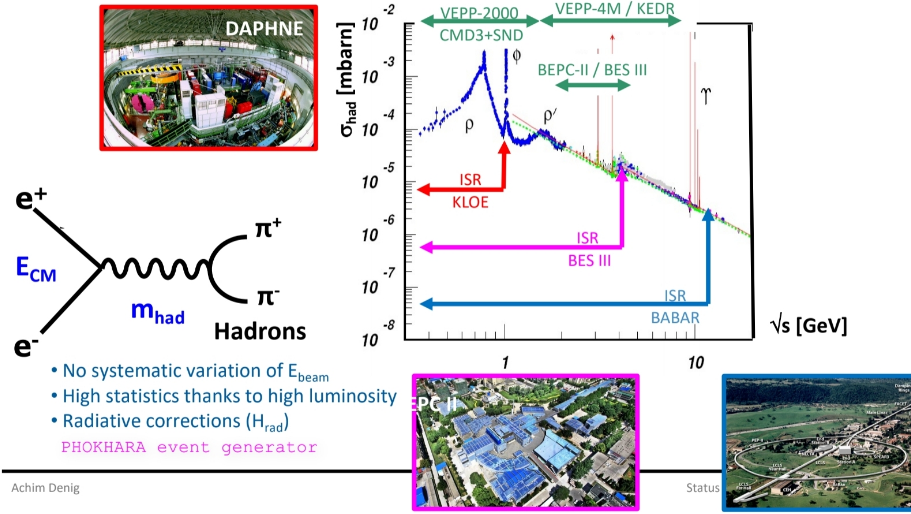
Bouchiat, Michel (61)



$$\Leftrightarrow \qquad a_\mu^{\mathrm{hvp}} = \frac{\alpha^2}{3\pi^2}\int_{s_{th}}^{\infty}\frac{ds}{s}K(s)R(s)$$

$K(s)$ known, depends on m_μ and $K(s) \sim \frac{1}{s}$ for large s

HVP contribution: Master Formula



Slide by Achim Denig, MUonE Workshop, June 2024

Comparison between DHMZ19 and KNT19

	DHMZ19	KNT19	Difference
$\pi^+\pi^-$	507.85(3.38)	504.23(1.90)	3.62
$\pi^+\pi^-\pi^0$	46.21(1.45)	46.63(94)	−0.42
$\pi^+\pi^-\pi^+\pi^-$	13.68(0.30)	13.99(19)	−0.31
$\pi^+\pi^-\pi^0\pi^0$	18.03(0.55)	18.15(74)	−0.12
K^+K^-	23.08(0.44)	23.00(22)	0.08
K_SK_L	12.82(0.24)	13.04(19)	−0.22
$\pi^0\gamma$	4.41(0.10)	4.58(10)	−0.17
Sum of the above	626.08(3.90)	623.62(2.27)	2.46
$[1.8, 3.7]\,\mathrm{GeV}$ (without $c\bar{c}$)	33.45(71)	34.45(56)	−1.00
J/ψ, $\psi(2S)$	7.76(12)	7.84(19)	−0.08
$[3.7, \infty)\,\mathrm{GeV}$	17.15(31)	16.95(19)	0.20
Total $a_\mu^{\text{HVP, LO}}$	694.0(4.0)	692.8(2.4)	1.2

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For the dominant $\pi\pi$ channel more theory input can be used

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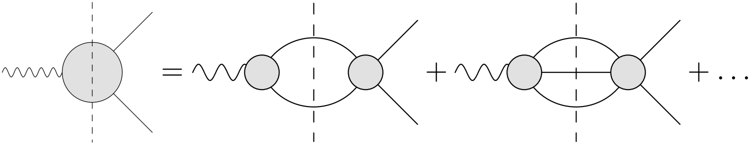
For the 3π and KK channels also Hoferichter, Hoid, Kubis, Stamen, Hariharan, Stoffer

Omnès representation including isospin breaking



$$F_V(s) = \Omega_{\pi\pi}(s) \cdot G_\omega(s) \cdot \Omega_{\text{in}}(s)$$

Omnès representation including isospin breaking

- Omnès representation

$$F_V^\pi(s) = \exp\left[\frac{s}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{\delta(s')}{s'(s'-s)}\right] \equiv \Omega(s)$$

- Split elastic ($\leftrightarrow \pi\pi$ phase shift, δ_1^1) from inelastic phase

$$\delta = \delta_1^1 + \delta_{\text{in}} \quad \Rightarrow \quad F_V^\pi(s) = \Omega_1^1(s)\Omega_{\text{in}}(s)$$

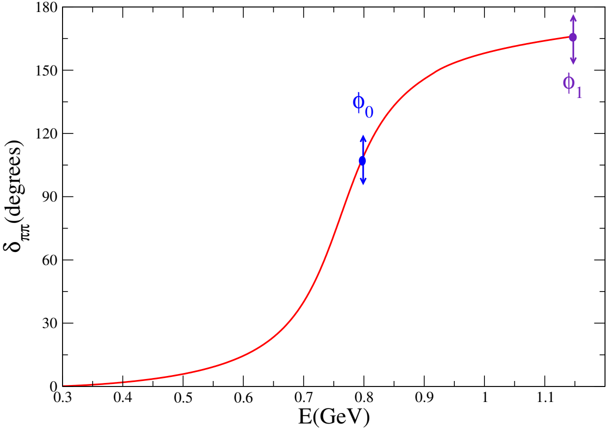
Eidelman-Lukaszuk: unitarity bound on δ_{in}

$$\sin^2\delta_{\text{in}} \leq \frac{1}{2}\left(1-\sqrt{1-r^2}\right), \; r = \frac{\sigma^{I=1}_{e^+e^-\to\neq 2\pi}}{\sigma_{e^+e^-\to 2\pi}} \Rightarrow s_{\text{in}} = (M_\pi+M_\omega)^2$$

- $\rho-\omega$–mixing $\qquad F_V(s) = \Omega_{\pi\pi}(s)\cdot\Omega_{\text{in}}(s)\cdot G_\omega(s)$

$$G_\omega(s) = 1+\epsilon\frac{s}{s_\omega - s} \qquad \text{where} \qquad s_\omega = (M_\omega - i\Gamma_\omega/2)^2$$

Essential free parameters

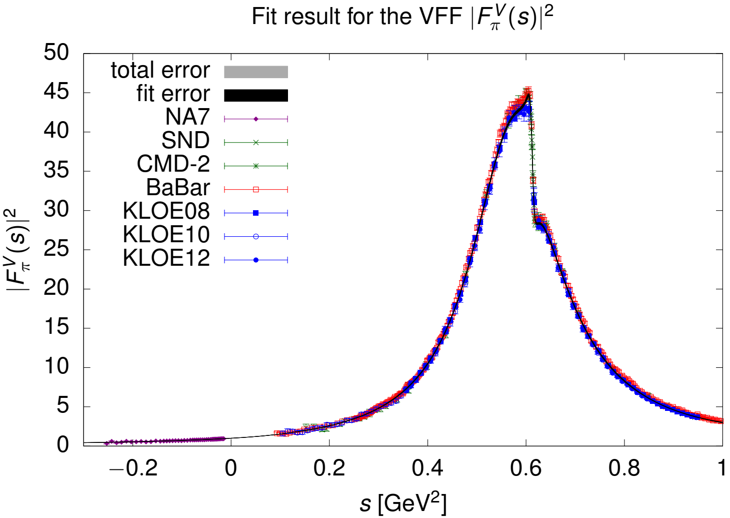


Estimated range ($\pi N \to \pi\pi N$): Caprini, GC, Leutwyler (12)

$$\phi_0 = 108.9(2.0)^\circ \qquad \phi_1 = 166.5(2.0)^\circ$$

Fit results

GC, Hoferichter, Stoffer (18)



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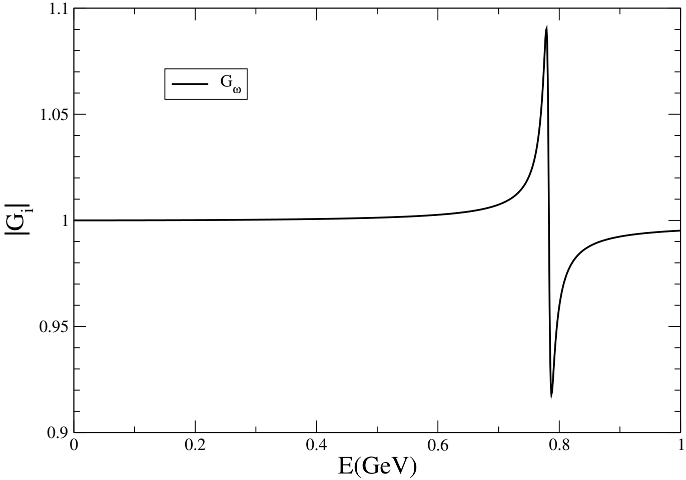

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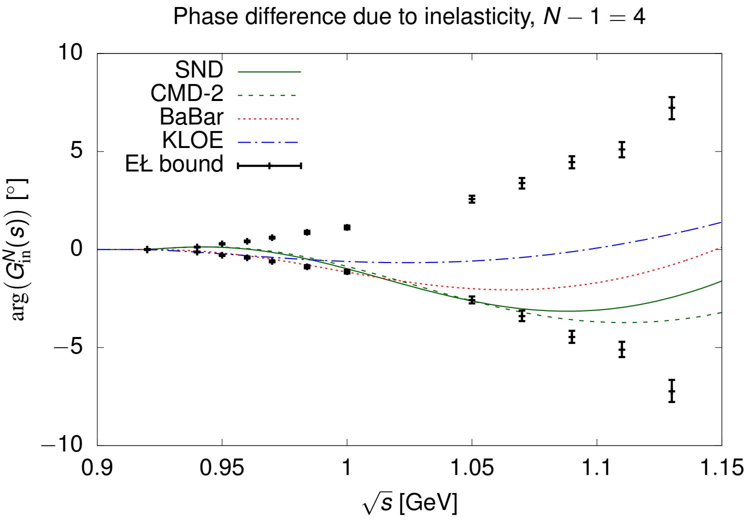


2π: comparison with the dispersive approach

2π channel described dispersively $\Rightarrow$ more theory constraints

Ananthanarayan, Caprini, Das (19), GC, Hoferichter, Stoffer (18) WP(20)

Energy range	CHS18	DHMZ19	KNT19
$\leq 0.6\,\mathrm{GeV}$	110.1(9)	110.4(4)(5)	108.7(9)
$\leq 0.7\,\mathrm{GeV}$	214.8(1.7)	214.7(0.8)(1.1)	213.1(1.2)
$\leq 0.8\,\mathrm{GeV}$	413.2(2.3)	414.4(1.5)(2.3)	412.0(1.7)
$\leq 0.9\,\mathrm{GeV}$	479.8(2.6)	481.9(1.8)(2.9)	478.5(1.8)
$\leq 1.0\,\mathrm{GeV}$	495.0(2.6)	497.4(1.8)(3.1)	493.8(1.9)
$[0.6, 0.7]\,\mathrm{GeV}$	104.7(7)	104.2(5)(5)	104.4(5)
$[0.7, 0.8]\,\mathrm{GeV}$	198.3(9)	199.8(0.9)(1.2)	198.9(7)
$[0.8, 0.9]\,\mathrm{GeV}$	66.6(4)	67.5(4)(6)	66.6(3)
$[0.9, 1.0]\,\mathrm{GeV}$	15.3(1)	15.5(1)(2)	15.3(1)
$\leq 0.63\,\mathrm{GeV}$	132.8(1.1)	132.9(5)(6)	131.2(1.0)
$[0.6, 0.9]\,\mathrm{GeV}$	369.6(1.7)	371.5(1.5)(2.3)	369.8(1.3)
$\left[\sqrt{0.1}, \sqrt{0.95}\right]\,\mathrm{GeV}$	490.7(2.6)	493.1(1.8)(3.1)	489.5(1.9)

Combination method and final result

Complete analyses DHMZ19 and KNT19, as well as CHS19 (2π) and HHK19 (3π), have been so combined:

HHK=Hoferichter, Hoid, Kubis

- central values are obtained by simple averages (for each channel and mass range)
- the largest experimental and systematic uncertainty of DHMZ and KNT is taken
- $1/2$ difference DHMZ$-$KNT (or BABAR$-$KLOE in the 2π channel, if larger) is added to the uncertainty

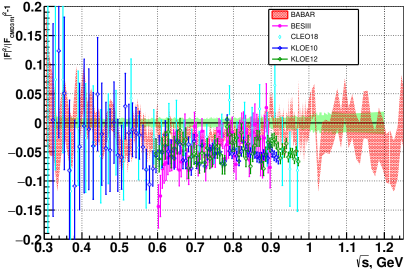
Final result:

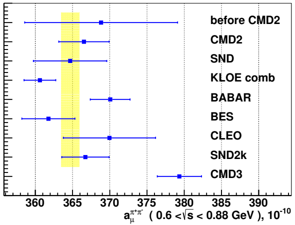
$$a_\mu^{\text{HVP, LO}} = 693.1(2.8)_{\text{exp}}(2.8)_{\text{sys}}(0.7)_{\text{DV+QCD}} \times 10^{-10}$$
$$= 693.1(4.0) \times 10^{-10}$$

WP(20)

CMD-3 measurement of $e^+e^- \to \pi^+\pi^-$

F. Ignatov et al., CMD-3, `arXiv:2302.08834`

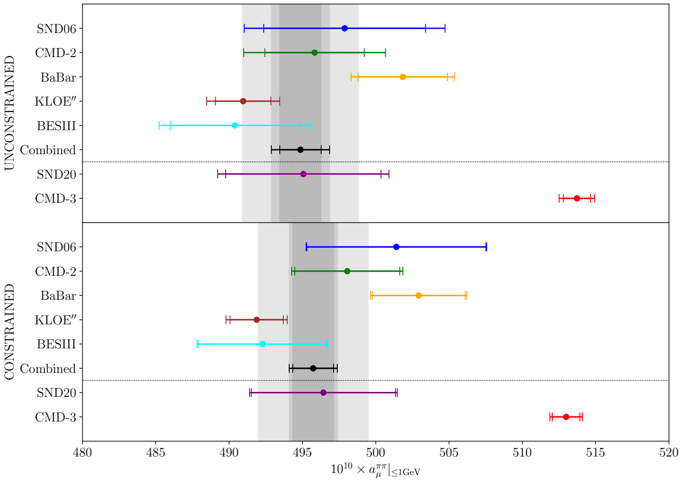




The comparison of pion form factor measured in this work with the most recent ISR experiments (BABAR [21], KLOE [18, 19], BES [22]) is shown in Fig. 34. The comparison with the most precise previous energy scan experiments (CMD-2 [12, 13, 14, 15], SND [16] at the VEPP-2M and SND [23] at the VEPP-2000) is shown in Fig. 35. The new result generally shows larger pion form factor in the whole energy range under discussion. The most significant difference to other energy scan measurements, including previous CMD-2 measurement, is observed at the left side of ρ-meson ($\sqrt{s} = 0.6 - 0.75$ GeV), where it reach up to 5%, well beyond the combined systematic and statistical errors of the new and previous results. The source of this difference is unknown at the moment.

Comparison between CMD-3 and other experiments

Leplumey and Stoffer, arXiv:2501.09643



Comparison between CMD-3 and other experiments

Leplumey and Stoffer, arXiv:2501.09643

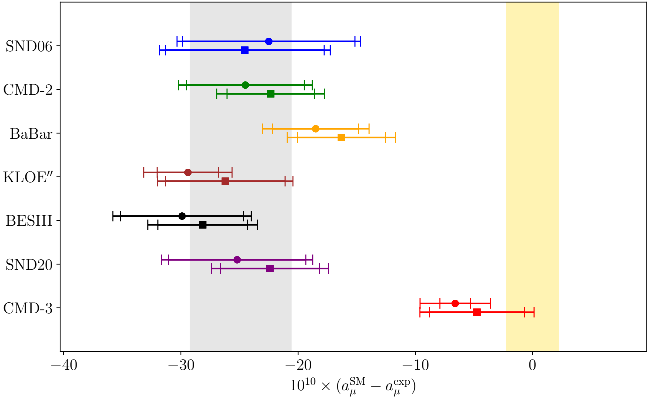
Discrepancy w/ CMD-3	$a_\mu^{\pi\pi}\big\vert_{\leq 1\,\mathrm{GeV}}$	
	unconstrained	constrained
SND06	2.0σ	1.8σ
CMD-2	3.3σ	3.7σ
BaBar	2.9σ	2.8σ
KLOE″	7.4σ	8.9σ
BESIII	4.2σ	4.5σ
SND20	3.0σ	3.2σ
Combination	4.4σ [7.3σ]	4.4σ [8.1σ]

Uncertainties in brackets exclude KLOE-BaBar systematic eff.

Combination: NA7 + all data sets other than SND20 and CMD-3

Comparison between different experiments

Figure courtesy of Thomas Leplumey



Circles: F_π^V dispersive analysis GC, Hoferichter, Stoffer (18), Leplumey, Stoffer (25)

Squares: integral over data Keshavarzi, Nomura, Teubner, Wright; DHMZ will be added

Yellow band: experimental uncertainty

Updates on IB corrections from $(g-2)_7$ @KEK 2024

- KLOE and BESIII have rebutted claims that higher-order radiative corrections might have solved the puzzle

  talks by A. Denig and G. Venanzoni @KEK24

- claim that initial/final radiation interference on the box diagram might impact significantly radiative-return experiments is under scrutiny F. Ignatov @STRONG2020 Zürich (23)

- reconsideration of τ decays as input for HVP has been advocated by DHMZ TI Virtual workshops on Nov. 8 and Dec. 9

- analysis of IB for τ decays on the lattice is ongoing

  talk by M. Bruno @KEK24

- dispersive analysis of IB for τ decays is ongoing

  talk by M. Cottini @KEK24

The 2 × 2 matrix of Hadronic Contributions

	dispersive	lattice
HLbL	✓	✓
HVP	?!?!?!	??

Lattice methods to calculate HVP

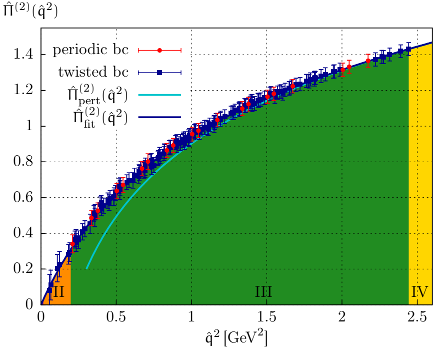
- Direct:

$$\Pi_{\mu\nu}(Q) = \int d^4x e^{iQx} \langle J_\mu(x) J_\nu(0) \rangle = (Q_\mu Q_\nu - \delta_{\mu\nu} Q^2) \Pi(Q^2)$$

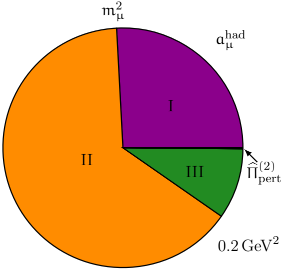
$$a_\mu^{\text{hvp}} = 4\alpha^2 \int_0^\infty dQ^2 K(Q^2; m_\mu^2) \left[\Pi(Q^2) - \Pi(0) \right]$$

Disadvantage: integrand peaked near $Q^2 = m_\mu^2 \Rightarrow$ need many points at small momenta $\Rightarrow$ large volumes or twisted boundary conditions

Lattice methods to calculate HVP



Della Morte et al. (Mainz) JHEP 2012



Region I ($0 < Q^2 < m_\mu^2$) is invisible on the left plot

Lattice methods to calculate HVP

- Direct:

$$\Pi_{\mu\nu}(Q) = \int d^4x e^{iQx} \langle J_\mu(x) J_\nu(0) \rangle = (Q_\mu Q_\nu - \delta_{\mu\nu} Q^2)\Pi(Q^2)$$

$$a_\mu^{\text{hvp}} = 4\alpha^2 \int_0^\infty dQ^2 K(Q^2; m_\mu^2) \left[\Pi(Q^2) - \Pi(0)\right]$$

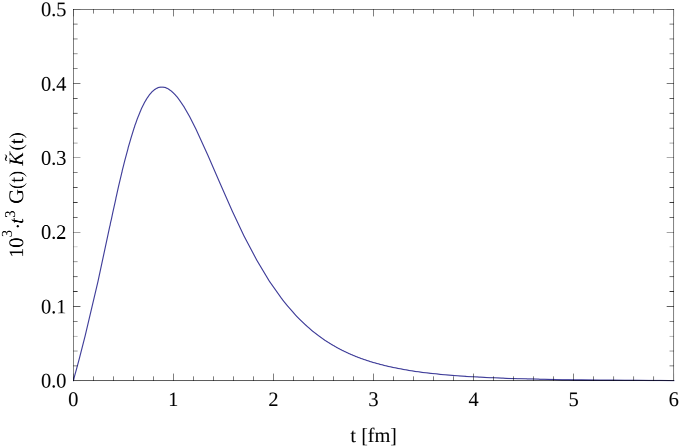
- Time-Momentum Representation (TMR): Bernecker, Meyer (11)

$$-G(t)\delta_{kl} = \int d^3x \langle J_k(x) J_l(0) \rangle, \quad a_\mu^{\text{hvp}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, t^3 G(t) \tilde{K}(t, m_\mu)$$

Disadvantage: noise grows quickly with t for small quark masses

Lattice methods to calculate HVP

Bernecker, Meyer 2011



All modern calculations have adopted this approach

Complete Lattice calculations of a_μ^{HVP}

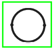
- BMW (20)
  Staggered fermions, physical m_q, all IB effects included
- BMW-DMZ (24)
  Staggered fermions, physical m_q, all IB effects included, long-time region evaluated w/ data (hybrid approach)
- Mainz/CLS (23-24)
  Wilson fermions, near physical m_q, connected IB contr. included
- RBC/UKQCD (24)
  Domain-wall fermions, physical m_q, connected IB contr. included

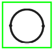
Partial results are available from
Fermilab/HPQCD/MILC (staggered fermions)
ETMC (Twisted-mass fermions)

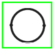
Some details about the BMW calculation

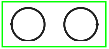
Borsanyi et al. Nature 2021

Isospin-symmetric

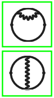

Connected light
$633.7(2.1)_{stat}(4.2)_{syst}$

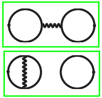

Connected strange
$53.393(89)_{stat}(68)_{syst}$


Connected charm
$14.6(0)_{stat}(1)_{syst}$

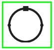

Disconnected
$-13.36(1.18)_{stat}(1.36)_{syst}$

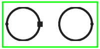
QED isospin breaking: valence


Connected $-1.23(40)_{stat}(31)_{syst}$

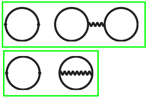

Disconnected $-0.55(15)_{stat}(10)_{syst}$

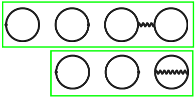
Strong–isospin breaking


Connected
$6.60(63)_{stat}(53)_{syst}$


Disconnected
$-4.67(54)_{stat}(69)_{syst}$

QED isospin breaking: sea


Connected $0.37(21)_{stat}(24)_{syst}$

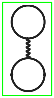

Disconnected $-0.040(33)_{stat}(21)_{syst}$

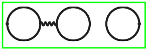
Other

Bottom; higher-order; perturbative

$0.11(4)_{tot}$

QED isospin breaking: mixed


Connected $-0.0093(86)_{stat}(95)_{syst}$


Disconnected $0.011(24)_{stat}(14)_{syst}$

Finite-size effects

Isospin-symmetric

$18.7(2.5)_{tot}$

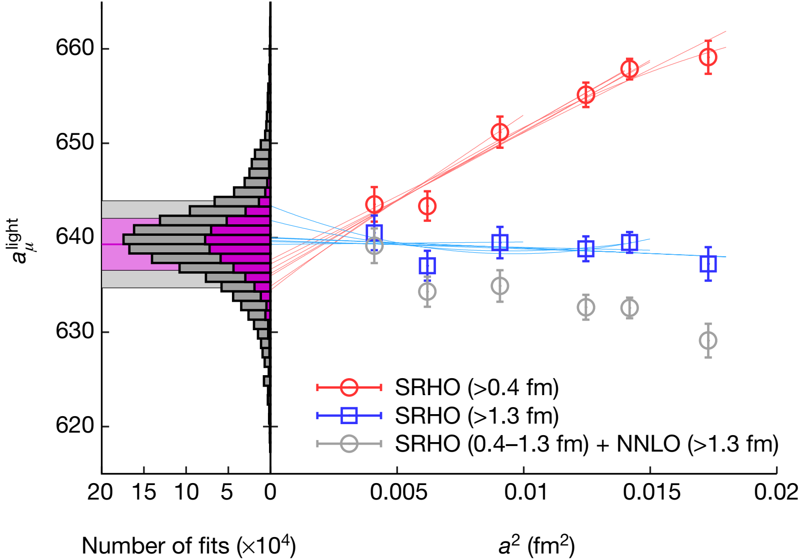
Isospin-breaking

$0.0(0.1)_{tot}$

$$a_\mu^{\text{LO-HVP}}\,(\times 10^{10}) = 707.5(2.3)_{stat}(5.0)_{syst}(5.5)_{tot}$$

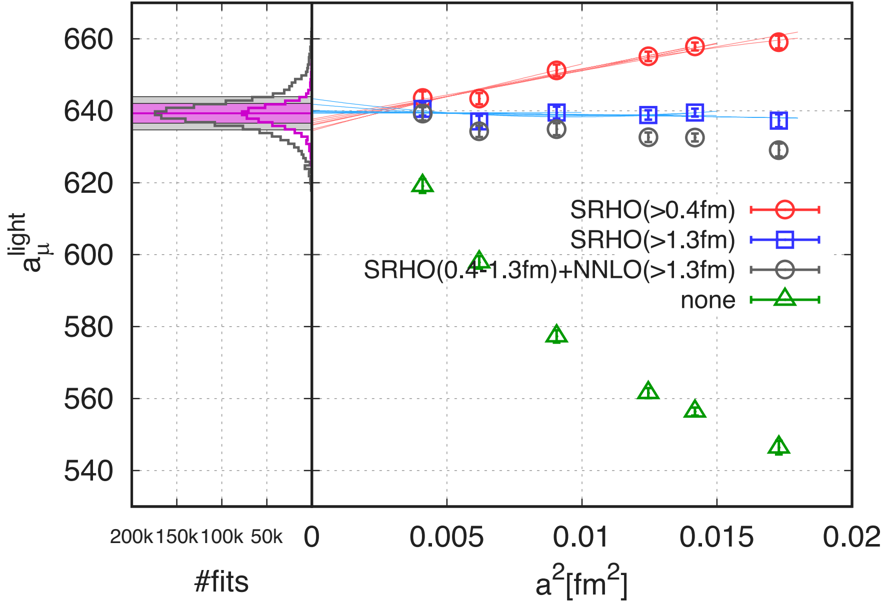
Some details about the BMW calculation

Borsanyi et al. Nature 2021

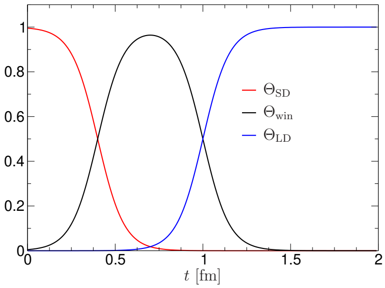


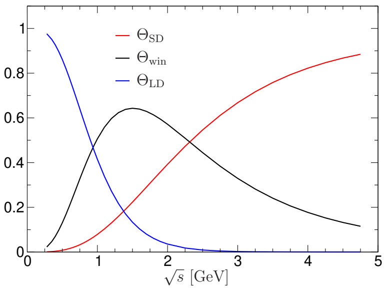
Some details about the BMW calculation

Borsanyi et al. Nature 2021



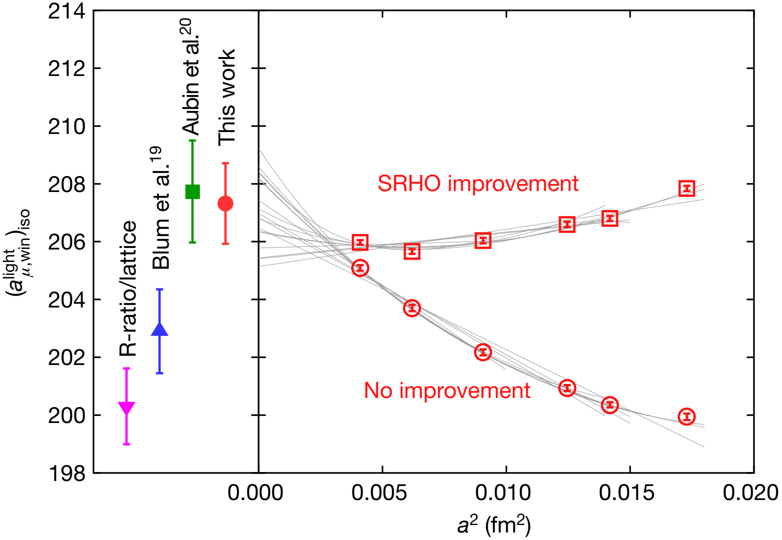
Window quantities are easier to calculate





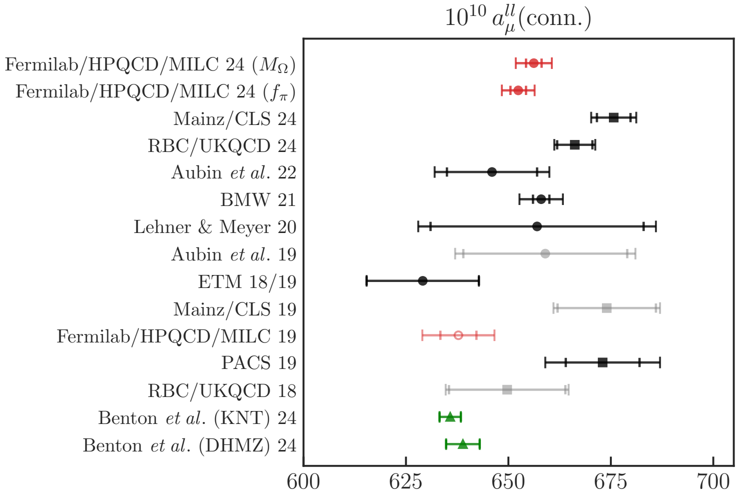
Window quantities are easier to calculate

Article



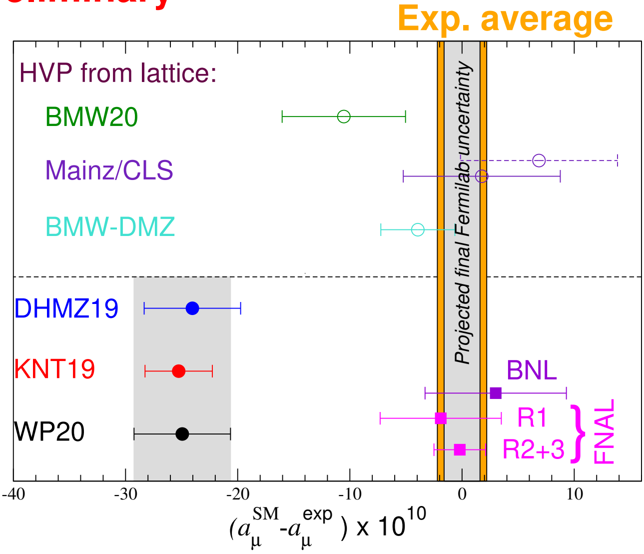
Comparison: light-quark connected contribution

Figure from Fermilab/HPQCD/MILC (24)



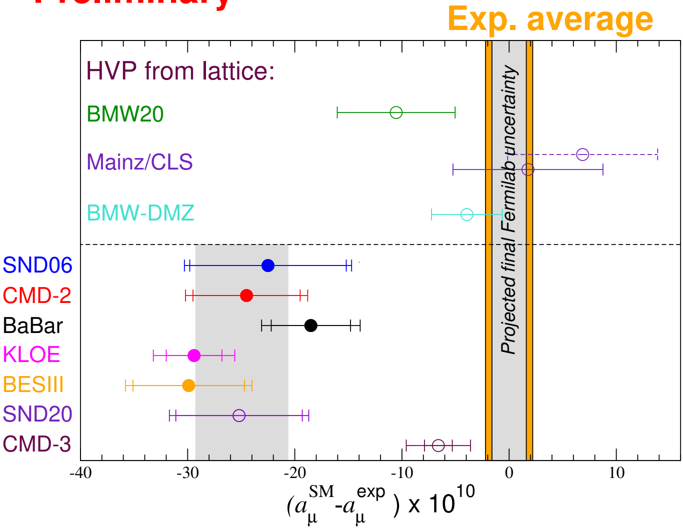
Lattice input for a_μ^{HVP} vs. Experiment

Preliminary



Lattice input for a_μ^{HVP} vs. Experiment

Preliminary



Data-driven data points: CHLS

The 2 × 2 matrix of Hadronic Contributions

	dispersive	lattice
HLbL	✓	✓
HVP	?!?!?!	(✓)

More details and final numbers in WP25 `(2504.xxxxx)`

Conclusions

- Dispersive evaluation of HLbL contribution:
  WP20 20% $\rightarrow$ WP25 $\sim$ 10% accuracy.
  Lattice calculations [Mainz/CLS(21), RBC/UKQCD(23), BMW24] agree with it

- WP20: 0.6% error of data-driven HVP contribution dominated the theory uncertainty

  Main contribution: $\pi\pi$ (<1 GeV) based on [CMD-2, SND, BaBar, KLOE, BES-III]
  Puzzle: results by CMD-3 (23) significantly higher!

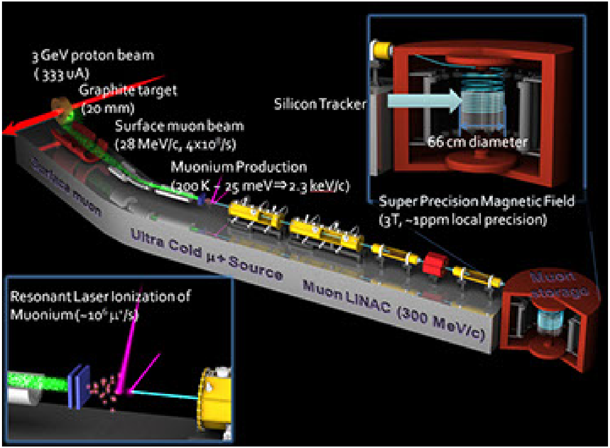
- Lattice calculations of HVP [BMW20, Mainz/CLS24, RBC/UKQCD24, and BMW-DMZ24]:
  similar precision, agree with each other and with CMD-3
  but differ from WP20 dispersive

  HVP from lattice and CMD-3: agreement with the a_μ measurement

Outlook

- The Fermilab experiment aims to reduce the BNL uncertainty by a factor four $\Rightarrow$ final result expected in a few months
- Improvements on the SM theory/data side:
  - Situation for HVP data-driven urgently needs to be clarified:
    - New CMD-3 result—after thorough scrutiny—is a puzzle
    - Forthcoming measur./analyses: BaBar, Belle II, BESIII, KLOE, SND
    - Model-independent evaluation of RadCorr underway
    - MuonE will provide an alternative way to measure HVP
  - HVP lattice:
    good agreement at present; more calculations are coming [Fermilab-MILC-HPQCD, ETMC]; IB evaluation needs to be improved
  - HLbL: goal of $\sim$ 10% uncertainty (data-driven and lattice) has been achieved. Further improvements underway

Future: Muon $g-2$/EDM experiment @ J-PARC



Credit: J-PARC