Beyond the Flavor Anomalies, Rome, April 9-11, 2025

Cabibbo anomaly

S. Simula and T. Tong

$$V_{\rm CKM} \equiv V_L^u V_L^{d\dagger} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$ describes quark flavor weak mixings

$V_{us} = \sin\theta_c$ plays a pivotal role for hierarchy of the Wolfenstein parameterization

first-row unitarity: $|V_u|^2 = |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$ in the SM

$|V_{ub}| = 0.00382\,(24)$ can be neglected

outline

- determinations of $|V_{us}|$ with the SM (SS)
- $|V_{ud}|$ and global fits within EFT (T. Tong)

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main determinations of $|V_{ud}|$ and $|V_{us}|$ from β decays within the SM

source: PDG review 2024 (use of LQCD results with $N_f = 2+1+1$)

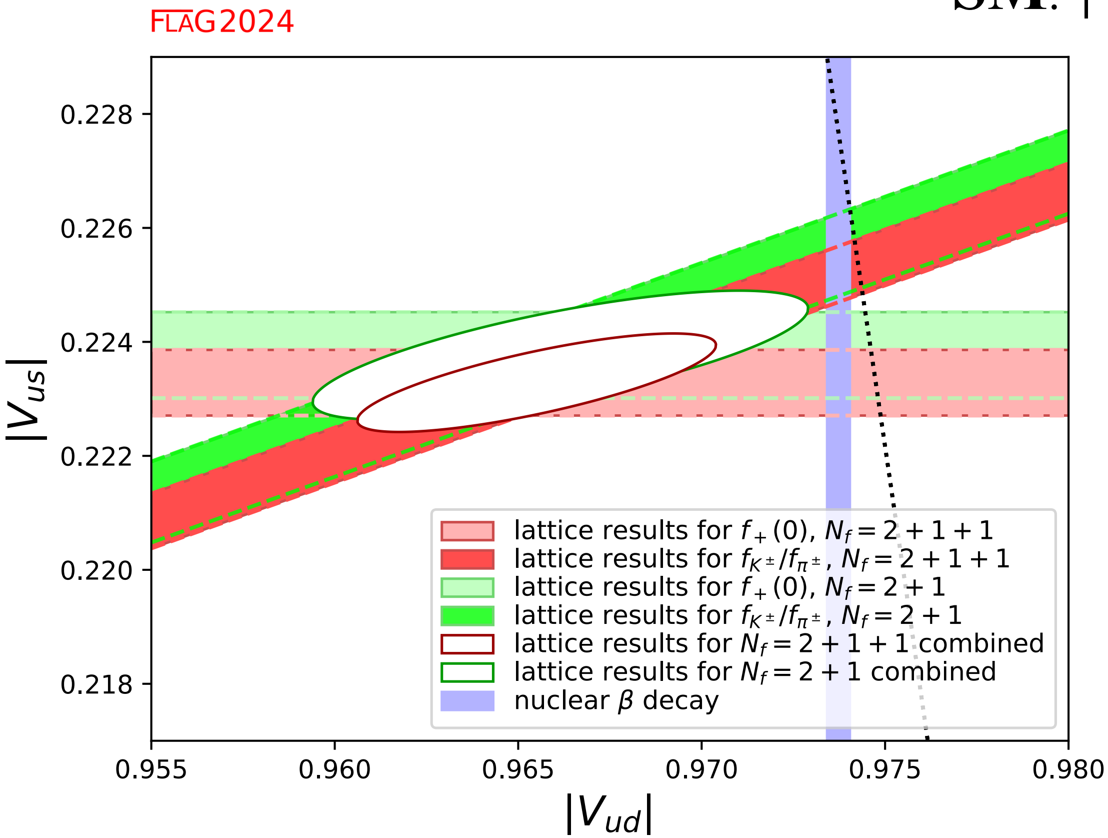
$\lvert V_{ud}\rvert$		$\lvert V_{us}\rvert / \lvert V_{ud}\rvert$		$\lvert V_{us}\rvert$	
0.97367 (32)	superallowed	0.23108 (51)	$K_{\mu2}/\pi_{\mu2}$	0.22330 (53)	$K_{\ell3}$
0.97441 (88)	neutron			0.2207 (14)	τ decay
0.9739 (27)	π_{e3}			0.2250 (27)	hyperon

first-row unitarity:

$$|V_u|^2 \equiv |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.99867\ (68) \quad \simeq 2.0\sigma \quad K_{\mu2}/\pi_{\mu2}$$

$$= 0.99791\ (67) \quad \simeq 3.1\sigma \quad K_{\ell3}$$

SM: $|V_u|^2 = 1$



FLAG-6 (2411.04268)

(use of LQCD results with $N_f = 2+1+1$)

$|V_{us}|/|V_{ud}| = 0.23126\ (50) \quad K_{\mu2}/\pi_{\mu2}$

$|V_{us}| = 0.22328\ (58) \quad K_{\ell3}$

using $|V_{ud}| = 0.97373\ (31)$

$$|V_u|^2 = 0.99888\ (67) \quad \simeq 1.7\sigma \quad K_{\mu2}/\pi_{\mu2}$$

$$= 0.99802\ (66) \quad \simeq 3.0\sigma \quad K_{\ell3}$$

first-row unitarity deficit

between 2σ ($K_{\mu2}/\pi_{\mu2}$) and 3σ ($K_{\ell3}$)

most precise determinations of the Cabibbo angle

=> ***leptonic*** $K_{\ell 2}/\pi_{\ell 2}$ ***decays***: $\frac{|V_{us}|}{|V_{ud}|}$ using LQCD determinations of $\frac{f_K}{f_\pi}$ (**axial** weak current in SM)

=> ***semileptonic*** $K_{\ell 3}$ ***decays***: $|V_{us}|$ using LQCD determinations of $f_+(q^2=0)$ (**vector** weak current in SM)

results from these two sources have been consolidated by lattice QCD+QED simulations in recent years

other processes

=> ***hadronic τ-decays***: $|V_{us}|$ from determinations of $R_{us} = \frac{\Gamma(\tau \mapsto X_{us}\nu_\tau)}{\Gamma(\tau \mapsto e\bar{\nu}_e\nu_\tau)}$ (**V-A** weak current in SM)

there are quite interesting **news** from recent lattice QCD simulations of R_{us} (and also R_{ud})

=> ***hyperon decays***: $|V_{us}|$ from semileptonic hyperon decays with $\Delta S = 1$ (**V-A** weak current in SM)

$$|V_{us}| = 0.2250\,(27)$$ **[Cabibbo et al. hep-ph/0307214]**

extraction of $|V_{us}|/|V_{ud}|$ from leptonic $K_{\ell 2}$ and $\pi_{\ell 2}$ decays

$$\Gamma(PS^+ \to \ell^+ \nu_\ell) = \frac{G_F^2}{8\pi} |V_{q_1 q_2}|^2 m_\ell^2 \left(1 - \frac{m_\ell^2}{M_{PS^+}^2} \right)^2 M_{PS^+} S_{EW} f_{PS}^2 \left(1 + \delta_{SU(2)}^{PS^+} + \delta_{EM}^{PS^+} \right)$$

S_{EW} = universal short-distance EW correction $\simeq 1.0232\,(3)$

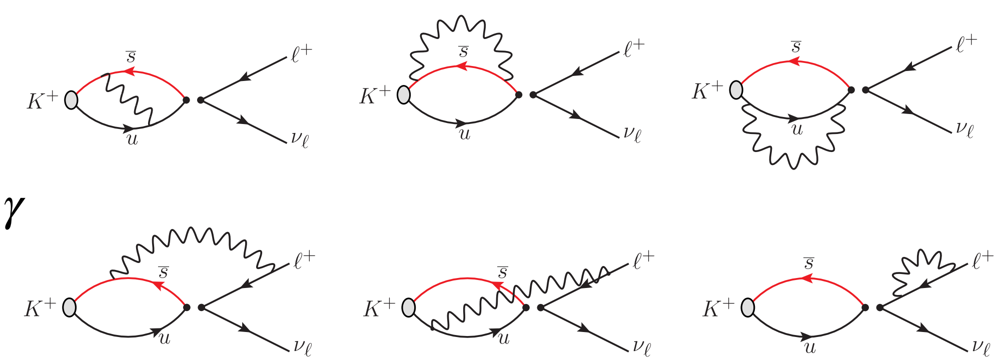
$f_{PS} \equiv p^\mu \langle 0 | \overline{q}_2 A_\mu q_1 | PS(p^\mu) \rangle / M_{PS}^2$ = meson decay constant in isoQCD ($m_u = m_d$ and $\alpha_{EM} = 0$)

$\delta_{SU2}^{PS^+}$ = strong SU(2)-breaking corrections (due to $m_u \neq m_d$)

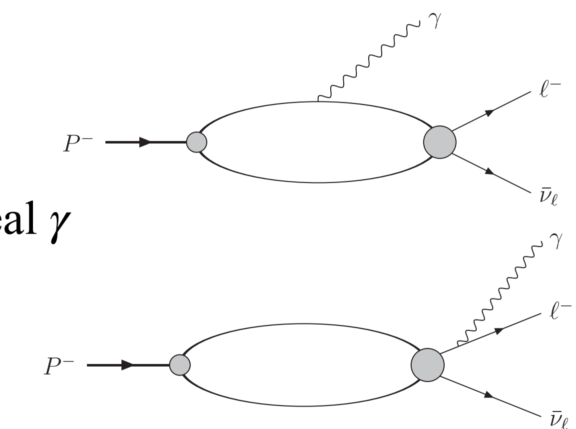
see FLAG-6 (**2411.04268**) for the prescription defining the isoQCD physical point

$\delta_{EM}^{PS^+}$ = EM corrections depending on the hadronic structure of the decaying meson ($\alpha_{EM} \neq 0$)

virtual γ



real γ



master relation:
$$\frac{\Gamma(K_{\ell 2})}{\Gamma(\pi_{\ell 2})} = \frac{|V_{us}|^2}{|V_{ud}|^2} \frac{M_{K^+}}{M_{\pi^+}} \left(\frac{1 - m_\ell^2 / M_{K^+}^2}{1 - m_\ell^2 / M_{\pi^+}^2} \right)^2 \frac{f_K^2}{f_\pi^2} \left(1 + \delta^{K^+} - \delta^{\pi^+} \right)$$

$$\delta^{PS^+} = \delta_{SU(2)}^{PS^+} + \delta_{EM}^{PS^+}$$

FLAG and PDG reviews make use of ChPT estimates **[see Cirigliano and Neufeld 1102.0563]**

$$\delta^{K^+}_{SU(2)} - \delta^{\pi^+}_{SU(2)} = -0.0043\,(12) \qquad \delta^{K^+}_{EM} - \delta^{\pi^+}_{EM} = -0.0069\,(17)$$

ChPT

$$\delta^{K^+} - \delta^{\pi^+} = -0.0112\,(21)$$

in 2018/2019 the first QCD+QED determination of the isospin-breaking corrections on the lattice by the RM123+Soton collaboration using ETMC gauge configurations **[see arXiv:1711.06537 and arXiv:1904.08731]**

$$\delta^{K^+}_{SU(2)} - \delta^{\pi^+}_{SU(2)} = -0.0064\,(7) \qquad \delta^{K^+}_{EM} - \delta^{\pi^+}_{EM} = -0.0062\,(12)$$

lattice QCD+QED$_L$

$$\delta^{K^+} - \delta^{\pi^+} = -0.0126\,(14)$$

reassuring agreement …

- another recent lattice result: $\delta^{K^+} - \delta^{\pi^+} = -0.0086\,(39)$ **from RBC/UKQCD 2211.12865** (power-law FVEs)
- an interesting approach is QED$_\infty$ + QCD with IVR **[see Christ et al. 2304.08026]** (exponentially small FVEs)

it is customary to include strong SU(2)-breaking corrections as $\dfrac{f_{K^+}}{f_{\pi^+}} = \dfrac{f_K}{f_\pi}\sqrt{1 + \delta^{K^+}_{SU(2)} - \delta^{\pi^+}_{SU(2)}}$

$$\left.\frac{\Gamma(K_{\mu 2})}{\Gamma(\pi_{\mu 2})}\right|_{exp.} \to \frac{|V_{us}|}{|V_{ud}|}\frac{f_{K^+}}{f_{\pi^+}} = 0.27599\,(41) \quad (\simeq 0.15\%)$$

[see Moulson 1704.04104] adopted by FLAG-6 (2411.04268)

using δ_{EM} from lattice QCD+QED: $\dfrac{|V_{us}|}{|V_{ud}|}\dfrac{f_{K^+}}{f_{\pi^+}} = 0.27683\,(29)_{exp}\,(20)_{th}$ [35] $(\simeq 0.13\%)$

FLAG-6 (2411.04268)

Collaboration	Ref.	N_f	publication status	chiral extrapolation	continuum extrapolation	finite-volume errors	f_K/f_π	$f_{K^\pm}/f_{\pi^\pm}$
ETM 21	[45]	2+1+1	A	★	★	★	1.1995(44)(7)	1.1957(44)(7)
CalLat 20	[44]	2+1+1	A	★	★	★	1.1964(32)(30)	1.1942(32)(31)
FNAL/MILC 17	[20]	2+1+1	A	★	★	★	1.1980(12)($^{+5}_{-15}$)	1.1950(15)($^{+6}_{-18}$)
ETM 14E	[43]	2+1+1	A	○	★	○	1.188(11)(11)	1.184(12)(11)
FNAL/MILC 14A	[21]	2+1+1	A	★	★	★		1.1956(10)($^{+26}_{-18}$)
ETM 13F	[356]	2+1+1	C	○	★	○	1.193(13)(10)	1.183(14)(10)
HPQCD 13A	[42]	2+1+1	A	★	○	★	1.1948(15)(18)	1.1916(15)(16)
MILC 13A	[357]	2+1+1	A	★	★	★		1.1947(26)(37)
MILC 11	[358]	2+1+1	C	○	○	○		$1.1872(42)^{\dagger}_{\mathrm{stat.}}$
ETM 10E	[359]	2+1+1	C	○	○	○	$1.224(13)_{\mathrm{stat}}$	
CLQCD 23	[10]	2+1	A	★	★	★		1.1907(76)(17)
QCDSF/UKQCD 16	[50]	2+1	A	○	★	○	1.192(10)(13)	1.190(10)(13)
BMW 16	[49, 360]	2+1	A	★	★	★	1.182(10)(26)	1.178(10)(26)
RBC/UKQCD 14B	[12]	2+1	A	★	★	★	1.1945(45)	
RBC/UKQCD 12	[229]	2+1	A	★	○	★	1.199(12)(14)	
Laiho 11	[54]	2+1	C	○	★	○		$1.202(11)(9)(2)(5)^{\dagger\dagger}$
MILC 10	[47]	2+1	C	○	★	★		1.197(2)($^{+3}_{-7}$)
JLQCD/TWQCD 10	[361]	2+1	C	○	■	★	1.230(19)	
RBC/UKQCD 10A	[119]	2+1	A	○	○	★	1.204(7)(25)	
BMW 10	[48]	2+1	A	★	★	★	1.192(7)(6)	
MILC 09A	[19]	2+1	C	○	★	★		1.198(2)($^{+6}_{-8}$)
MILC 09	[196]	2+1	A	○	★	★		1.197(3)($^{+6}_{-13}$)
Aubin 08	[362]	2+1	C	○	○	○		1.191(16)(17)
RBC/UKQCD 08	[236]	2+1	A	○	■	★	1.205(18)(62)	
HPQCD/UKQCD 07	[46]	2+1	A	○	○	○	1.189(2)(7)	
MILC 04	[239]	2+1	A	○	○	○		1.210(4)(13)

only results with A and no red tags enter the FLAG averages

three methods to include strong SU(2)-breaking corrections

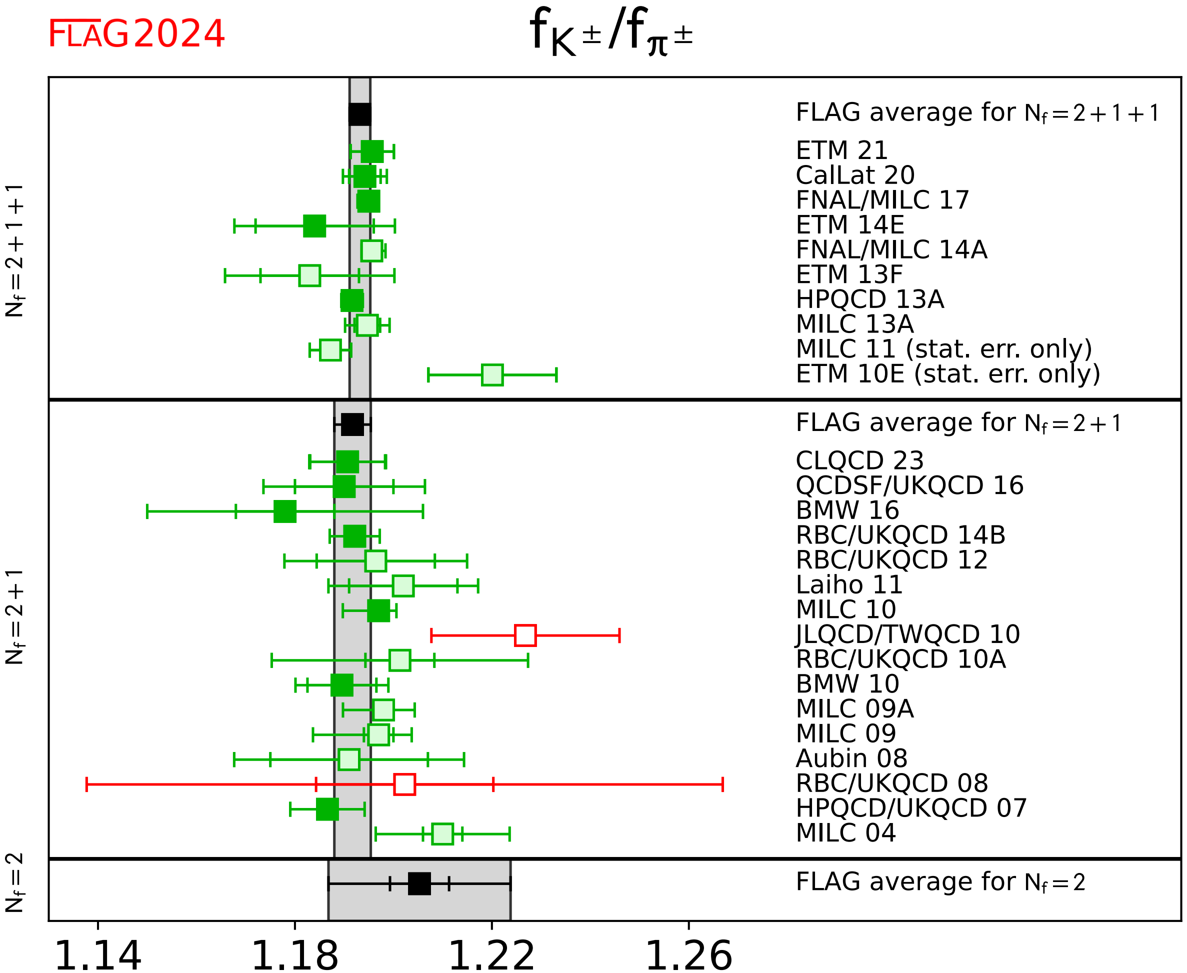
- extrapolation to m_u or m_d
- insertion of the scalar density (RM123 method)
- estimate using ChPT

$\delta_{SU(2)}^{extrapolation} = -0.0054\ (14)$ HPQCD

$\delta_{SU(2)}^{extrapolation} = -0.0052\ (9)$ FNAL/MILC

$\delta_{SU(2)}^{insertion} = -0.0064\ (7)$ ETMC

$\delta_{SU(2)}^{ChPT} = -0.0043\ (12)$



$$\frac{f_{K^+}}{f_{\pi^+}} = 1.1934\,(19) \qquad N_f = 2+1+1 \qquad (\simeq 0.16\%)$$

CalLat, ETMC, FNAL/MILC, HPQCD

$$\frac{f_{K^+}}{f_{\pi^+}} = 1.1916\,(34) \qquad N_f = 2+1 \qquad (\simeq 0.29\%)$$

BMW, CLQCD, HPQCD, MILC, QCDSF, RBC/UKQCD

$$\frac{|V_{us}|}{|V_{ud}|} = 0.23126\,(50) \qquad N_f = 2+1+1 \qquad (\simeq 0.22\%)$$

$$\frac{|V_{us}|}{|V_{ud}|} = 0.23161\,(75) \qquad N_f = 2+1 \qquad (\simeq 0.32\%)$$

open issues

experiment: present database dominated by a single experiment (KLOE)

theory: removal of the electroquenched approximation (null electric charges for sea quarks)

extraction of $|V_{us}|$ from leptonic $K_{\ell 3}$ decays

$$\Gamma(K^{+,0} \to \pi^{0,-} \ell^+ \nu_\ell) = \frac{G_F^2 M_{PS^+}^5}{192\pi^3} C_{K^{+,0}}^2 S_{EW} |V_{us} f_+^{K^0\pi^-}(0)|^2 I_{K\ell}^{(0)} \left(1 + \delta_{EM}^{K^{+,0}\ell} + \delta_{SU(2)}^{K^{+,0}\pi}\right)$$

$C_{K^{+,0}}$ = Clebsch-Gordan coefficient ($C_{K^+} = 1/\sqrt{2}$, $C_{K^0} = 1$)

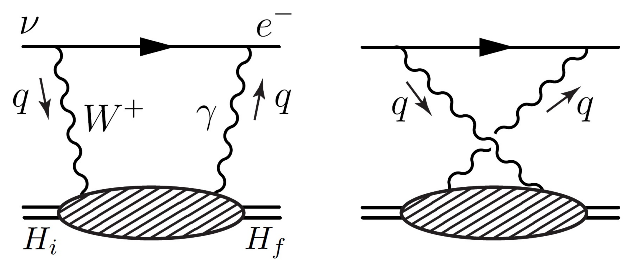
$f_+(0) \equiv f_+^{K^0\pi^-}(0)$ = vector form factor at zero momentum transfer

$I_{K\ell}^{(0)}$ = phase-space integral sensitive to the momentum dependence of vector (and scalar) form factor

$\delta_{EM}^{K^{+,0}\ell}$, $\delta_{SU2}^{K^{+,0}\pi}$ = strong SU(2)-breaking and long-distance EM corrections

$\delta_{EM}^{K^{+,0}\ell}$		**Cirigliano et al. 0807.4507 ChPT**	**Seng et al. 2203.05217 hybrid with LQCD for the γW box**
	K_{e3}^0	0.495 ± 0.110	0.580 ± 0.016
	$K_{e3}^\pm$	0.050 ± 0.125	0.105 ± 0.024
	$K_{\mu 3}^0$	0.700 ± 0.110	0.770 ± 0.019
	$K_{\mu 3}^\pm$	0.008 ± 0.125	0.025 ± 0.027

γW box



		$[\gamma W]_K$
	Feng et al. 2003.09798 (pion)	
LQCD:	**Ma et al. 2102.12048 (kaon)**	$2.437\ (44) \cdot 10^{-3}$
	Yoo et al. 2305.03198 (pion, kaon)	$2.389\ (17) \cdot 10^{-3}$

$\delta_{SU(2)}^{K^0\pi^-} = 0$

$\delta_{SU(2)}^{K^+\pi^0} \propto_{ChPT} \ Q^{-2} = \dfrac{m_d^2 - m_u^2}{m_s^2 - m_{ud}^2}$

$\delta_{SU(2)}^{K^+\pi^0} = (2.61 \pm 0.17)\ \%$ Q from $\eta \to 3\pi$ **Colangelo et al. 1807.11937**

$= (2.52 \pm 0.11)\ \%$ Q from LQCD **Cirigliano et al. 2208.11707**

nice consistency between the channels $K^+ \to \pi^0$ and $K^0 \to \pi^-$

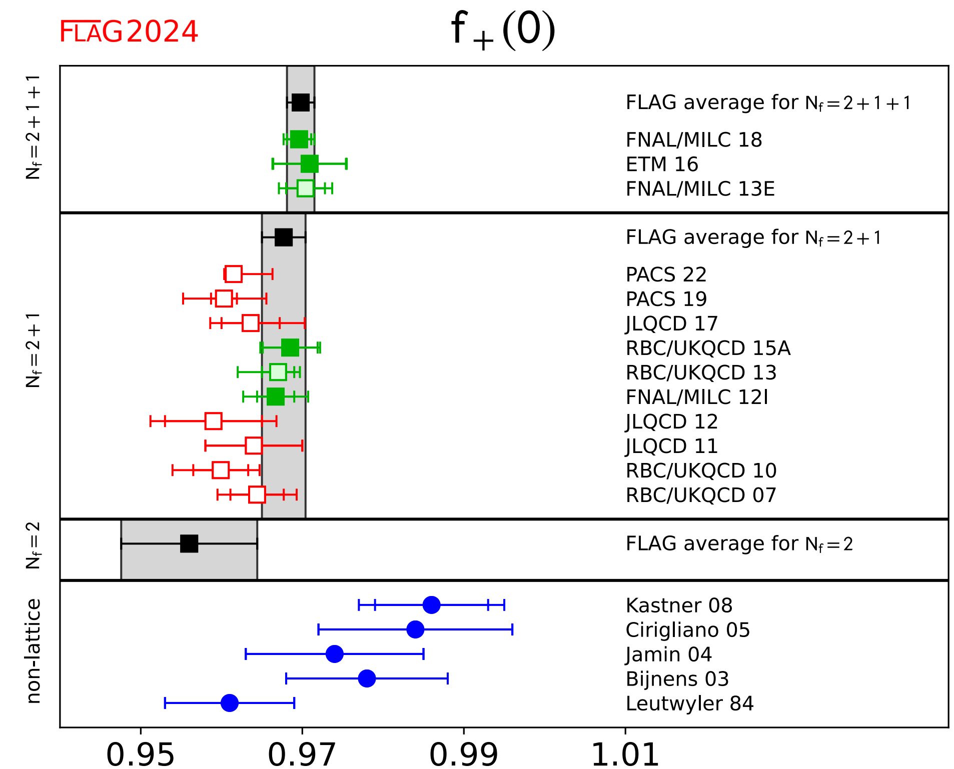
$$\Gamma(K^{+,0} \to \pi^{0,-} \ell^+ \nu_\ell)\Big|_{exps} \longrightarrow |V_{us}| f_+(0) = 0.21654\,(41) \qquad (\simeq 0.19\%)$$

[see Moulson 1704.04104]
adopted by FLAG-6 (2411.04268)

six modes, several exps.
KTeV, KLOE, ISTRA+, NA48/2, …

$\to$ 0.21634 (38) **Seng et al. 2203.05217**

Collaboration	Ref.	N_f	publication status	chiral extrapolation	continuum extrapolation	finite-volume errors	$f_+(0)$
FNAL/MILC 18	[39]	2+1+1	A	★	★	★	0.9696(15)(12)
ETM 16	[38]	2+1+1	A	○	★	○	0.9709(45)(9)
FNAL/MILC 13E	[341]	2+1+1	A	★	★	★	0.9704(24)(22)
PACS 22	[342]	2+1	A	○	■	★	$0.9615(10)(^{+47}_{-6})$
PACS 19	[343]	2+1	A	○	■	★	$0.9603(16)(^{+50}_{-48})$
JLQCD 17	[336]	2+1	A	○	■	○	$0.9636(36)(^{+57}_{-35})$
RBC/UKQCD 15A	[41]	2+1	A	★	○	○	0.9685(34)(14)
RBC/UKQCD 13	[344]	2+1	A	★	○	○	$0.9670(20)(^{+18}_{-46})$
FNAL/MILC 12I	[40]	2+1	A	○	○	★	0.9667(23)(33)
JLQCD 12	[345]	2+1	C	○	■	★	0.959(6)(5)
JLQCD 11	[346]	2+1	C	○	■	★	0.964(6)
RBC/UKQCD 10	[347]	2+1	A	○	■	★	$0.9599(34)(^{+31}_{-47})(14)$
RBC/UKQCD 07	[348]	2+1	A	○	■	★	0.9644(33)(34)(14)



$f_+(0) = 0.9698\,(18) \qquad N_f = 2+1+1 \qquad (\simeq 0.19\%)$ **ETMC, FNAL/MILC**

$f_+(0) = 0.9677\,(27) \qquad N_f = 2+1 \qquad (\simeq 0.28\%)$ **FNAL/MILC, RBC/UKQCD**

$|V_{us}| = 0.22328\,(58) \qquad N_f = 2+1+1 \qquad (\simeq 0.26\%)$

$|V_{us}| = 0.22377\,(75) \qquad N_f = 2+1 \qquad (\simeq 0.34\%)$

open issue

isospin-breaking corrections, both $\delta_{SU2}^{K^{+,0}\pi}$ and $\delta_{EM}^{K^{+,0}\ell}$, not yet available from lattice QCD+QED

isospin-breaking corrections in $K_{\ell 3}$ decays

leptonic decays: Γ_0 = virtual photon rate $\Longrightarrow$ infrared divergent **RM123+Soton 1502.00257**

$\Gamma_1(\Delta E_\gamma)$ = real photon emission up to ΔE_γ $\Longrightarrow$ infrared divergent

$\Gamma(\Delta E_\gamma) = \Gamma_0 + \Gamma_1(\Delta E_\gamma)$ = infrared safe **[Block&Nordsiek '37]**

* infrared divergence universal $\Longrightarrow$ structure independent (soft photons) *

on a lattice $\Longrightarrow$
$$\Gamma(\Delta E_\gamma) = \lim_{V\to\infty}\left[\Gamma_0(V) - \Gamma_0^{pt}(V)\right] + \lim_{\mu_\gamma\to 0}\left[\Gamma_0^{pt}(\mu_\gamma) + \Gamma_1(\mu_\gamma, \Delta E_\gamma)\right]$$

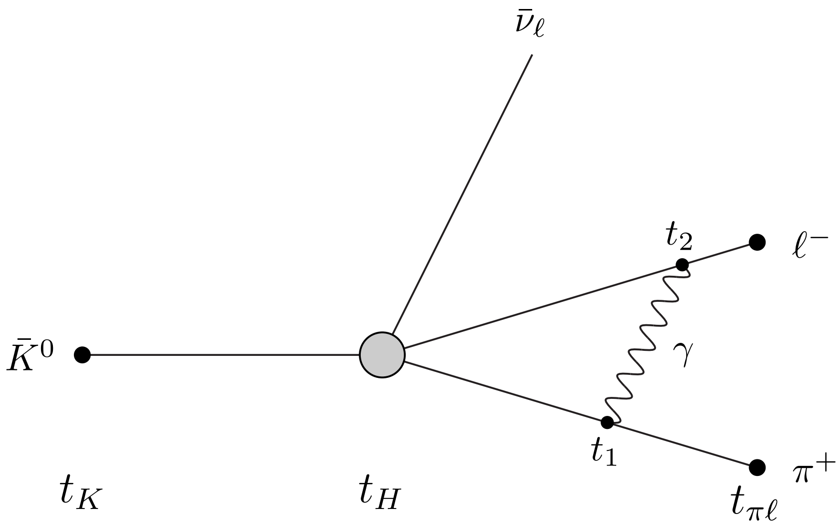
virtual photon regulated by the lattice volume V

virtual and real photons regulated by a small mass μ_γ

pt = point-like & perturbative calculable

$K_{\ell 3}$ decays:
$$\frac{d^2\Gamma(\Delta E_\gamma)}{dq^2 ds_{\pi\ell}} = \lim_{V\to\infty}\left[\frac{d^2\Gamma_0(V)}{dq^2 ds_{\pi\ell}} - \frac{d^2\Gamma_0^{pt}(V)}{dq^2 ds_{\pi\ell}}\right] + \lim_{\mu_\gamma\to 0}\left[\frac{d^2\Gamma_0^{pt}(\mu_\gamma)}{dq^2 ds_{\pi\ell}} + \frac{d^2\Gamma_1(\mu_\gamma, \Delta E_\gamma)}{dq^2 ds_{\pi\ell}}\right]$$

$q^2 = (p_K - p_\pi)^2$ and $s_{\pi\ell} = (p_\pi + p_\ell)^2$ **see Sachrajda@Lat '19 [1910.07342]**



- presence of unphysical terms growing exponentially in Euclidean time $E_{\pi\ell}^{int} < E_{\pi\ell}^{ext}$

  their number depends on $s_{\pi\ell}$ and on BCs

- finite-volume corrections of order $\mathcal{O}(1/L)$ in QED_L depend on $f_{+,0}(q^2)$ and their derivatives $df_{+,0}(q^2)/dq^2$

- an interesting approach is QED_∞ + QCD with IVR **see Christ et al. 2304.08026 and Christ@Lattice '23 [2402.08915]**

determination of $|V_{us}|/|V_{ud}|$ from semileptonic $K_{\ell 3}/\pi_{e3}$

Seng et al. 2107.14798

* semileptonic π_{e3} ($\pi^- \to \pi^0 e \bar{\nu}_e$) decays are a (theoretically clean) way to determine $|V_{ud}|$

however, present experiments on $Br(\pi_{e3})$ lead to $|V_{ud}|_{\pi_{e3}} = 0.9739\ (27)$, ten times less precise than $|V_{ud}|$ from superallowed nuclear beta decays → next generation of pion rare decays (**PIONEER**, …)

* taking $|V_{us}|$ from semileptonic $K_{\ell 3}$ decays and $|V_{ud}|$ from semileptonic π_{e3} decays one gets

$$\frac{|V_{us}|_{K_{\ell 3}}}{|V_{ud}|_{\pi_{e3}}} = 0.22928\ (84) \quad (\simeq 0.36\%) \qquad \text{against} \qquad \left.\frac{|V_{us}|}{|V_{ud}|}\right|_{K_{\ell 2}/\pi_{\ell 2}} = 0.23126\ (50) \quad (\simeq 0.22\%)$$

"vector" ratio — * $\simeq 2.0\sigma$ **difference** * — "axial" ratio

impact of precise measurements of $Br(K_{\mu 3})/Br(K_{\mu 2})$

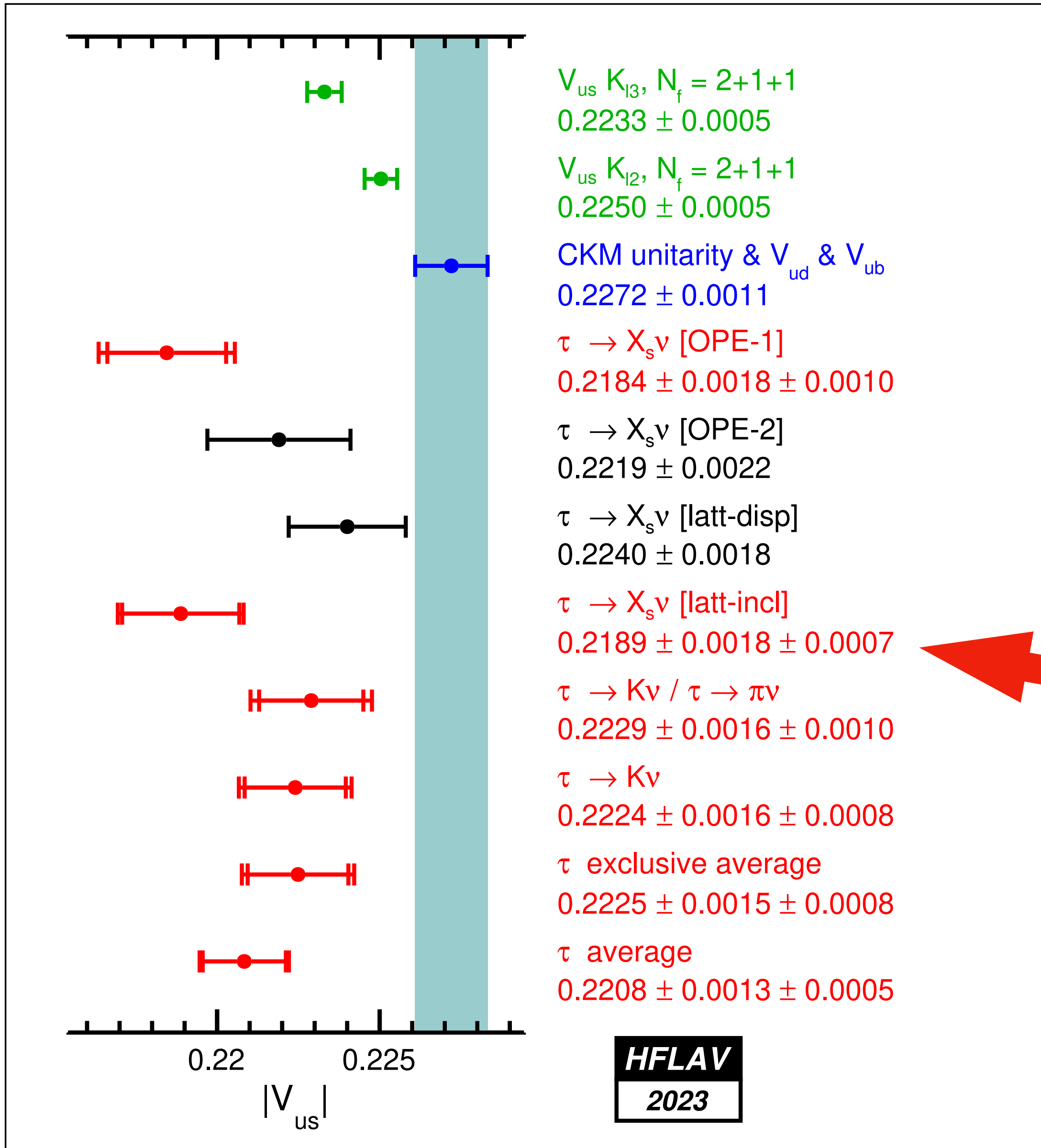
Cirigliano et al. 2208.11707

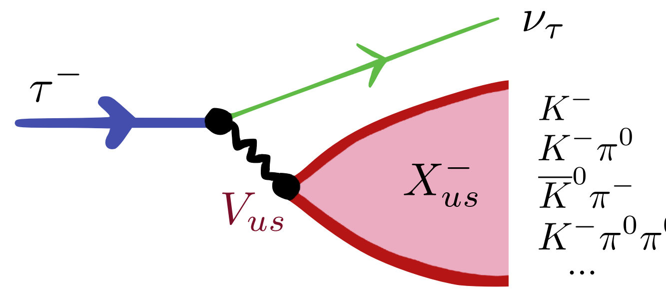
present experimental value: $Br(K_{\mu 3})/Br(K_{\mu 2}) = 0.05294\ (51) \quad (\simeq 0.96\%)$

bringing the precision down to $\simeq 0.2\ \%$ (for instance NA62) may help clarifying the experimental situation between the semileptonic and leptonic kaon sectors

hadronic τ-lepton decays and $|V_{us}|$

summary of $|V_{us}|$ from τ decays **[HFLAV 2411.18639]**





inclusive $\tau \to X_s\nu$ decays

$$|V_{us}| = \sqrt{\frac{R_{us}}{R_{ud}/|V_{ud}|^2 - \delta R_{SU(3)}}}$$

$$R_{us(d)} = \frac{\Gamma(\tau \mapsto X_{us(d)}\nu_\tau)}{\Gamma(\tau \mapsto e\bar{\nu}_e\nu_\tau)}$$ from experiments

$\delta R_{SU(3)}$ evaluated via OPE

OPE-1 **[Gamiz et al. hep-ph/0612154]**
OPE-2 **[Maltman et al. 1510.06954 and 2019]**
latt-disp **[RBC/UKQCD 1803.07226]**
latt-incl **[ETMC 2403.05404]**

no long-distance SU(2)-breaking corrections

exclusive $\tau \to K(\pi)\nu$ decays

require $f_{K(\pi)^\pm}$ from LQCD and long-distance RCs

[see Arroyo-Ureña et al. 2107.04603]

VMD model for SD FFs

* Fermi effective theory, optical theorem, Lorentz invariance

$$R_{us} \equiv \frac{\Gamma(\tau \to X_{us}\nu_\tau)}{\Gamma(\tau \to e\bar{\nu}_e\nu_\tau)} = 6\pi S_{EW} |V_{us}|^2 \int_0^1 ds\,(1-s)^2\, \left[\rho_L(s) + (1+2s)\,\rho_T(s)\right]$$

$s = q^2/m_\tau^2$

$S_{EW} = 1.0201\,(3)$

* L and T components of the spectral density for the weak (us) hadronic current

$$\rho^{\mu\nu}_{us}(q) = (2\pi)^4 \langle 0 | J^\mu_{us}(0)\, \delta^4(\mathscr{P} - q)\, [J^\nu_{us}(0)]^\dagger | 0\rangle = q^\mu q^\nu \rho_L(q^2) + \left(g^{\mu\nu} q^2 - q^\mu q^\nu\right) \rho_T(q^2)$$

* through lattice QCD simulations we can access Euclidean correlators

$$C^{\mu\nu}_{us}(t, \vec{q}) = \int d\vec{x}\, e^{-i\vec{q}\cdot\vec{x}} \langle 0 | T\left\{ J^\mu_{us}(-it, \vec{x}) \left[J^\nu_{us}(0)\right]^\dagger \right\} | 0 \rangle$$

$$\xrightarrow[t>0]{} \frac{1}{2\pi} \int_0^\infty dE\, e^{-Et}\, \rho^{\mu\nu}_{us}(E, \vec{q})$$

$$C_L(t) = C^{00}_{us}(t, \vec{0}) = \frac{1}{2\pi} \int_0^\infty dE\, e^{-Et} E^2 \rho_L(E^2)$$

$$C_T(t) = \frac{1}{3} C^{ii}_{us}(t, \vec{0}) = \frac{1}{2\pi} \int_0^\infty dE\, e^{-Et} E^2 \rho_T(E^2)$$

* inversion is ill-conditioned for kernels with non-smooth functions $\Longrightarrow$ **smearing** **[Gambino et al. 2005.13730]**

$$R^{L(T)}_{us} = \lim_{\sigma \to 0} R^{L(T)}_{us}(\sigma) = 12\pi S_{EW} \frac{|V_{us}|^2}{m_\tau^3} \int_0^\infty dE\, K_{L(T)}\left(\frac{E}{m_\tau}\right) \frac{1}{1 + e^{-\frac{E}{m_\tau \sigma}}} E^2 \rho_{L(T)}(E^2)$$

$$= \lim_{\sigma \to 0} 12\pi S_{EW} \frac{|V_{us}|^2}{m_\tau^3} \lim_{V\to\infty,\, a \to 0} \sum_{n=1}^{N} g^{L(T)}_n(\sigma)\, C_{L(T)}(na)$$

$$K_L(x) = \frac{1}{x}(1 - x^2)^2$$

$$K_T(x) = K_L(x)\,(1 + 2x^2)$$

$K_{L(T)}\left(\frac{E}{m_\tau}\right) \frac{1}{1 + e^{-\frac{E}{m_\tau \sigma}}} \to \sum_{n=1}^{N} g^{L(T)}_n(\sigma)\, e^{-naE}$ evaluated using the Hansen-Lupo-Tantalo (HLT) method **1903.06476**

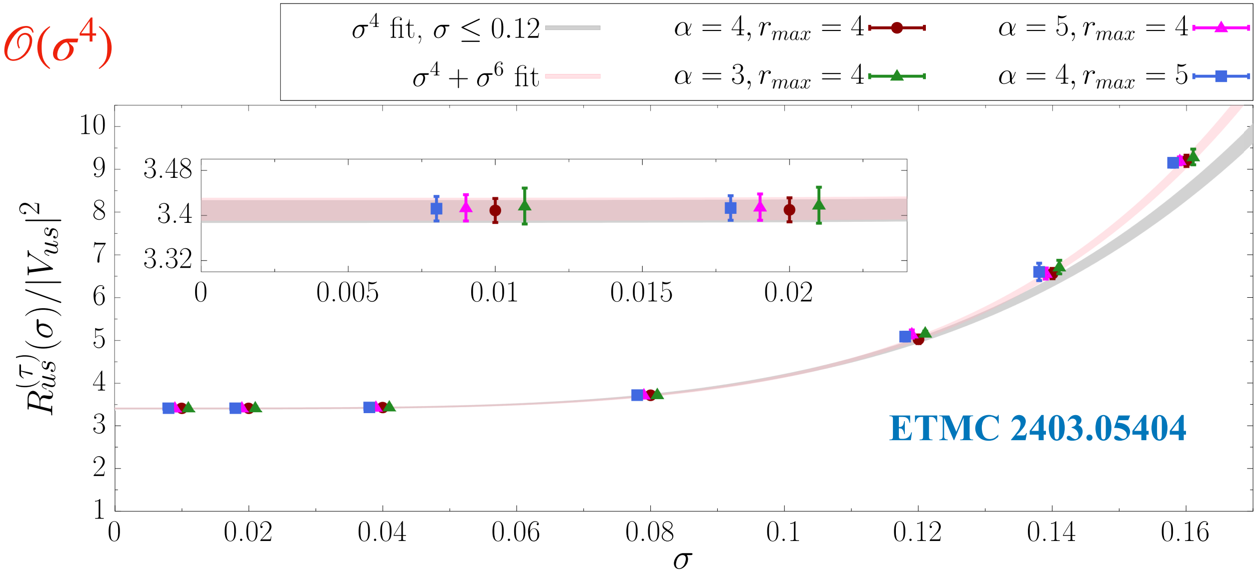
↓

minimization of an appropriate functional of syst. and stat. errors

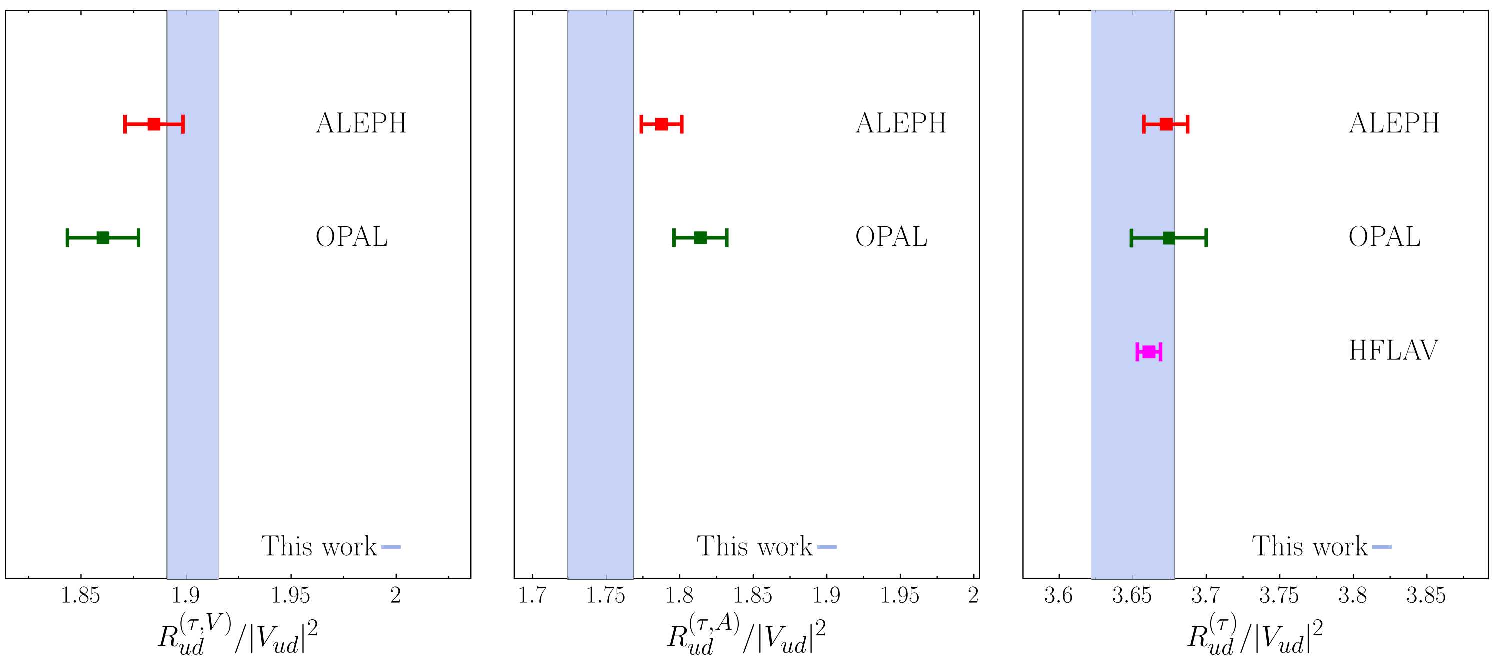
* it can be shown that $R^{L(T)}_{ud(s)}(\sigma) = R^{L(T)}_{ud(s)} + \mathcal{O}(\sigma^4)$

ETMC 2308.03125

extrapolation to $\sigma \to 0$ under well control



non-strange hadronic final states **[ETMC 2308.03125]**



$$\left.\frac{R_{ud}}{|V_{ud}|^2}\right|_{LQCD} = 3.650\,(28)$$
(~0.8 %)

$$\left.R_{ud}\right|_{HFLAV} = 3.471\,(7)$$
(~0.2 %)

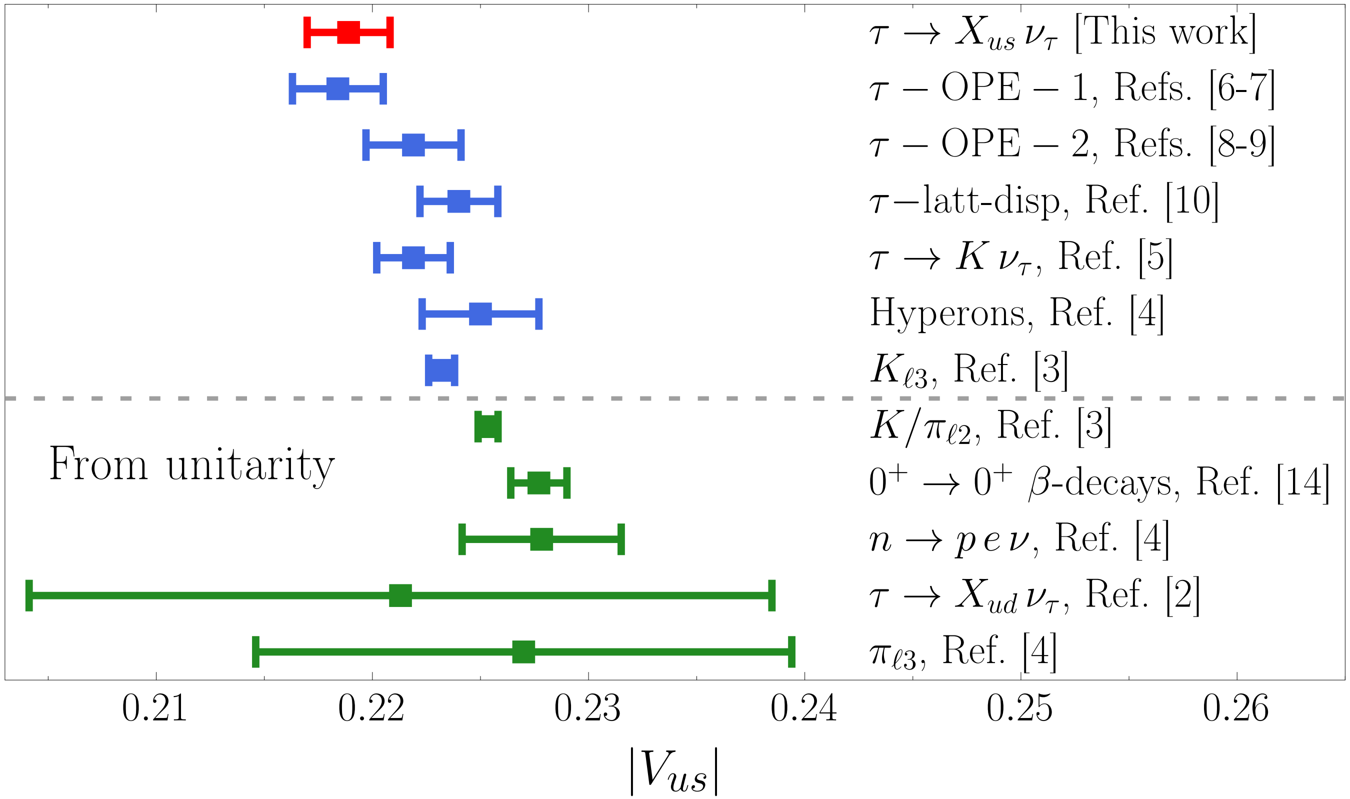
$$|V_{ud}| = 0.9752\,(37)_{th}\,(10)_{exp}$$

compatible with superallowed, but not competitive

$$\left.\frac{R^V_{ud} - R^A_{ud}}{R_{ud}}\right|_{LQCD} = 0.042\,(5)$$

0.026 (7) ALEPH **[hep-ex/0506072]**
0.013 (7) OPAL **[hep-ex/9808019]**

strange hadronic final states [ETMC 2403.05404]



$$\left.\frac{R_{us}}{|V_{us}|^2}\right|_{isoQCD} = 3.407\,(22)$$

(~0.7 %)

$$R_{us}\Big|_{HFLAV} = 0.1632\,(27)$$

(~1.7 %)

$$|V_{us}| = 0.2189\,(7)_{th}\,(18)_{exp}$$

2.2σ from $K_{\ell 3}$

3.3σ from $K_{\ell 2}/\pi_{\ell 2}$

using both (us) and (ud) channels

$$\frac{|V_{us}|}{|V_{ud}|} = 0.2244\,(11)_{th}\,(19)_{exp}$$

3.3σ from $K_{\ell 2}/\pi_{\ell 2}$

V/A decomposition and strange/non-strange ratios

$$\left.\frac{R^V_{us} - R^A_{us}}{R_{us}}\right|_{isoQCD} = 0.079\,(8)$$

preliminary

$$\left.\frac{|V_{ud}|^2}{|V_{us}|^2}\frac{R^V_{us}}{R^V_{ud}}\right|_{isoQCD} = 0.967\,(10) \quad \text{vector channel}$$

$$\left.\frac{|V_{ud}|^2}{|V_{us}|^2}\frac{R^A_{us}}{R^A_{ud}}\right|_{isoQCD} = 0.900\,(16) \quad \text{axial channel}$$

thanks to G. Gagliardi

open issues

- better precision for the **experimental result** of R_{us} (presently 1.7 %)

- **isospin-breaking corrections** δR_{us} not yet available from lattice QCD+QED (expected at the percent level)

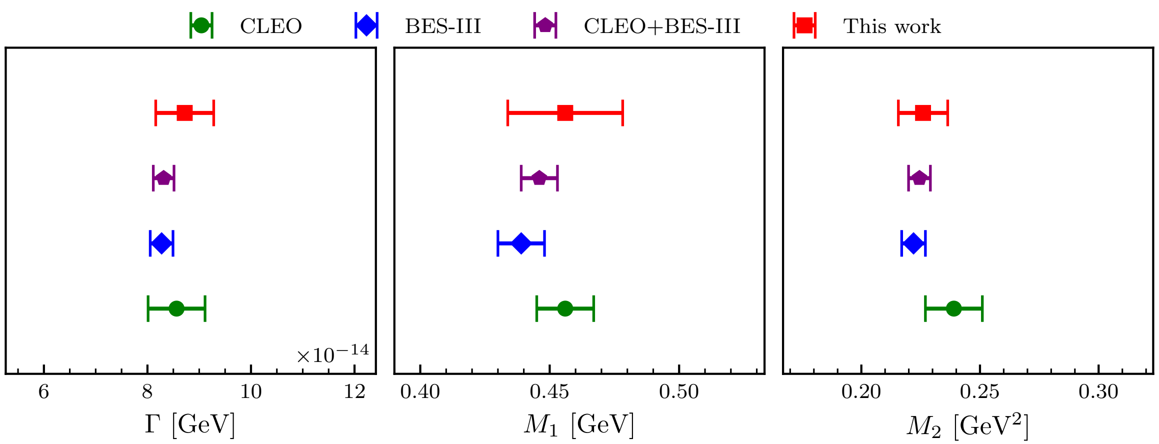
$$R_{us} = R_{us}^{(iso)} \left[1 + \delta R_{us}\right]$$

- the 3.3σ difference with $|V_{us}|$ from $K_{\ell 2}/\pi_{\ell 2}$ would require a fractional shift $\delta R_{us} = -0.058\,(18)$

- the evaluation of δR_{us} from first-principles is mandatory

work in progress by a collaboration among people from CERN, Cyprus Institute, Helmholtz Institut (Mainz), Humboldt Universität (Berlin), Universities of RM-ToV and RM3, …

a bit of advertising: a first-principle lattice QCD calculation of the inclusive semileptonic decay of the Ds-meson **[A. De Santis et al. (ETMC) 2504.06063 and 2504.06064]** with the HLT method



accuracy relevant for phenomenological studies

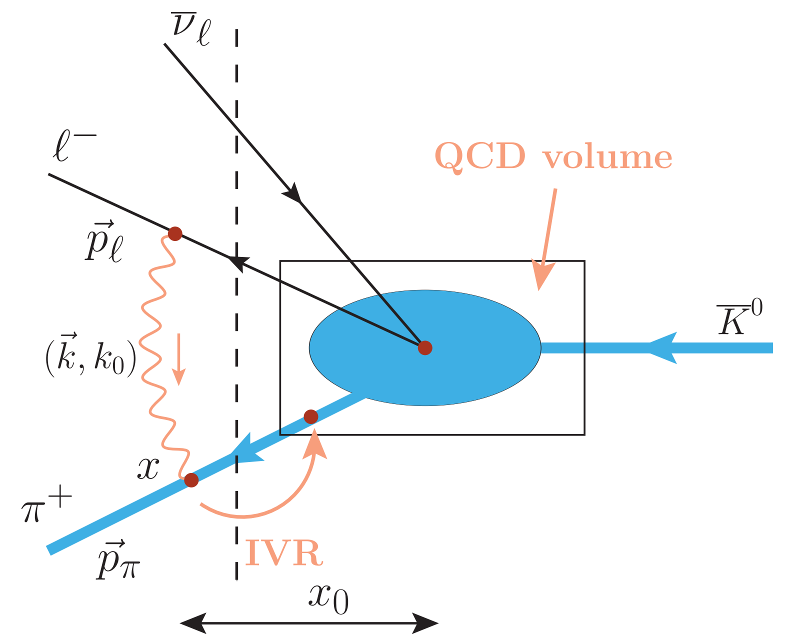
extension to the b-sector

backup slides

QED$_\infty$ + QCD with IVR [Christ et al. 2304.08026 and Christ@Lattice '23 2402.08915]

single pion contribution at large x_0

$$
\begin{aligned}
\mathcal{A}_\pi^{\mu\nu}(\vec{p}_\pi, \vec{x}, x_0) &= \langle \pi(\vec{p}_\pi)| J^\mu_{\rm EM}(\vec{x}, x_0) \left[\int d^3p |\pi(\vec{p})\rangle\langle\pi(\vec{p})| \right] J^\nu_{\rm W}(0)|K(\vec{0})\rangle_M . \\
&= \int d^3p e^{-i(x_0+it_s)(E_{\vec{p}}-E_\pi)} \langle \pi(\vec{p}_\pi)| J^\mu_{\rm EM}(\vec{x}, -it_s)|\pi(\vec{p})\rangle\langle\pi(\vec{p})| J^\nu_{\rm W}(0)|K\vec{0}\rangle_M \\
&= \int d^3p e^{-i(x_0+it_s)(E_{\vec{p}}-E_\pi)} \int \frac{d^3y}{(2\pi)^3} e^{i(\vec{p}-\vec{p}_\pi)\cdot(\vec{x}-\vec{y})} \\
&\qquad h^{\mu\rho} h^{\nu\sigma} \langle \pi(\vec{p}_\pi)| J^\rho_{\rm EM}(\vec{y}, t_s) J^\sigma_{\rm W}(0)|K\vec{0}\rangle_E ,
\end{aligned}
$$



the Minkowski-space single-pion contribution is expressed as the Fourier transform of a Euclidean amplitude calculable with lattice QCD

- pion-photon scattering in the continuum and infinite volume
- exponentially decreasing FSEs (not power-like)

impact of precise measurements of $Br(K_{\mu 3})/Br(K_{\mu 2})$

Cirigliano et al. 2208.11707

	current fit	$K_{\mu3}/K_{\mu2}$ BR at 0.5%			$K_{\mu3}/K_{\mu2}$ BR at 0.2%		
		central	$+2\sigma$	-2σ	central	$+2\sigma$	-2σ
χ^2/dof	25.5/11	25.5/12	31.8/12	32.1/12	25.5/12	35.6/12	35.9/12
p-value [%]	0.78	1.28	0.15	0.13	1.28	0.04	0.03
BR($\mu\nu$) [%]	63.58(11)	63.58(09)	63.44(10)	63.72(11)	63.58(08)	63.36(10)	63.80(11)
$S(\mu\nu)$	1.1	1.1	1.3	1.4	1.2	1.6	1.7
BR($\pi\pi^0$) [%]	20.64(7)	20.64(6)	20.73(7)	20.55(8)	20.64(6)	20.78(7)	20.50(10)
$S(\pi\pi^0)$	1.1	1.2	1.3	1.5	1.2	1.5	2.0
BR($\pi\pi\pi$) [%]				5.56(4)			
$S(\pi\pi\pi)$				1.0			
BR(K_{e3}) [%]	5.088(27)	5.088(24)	5.113(25)	5.061(31)	5.088(23)	5.128(24)	5.046(32)
$S(K_{e3})$	1.2	1.2	1.2	1.6	1.3	1.3	1.8
BR($K_{\mu3}$) [%]	3.366(30)	3.366(13)	3.394(16)	3.336(27)	3.366(7)	3.411(13)	3.320(18)
$S(K_{\mu3})$	1.9	1.2	1.5	2.6	1.1	2.2	3.1
BR($\pi\pi^0\pi^0$) [%]				1.764(25)			
$S(\pi\pi^0\pi^0)$				1.0			
$\tau_\pm$ [ns]	12.384(15)	12.384(15)	12.382(15)	12.385(15)	12.384(15)	12.381(15)	12.386(15)
$S(\tau_\pm)$				1.2			
$\frac{V_{us}}{V_{ud}}\Big\rvert_{K_{\ell2}/\pi_{\ell2}}$	0.23108(51)	0.23108(50)	0.23085(51)	0.23133(51)	0.23108(49)	0.23071(51)	0.23147(52)
$V_{us}^{K_{\ell3}}$	0.22330(53)	0.22337(51)	0.22360(52)	0.22309(54)	0.22342(49)	0.22386(52)	0.22287(52)
$\frac{F_K}{F_\pi}\frac{V_{us}}{V_{ud}}\Big\rvert_{K_{\ell2}/\pi_{\ell2}}$	0.27679(34)	0.27679(31)	0.27651(35)	0.27709(34)	0.27679(30)	0.27634(33)	0.27726(35)
$f_+(0)V_{us}^{K_{\ell3}}$	0.21656(35)	0.21662(31)	0.21685(33)	0.21636(35)	0.21667(28)	0.21710(32)	0.21614(34)
$\Delta^{(1)}_{\text{CKM}}$	−0.00176(56)	−0.00173(55)	−0.00162(56)	−0.00185(56)	−0.00171(55)	−0.00151(56)	−0.00195(56)
	-3.1σ	-3.1σ	-2.9σ	-3.3σ	-3.1σ	-2.7σ	-3.5σ
$\Delta^{(2)}_{\text{CKM}}$	−0.00098(58)	−0.00098(58)	−0.00108(58)	−0.00087(58)	−0.00098(58)	−0.00114(58)	−0.00081(58)
	-1.7σ	-1.7σ	-1.9σ	-1.5σ	-1.7σ	-2.0σ	-1.4σ
$\Delta^{(3)}_{\text{CKM}}$	−0.0164(63)	−0.0157(60)	−0.0118(62)	−0.0202(63)	−0.0153(59)	−0.0083(62)	−0.0233(62)
	-2.6σ	-2.6σ	-1.9σ	-3.2σ	-2.6σ	-1.4σ	-3.8σ

$$\Delta^{(1)}_{\text{CKM}} = \left|V_{ud}^{\beta}\right|^2 + \left|V_{us}^{K_{\ell3}}\right|^2 - 1,$$
$$\Delta^{(2)}_{\text{CKM}} = \left|V_{ud}^{\beta}\right|^2 + \left|V_{us}^{K_{\ell2}/\pi_{\ell2},\beta}\right|^2 - 1,$$
$$\Delta^{(3)}_{\text{CKM}} = \left|V_{ud}^{K_{\ell2}/\pi_{\ell2},\,K_{\ell3}}\right|^2 + \left|V_{us}^{K_{\ell3}}\right|^2 - 1,$$

Table 1: Fit results for the current global fit as well as variants including a new measurement of the $K_{\mu3}/K_{\mu2}$ branching fraction, with uncertainty of 0.5% and 0.2%, respectively, and central value either as expected from the current fit, $\text{BR}(K_{\mu3})/\text{BR}(K_{\mu2}) = 0.05294(51)$, or shifted by $\pm 2\sigma$ of the current fit error. In each channel, the scale factors are given to quantify the tension as originating therefrom [3]. Note that the branching ratios for $\pi\pi\pi$ and $\pi\pi^0\pi^0$ are virtually unaffected by the new measurement due to very few correlated ratios with the (semi-) leptonic channels in the data base (in cases in which no significant changes occur, only a single entry is given that applies to all columns). The values of V_{us} and V_{us}/V_{ud} are extracted using the same input as described in the main text, adding in quadrature all uncertainties given in Eq. (7). $\Delta^{(1,2,3)}_{\text{CKM}}$ are defined in Eq. (8), and $\Delta^{(1,2)}_{\text{CKM}}$ are evaluated using V_{ud}^{β} from Eq. (5).

The smeared-ratio from a Backus-Gilbert-like approach

We however still need a regularization mechanism to tame the oscillations of the $\boldsymbol{g}_{\mathrm{I}}$ coefficients (that would blow up our uncertainties).

The Hansen-Lupo-Tantalo (HLT) method provides the coefficients $\boldsymbol{g}_{\mathrm{I}}(\sigma)$ minimizing a functional $W_{\mathrm{I}}^{\alpha}[\boldsymbol{g}]$ which balances syst. and stat. errors of reconstructed $R_{ud}^{(\tau,\mathrm{I})}(\sigma)$

$$W_{\mathrm{I}}^{\alpha}[\boldsymbol{g}] = \frac{A_{\mathrm{I}}^{\alpha}[\boldsymbol{g}]}{A_{\mathrm{I}}^{\alpha}[\boldsymbol{0}]} + \lambda B_{\mathrm{I}}[\boldsymbol{g}]\,, \qquad \left.\frac{\partial W_n[\boldsymbol{g}]}{\partial g}\right|_{\boldsymbol{g}=\boldsymbol{g}_{\mathrm{I}}} = 0$$

$$A_{\mathrm{I}}^{\alpha}[\boldsymbol{g}] = \int_{E_{\mathrm{min}}}^{r_{\mathrm{max}}/a} \mathrm{d}E\; e^{aE\alpha} \left| K_{\mathrm{I}}^{\sigma}\left(\frac{E}{m_\tau}\right) - \sum_{n=1}^{N} g_n e^{-naE} \right|^2 \Longleftarrow \text{(syst.)}^2 \text{ error due to reconstruction}$$

$$B_{\mathrm{I}}[\boldsymbol{g}] \propto \sum_{n_1,n_2=1}^{N} g_{n_1} g_{n_2} \,\mathrm{Cov}\left(C_{\mathrm{I}}(an_1), C_{\mathrm{I}}(an_2)\right) \Longleftarrow \text{(stat.)}^2 \text{ error of reconstructed } R_{ud}^{(\tau,\mathrm{I})}(\sigma)$$

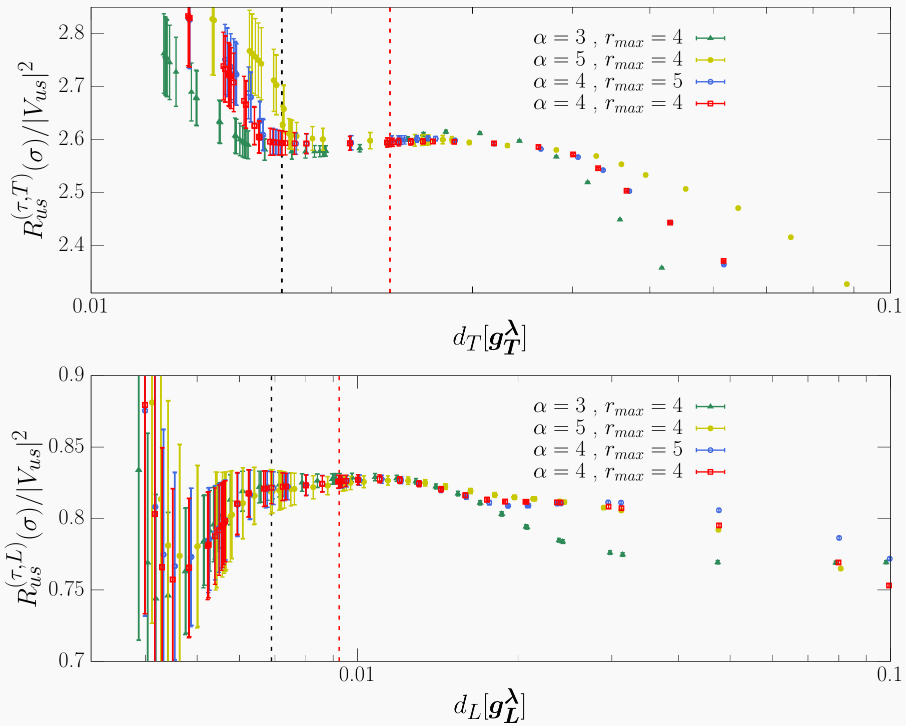
- λ is trade-off parameter $\Longrightarrow$ tuned for optimal balance of syst. and stat. errors. $\{\alpha, E_{\mathrm{min}}, r_{\mathrm{max}}\}$ algorithmic params. to tune for optimal performance.

courtesy by G. Gagliardi talk @ Lattice '24 (Liverpool)

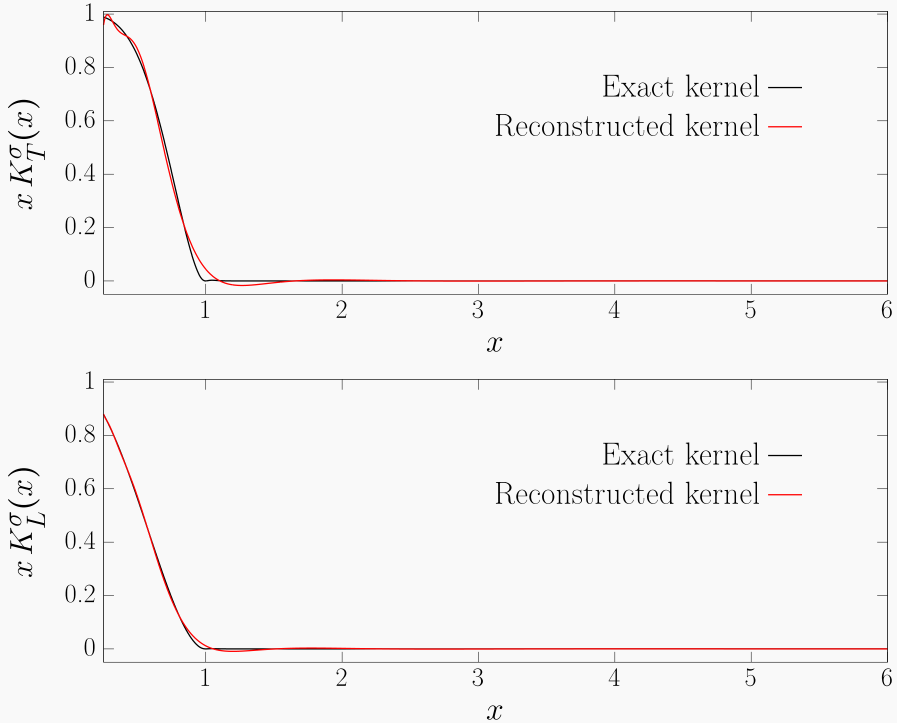
Stability analysis [Bulava et al, JHEP07 (2022)] $(\sigma = 0.02)$

For each contribution and σ, perform a scan in λ to find the region where stat. errors dominate over systematics due to incorrect reconstruction of kernel functions.

- Goodness of reconstruction *measured* by $d_{\mathrm{I}}[\boldsymbol{g}_{\mathbf{I}}^{\boldsymbol{\lambda}}] \equiv \sqrt{A_{\mathrm{I}}^{0}[\boldsymbol{g}_{\mathbf{I}}^{\boldsymbol{\lambda}}]/A_{\mathrm{I}}^{0}[\mathbf{0}]}$



Optimal λ (rightmost vertical line) chosen in the region of small $d_{\mathrm{I}}[\boldsymbol{g}_{\mathbf{I}}^{\boldsymbol{\lambda}}]$ where a plateaux is visible.



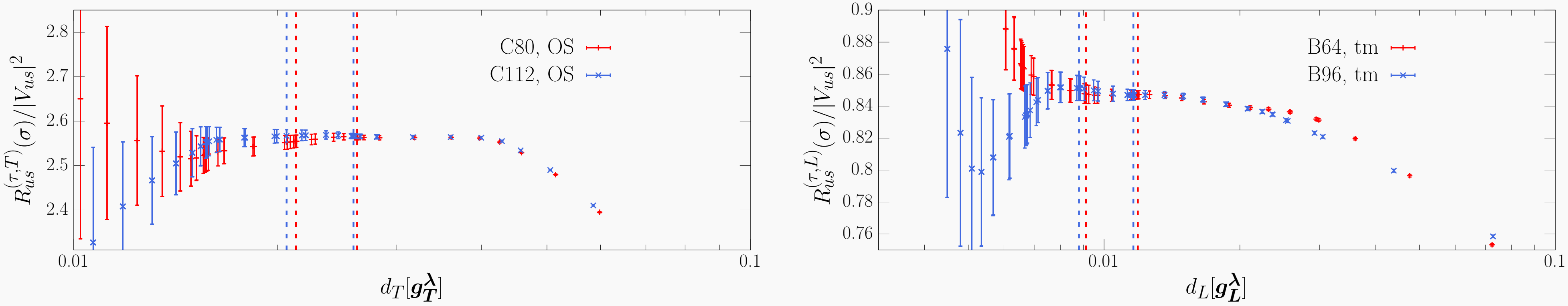
Comparison between exact and reconstructed kernel at optimal λ.

Exponential penalty $\exp(\alpha a E)$ for errors at large E drastically improves stability.

courtesy by G. Gagliardi talk @ Lattice '24 (Liverpool)

Data-driven estimate of FSEs ($\sigma = 0.02$)

FSEs estimated from observed spread on B64/B96 and C80/C112 ensembles.



- FSEs typically very tiny...larger than $2\sigma_{\rm stat}$ in only 1% of the cases.
- We associate to our results at $L \sim 5.5$ fm a systematic error due to FSEs estimated as

$$\Sigma_{\rm I}^{\rm FSE}(\sigma) = \max_{\rm r=\{tm,OS\}} \left\{ \Delta_{\rm I}^{\rm r}(\sigma)\, {\rm erf}\left(\frac{1}{\sqrt{2}\sigma_{\Delta_{\rm I}^{\rm r}(\sigma)}} \right) \right\}$$

$$\Delta_{\rm I}^{\rm r}(\sigma) = \left| R_{us}^{(\tau,{\rm I}),{\rm r}}(\sigma, {\rm C80}) - R_{us}^{(\tau,{\rm I}),{\rm r}}(\sigma, {\rm C112}) \right|, \quad \sigma_{\Delta_I^r(\sigma)} \text{ is relative uncertainty of } \Delta_{\rm I}^{\rm r}(\sigma)$$

courtesy by G. Gagliardi talk @ Lattice '24 (Liverpool)

Relation between spectral density and Euclidean correlator

$$C^{\alpha\beta}(t,\boldsymbol{q}) = \int d^3x\, e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \langle 0|\, T\left(J^{\alpha}_{ud}(-it,\boldsymbol{x})\, J^{\beta}_{ud}(0)^{\dagger}\right)\, |0\rangle$$

Let's find the relation between $C^{\alpha\beta}(t,\boldsymbol{q})$ and the spectral density $\rho^{\alpha,\beta}(E,\boldsymbol{q})$:

$$C^{\alpha\beta}(t,\boldsymbol{q}) \overset{t\geq 0}{=} \int d^3x e^{-i\boldsymbol{q}\boldsymbol{x}} \langle 0|J^{\alpha}_{ud}(0) e^{-\mathcal{H}t+i\boldsymbol{\mathcal{P}}\boldsymbol{x}} J^{\beta}_{ud}(0)^{\dagger}|0\rangle$$

$$= \langle 0|J^{\alpha}_{ud}(0) e^{-\mathcal{H}t}(2\pi)^3\delta^3(\boldsymbol{\mathcal{P}}-\boldsymbol{q}) J^{\beta}_{ud}(0)^{\dagger}|0\rangle$$

$$= \int_{-\infty}^{\infty} \frac{\mathrm{d}E}{2\pi} e^{-Et} \langle 0|J^{\alpha}_{ud}(0)(2\pi)^4 \underbrace{\delta(\mathcal{H}-E)\,\delta^3(\boldsymbol{\mathcal{P}}-\boldsymbol{q})}_{\delta^4(\mathcal{P}-q_E)\ ,\ q_E=(E,\boldsymbol{q})} J^{\beta}_{ud}(0)^{\dagger}|0\rangle$$

where we just used the relation $e^{-\mathcal{H}t} = \int_{-\infty}^{\infty} \frac{\mathrm{d}E}{2\pi}\, e^{-Et}\, 2\pi\, \delta(\mathcal{H}-E)$

Recalling the definition of the spectral density one has

$$C^{\alpha\beta}(t,\boldsymbol{q}) \overset{t\geq 0}{=} \int_0^{\infty} \frac{\mathrm{d}E}{2\pi} e^{-Et} \rho^{\alpha\beta}_{ud}(E,\boldsymbol{q})$$

courtesy by G. Gagliardi talk @ Lattice '24 (Liverpool)

inclusive τ-lepton decays and $|V_{us}|$

[RBC/UKQCD 1803.07226]

new sum rule:

$$\int_0^\infty \tilde{\rho}_{us}(s)\omega_N(s)ds = \sum_{k=1}^{N} \underset{s=-Q_k^2}{\text{Res}} \left[\tilde{\Pi}_{us}(-s)\omega_N(s)\right]$$

$$= \sum_{k=1}^{N} \frac{\tilde{\Pi}_{us;V+A}(Q_k^2)}{\prod_{j\neq k}(Q_j^2 - Q_k^2)} \equiv \tilde{F}_{\omega_N}$$

weight function $\omega_N(s) = \Pi_{k=1}^N \frac{1}{s+Q_k^2}$ (Q_k^2 space-like poles $\lesssim 1$ GeV^2)

$\tilde{\Pi}_{us;V+A}(Q_k^2) =$ HVP calculated in the lattice

$$|V_{us}| = \sqrt{\tilde{R}_{us;w_N} / \left(\tilde{F}_{\omega_N} - \int_{m_\tau^2}^\infty \tilde{\rho}_{us}^{\rm pQCD}(s)\omega_N(s)ds\right)}$$

$\tilde{R}_{us;w_N}$ → integrated data with the weight function

$\tilde{\rho}_{us}^{\rm pQCD}$ → pQCD spectral density

weight function tailored to minimize the high-s region

⬇

$|V_{us}| = 0.2240\,(18)$

consistent with $|V_{us}| = 0.2224\,(18)$ from $\tau \to K\nu_\tau$

TABLE I. Sample relative spectral integral contributions.

Contribution	Value [%]			
[N, C(GeV^2)]	[3, 0.3]	[3, 1]	[4, 0.7]	[5, 0.9]
K	65.5	30.9	61.7	66.9
$K\pi$	21.4	28.6	26.4	25.2
$K^-\pi^+\pi^-$	2.4	5.6	2.8	2.1
$\bar{K}^0\pi^-\pi^0$	3.1	7.3	3.6	2.7
Residual	2.7	6.8	2.9	2.1
pQCD	4.9	20.8	2.7	1.1

inclusive hadronic τ-lepton decays

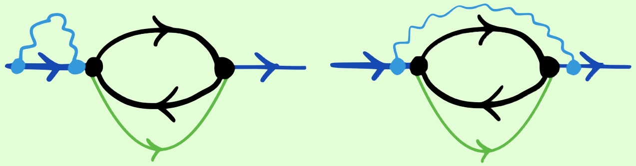
$$\Gamma(\tau \to X_{us}\nu_\tau) = \frac{\rho(m_\tau)}{2m_\tau} \qquad \rho(\omega) = \langle \tau^- | H_{\mathrm{w}}^{us}\,(2\pi)\delta(\mathbb{H}-\omega)\,H_{\mathrm{w}}^{us} | \tau^- \rangle$$

with X_{us} being inclusive in hadrons + photons:

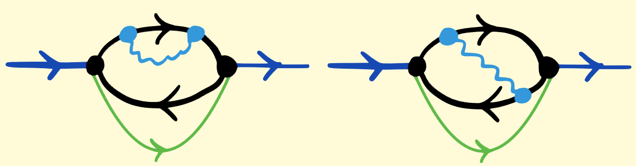
RM123 approach

$$\Gamma = \Gamma_{\text{lep}} + \Gamma_{\text{fact}} + \Gamma_{\text{non-fact}}$$

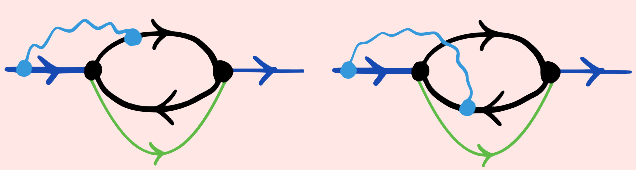
preliminary data look promising!



$$\Gamma_{\text{lep}} = \frac{G_F^2\, m_\tau^5}{(4\pi)^4} \int_0^\infty ds\, \left[\delta\mathcal{K}_{\mathrm{T}}(s)\rho_{\mathrm{T}}(s) + \delta\mathcal{K}_{\mathrm{L}}(s)\rho_{\mathrm{L}}(s)\right]$$



$$\Gamma_{\text{fact}} = \frac{G_F^2\, m_\tau^5}{32\pi^2} \int_0^\infty ds\, \left[\mathcal{K}_{\mathrm{T}}(s)\rho_{\mathrm{T}}^{\text{full}}(s) + \mathcal{K}_{\mathrm{L}}(s)\rho_{\mathrm{L}}^{\text{full}}(s)\right]$$



$$\Gamma_{\text{non-fact}} = \frac{G_F^2\, m_\tau^5}{64\pi^2} \int_0^\infty ds\, \mathcal{K}(s)\,\delta\rho(s)$$

courtesy by M. Di Carlo talk @ LatticeNET 2025 (Benasque)