# Progress with non-local form factors

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Beyond the Flavour Anomalies 2025, Centro Congressi Sapienza
Rome, 9th April 2025

Based on 2406.14608 and 2412.04388

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# What’s in this talk?



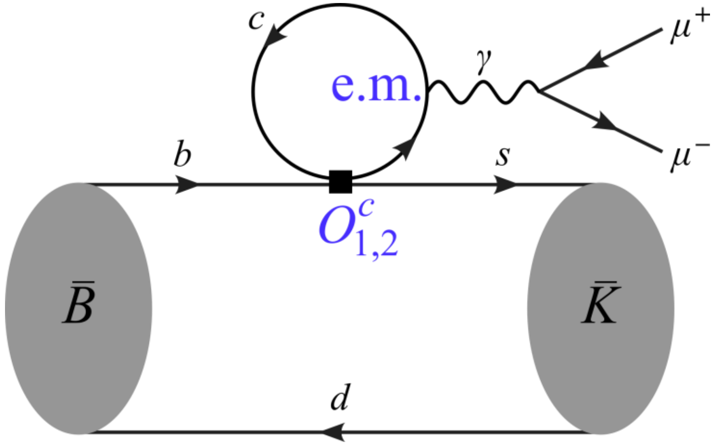
Definition of the **non-local form factors** (Nico)

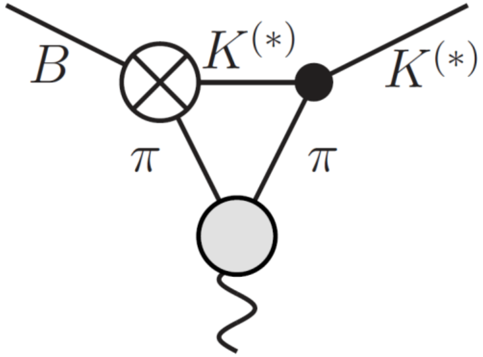
Methods to calculate non-local form factors
$\Longrightarrow$ **light-cone OPE** (Nico)

**Anomalous cuts** (Simon)

Calculation of **rescattering effects** (Simon)

**Parametrizations and unitarity bounds** in presence of subthreshold and anomalous cuts (Nico)



# Introduction

# $B \to K^{(*)}\ell^+\ell^-$ decay amplitude



calculate decay amplitudes precisely to probe the SM

$b \to s\mu^+\mu^-$ anomalies: NP or underestimated QCD uncertainties?

$$\mathcal{A}(B \to K^{(*)}\ell^+\ell^-) = \mathcal{N}\left[\underbrace{\left(C_9 L_V^\mu + C_{10} L_A^\mu\right)}\mathcal{F}_\mu - \underbrace{\frac{L_V^\mu}{q^2}}\left(C_7\, \mathcal{F}_{T,\mu} + \mathcal{H}_\mu\right)\right]$$

Wilson coefficients, leptonic matrix elements (and constants $\boldsymbol{\alpha}$, $\boldsymbol{V_{CKM}}$…)

perturbative objects, small uncertainties

# $B \to K^{(*)}\ell^+\ell^-$ decay amplitude



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local hadronic matrix elements (MEs)

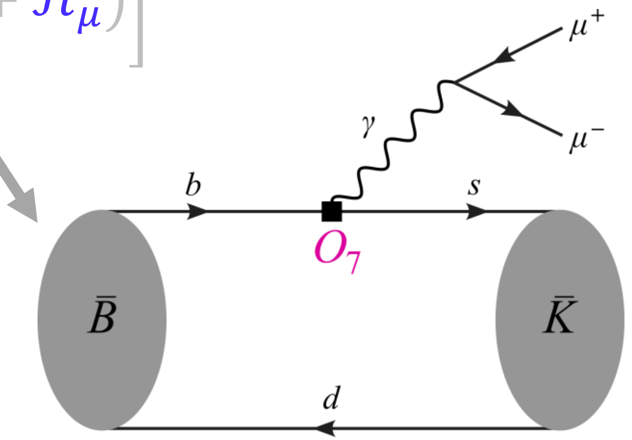
$$\mathcal{F}_\mu = \langle K^{(*)} | O_{7,9,10}^{\text{had}} | B \rangle \qquad O_{7,9,10}^{\text{had}} = (\bar{s}\,\Gamma\, b)$$

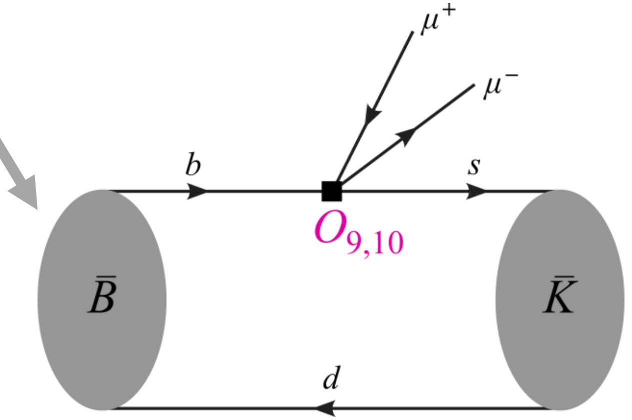
leading hadronic contributions

non-perturbative QCD objects

$\Longrightarrow$ calculate with lattice QCD (or LCSR)

moderate uncertainties ($3\% - 15\%$ )



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non-local hadronic MEs

$$\mathcal{H}_\mu = i\int d^4x\, e^{iq\cdot x}\langle K^{(*)}|T\{j_\mu^{\mathrm{em}}(x), O_{1,2}^c(0)\}|B\rangle$$

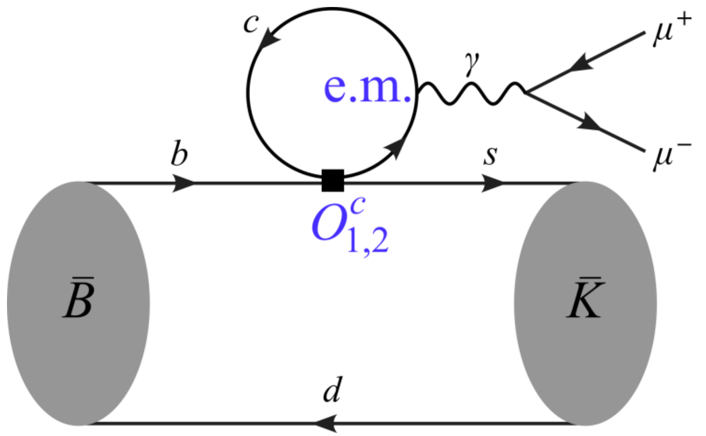
$$O_{1,2}^c = (\bar{s}\,\Gamma\, b)(\bar{c}\,\Gamma\, c)$$

subleading (?) hadronic contributions

non-perturbative QCD objects

$\Longrightarrow$ very hard to calculate

large uncertainties



# Methods to calculate non-local FFs



Parametrize hadronic matrix elements in terms of form factors (FFs)

$$\mathcal{H}_\mu(k,q) = \sum_\lambda \mathcal{S}_\lambda(k,q)\, \mathcal{H}_\lambda(q^2)$$

Non-perturbative techniques are needed to compute non-local FFs

- lattice QCD ⟹ please wait one more hour
- QCD factorization:
  factorize hard and soft contributions
  ⟹ double expansion in $1/m_b$ and $1/E_{K^{(*)}}$
  valid for $q^2 < 7\ \mathrm{GeV}^2$
  How to calculate power corrections? How extend to $\Lambda_b$ decays?
  Is the perturbative treatment of the charm loop reliable close to threshold?
- light-cone operator product expansion (LCOPE) ⟹ see next slide

# Light-cone OPE for non-local FFs



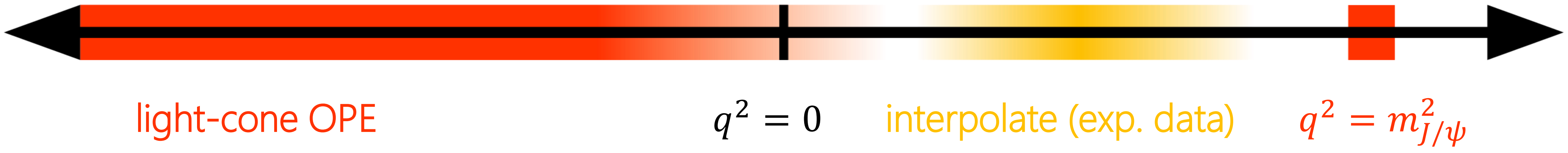
1. Calculate the non-local FFs $\mathcal{H}_\lambda$ using a LCOPE at **negative** $q^2$

$$\mathcal{H}_\lambda(q^2) = C_\lambda(q^2)\mathcal{F}_\lambda(q^2) + \tilde{C}_\lambda(q^2)\mathcal{V}_\lambda(q^2) + \cdots$$

2. Extract $\mathcal{H}_\lambda$ at $q^2 = m_{J/\psi}^2$ from $B \to K^{(*)}J/\psi$ measurements

3. **Interpolate** these two results to obtain theoretical predictions in the low $q^2$ $(0 < q^2 < 8\ \mathrm{GeV}^2)$ region $\Longrightarrow$ compare with experimental data

Need a parametrization to interpolate $\mathcal{H}_\lambda$ $\Longrightarrow$ see end of this talk

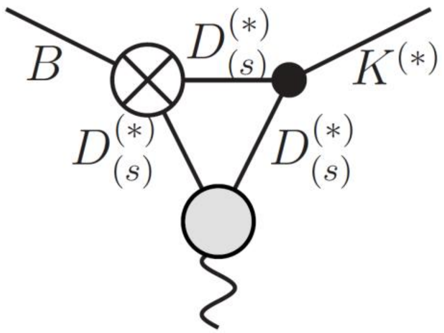


# Rescattering effects



**Missing contributions?**

Ciuchini et al. 2022 (also way before) claim that $B \to \bar{D}D_s \to K^{(*)}\ell^+\ell^-$ rescattering might have a sizable contribution $\Longrightarrow O(20\%)$ at amplitude level




**LCOPE contains (implicitly) rescattering effects**

**partonic** calculation does not yield large contribution (LP OPE and NLO $\alpha_s$)

$$\mathcal{H}_\lambda(q^2) = C_\lambda(q^2)\mathcal{F}_\lambda(q^2) + \tilde{C}_\lambda(q^2)\mathcal{V}_\lambda(q^2) + \cdots$$

$C_\lambda$ is complex valued for any $q^2$ value due to branch cut in $p^2 = M_B^2$ as expected

[Asatrian/Greub/Virto 2019]

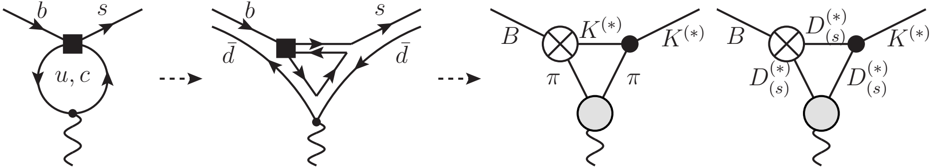
Large quark-hadron duality violation?

Slow convergence of the LCOPE?

Alternative approach $\Longrightarrow$ directly calculate rescattering effects using hadronic methods

# Anomalous thresholds

- **Triangle loop** contributions to non-local form factors:




- Start with ***u*-quark loop** and **$\pi\pi$ intermediate states**:
  - CKM-suppressed $\sim \lambda^4$ compared to ***c*-quark loop** $\sim \lambda^2$
  - Input (Form factors, branching ratios, polarization fractions . . . ) well known
  - Sizable energy gap to next state $\pi\omega$
  
    $\hookrightarrow$ cf. various $D_{(s)}^{(*)}\bar{D}_{(s)}^{(*)}$ for hadronization of charm loop within close proximity
- Build **dispersive framework**

# Analytic structure of form factors

- Fundamental principles: **analyticity** (causality) and **unitarity** (probability conservation)
- Start with **analyticity**: amplitudes are analytic in all **kinematic invariants**
  - **Meson masses** $(q+k)^2 = M_B^2$, $k^2 = M_{K^{(*)}}^2$

    $\hookrightarrow$ only defined on-shell
  - **Photon virtuality** $q^2$

    $\hookrightarrow$ can define analytic continuation for arbitrary $q^2$ in the complex plane
- **Singularities in $q^2$**
  - **Poles**: (infinitely) narrow states

    $\hookrightarrow q^2 = M_{J/\psi}^2, M_{\psi(2S)}^2$
  - **Thresholds**: branch points of $\gamma^* \to \{\pi^+\pi^-, D\bar{D}, \ldots\}$ cuts

    $\hookrightarrow q^2 = \{4M_\pi^2, 4M_D^2, \ldots\}$

# Dispersion relations

- Next up: **unitarity** of the $S$-matrix implies **unitarity relation** (set $t = q^2$)

$$\text{disc}\,\mathcal{M}_{if}(t) \equiv \lim_{\varepsilon\to 0}\left[\mathcal{M}_{if}(t+\mathrm{i}\varepsilon) - \mathcal{M}_{if}(t-\mathrm{i}\varepsilon)\right] = \mathrm{i}\sum_n \mathcal{M}^*_{fn}\mathcal{M}_{in}$$

$\hookrightarrow$ summing over intermediate states $n \in \{\pi^+\pi^-, D\bar{D}, \ldots\}$

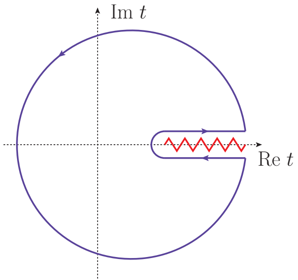
- Amplitudes are analytic with **branch cuts** along real axis

$\hookrightarrow$ starting at thresholds $t_{\text{thr}} = \{4M_\pi^2, 4M_D^2, \ldots\}$

- Know **discontinuity** along cuts from **unitarity relation**
- Reconstruct from discontinuity via **dispersion relation**:

$$\mathcal{M}_{if}(t) = \frac{1}{2\pi\mathrm{i}}\oint \mathrm{d}t'\,\frac{\mathcal{M}_{if}(t')}{t'-t} = \frac{1}{2\pi\mathrm{i}}\int_{t_{\text{thr}}}^{\infty}\mathrm{d}t'\,\frac{\text{disc}\,\mathcal{M}_{if}(t')}{t'-t}$$

$\hookrightarrow$ using Cauchy's theorem



# Form factor dispersion relation

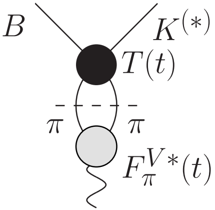
- **Unitarity relation** for $B \to K^{(*)}\gamma^*$ form factor with intermediate $\pi\pi$

$$\text{disc}\,\Pi(t) = 2i\, t\, \sigma_\pi(t)^3\, T(t)\, F_\pi^{V*}(t)$$

$\hookrightarrow$ pion vector form factor $F_\pi^V(t)$, $B \to K^{(*)}\pi\pi$ P-wave amplitude $T(t)$

- Form factor **dispersion relation**

$$\Pi(t) = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} \mathrm{d}t' \, \frac{t'\, \sigma_\pi(t')^3\, T(t')\, F_\pi^{V*}(t')}{t' - t}$$



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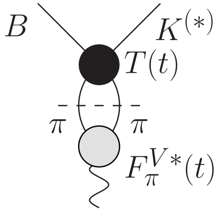
- Consider left-hand cut from crossed-channel $K^{(*)}$-exchange in $T(t)$

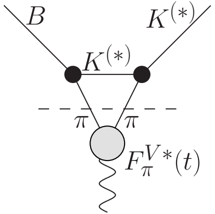
$\hookrightarrow$ leads to **triangle topology**

- Simple Born amplitude violates unitarity (Watson's theorem)

$\hookrightarrow$ need to include **$\pi\pi$-rescattering**

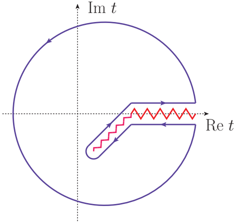
$\hookrightarrow$ unitarize via Muskhelishvili–Omnès representation



# Analytic structure of form factors

- **Kinematic invariants**
  - **Meson masses** $(q+k)^2 = M_B^2$, $k^2 = M_{K^{(*)}}^2$
    $\hookrightarrow$ only defined on-shell
  - **Photon virtuality** $q^2$
    $\hookrightarrow$ can define analytic continuation for arbitrary $q^2$ in the complex plane
- **Singularities in $q^2$**
  - **Poles**: (infinitely) narrow states
    $\hookrightarrow$ $q^2 = M_{J/\psi}^2, M_{\psi(2S)}^2$
  - **Normal thresholds**: branch points of $\gamma^* \to \{\pi^+\pi^-, D\bar{D}, \ldots\}$ cuts
    $\hookrightarrow$ $q^2 = \{4M_\pi^2, 4M_D^2, \ldots\}$
  - **Anomalous thresholds**: anomalous branch points
    $\hookrightarrow$ kinematic singularity of the triangle diagram
    $\hookrightarrow$ position depends on left-hand-cut structure of $B \to K^{(*)}\pi\pi$ amplitude




- Need to modify dispersion relation in presence of additional singularities!

  $\hookrightarrow$ **anomalous threshold** leading to additional cuts

$$\mathcal{M}_{if}(t) = \frac{1}{2\pi i}\int_{t_{thr}}^{\infty} dt' \frac{\text{disc}\,\mathcal{M}_{if}(t')}{t'-t} + \frac{1}{2\pi i}\int_0^1 dx\,\frac{\partial t_x}{\partial x}\frac{\text{disc}_{an}\,\mathcal{M}_{if}(t_x)}{t_x - t}$$

  $\hookrightarrow$ with integration contour $t_x = x\,t_{thr} + (1-x)\,t_{anom}$

- Three cases:

  (1) $t_{anom}$ on normal cut

  $\hookrightarrow$ analytic continuation of normal discontinuity

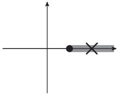
  (2) $t_{anom}$ on negative real axis

  $\hookrightarrow$ integration deformed along real axis

  (3) $t_{anom}$ in complex plane

  $\hookrightarrow$ integration deformed into complex plane

(1)



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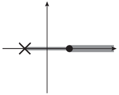

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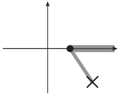
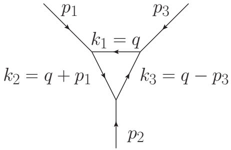


# Anomalous thresholds: where do they come from?

- **Landau equations**: singularities of general loop integral
- **Triangle diagram**:

$$\alpha_i(k_i^2 - m_i^2) = 0 \qquad \sum_{i=1}^{3} \alpha_i k_i = 0$$

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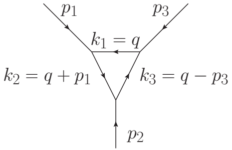


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- **Normal thresholds**: e.g., $\alpha_1 = 0 \Rightarrow p_2^2 = (m_2 \pm m_3)^2$

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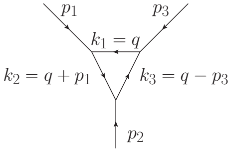



$\hookrightarrow$ "Leading singularity" $\Leftrightarrow$ all Feynman parameters $\alpha_i \neq 0$

- **Normal thresholds**: e.g., $\alpha_1 = 0 \Rightarrow p_2^2 = (m_2 \pm m_3)^2$
- **Anomalous threshold**: all $\alpha_i \neq 0$

  $\hookrightarrow$ $p_2^2 = t_\pm \equiv p_1^2 \frac{m_1^2+m_3^2}{2m_1^2} + p_3^2 \frac{m_1^2+m_2^2}{2m_1^2} - \frac{p_1^2 p_3^2}{2m_1^2} - \frac{(m_1^2-m_2^2)(m_1^2-m_3^2)}{2m_1^2} \pm \frac{1}{2m_1^2}\sqrt{\lambda(p_1^2, m_1^2, m_2^2)\lambda(p_3^2, m_1^2, m_3^2)}$

  $\hookrightarrow$ can be complex-valued

# Anomalous thresholds: when do they matter?

- **Anomalous threshold**:

$$t_\pm \equiv p_1^2 \frac{m_1^2 + m_3^2}{2m_1^2} + p_3^2 \frac{m_1^2 + m_2^2}{2m_1^2} - \frac{p_1^2 p_3^2}{2m_1^2} - \frac{(m_1^2 - m_2^2)(m_1^2 - m_3^2)}{2m_1^2} \pm \frac{1}{2m_1^2} \sqrt{\lambda(p_1^2, m_1^2, m_2^2)\lambda(p_3^2, m_1^2, m_3^2)}$$

- For sufficiently small $p_1^2$, $p_3^2$, $t_\pm$ lie on the **second sheet**
- But $p_1^2 = M_B^2$ large: anomalous threshold moves onto **first sheet** for

$$m_3 p_1^2 + m_2 p_3^2 - (m_2 + m_3)(m_1^2 + m_2 m_3) > 0$$

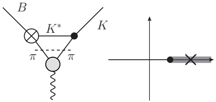
$\hookrightarrow$ condition simplifies to $M_B^2 + M_{K^{(*)}}^2 > 2\left(M_\pi^2 + M_{K^{(*)}}^2\right)$ (fulfilled!)

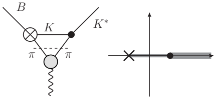
- Anomalous term in form factor dispersion relation

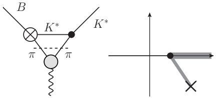
$$\Pi(t) = \underbrace{\frac{1}{\pi} \int_{4M_\pi^2}^{\infty} \mathrm{d}t' \, \frac{t' \, \sigma_\pi(t')^3 \, T(t') \, F_\pi^{V*}(t')}{t' - t}}_{\equiv \Pi^{\text{norm}}(t)} + \underbrace{\frac{1}{\pi} \int_{t_+}^{4M_\pi^2} \mathrm{d}t' \, \frac{t' \, \sigma_\pi(t')^3 \, \text{disc}\, T(t') \, F_\pi^{V}(t')}{t' - t}}_{\equiv \Pi^{\text{anom}}(t)}$$

$\hookrightarrow$ how important is the anomalous part? Examine anomalous fraction $|\Pi^{\text{anom}}(t)/\Pi^{\text{norm}}(t)|$

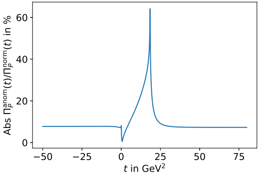
# Anomalous fractions for $B^0 \to K^{(*)0}\gamma^*$

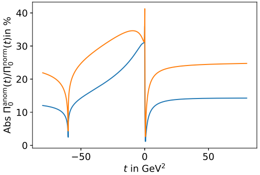


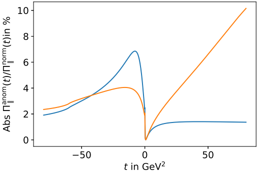



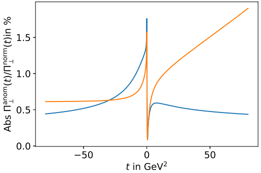




| $t_+$ / GeV$^2$ = 18.6 (case 1) | $-57.8$ (case 2) | $0.5 - 4.2i$ (case 3) |
|---|---|---|




SM, Hoferichter, Kubis 2024

| | $\lambda = 0$ | $\lambda = \parallel$ | $\lambda = \perp$ |
|---|---|---|---|

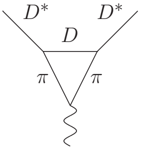
- All parameters fixed from **data**! One ambiguity left due to lack of Dalitz plot data (blue and orange curve)
- **Anomalous contributions** can be $\gtrsim$ **10%** away from thresholds, resonances
- **Hierarchy** between cases 1,2,3? Or between helicities? $\to$ only have case 3 in $c$-loops

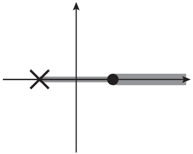
# Comparison: $D^*$ form factors

- Study different scenario with same analytic structure: ***$D^*$* form factors** at low $q^2$

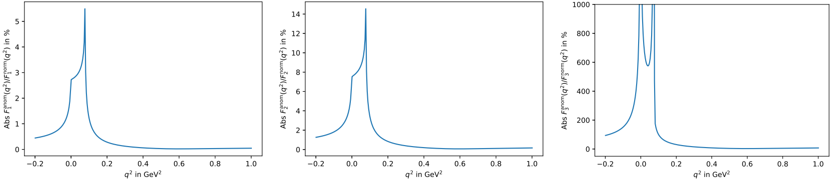
  $\hookrightarrow$ anomalous threshold at $s_+ = -\frac{\lambda(M_\pi^2, M_D^2, M_{D^*}^2)}{M_D^2} = -0.0012\,\text{GeV}^2$ on first sheet

  $\hookrightarrow$ small scale due to $M_{D^*} \gtrsim M_D + M_\pi$



- Three form factors ($F_1(q^2) \sim$ electric, $F_2(q^2) \sim$ magnetic, $F_3(q^2) \sim$ quadrupole)

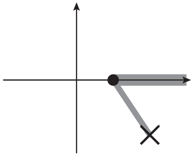



SM et al., in progress

- For $F_3(q^2)$ the anomalous part dominates completely $\rightarrow$ indication for hierarchy between helicities

# Towards the charm loops

- Expect impact of anomalous thresholds to be qualitatively similar
- All anomalous threshold in lower complex plane (case 3)



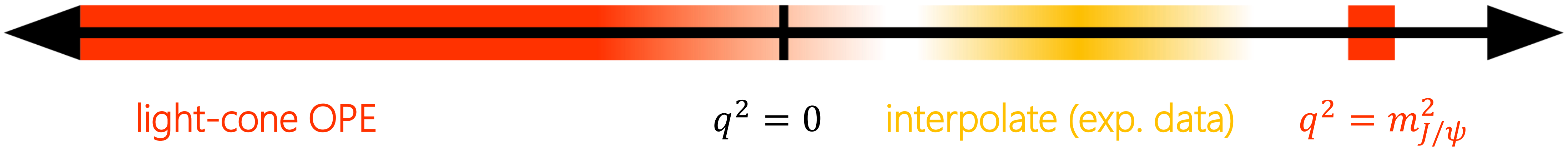
- Cannot assess size yet, but could expect similar hierarchy between helicities
- Difficulties:
  - More intermediate states in close proximity: $\bar{D}D$, $\bar{D}D^*$, $\bar{D}^*D^*$, ...
  - Phenomenology of form factors and amplitudes less well understood
  - Need more precise data (branching ratios, polarization fractions, Dalitz plots, ...) for $B \to D_s^{(*)} D^{(*)}$ and $B \to K^{(*)} D_{(s)}^{(*)} \bar{D}_{(s)}^{(*)}$

# Parametrizations and unitarity bounds

# The need of a (bounded parametrization)

FFs are functions of $q^2$

FFs are known in a finite number of isolated $q^2$ points




Interpolate or extrapolate FF using a parametrization

$$\mathcal{H}(q^2) \propto \sum_{n=0}^{\infty} a_n\, g_n(q^2)$$

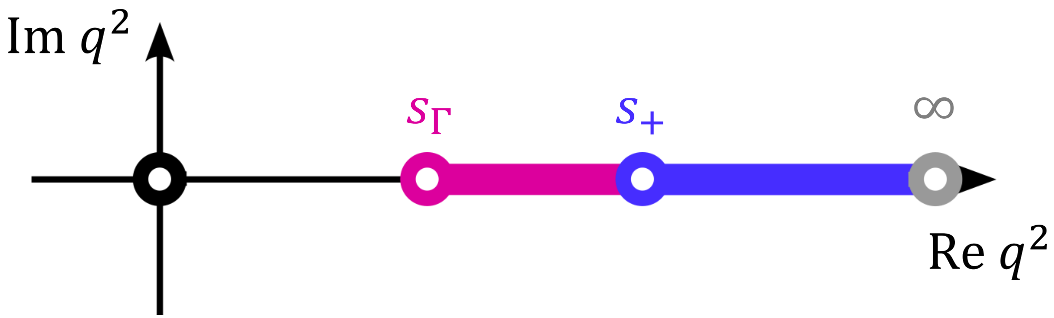
the functions $g_n(q^2)$ are conveniently chosen, fit (a few) the $a_n$

What's the best parametrization? How to estimate the truncation error?

# Analytic properties of local FFs



Study FF analytic structure to find a suitable parametrization. Example $B \to K$ local FFs




**FFs are analytic except for branch cuts** (i.e. lines of discontinuity) starting at

$s_+ = (m_B + m_K)^2$, process threshold

$s_\Gamma = \left(m_{B_s} + m_\pi\right)^2 < s_+$, subthreshold branch cut

Obtain a constrain on the FFs using unitarity (see [Okubo 1971])

$$\int_{s_+}^{\infty} dq^2 \, |\det J| \, |\phi(q^2)\mathcal{F}(q^2)|^2 < \chi$$

**calculate** $\chi$ perturbatively, $\phi$ known function

# Traditional approach: BGL



[Boyd/Grinstein/Lebed 1994 and 1997]

Perform the conformal mapping

$$z(q^2) = \frac{\sqrt{s_+ - q^2} - \sqrt{s_+}}{\sqrt{s_+ - q^2} + \sqrt{s_+}}$$

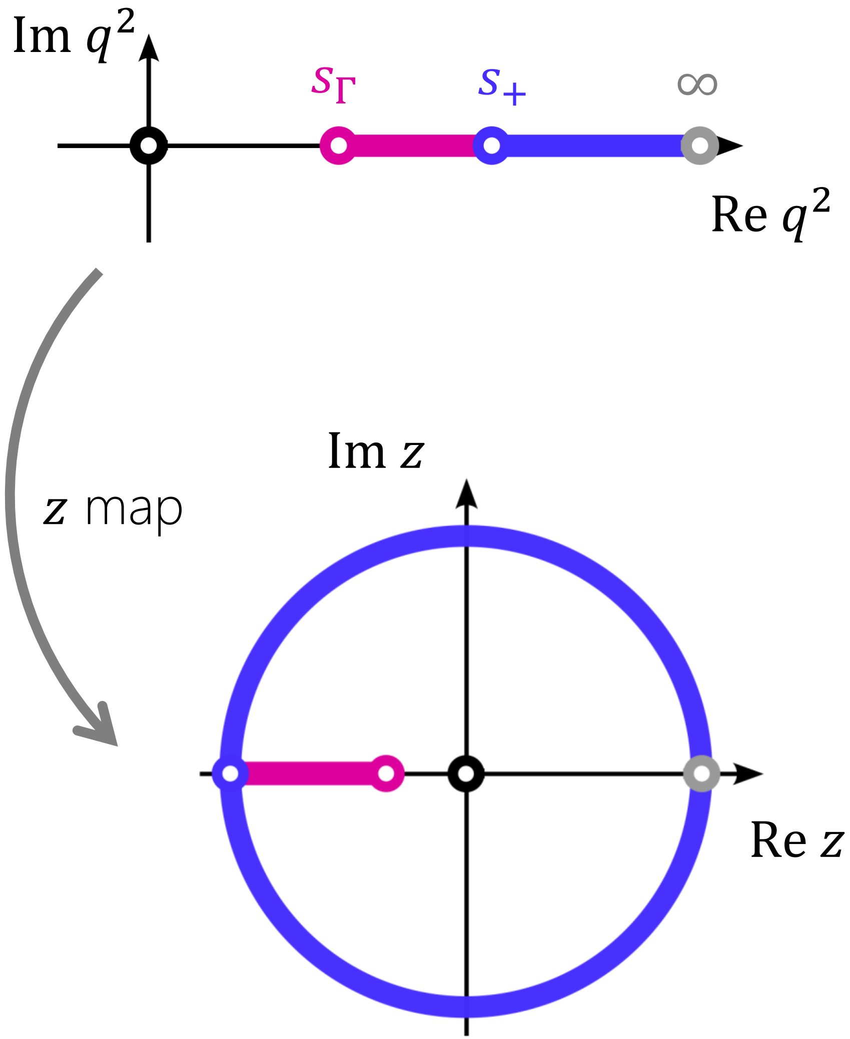
expand FFs for $|z| < 1$ as

$$\mathcal{F}(z) = \frac{1}{\phi(z)} \sum_{n=0}^{\infty} a_n z^n$$

obtain a bound on the coefficients

$$\int_{s_+}^{\infty} dq^2 \, |\det J| \, |\phi(q^2) \mathcal{F}(q^2)|^2 < \chi \quad \Rightarrow \quad \sum_{n=0}^{\infty} a_n^2 < \chi$$

Problem! series is divergent due to the branch cut in $s_\Gamma$



# Problems with BGL



Having a branch cut invalidate the expansion for $|z| < 1$

$$\mathcal{F}(z) \neq \frac{1}{\phi(z)} \sum_{n=0}^{\infty} a_n z^n \quad \text{for some} \quad |z| < 1$$

same issue appears for FFs in $B \to D^{(*)}$, $\Lambda_b \to \Lambda$, ...

It is crucial to address this issue to accurately estimate uncertainties in $b$-hadron decays

Issue discussed in the literature, but solutions are unsatisfactory:
they do not allow a rigorous estimate of the truncation error (see next slides)

Find a way to recover the unitarity bound:

[Boyd/Grinstein/Lebed 1995]
[Caprini/Neubert 1996]
[NG/van Dyk/Virto 2020]
[Flynn/Jüttner/Tsang 2023]

$$\sum_{n=0}^{\infty} a_n^2 < \chi$$

**Essential to estimate truncation error**! (we can only fit a finite number of $a_n$)

# Our approach: GG



Just a reminder: $s_+ = (m_B + m_K)^2$ , $s_\Gamma = (m_{B_s} + m_\pi)^2$

Modify the conformal mapping ($s_+ \mapsto s_\Gamma$)

$$\hat{z}(q^2) = \frac{\sqrt{s_\Gamma - q^2} - \sqrt{s_\Gamma}}{\sqrt{s_\Gamma - q^2} + \sqrt{s_\Gamma}}$$

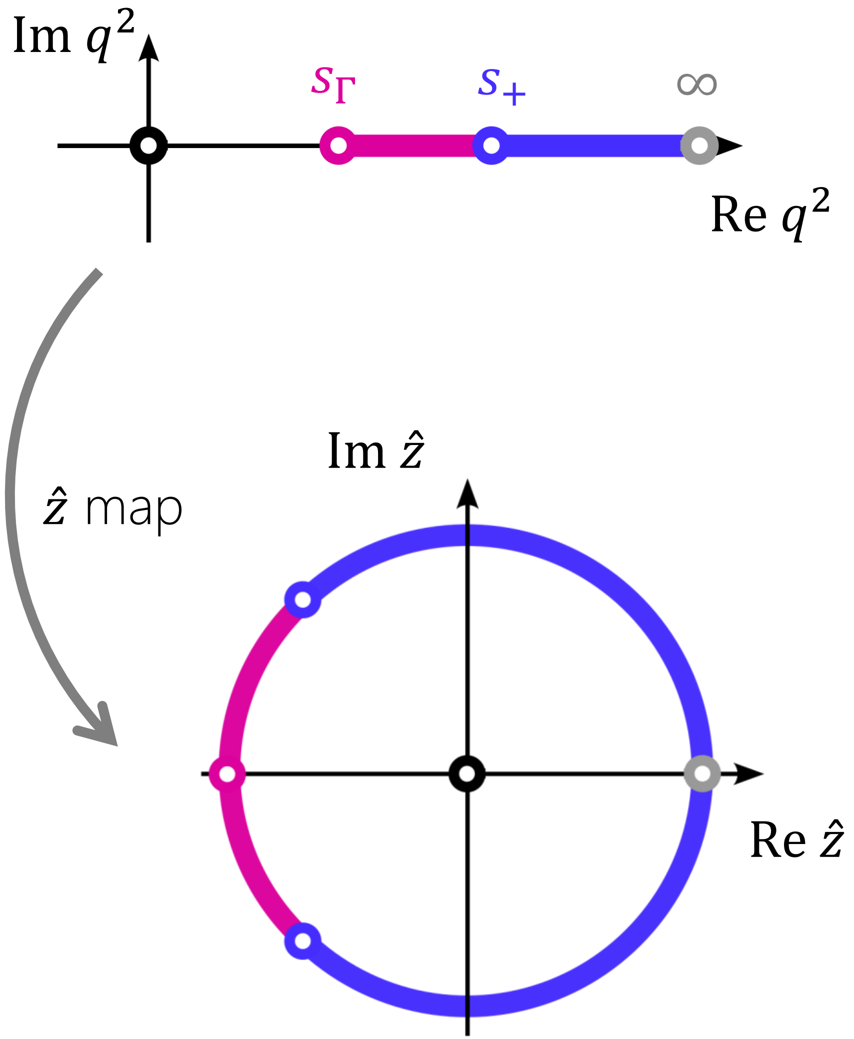
expand FFs for $|\hat{z}| < 1$ (no singularities now!) as

$$\mathcal{F}(\hat{z}) = \frac{1}{\phi(\hat{z})} \sum_{n=0}^{\infty} b_n \hat{z}^n$$

however

$$\int_{s_+}^{\infty} dq^2 \,|\det J\,| |\phi(q^2)\mathcal{F}(q^2)|^2 < \chi \quad \not\Rightarrow \quad \sum_{n=0}^{\infty} b_n^2 < \chi$$

Integral must over the whole circle!



# Our derivation of the unitarity bound



Start from

$$\int_{s_+}^{\infty} dq^2\, |\det J|\, |\phi(q^2)\mathcal{F}(q^2)|^2 < \chi$$

add on both sides

$$\Delta\chi \equiv \int_{s_\Gamma}^{s_+} dq^2\, |\det J|\, |\phi(q^2)\mathcal{F}(q^2)|^2$$

Estimate $\Delta\chi$ using large $q^2$ scaling behaviour (for $B \to K$ FFs $\frac{\Delta\chi}{\chi} < 1\%$)

Obtain the unitarity bound [Gopal/NG 2024]

$$\int_{s_\Gamma}^{\infty} dq^2\, |\det J|\, |\phi(q^2)\mathcal{F}(q^2)|^2 < \chi + \Delta\chi \quad \Rightarrow \quad \sum_{n=0}^{\infty} b_n^2 < \chi + \Delta\chi$$

New parametrization for FFs that allows to calculate the truncation error!

# Problems with cut modelling and polynomials

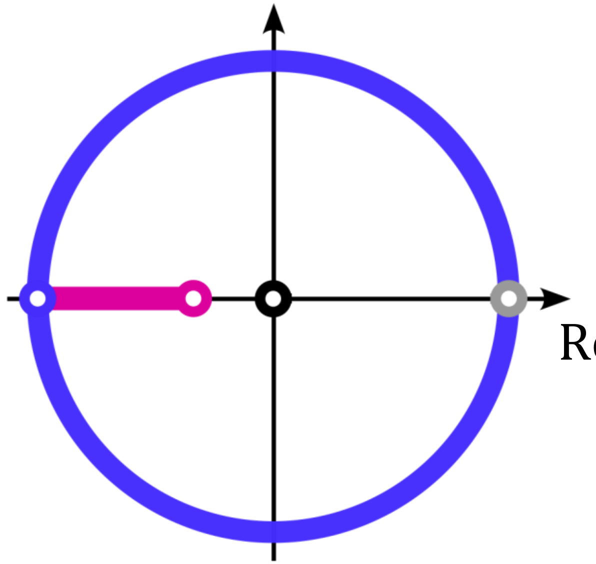


Model the branch cut and subtract it

$$\tilde{\mathcal{F}}(z) \equiv \mathcal{F}(z) - \mathcal{F}_{\mathrm{cut}}(z)$$

expand $\tilde{\mathcal{F}}(z)$ [Boyd/Grinstein/Lebed 1995] [Caprini/Neubert 1996]

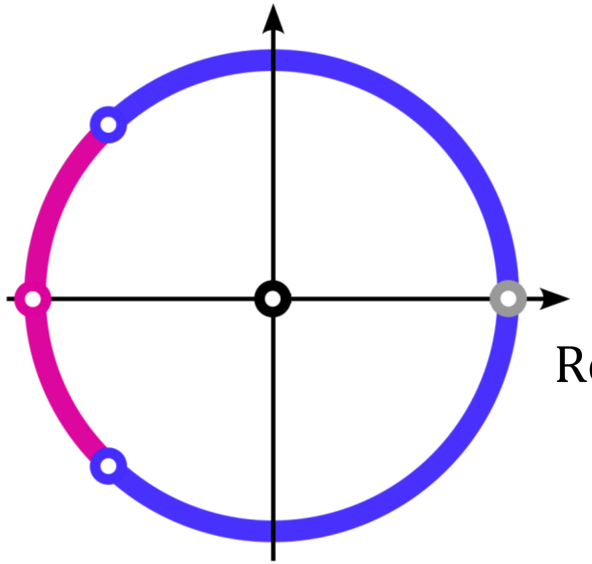
Problem: $\mathcal{F}_{\mathrm{cut}}(z)$ is not known
$\Longrightarrow$ cannot rely on exact numerical cancellation of singularities




Expand in polynomials orthogonal on the blue arc [NG/van Dyk/Virto 2020] [Flynn/Jüttner/Tsang 2023]

$$\mathcal{F}(\hat{z}) = \frac{1}{\phi(\hat{z})} \sum_{n=0}^{\infty} b_n \hat{z}^n$$

$|\boldsymbol{p_n(\hat{z})}| \to \infty$ for $\boldsymbol{n} \to \infty$ and some $\hat{z}$ in the unit disk



# Anomalous branch cuts



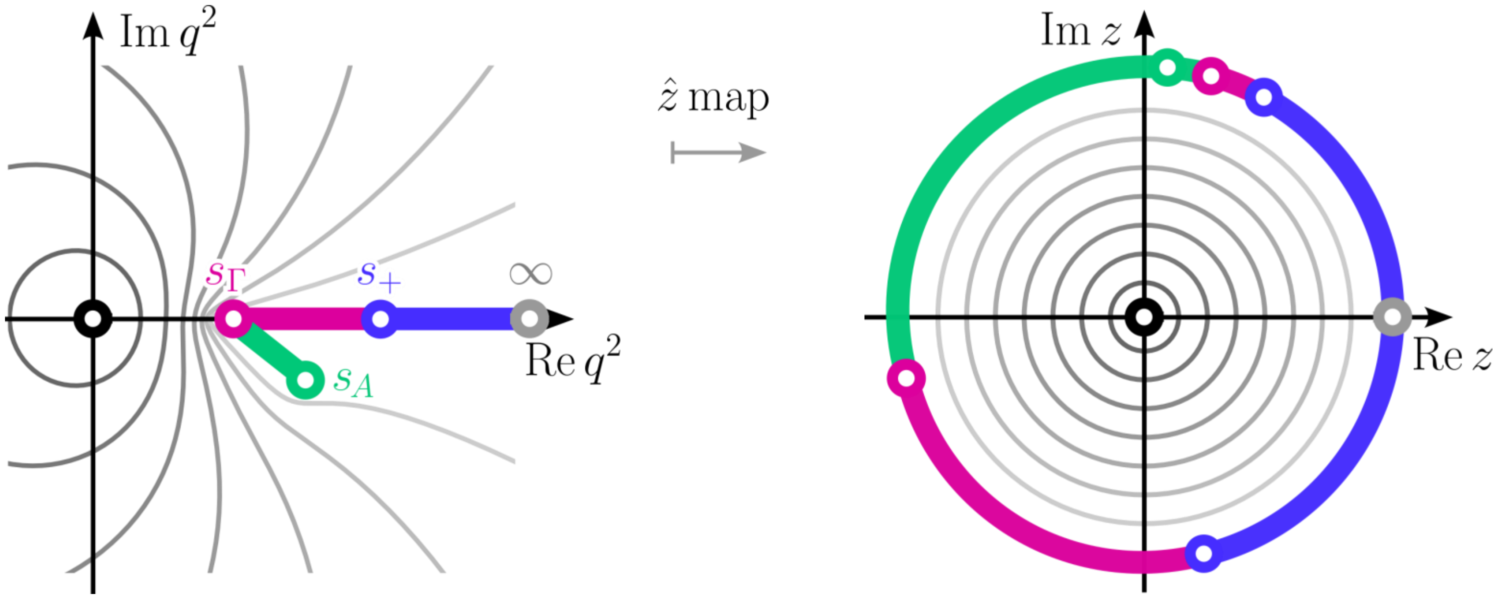
Non-local FFs may present have anomalous branch cuts that extend into the complex plane
Example $B \to DD_s^* \to K\ell^+\ell^-$ rescattering

$$s_+ = (m_B + m_K)^2 \qquad s_\Gamma = (2m_D)^2 \qquad s_A = 24.1 - 3.5i$$

[Mutke et al. 2024]

Apply the same procedure as for the subthreshold branch cuts, but:

- $\hat{z}$ map is very hard to obtain (existence guaranteed by the Riemann Mapping Theorem)
- $\Delta\chi$ calculation extremely challenging




[Gopal/NG 2024]

# Summary and conclusions

# Summary and conclusions



Contributions from anomalous thresholds can make **up $\gtrsim$ 10% of** (light-quark-loop-induced) **non-local FFs**

Precise **data needed** (branching ratios, polarization fractions, Dalitz plots,…) to quantify this **for charm loops**

The traditional (BGL) parametrization neglect subthreshold branch cuts, leading to systematic effects (polynomial expansion and non-orthogonal bounds do not fully resolve the issue)

We propose a new easy to implement parametrization to solve the issue

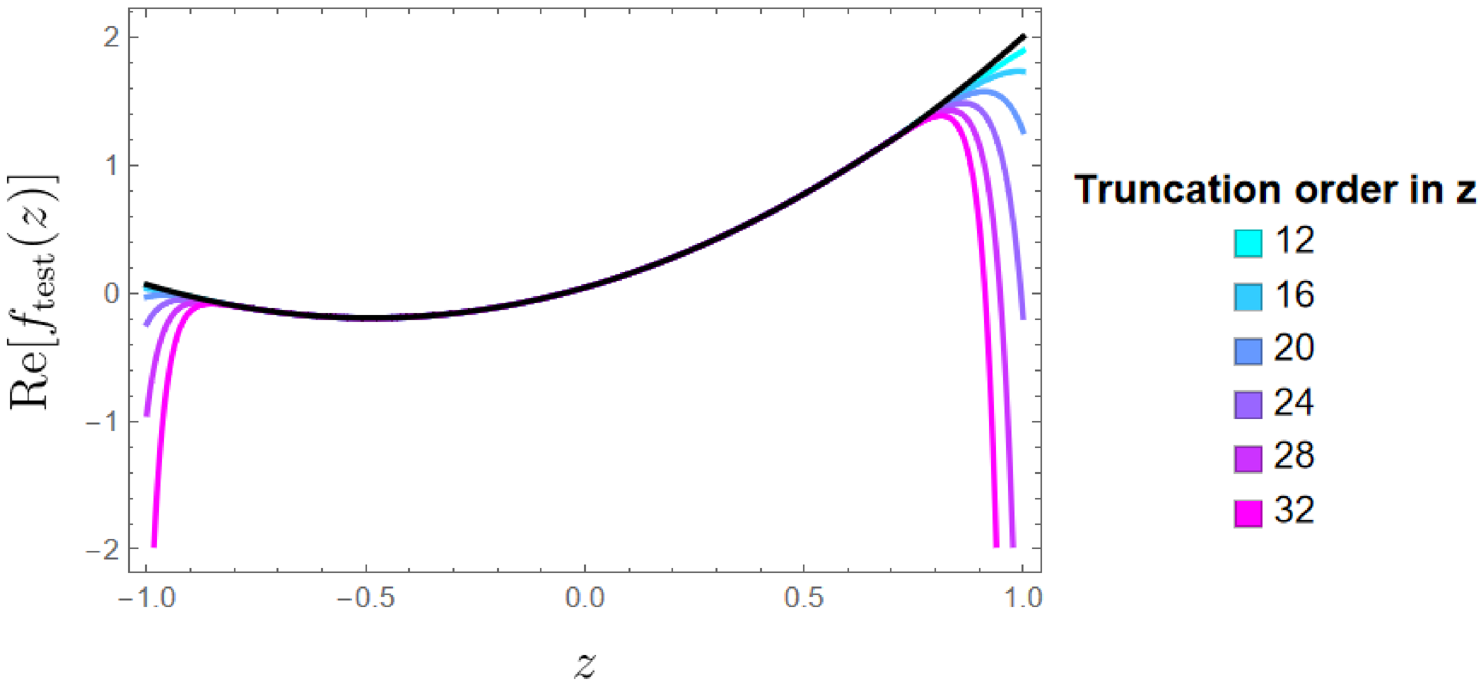
Our parametrization can account for both subthreshold and anomalous cuts

# Thank you!

# Backup slides

# Impact of branch cuts in a Taylor expansion

$$f_{\text{test}}(z) = z + z^2 + 0.05\sqrt{0.7 - z}$$




Even if the branch cut is suppressed it generates divergent coefficients. Hence:

$$\sum_{n=0}^{\infty} b_n^2 \not< \chi$$

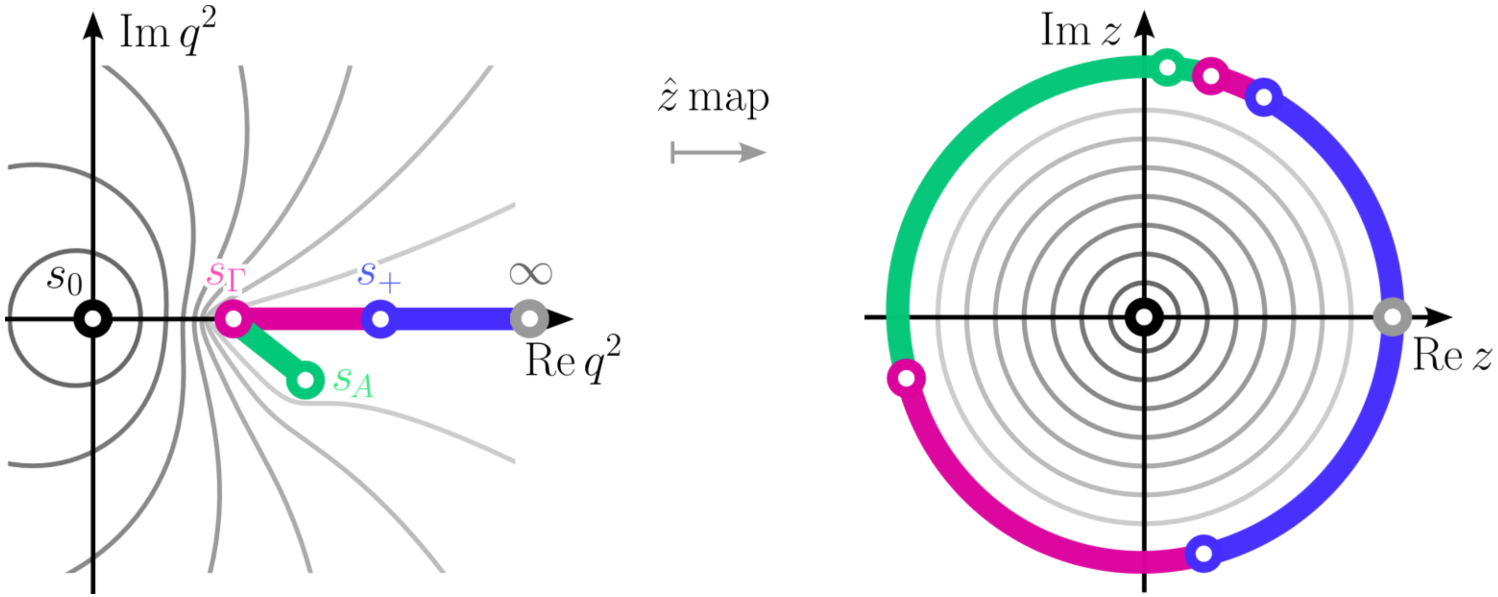
# Schwarz–Christoffel formula

Map the unit disk to the domain $\boldsymbol{\Omega}$

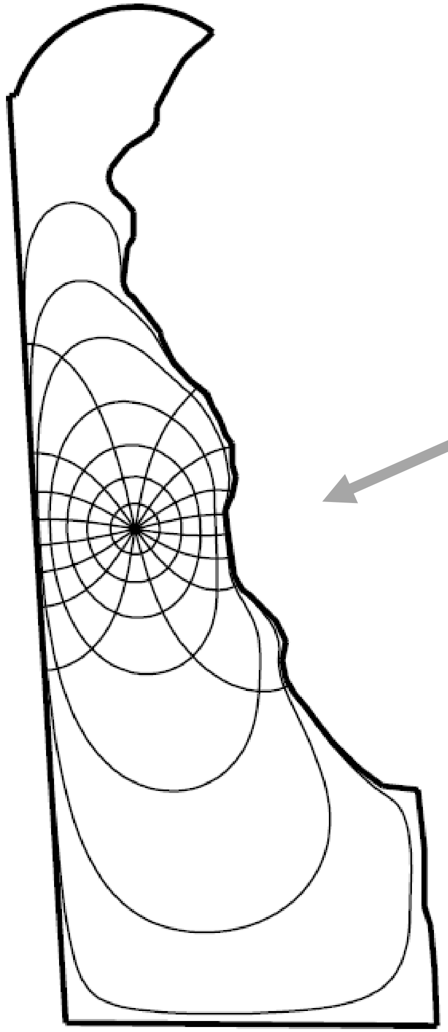
$$g(z) = A + C \int_0^z d\zeta \prod_{k=1}^{4} \left(1 - \frac{\zeta}{z_k}\right)^{\frac{\phi_k}{\pi} - 1}$$

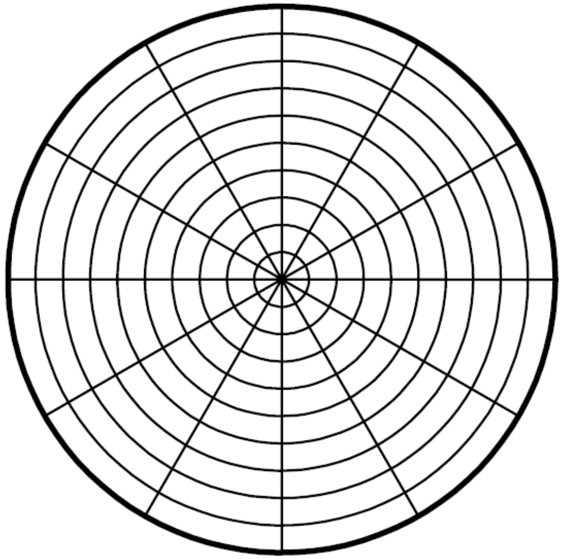
where

$$\Omega = \mathbb{C} \setminus \left( [s_\Gamma, \infty] \cup \{ (1-t)s_A + t\, s_\Gamma : \ t \in [0,1] \} \right)$$

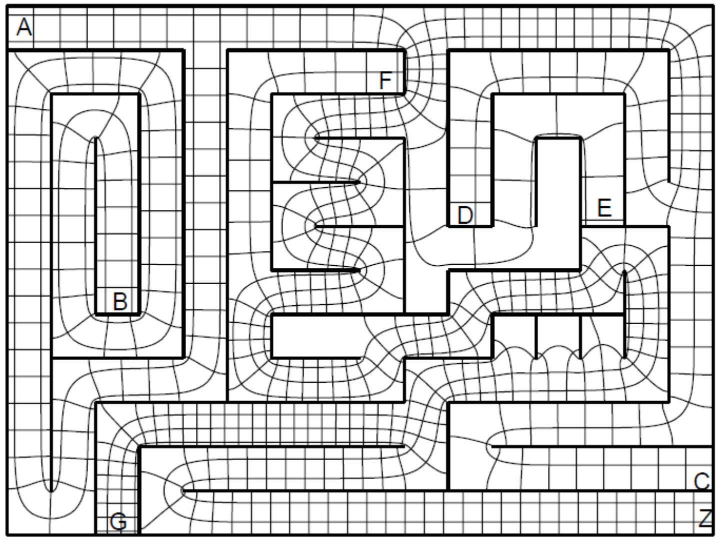


# Schwarz–Christoffel formula at work





[Driscoll/Trefethen 2002]



# $\Delta\chi$ calculation

Approximate FFs using their large \( $q^2$ \) scaling behaviour calculated in perturbative QCD

E.g. for $B \to K$

[Lepage/Brodsky 1980]
[Akhoury et al. 1994]

$$|\mathcal{F}_+(q^2)|^2 \simeq K\left(\frac{s_\Gamma}{q^2}\right)^2$$

According to [Becher/Hill 2005] $K{\sim}1$

Even assuming $K{\sim}100$

$$\frac{\Delta\chi}{\chi} \equiv \frac{1}{\chi}\int_{s_\Gamma}^{s_+} dq^2\, |\det J| |\phi(q^2)\mathcal{F}(q^2)|^2 \simeq 0.005$$

i.e. smaller than the uncertainty on$\chi$

This is due to the fact that $\frac{s_+ - s_\Gamma}{s_\Gamma} \ll 1$ and that $\chi$ is an inclusive quantity while $\Delta\chi$ is exclusive

# Polynomial parametrization

polynomial parametrization ($\hat{z}$ polynomials) [NG/van Dyk/Virto 2020]

$$\mathcal{H}_\lambda(\hat{z}) = \frac{1}{\mathcal{P}(z)\phi(z)} \sum_{n=0}^{\infty} \beta_n p_n(\hat{z}) \qquad \sum_{n=0}^{\infty} |\beta_n|^2 < 1$$

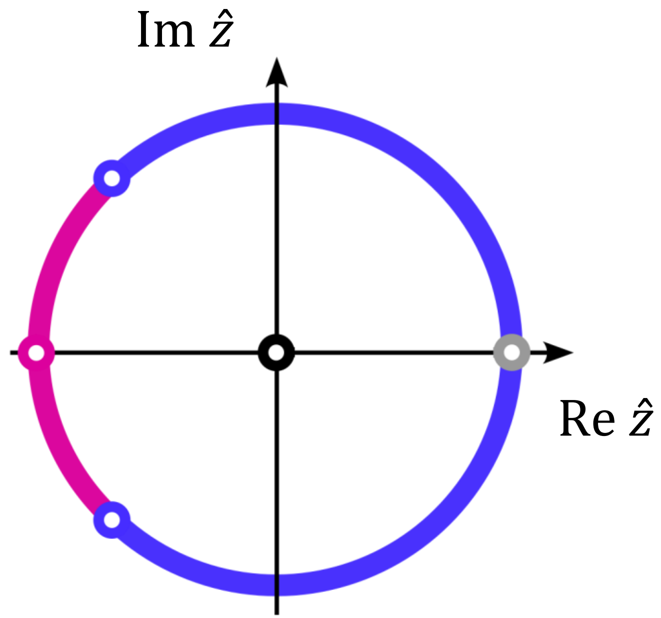
$|p_n(\hat{z})| \to \infty$ for $n \to \infty$ some $z$ in the unit disk

$$p_0^{B\to K}(\hat{z}) = \frac{1}{\sqrt{2\alpha_{BK}}}$$

$$p_1^{B\to K}(\hat{z}) = \left(\hat{z} - \frac{\sin(\alpha_{BK})}{\alpha_{BK}}\right)\sqrt{\frac{\alpha_{BK}}{2\alpha_{BK}^2+\cos(2\alpha_{BK})-1}}$$

$$p_2^{B\to K}(\hat{z}) = \left(\hat{z}^2 + \frac{\sin(\alpha_{BK})(\sin(2\alpha_{BK})-2\alpha_{BK})}{2\alpha_{BK}^2+\cos(2\alpha_{BK})-1}\hat{z} + \frac{2\sin(\alpha_{BK})(\sin(\alpha_{BK})-2\alpha_{BK})}{2\alpha_{BK}^2+\cos(2\alpha_{BK})-1}\right)$$

$$p_3^{B\to K}(\hat{z}) = \cdots$$



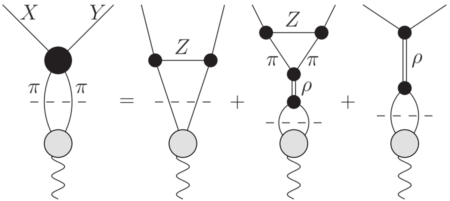
# Muskhelishvili–Omnès representation

- Unitarize $B \to K^{(*)}\pi\pi$ $P$-waves by including $\pi\pi$ rescattering (to fulfill Watson's theorem)

$$\text{disc}\, T(t) = 2i\, T(t) \sin\delta(t)\, e^{-i\delta(t)} = 2i\, T(t)\, \sigma_\pi(t)\, t_1^{1*}(t)$$

$\hookrightarrow$ $\pi\pi$ elastic scattering phase shift $\delta(t)$ $(I = 1, L = 1)$, $t_1^1(t) = \sin\delta(t)\, e^{i\delta(t)}/\sigma_\pi(t)$

$\hookrightarrow$ via Muskhelishvili–Omnès representation



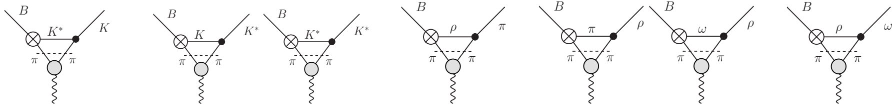

$\hookrightarrow$ dominated by $\rho$ resonance

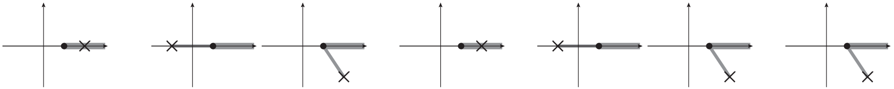
$$T^{\text{MO}}(t) = \Omega(t) \left[ \frac{t}{\pi} \int_{4M_\pi^2}^{\infty} \frac{dt'}{t'} \frac{T^{\text{Born}}(t') \sin\delta(t')}{|\Omega(t')|(t'-t)} + \frac{t}{\pi} \int_0^1 \frac{dx}{t_x} \frac{\partial t_x}{\partial x} \frac{\text{disc}\, T^{\text{Born}}(t_x)\, \sigma_\pi(t_x)\, t_1^1(t_x)}{\Omega(t_x)(t_x - t)} \right]$$

$$T^{\text{Omnès}}(t) = a\,\Omega(t), \qquad \Omega(t) = \exp\left[ \frac{t}{\pi} \int_{4M_\pi^2}^{\infty} \frac{dt'}{t'} \frac{\delta(t')}{t'-t} \right] = |\Omega(t)|\, e^{i\delta(t)}$$

# Anomalous thresholds for $B \to (P, V)\,\gamma^*$: list of processes

| $B \to K\gamma^*$ | $B \to K^*\gamma^*$ | | $B \to \pi\gamma^*$ | $B \to \rho\gamma^*$ | | $B \to \omega\gamma^*$ |
|---|---|---|---|---|---|---|

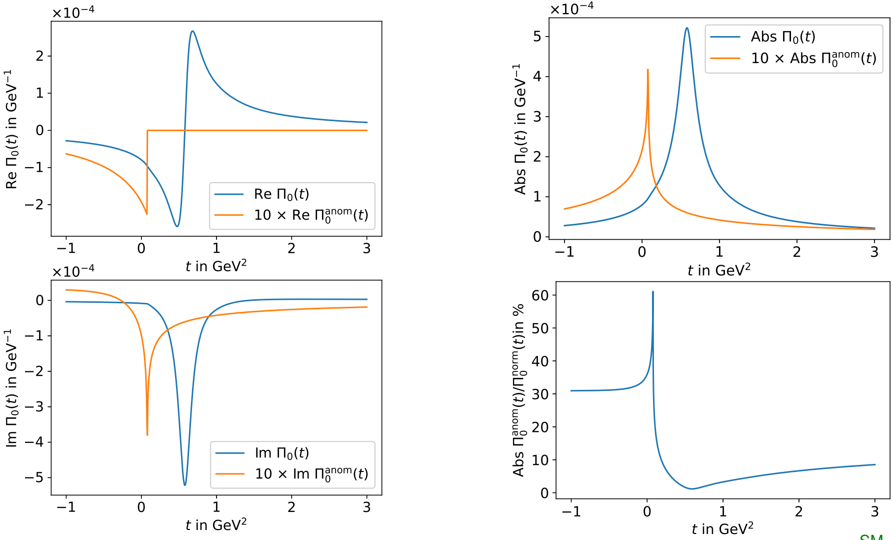


| | | | | | | |
|---|---|---|---|---|---|---|
| $t_+$ / GeV$^2$ = 18.6 | $-57.8$ | $0.5 - 4.2i$ | 26.4 | $-859.3$ | $0.7 - 4.8i$ | $0.2 - 4.9i$ |
| Br[$B \to K^*\pi$]<br>Br[$B \to K\pi\pi$] | Br[$B \to K^{(*)}\pi$]<br>Br[$B \to K^*\pi\pi$] | | Br[$B \to \rho\pi$]<br>Br[$B \to 3\pi$] | Br[$B \to \pi\pi, \pi\omega$]<br>Br[$B \to \rho\pi\pi$] | | Br[$B \to \rho\pi$]<br>Br[$B \to \omega\pi\pi$] |

- $t_{\rm thr} = 4M_\pi^2 = 0.08\,\text{GeV}^2$
- Large energy scales in limit $M_B \to \infty$, $M_\pi \to 0$: $t_+ \simeq -\frac{M_B^2 M_\rho^2}{M_\pi^2} \simeq -860\,\text{GeV}^2$

# Example: anomalous contribution to the longitudinal $B^0 \to K^{*0}\gamma^*$ FF




SM, Hoferichter, Kubis 2024