Università degli Studi di Bari *Aldo Moro*
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Bari Theory Xmas Workshop 2024

Work fluctuations, Singular Distributions and Big Jumps for a harmonically confined Active Particle

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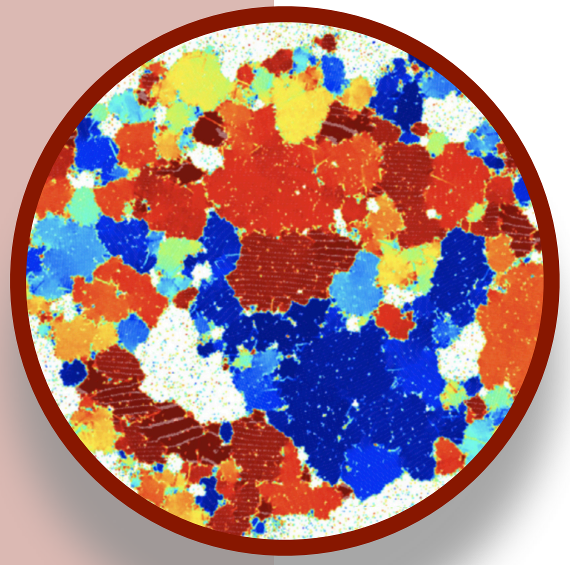
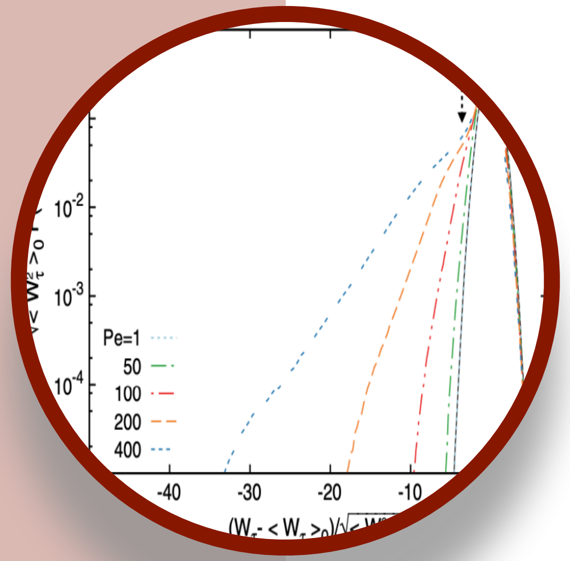
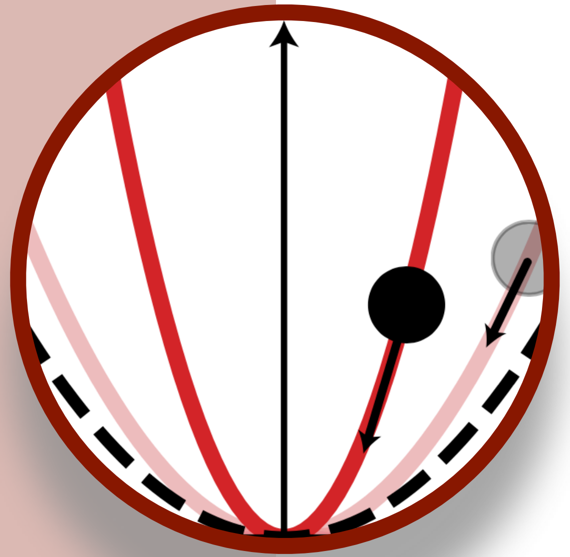
Outline



Active Matter



Singular Distributions

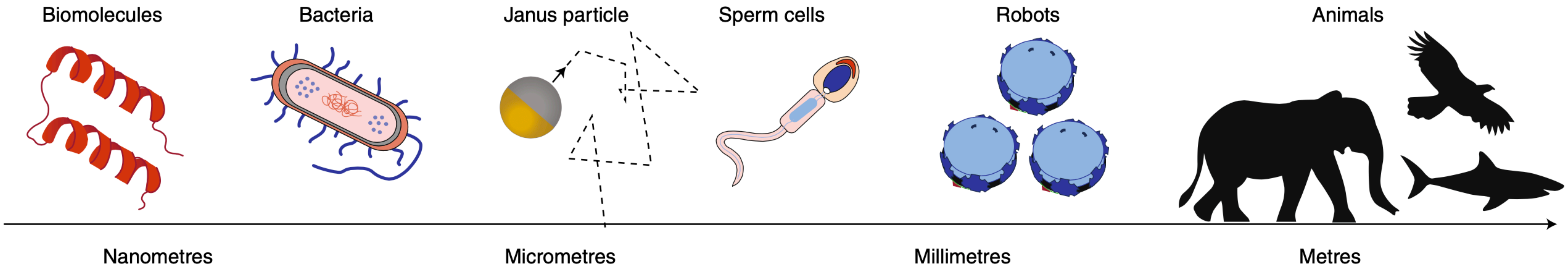


Active Work and Big Jumps

Take-home messages

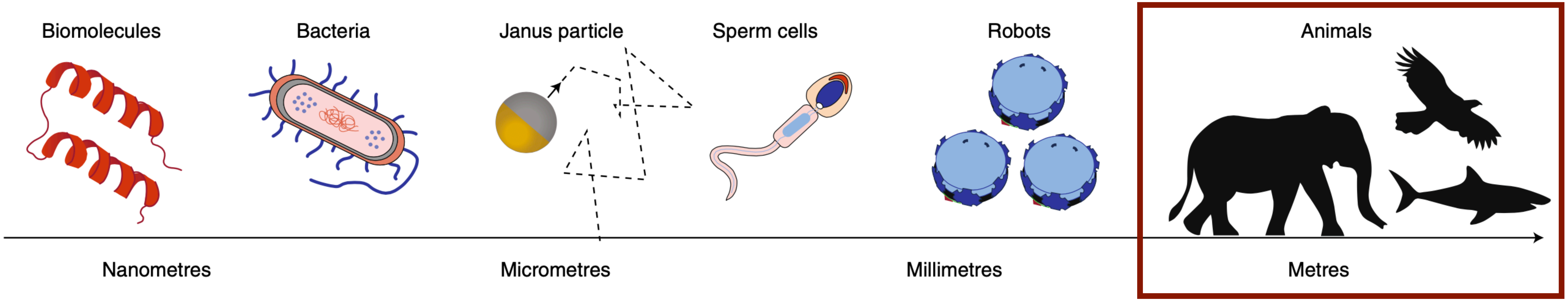
Active Matter

- **Active Matter:** single components transform energy from internal reservoirs or from the surrounding environment to **self propel**



Active Matter

- **Active Matter:** single components transform energy from internal reservoirs or from the surrounding environment to **self propel**



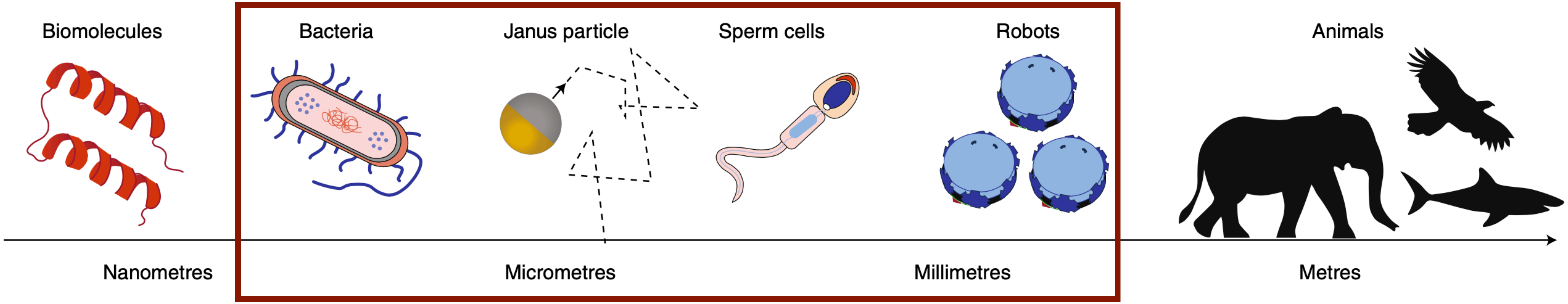
schools of fishes

herds of buffalos

flocks of birds

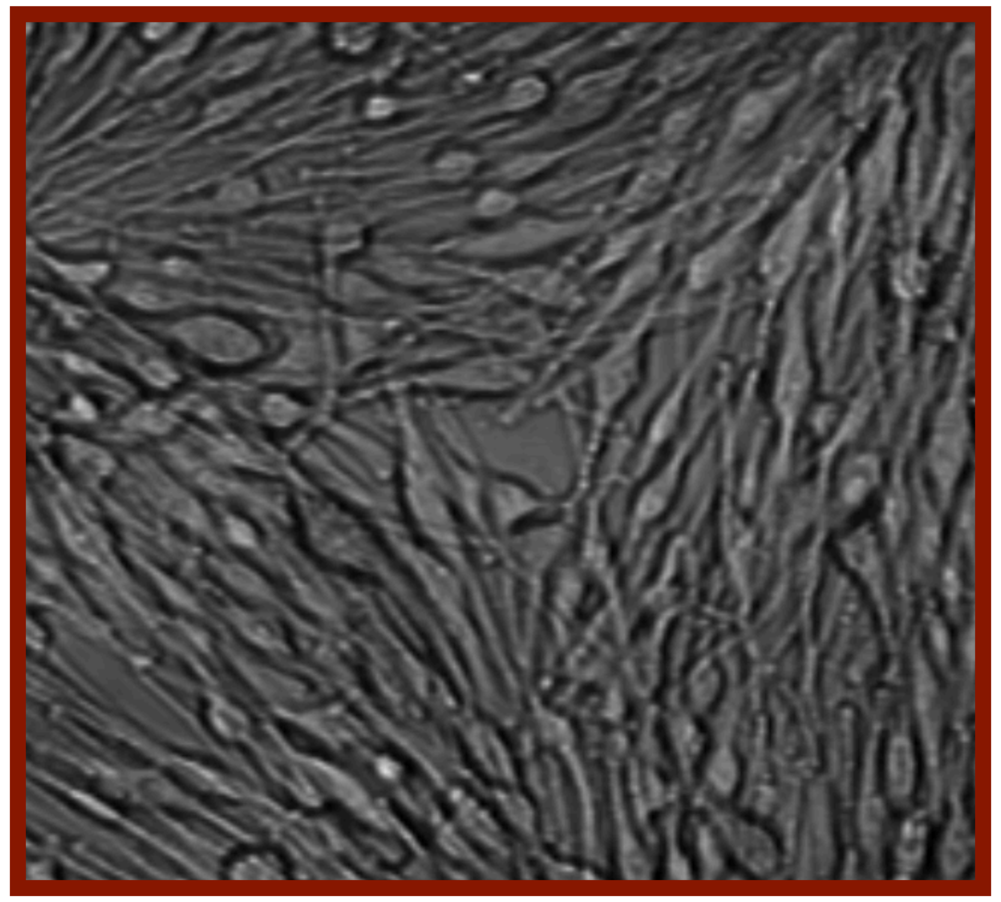
Active Matter

- **Active Matter:** single components transform energy from internal reservoirs or from the surrounding environment to **self propel**

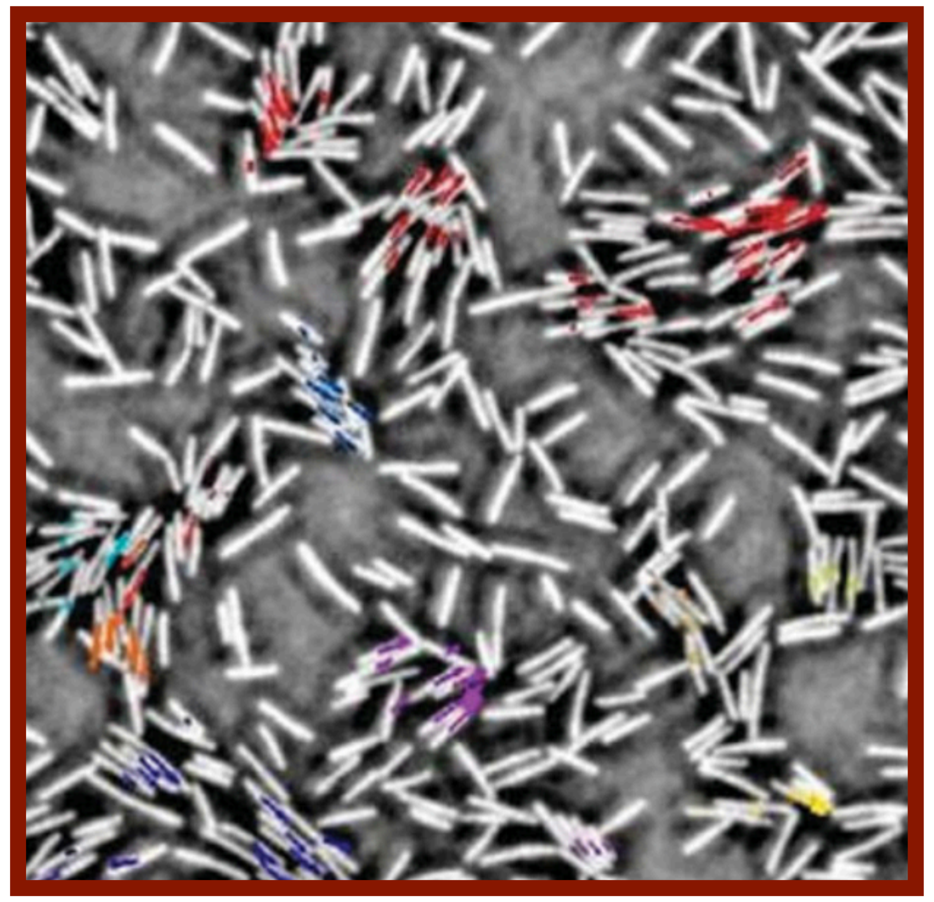


nature

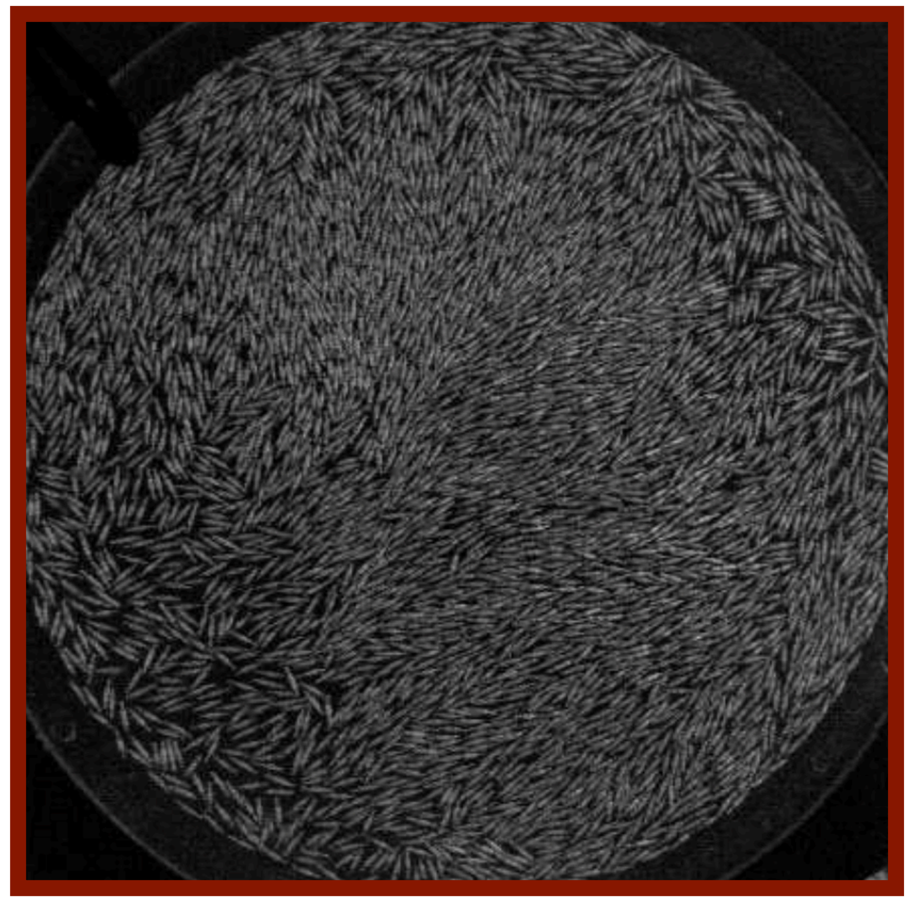
experiments



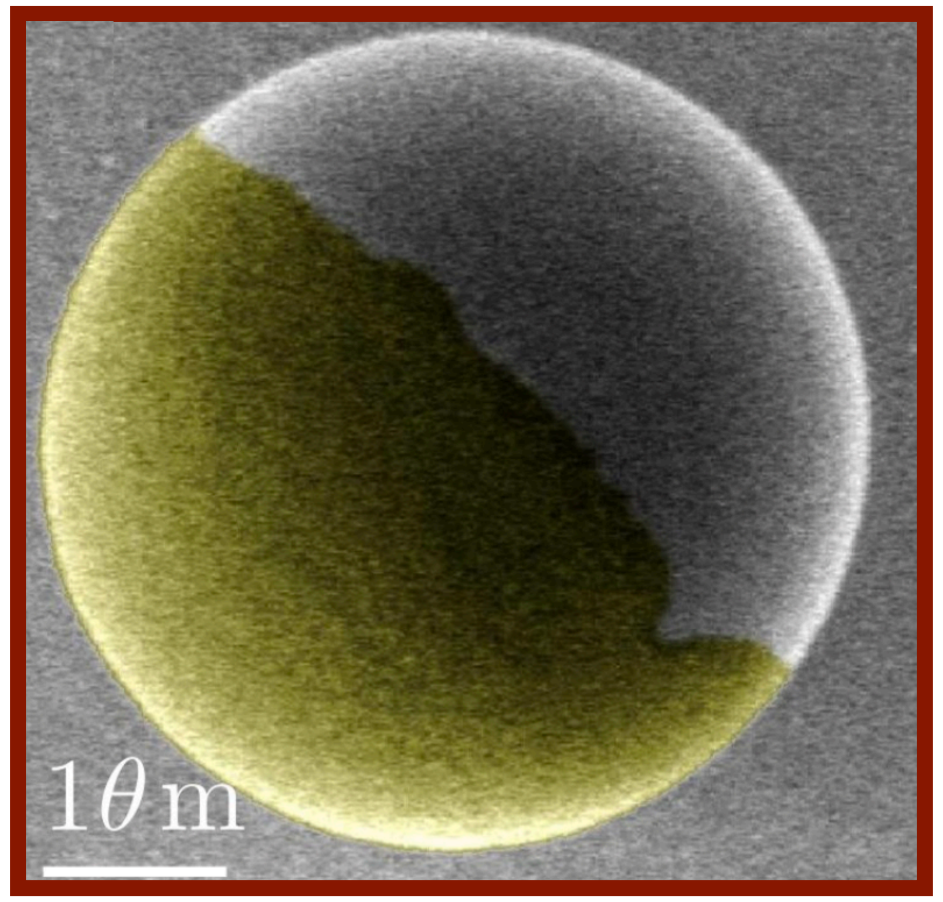
melanocytic cells



colonies of bacteria



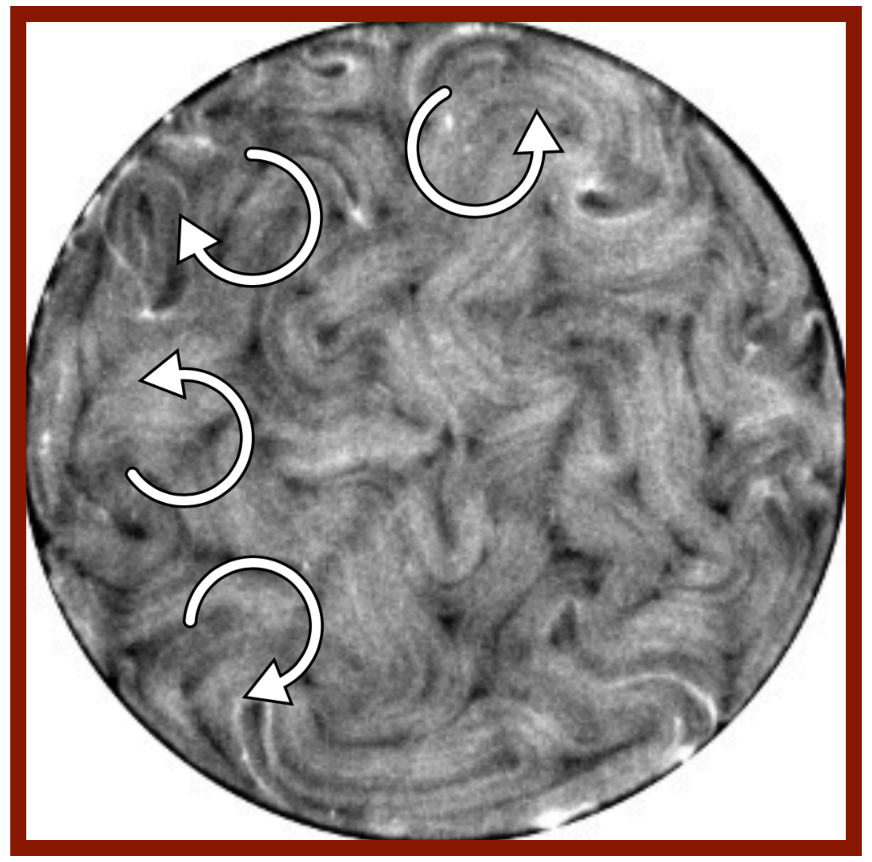
vibrated nanorods



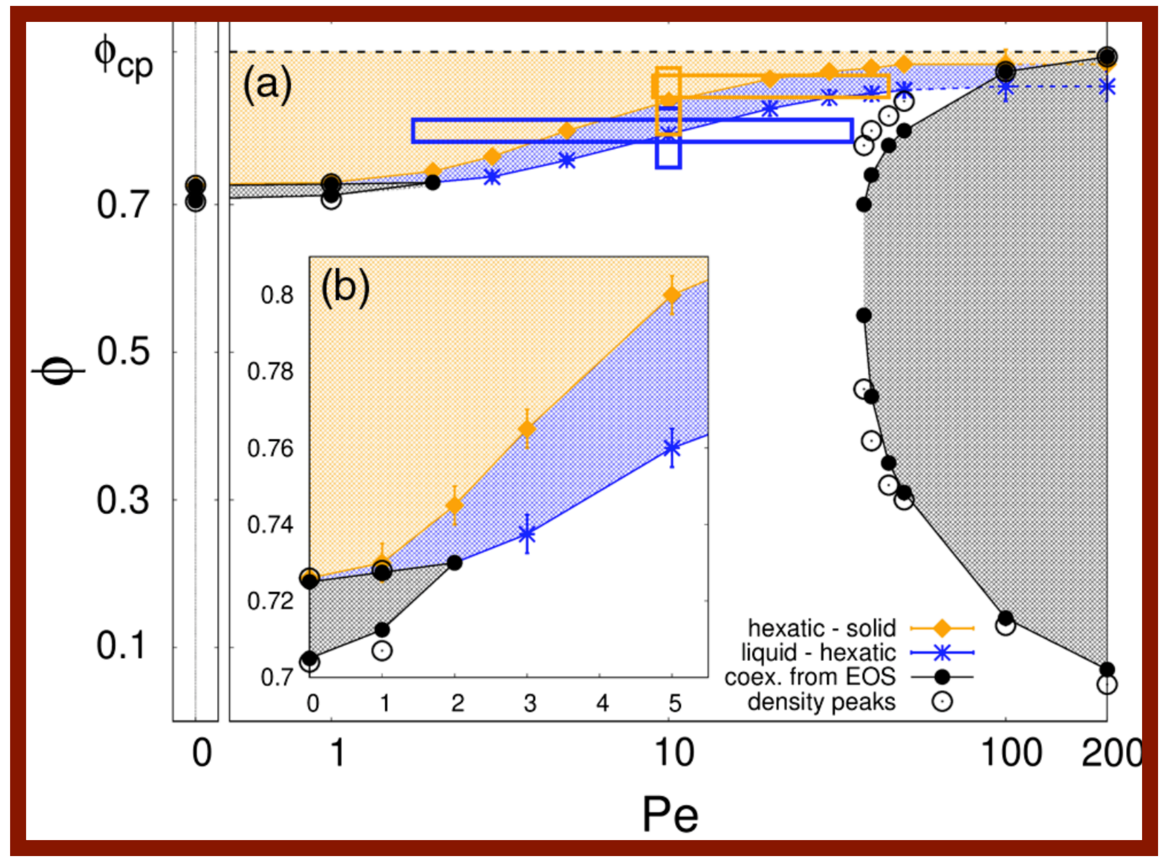
Janus particle

Active Matter

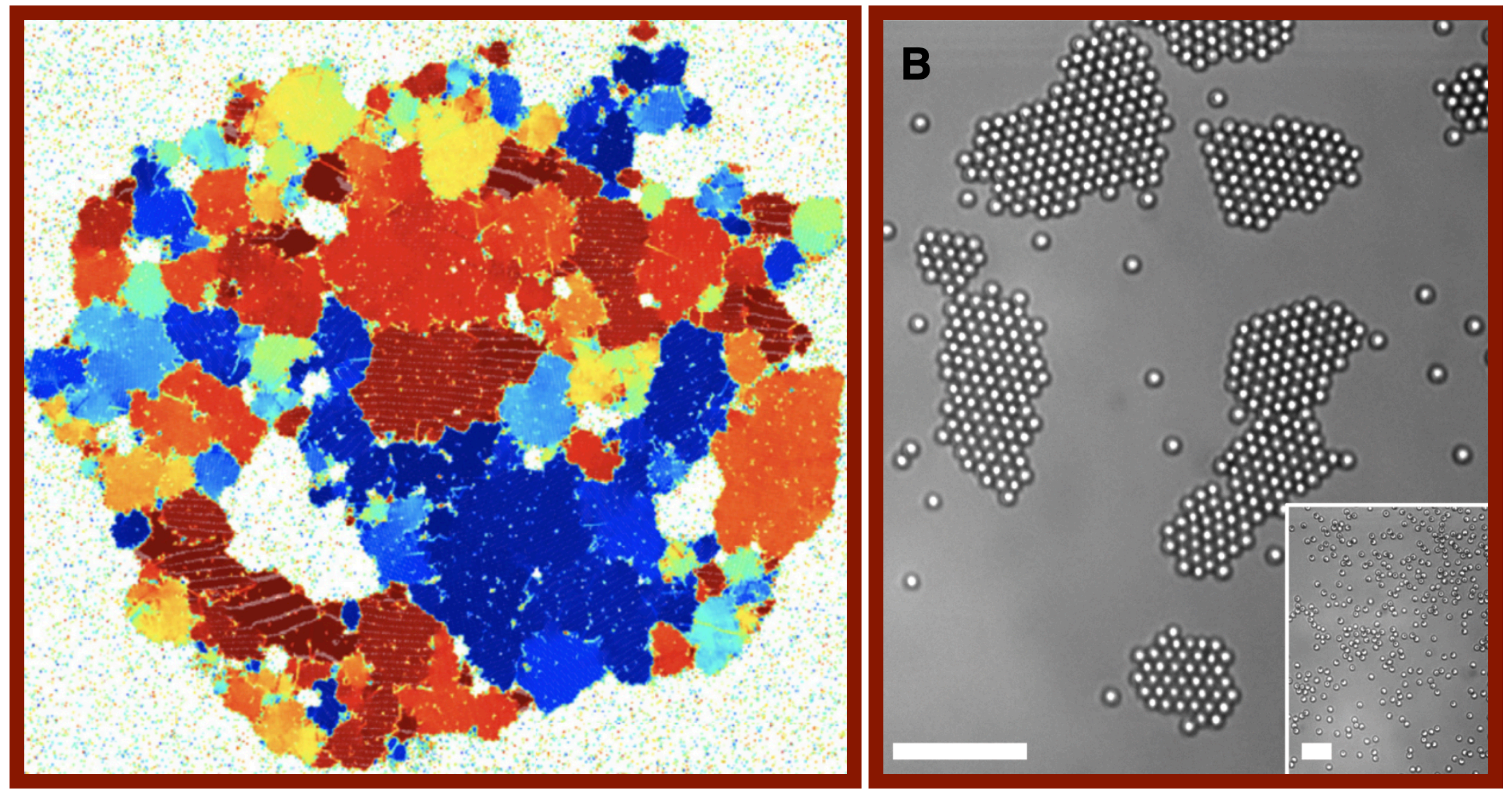
- **Theoretical interest**
  - emergence of new features with no counterpart in passive systems
  - strong connection with biological systems
  - new paradigm of out-of-equilibrium systems



spontaneous flows

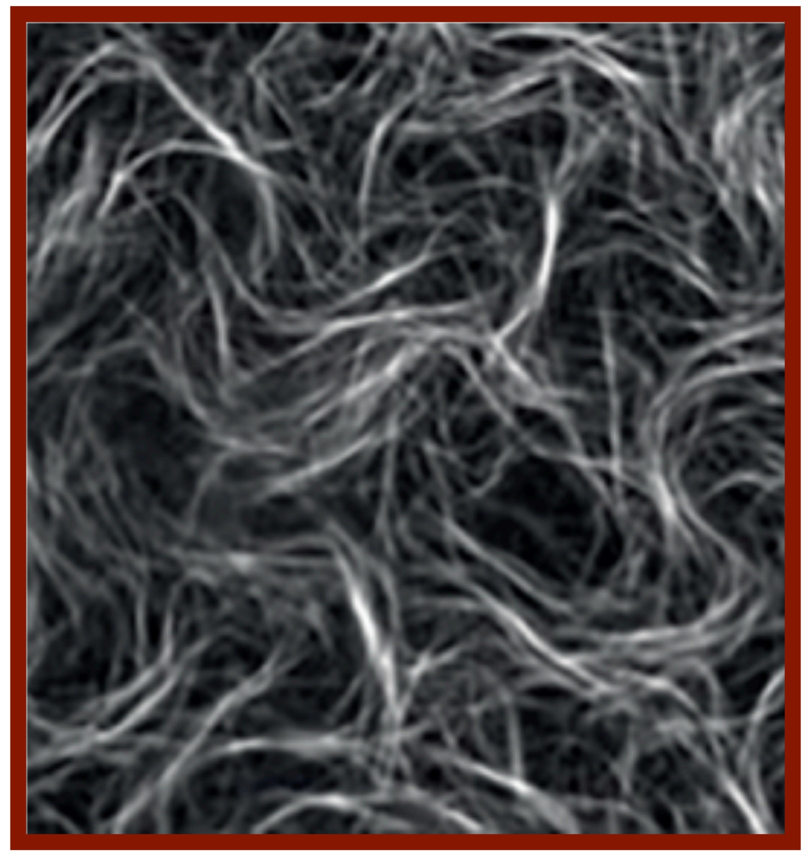


active, liquid, hexatic and solid phases

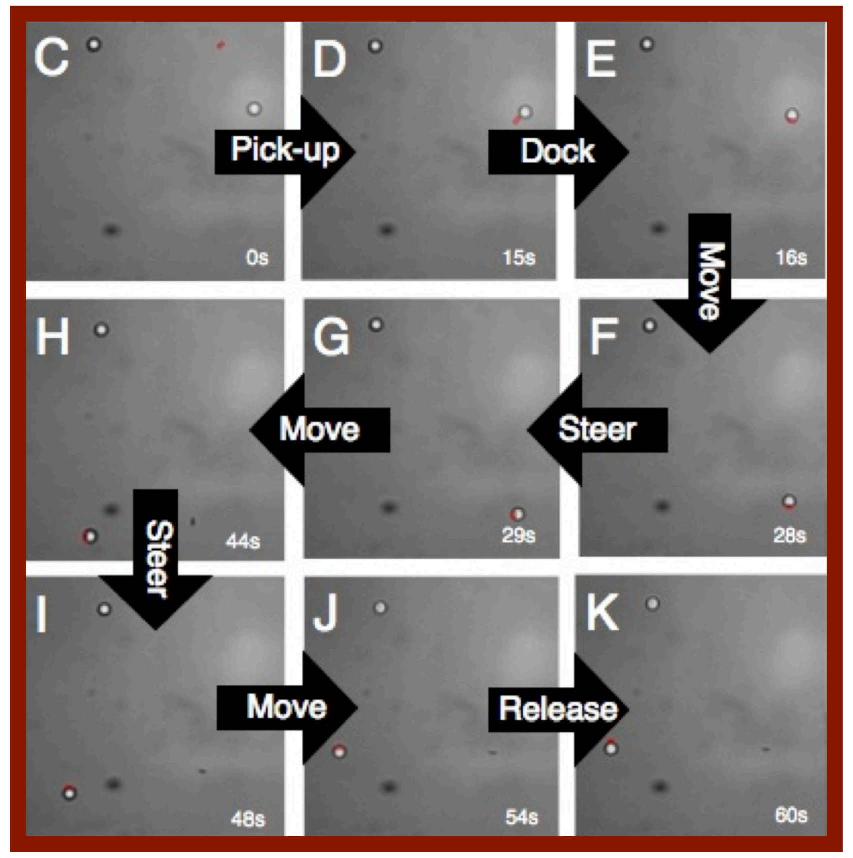


motility-induced phase separation

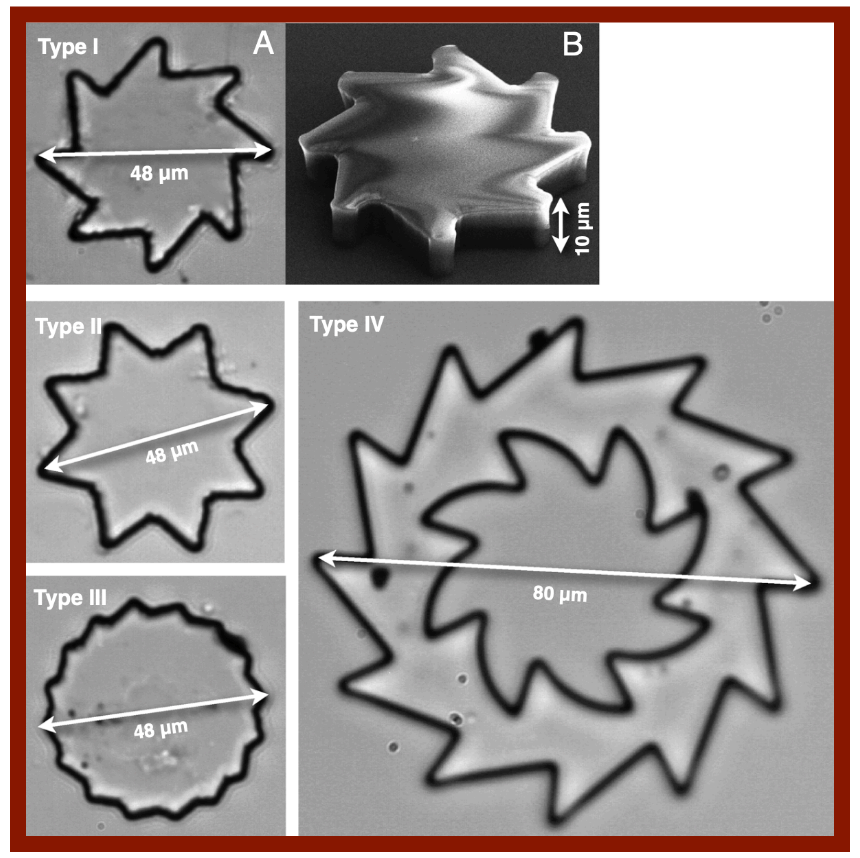
- **Experimental and technological interest**



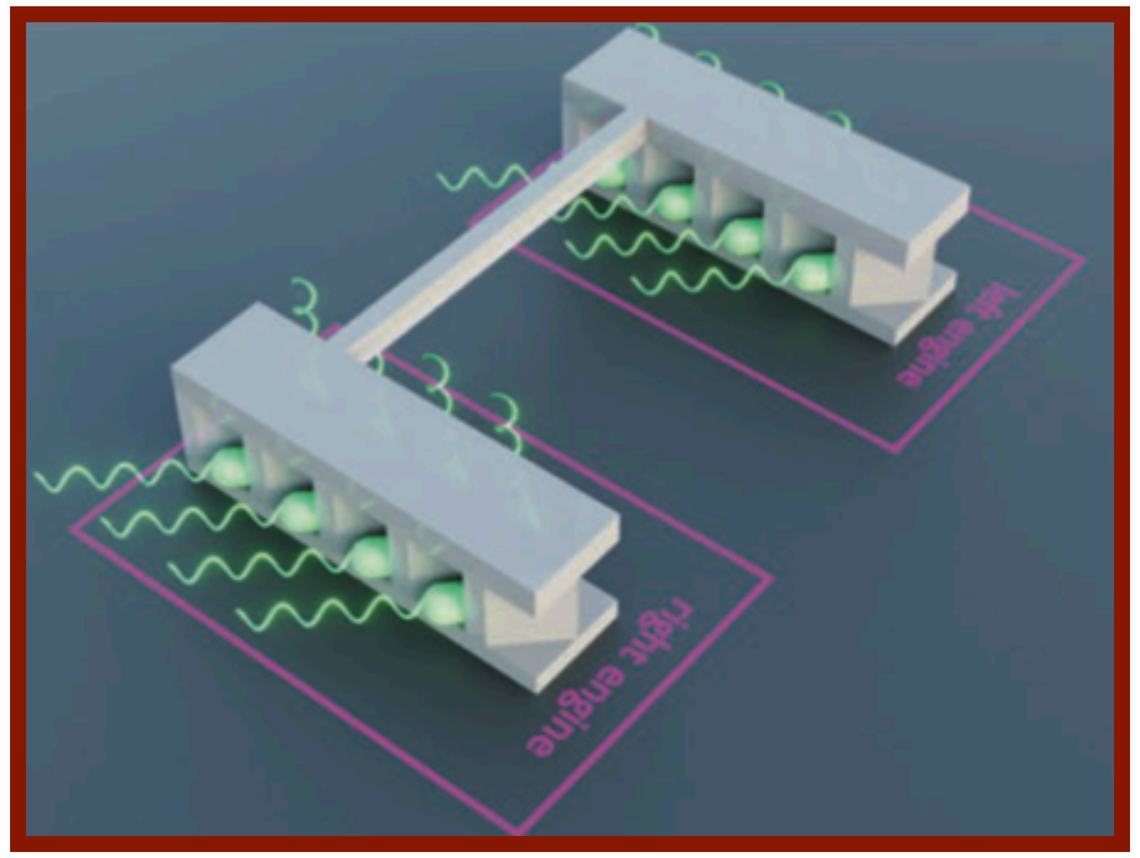
new bio-inspired materials



targeted delivery



active engines



nanorobots

Brownian motion

- **Langevin equations**

$$\ddot{x}_i(t) = -\underbrace{\frac{dU(x(t))}{dx_i(t)} + f_i(t)}_{\text{external and interaction forces (conservative + non-conservative)}} \underbrace{- \gamma \dot{x}_i(t) + \sqrt{2D}\ \xi_i(t)}_{\text{thermal bath (friction + noise)}}$$

Active particles models

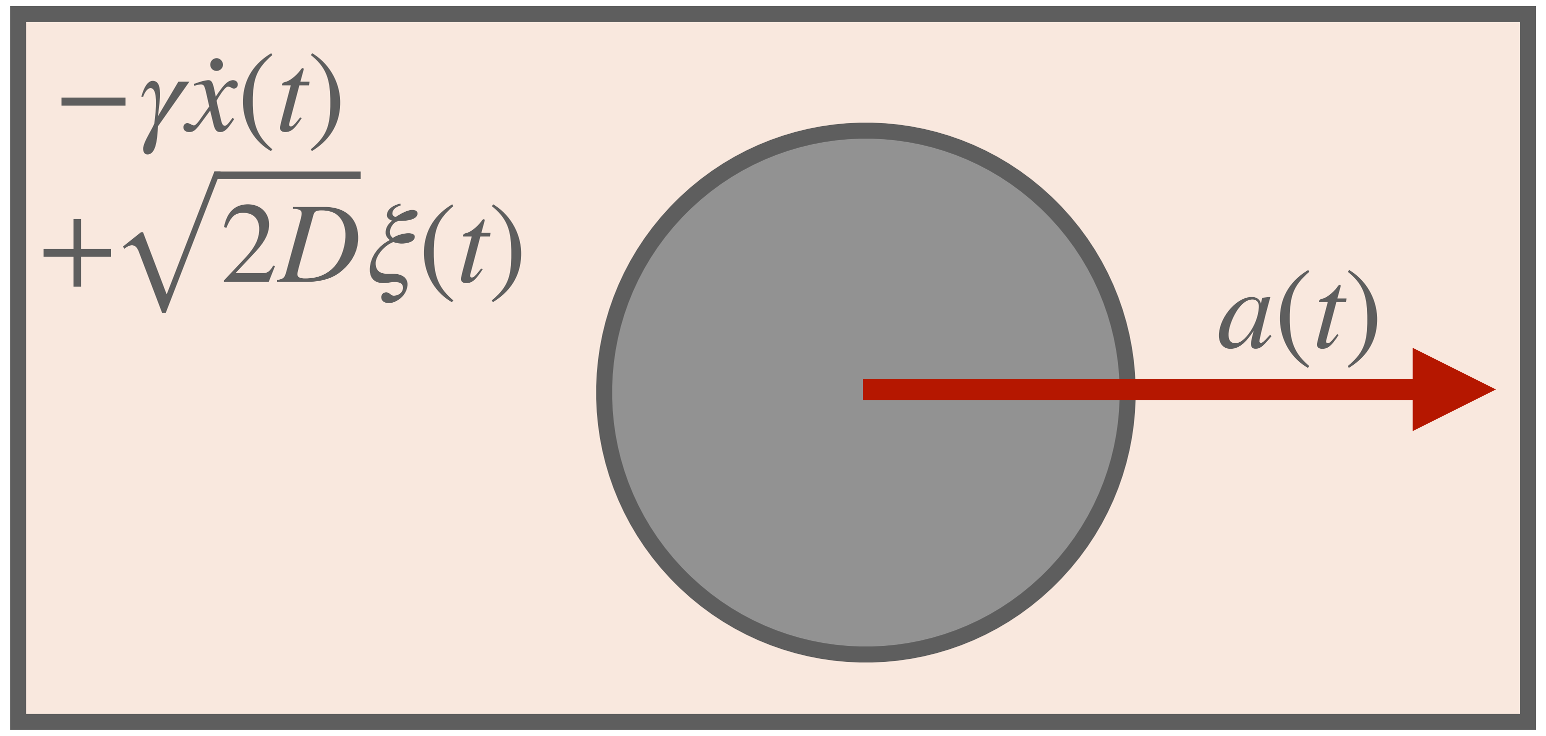
- **Langevin equations**

$$\ddot{x}_i(t) = -\frac{dU(x(t))}{dx_i(t)} + f_i(t) - \gamma \dot{x}_i(t) + \sqrt{2D}\;\xi_i(t) \quad \boxed{+a_i(t)}$$

external and interaction forces
(conservative + non-conservative)

thermal bath
(friction + noise)

active force
(self-propulsion)



Active particles models

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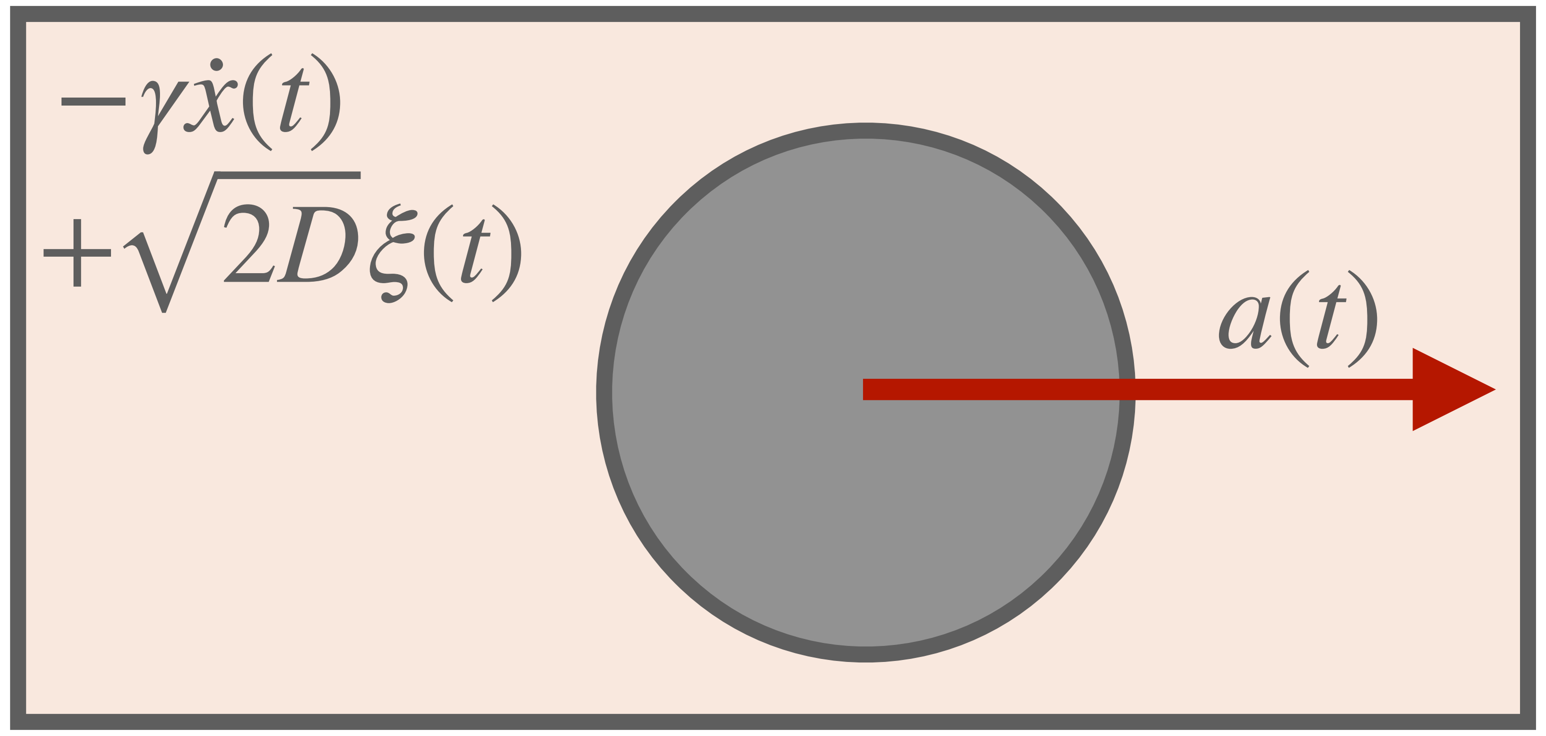
external and interaction forces (conservative + non-conservative)

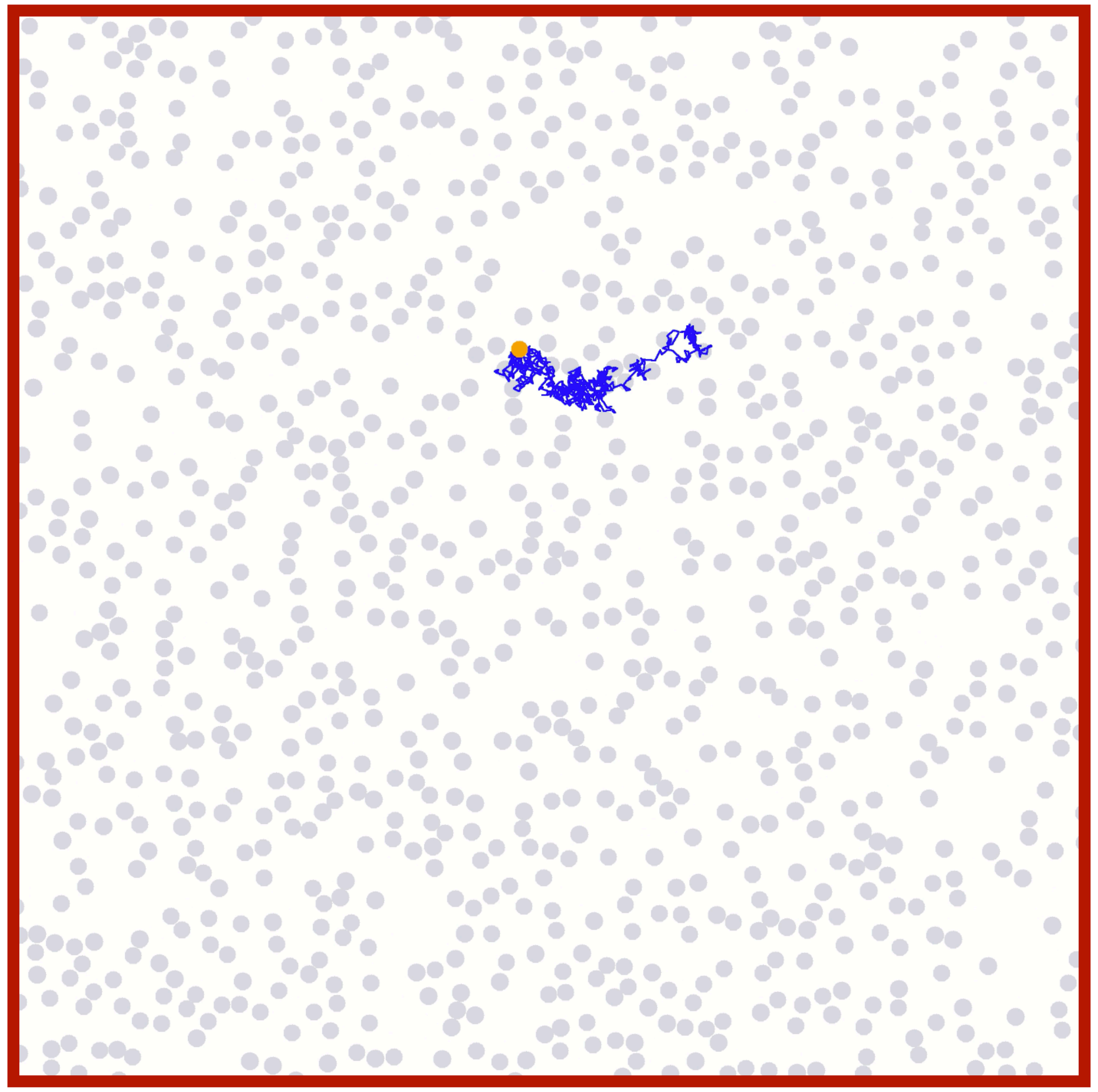
thermal bath (friction + noise)

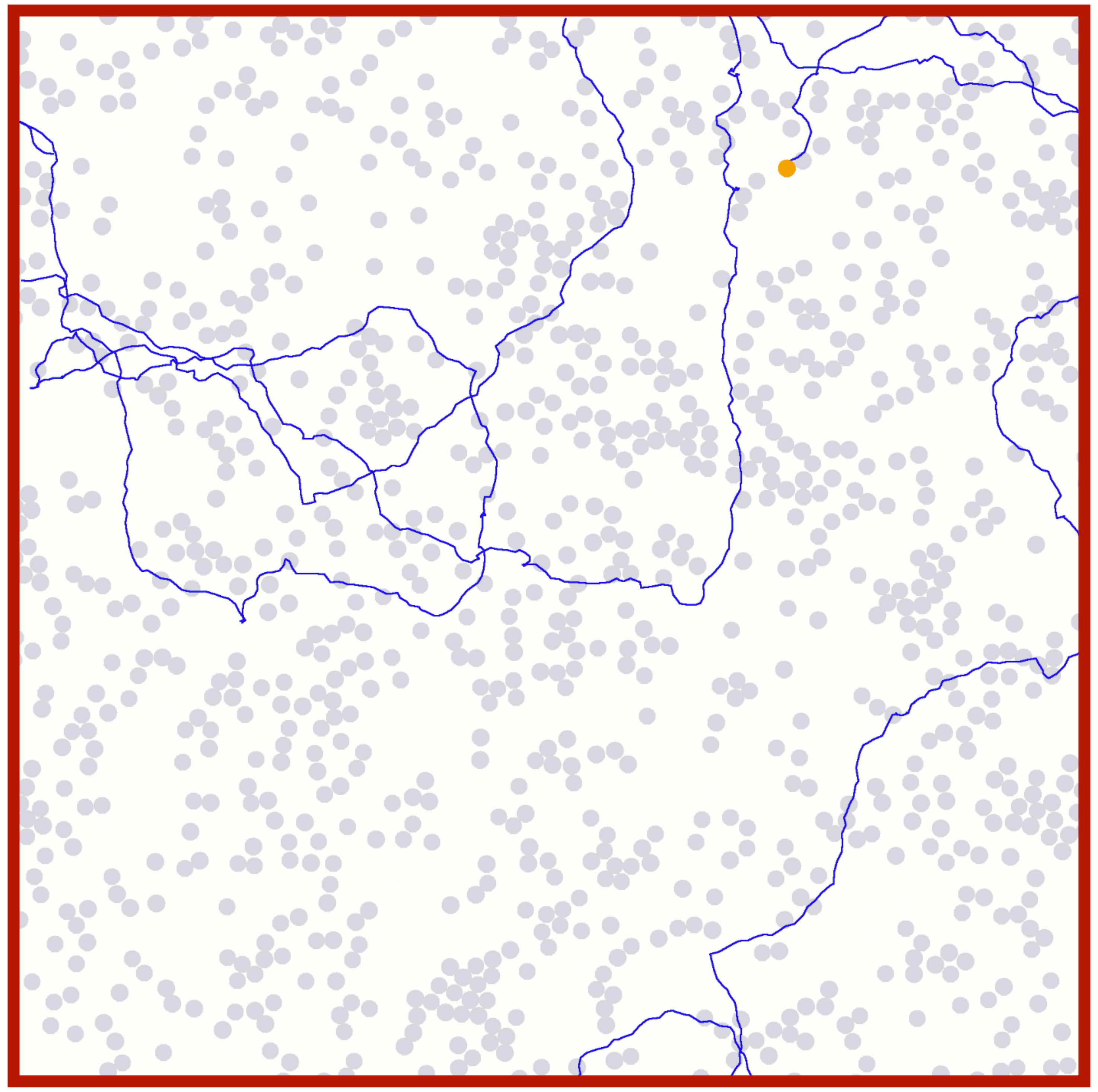
active force (self-propulsion)



- **Passive vs Active motion**

Passive Brownian Particle





Active Brownian Particle

Work Observables and Distributions

- **Dynamical observables**

$$\mathscr{W}_\tau = \frac{1}{\tau}\int_0^\tau G(x(s), \dot{x}(s), a(s))\dot{x}(s)\ ds$$

- integrated observables measured along particle trajectories
- G generic function of positions velocity and active force
- $1/\tau$ essential to make $\mathscr{W}_\tau$ intensive in time

- **Active Work**

$$\mathscr{W}_\tau = \frac{1}{\tau}\int_0^\tau a(s)\dot{x}(s)\ ds$$

- it captures the energy cost to sustain self propulsion
- important in applications: thermodynamical efficiency

- **Large Deviations Theory**

$$p(\mathscr{W}_\tau = w) \asymp e^{-\tau I(w)}$$

- asymptotic equivalence

$I(w)$ **Rate Function (RF)**

- extension of thermodynamic potentials to out of equilibrium configurations

$$\phi(\lambda) = \lim_{\tau\to\infty} \frac{1}{\tau}\ln(\langle e^{\lambda \mathscr{W}_\tau}\rangle)$$

Scaled Cumulant Generating Function (SCGF)

- function whose derivatives generate the moments of the distribution

$$I(w) = \sup_{\lambda\in O}\{\lambda w - \phi(\lambda)\}$$

- RF and SCGF often related through Legendre-Fenchel transform

Singular distributions

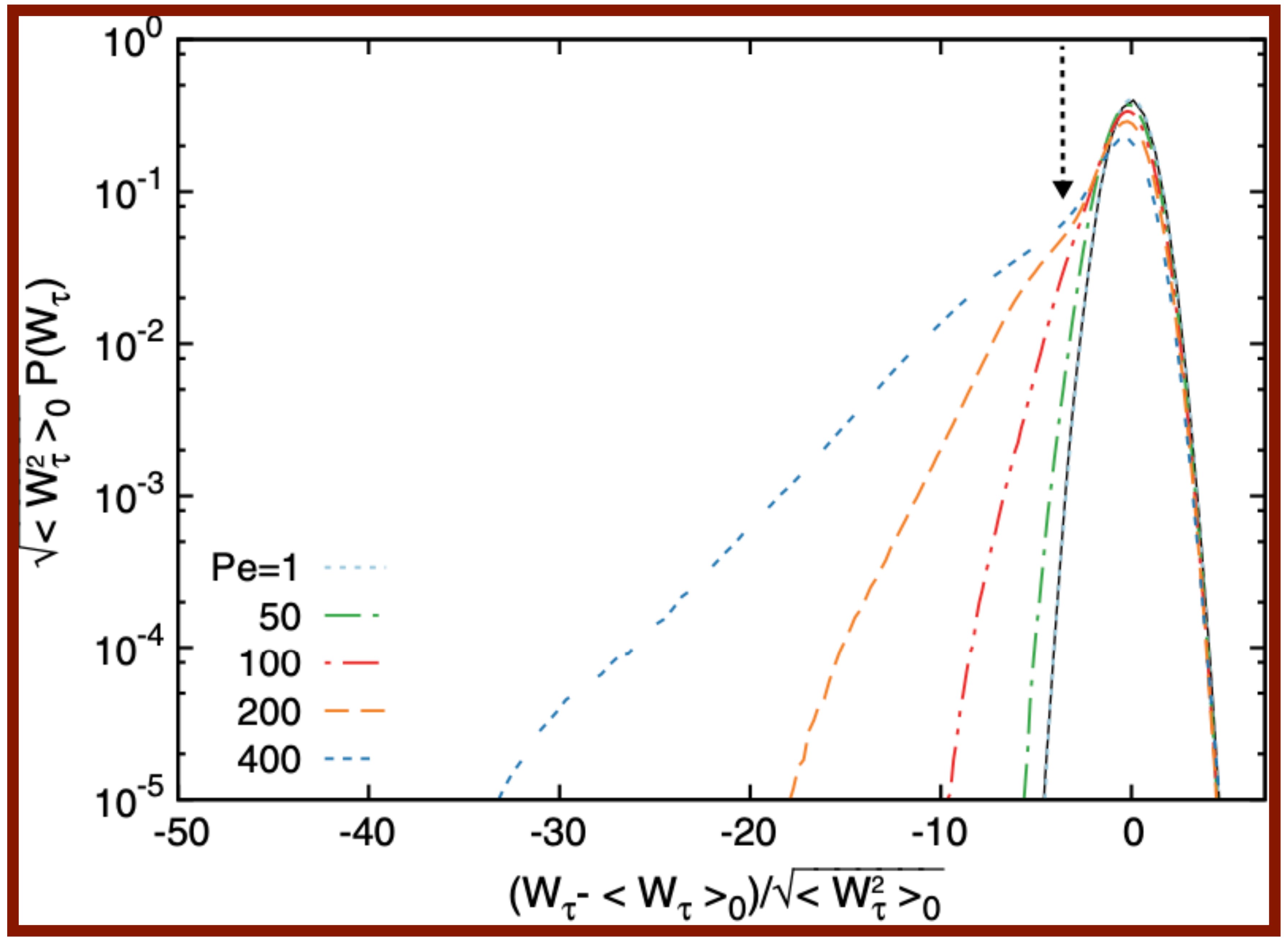
- Rate Functions can be singular
  - **Dynamical Phase Transitions**
    change in the physical mechanism producing fluctuations
  - **Trajectory Separation**
    trajectories in different regions of the RF behave dynamically different

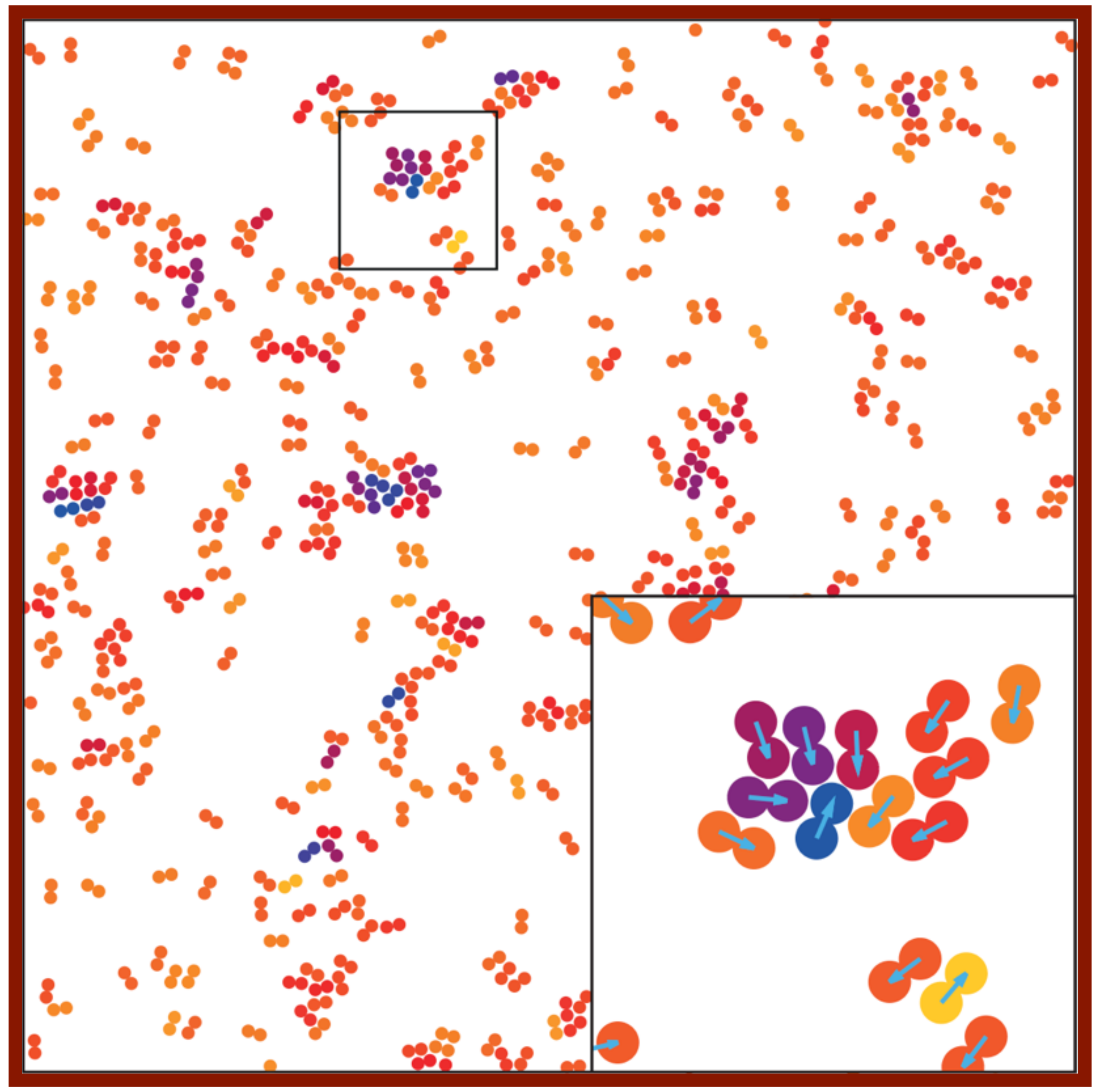
- Many examples in the context of Langevin models, urn model, ferromagnets, glassy systems

- Active Work in a system of interacting active particles

$$\mathscr{W}_\tau = \frac{1}{\tau}\int_0^\tau a(s)\dot{x}(s)\ ds$$

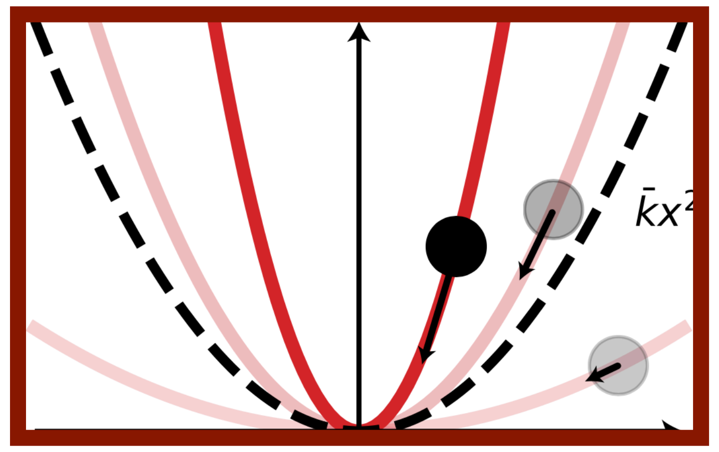
- In singular trajectories particles dragged against their active force





Analytical study of fluctuations of Active Work

- **Setting**
  single particle with external potential



- confining potentials mimic the trapping of other particles at finite density
- experimental realisations
- analytical results feasible

- Active Ornstein-Uhlenbeck Particle **(AOUP)**
  free or with external harmonic potential

$$\begin{cases} \dot{x}(t) = F_a \gamma^{-1} a(t) - kx(t) + \sqrt{2T/\gamma}\ \xi(t) \\ \dot{a}(t) = -\gamma_R a(t) + \sqrt{2D_R}\ \eta(t) \end{cases}$$

$$< a(t)a(t') > \simeq \left(e^{-\gamma_R(t-t')} - e^{-\gamma_R(t+t')}\right)$$

- **Approach**
  analytical evaluation of the Rate Function through Large Deviations techniques

- **Interest**
  - **Theoretical**
    - energy cost to sustain self propulsion
  - **Practical**
    - thermodynamic efficiency of Active Engines

- **Scope**
  - investigation of distribution singularities and Dynamical Phase Transitions

Free AOUP

J Stat Mech 2021,
Semeraro, Suma, Petrelli, Cagnetta and Gonnella

- Free AOUP in d dimensions

$$\begin{cases} \dot{x}(t) = F_a \gamma^{-1} a(t) + \sqrt{2T/\gamma}\ \xi(t) \\ \dot{a}(t) = -\gamma_R a(t) + \sqrt{2D_R}\ \eta(t) \end{cases}$$

- Active work $\mathscr{W}_a = \frac{1}{\tau}\int_0^\tau a(t)\dot{r}(t)\,dt$

- Probability distribution $p(w) = \langle(\delta(\mathscr{W}_a - w))\rangle \asymp e^{-\tau I(w)}$ evaluated through **path integral** techniques

  - Trajectory path probability

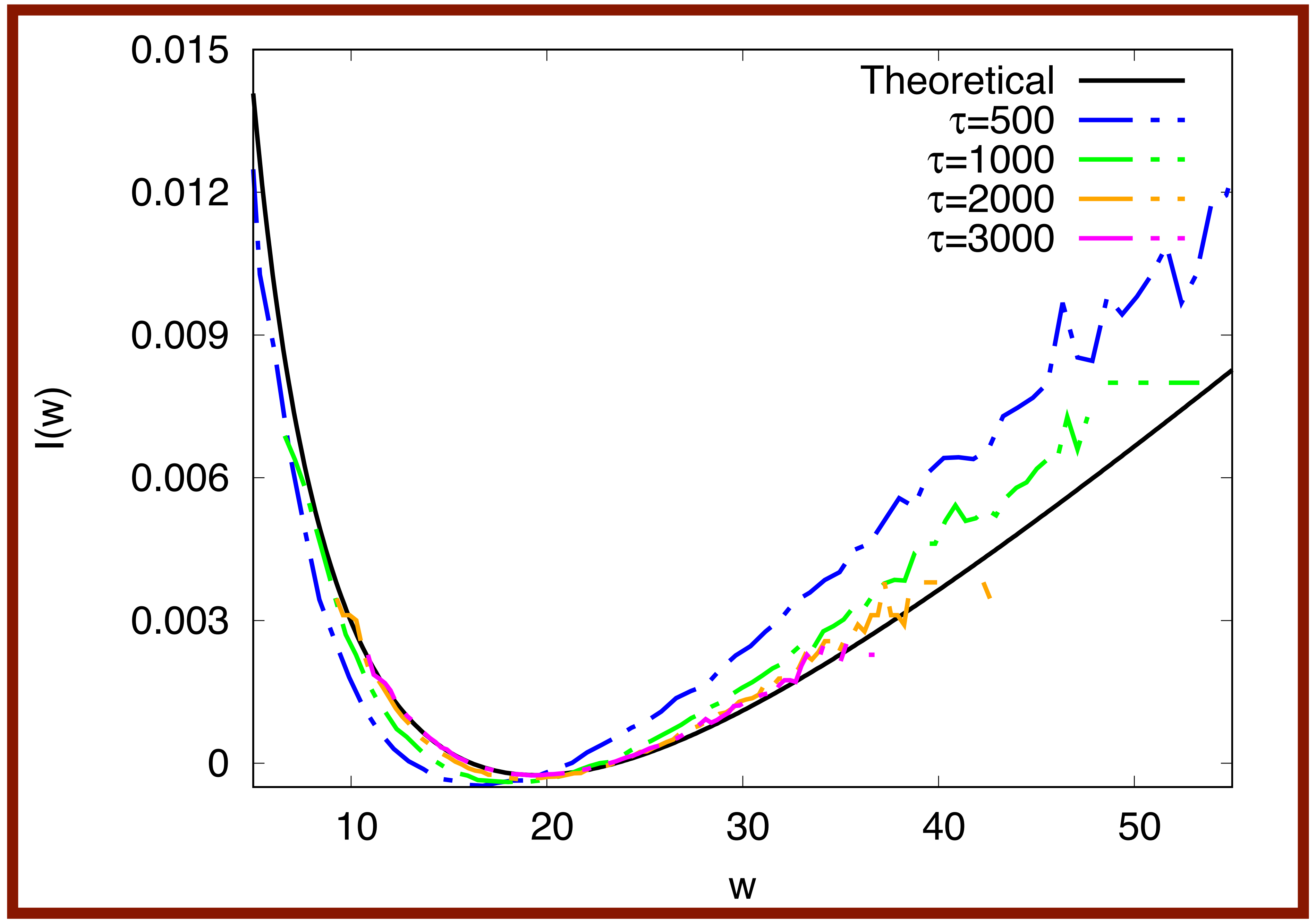
initial conditions distribution

$$\mathscr{P}_\tau(x(\tau), a(\tau)) \propto p(x_0, a_0) \times$$

$$\times\, exp\left\{ -\frac{1}{4D_T}\int_0^\tau [\dot{x}(s) - F_a\gamma^{-1}a(s)]^2\, ds \right\} exp\left\{ -\frac{1}{4D_R}\int_0^\tau [\dot{a}(s) + \gamma_R a(s)]^2\, ds \right\}$$

Onsager-Machlup weight for trajectories

  - Laplace representation of the δ function $p(w) = \frac{1}{2\pi\imath}\int_{-\imath\infty}^{+\imath\infty} e^{-\tau\lambda w}\langle e^{\lambda\mathscr{W}_a}\rangle$
  - Cumulant Generating Function $\phi(\lambda) = \langle e^{\lambda\mathscr{W}_a}\rangle \sim e^{\frac{\tau}{2}(\gamma_R - \sqrt{\gamma_R^2 - 4D_R\lambda\gamma(1+T\lambda)})}$
  - Saddle-point estimation of the RF



- no singularities in $I(w)$

Harmonically confined AOUP

- Harmonically-confined AOUP in 1 d

$$\begin{cases} \gamma \dot{x}(t) = a(t) - kx(t) + \sqrt{2\gamma T}\ \xi(t) \\ \dot{a}(t) = -\nu a(t) + F\sqrt{2\nu}\ \eta(t) \end{cases}$$

- Direct evaluation of $p(w)$ through **path integral** techniques becomes difficult

  - Trajectory path probability ✔

$$\mathscr{P}_\tau \propto \underbrace{\left\{-\frac{1}{2}(x(0)\ a(0))\Sigma_0^{-1}\begin{pmatrix}x(0)\\a(0)\end{pmatrix}\right\}}_{\text{initial conditions distribution}} \underbrace{exp\left\{-\frac{1}{4}\int_0^\tau [\dot{x}(s) - a(s) + \kappa x(s)]^2\ ds\right\} exp\left\{-\frac{1}{4Pe^2}\int_0^\tau [\dot{a}(s) + a(s)]^2\ ds\right\}}_{\text{Onsager-Machlup weight for trajectories}}$$

  - Laplace representation of the δ function ✔

$$p(w) = \frac{1}{2\pi\imath}\int_{-\imath\infty}^{+\imath\infty} e^{-\tau\lambda w}\ \langle e^{\lambda \mathscr{W}_a}\rangle$$

  - Cumulant Generating Function ✘
  - Saddle-point estimation of the RF

- New Large Deviations results for **quadratic functionals** of **Gauss-Markov chains**

J Math Phys 2023, Zamparo and Semeraro

  - Time-discretization procedure
  - Evaluation of the SCGF functional form
  - Evaluation of the SCGF domain
  - Continuum limit
  - Evaluation of the RF through Legendre-Fenchel transform

LDT for quadratic functionals of Gauss-Markov chains

J Math Phys 2023, Zamparo and Semeraro

- Time-discretizatioin procedure

Langevin Equations→ Markov chain $X_{n+1} = SX_n + G_n$

$W_\tau \cdot \tau \rightarrow$ quadratic functional

$$W_N = \underbrace{\frac{1}{2} < X_0, LX_0 > + \frac{1}{2} < X_N, RX_N >}_{\text{boundary terms}} + \underbrace{\frac{1}{2}\sum_{n=1}^{N} < X_n, UX_n > + \frac{1}{2}\sum_{n=2}^{N} < X_n, VX_{n-1} >}_{\text{bulk contributions}}$$

- Evaluation of the **Scaled Cumulant Generating Function**

$$\varphi(\mu) = \lim_{N\to\infty} \frac{1}{N} \log < e^{\lambda W_N} > = -\frac{1}{4\pi}\int_0^{2\pi} \ln\det F_\lambda(\theta)\ d\theta$$

$$F_\lambda(\theta) = (I - S^T e^{\iota\theta})(I - S^T e^{-\iota\theta}) - \lambda(U + Ve^{-\iota\theta} + V^T e^{\iota\theta})$$

Primary domain P:
$F_\lambda(\theta)$ is positive definite for all $\theta \in (0,2\pi)$

Effective domain E:
the matrices $\mathscr{L}_\lambda$ and $\mathscr{R}_\lambda$ related to the initial conditions (Σ_0) and boundary terms (L, R) are positive definite

- Evaluation of the **Rate Function**

continuum limit $\phi(\lambda) = \lim_{\epsilon\to 0} \frac{\varphi(\mu)}{\epsilon}$

Legendre-Fenchel transform $I(w) = \sup_{\lambda\in E}\{w\lambda - \phi(\lambda)\}$

Singular Rate Function

PRL 2023,
Semeraro, Gonnella,
Suma and Zamparo

- SCGF

$$\phi(\lambda) = \frac{1+\kappa}{2} - \frac{1}{2}\sqrt{(1+\kappa)^2 - 4Pe^2\lambda(1+\lambda)}$$

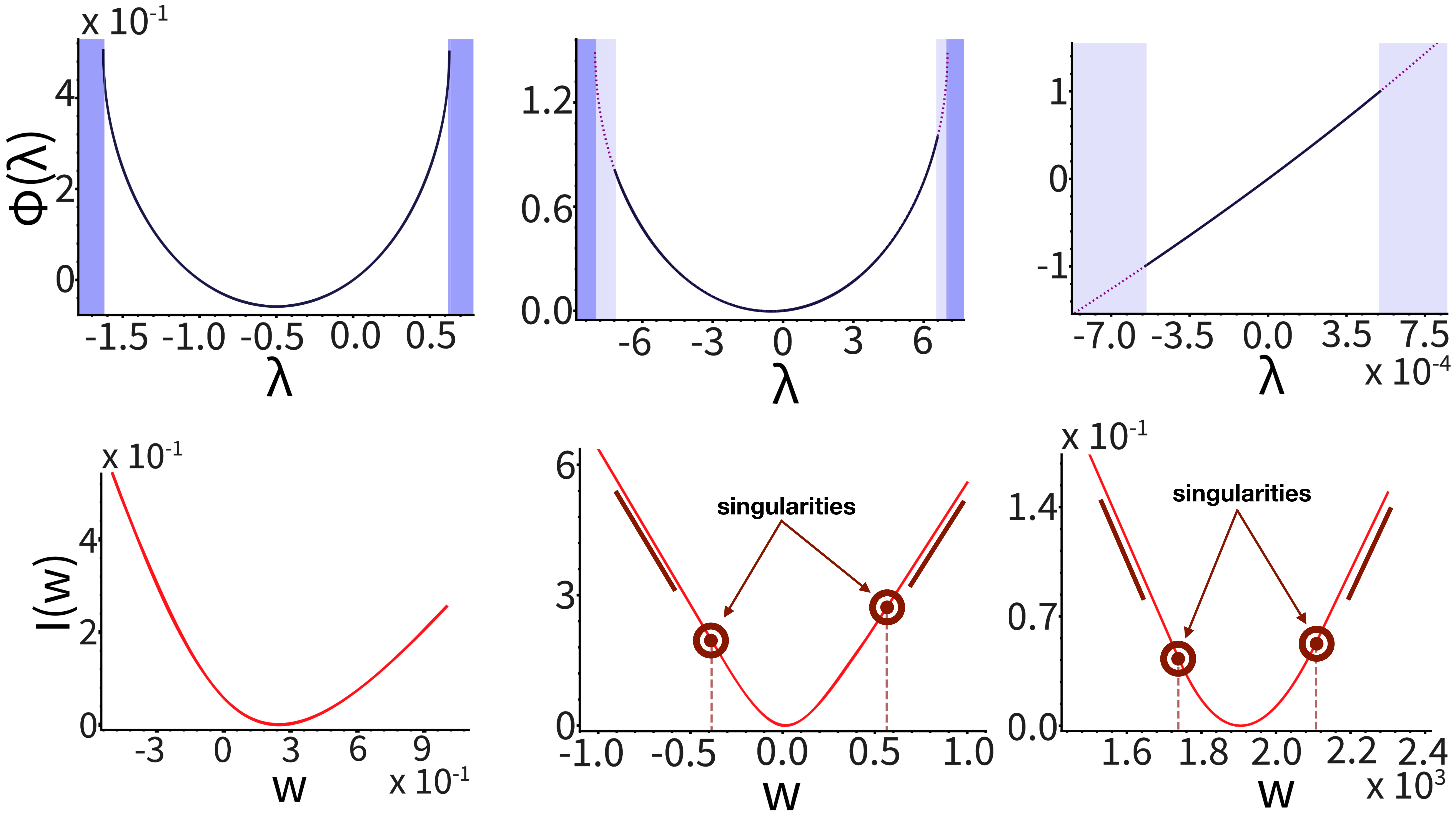
$$Pe = \frac{Fd}{k_BT} \quad \kappa = \frac{kd^2}{k_BT}$$

- Rate function

$$I(w) = \begin{cases} (w-w_-)\lambda_- + i(w) & w \le w_- \\ i(w) & w_- < w < w_+ \\ (w-w_+)\lambda_+ - i(w) & w \ge w_+ \end{cases}$$

$$i(w) = \frac{1}{2}\left(\sqrt{1+\left(\frac{w}{Pe}\right)^2} + \sqrt{(1+\kappa)^2 + Pe^2 - 1 - \kappa - w}\right)$$

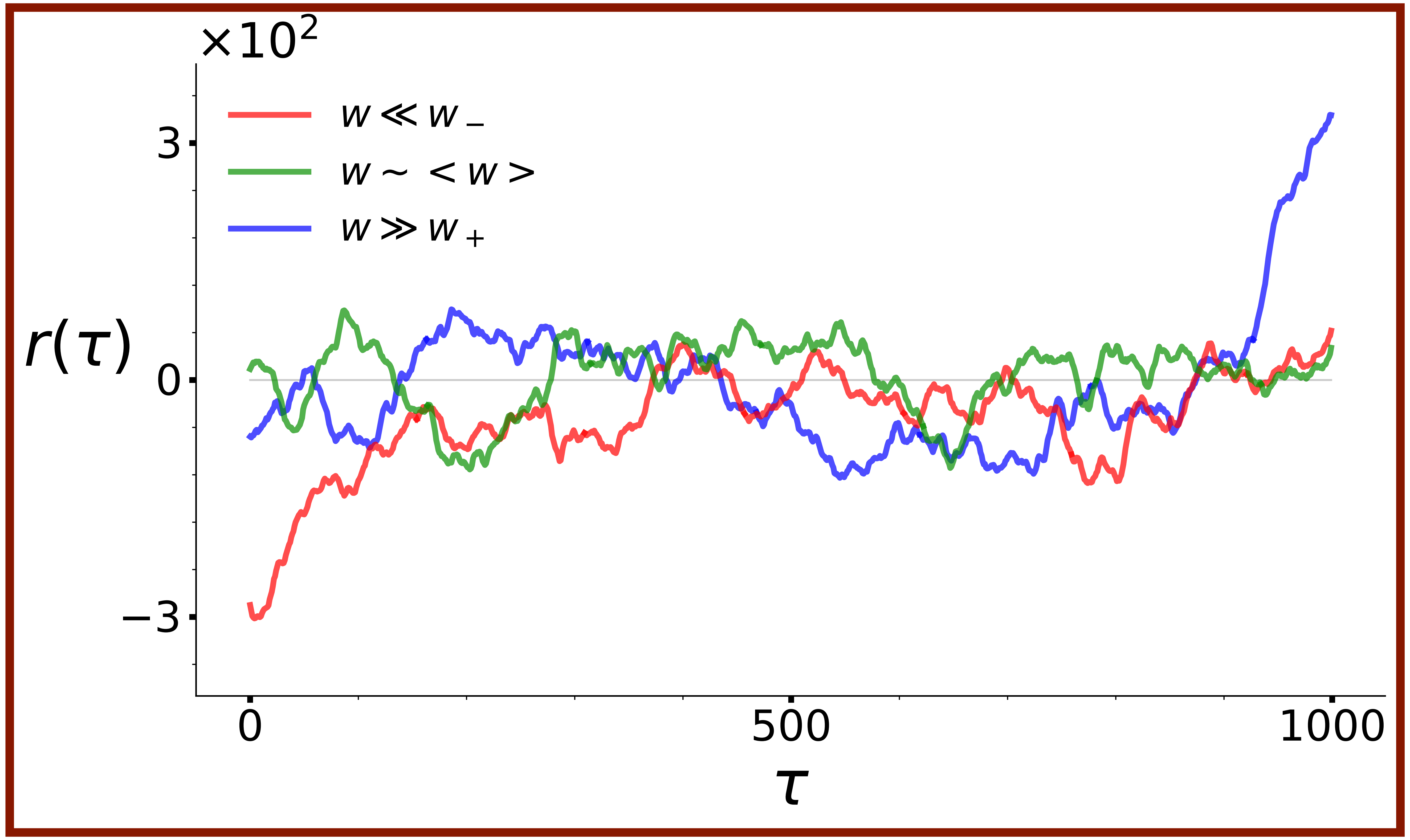
small Pe, κ | intermediate Pe, κ | large Pe, κ

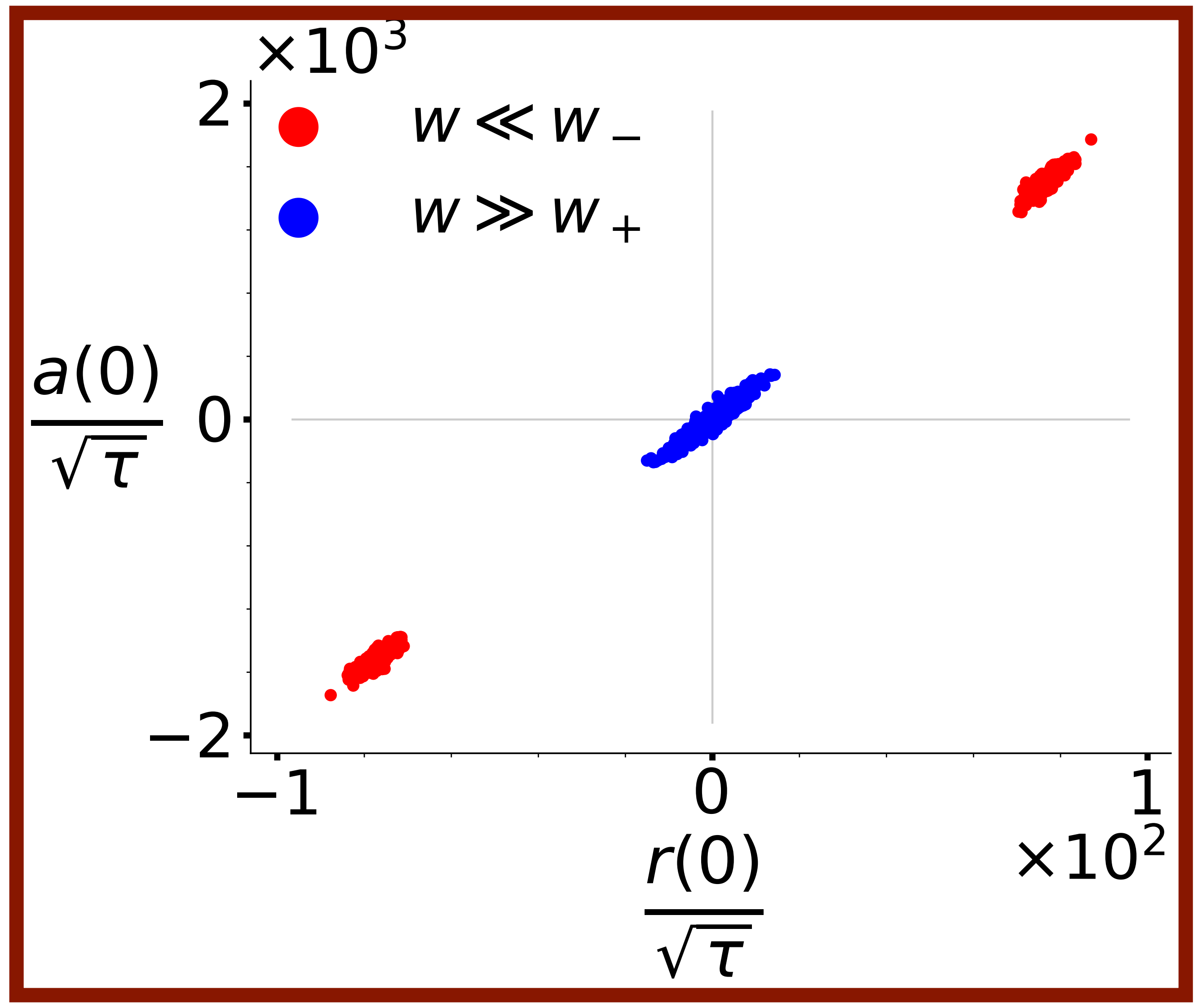


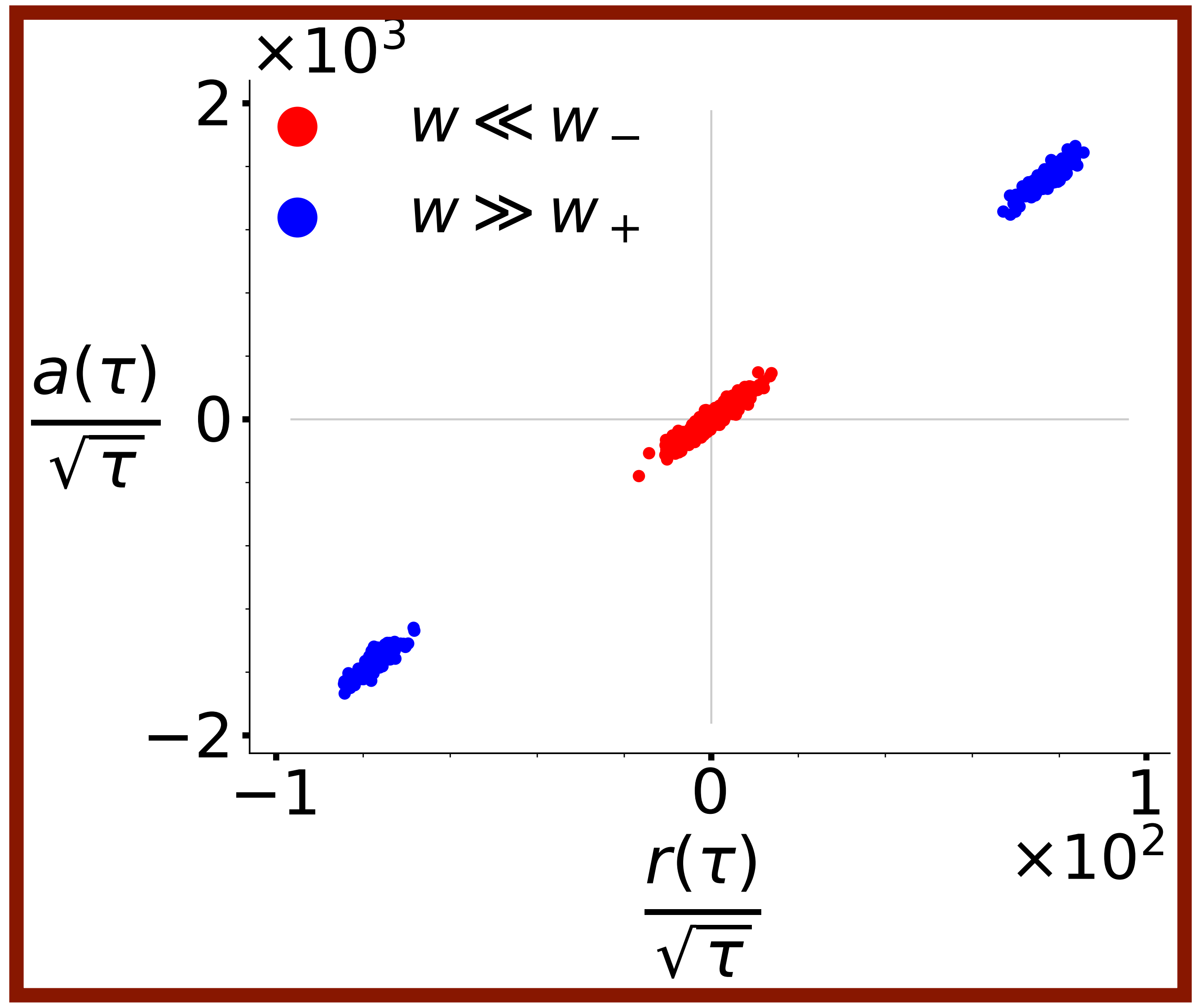
Linear Tails and Trajectory Separation

PRL 2023, Semeraro, Gonnella, Suma and Zamparo

- **Physical Mechanism:**
  singular trajectories are characterised and selected by **big jumps** in the initial ($w \ll w_{-}$) or final ($w \gg w_{+}$) values







Fluctuations of Injected power

J Phys A,
Carollo, Semeraro,
Gonnella and Zamparo

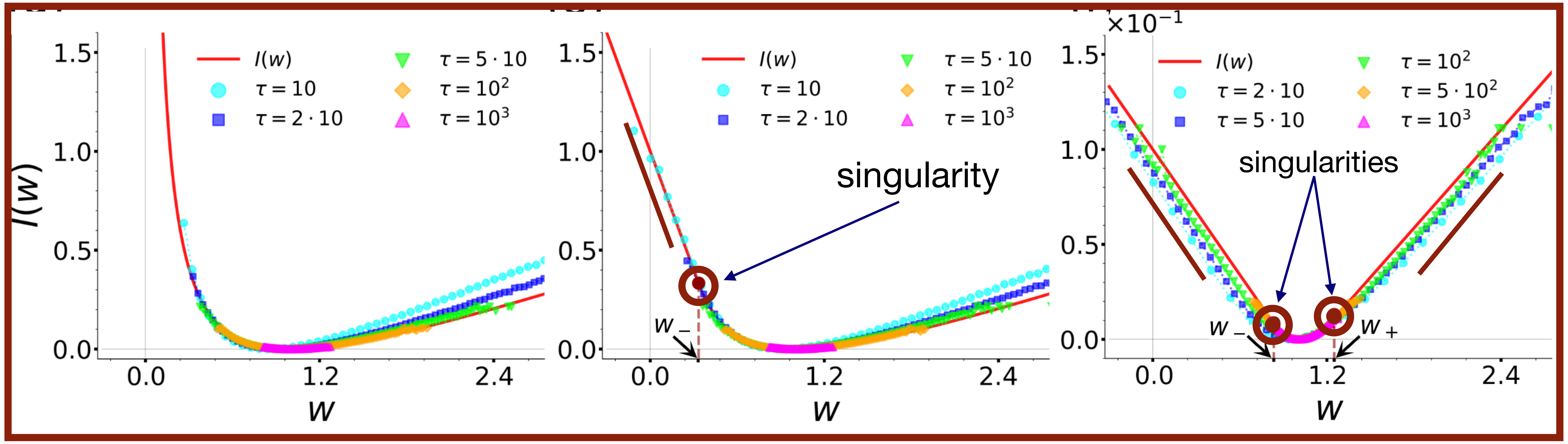
- Underdamped Brownian particle with external harmonic potential

$$\begin{cases} \dot{x} = v(t) \\ \dot{v}(t) = -\gamma v(t) - kx(t) + \sqrt{2D}\ \xi(t) \end{cases}$$

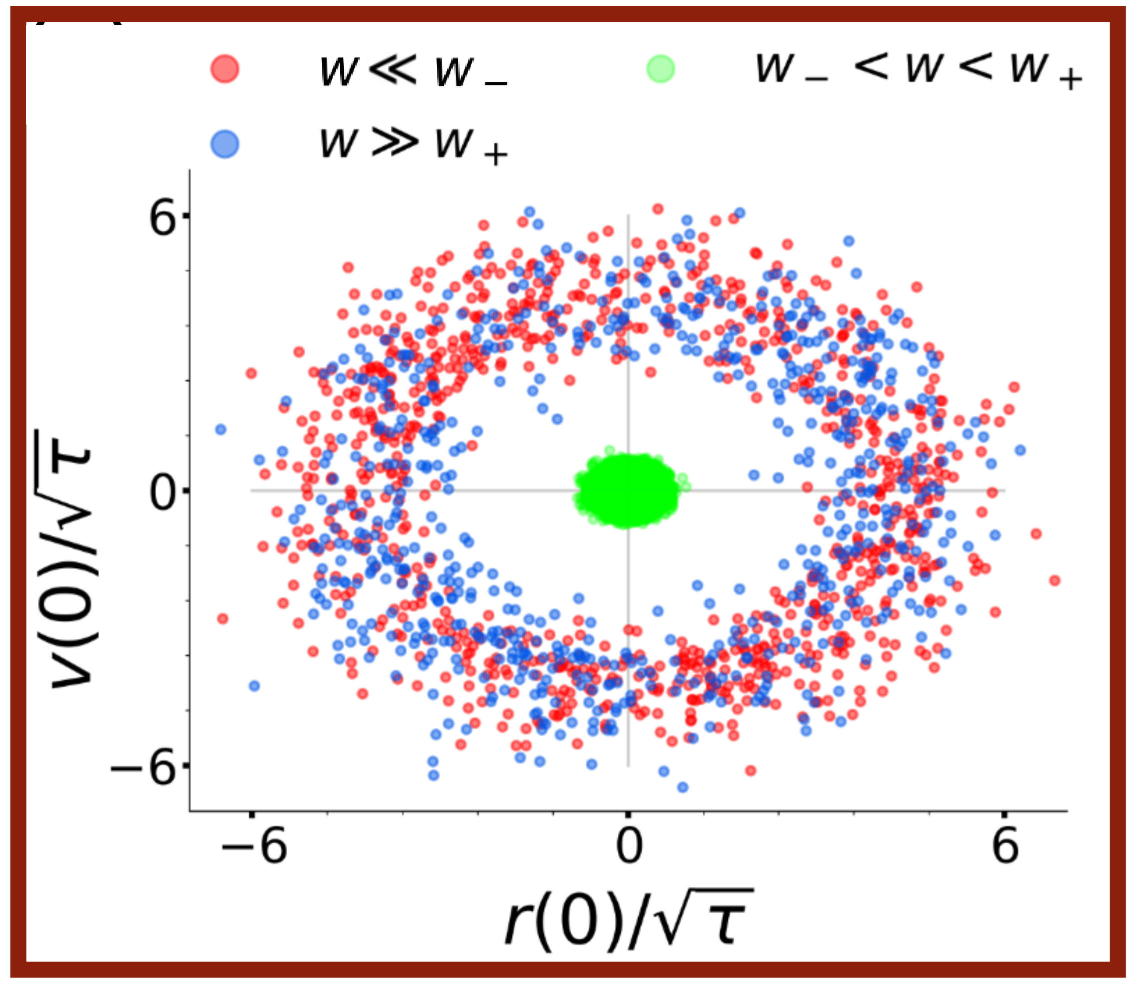
- Power injected by the random force

$$W_\tau = \frac{\sqrt{2D}}{\tau}\int_0^\tau \xi(t)\dot{x}(t)dt = \frac{1}{2\tau}[v^2(\tau) - v^2(0)] + \frac{k}{2\tau}[x^2(\tau) - x^2(0)] + \frac{\gamma}{\tau}\int_0^\tau v(t)\dot{x}(t)\ dt$$

- Singular Rate Functions
  - Fixed initial conditions
  - Stationary initial conditions
  - Generic uncorrelated initial conditions



- **Big jumps** in the initial conditions



Take-home messages

- **Active matter** is made of single components which transform energy to **self propel** → many examples from nature and experiments

- **Active Work** (and in general all work and work-related observables) play a major role on theoretical and experimental level → fluctuations described through Large Deviation Theory by **Rate Functions**

- **Peculiar tail structures** of Rate Functions signal peculiar dynamical behaviours → singularities and linear tails → **big jumps** (general mechanism)

Acknowledgements

Prof. Giuseppe Gonnella
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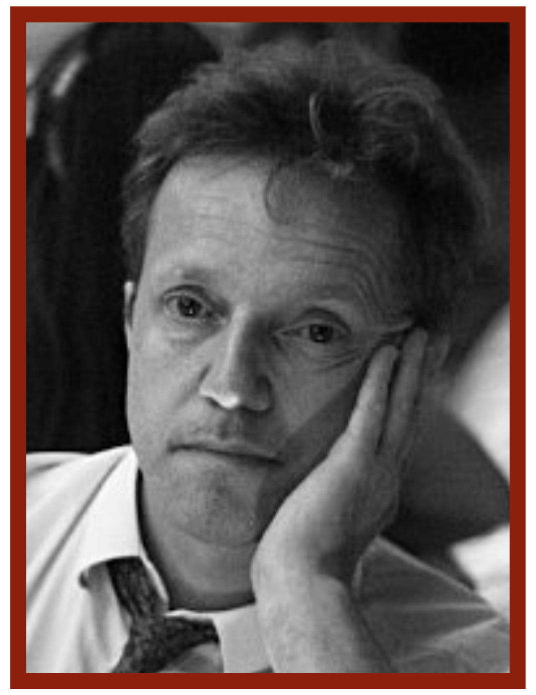

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Fluctuations of Entropy Production

EPL 2024,
Semeraro, Negro, Suma,
Corberi and Gonnella

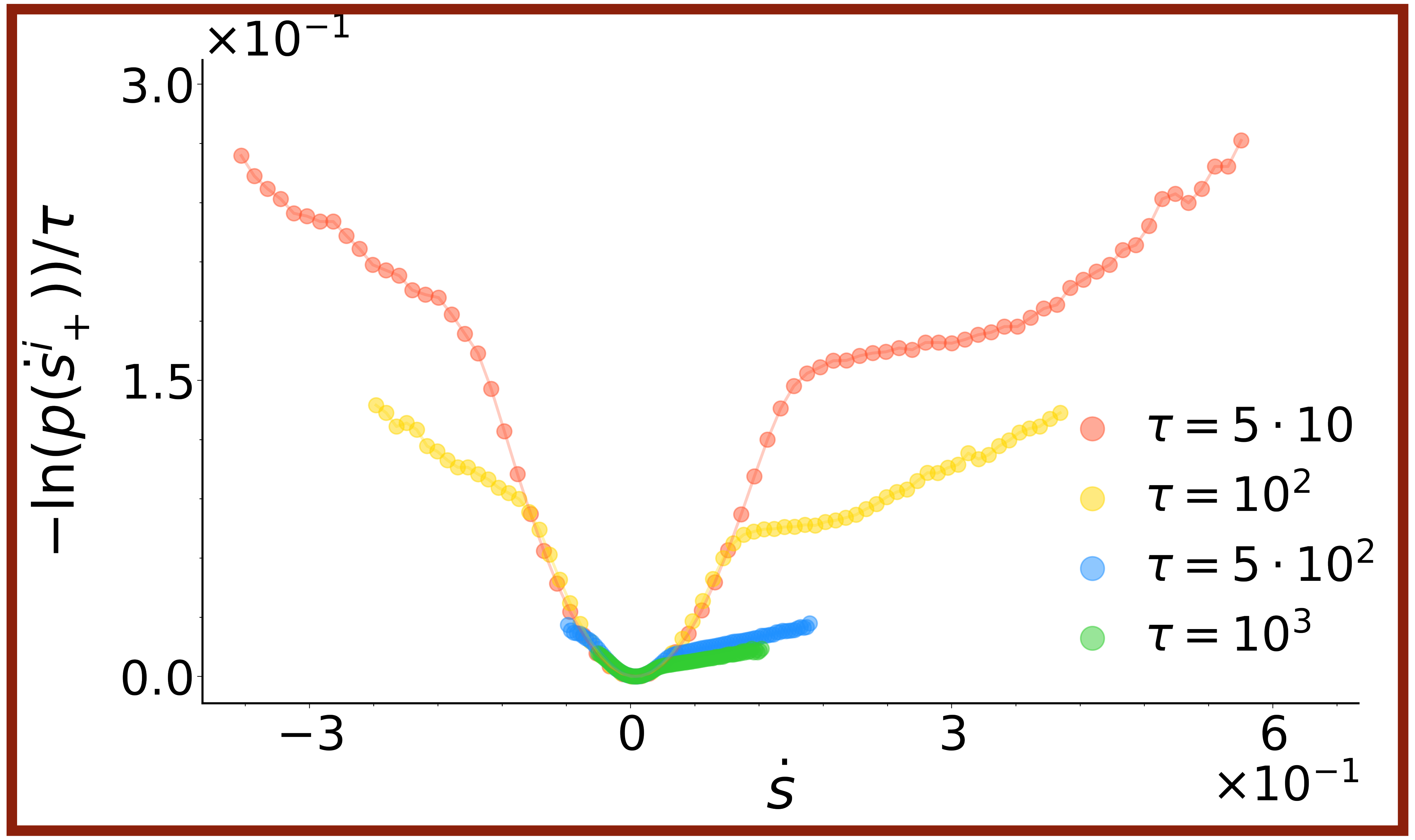
- Ensemble of interacting Active Brownian Particles

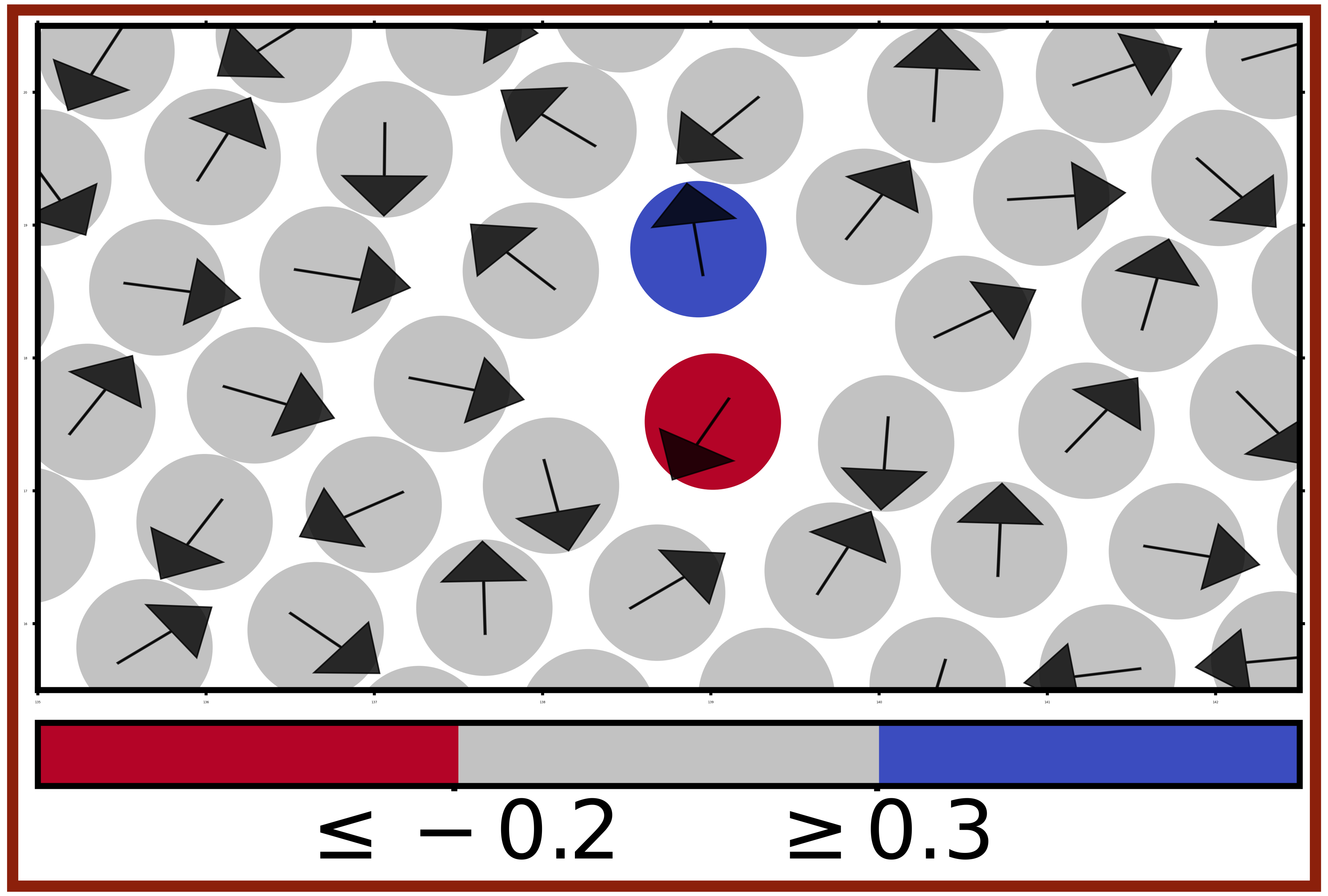
$$\begin{cases} \ddot{x}(t) = -\gamma\dot{x}(t) + F_a\hat{n}_i \sum_{i\neq j} \nabla U_i(r_{ij}) \sqrt{2\gamma k_B T}\ \xi_i(t) \\ \dot{\theta}_i(t) = \sqrt{2D_\theta}\ \eta_i \end{cases}$$

- Entropy production (similar to Active Work)

$$\mathcal{S}_\tau = \lim_{\tau\to\infty} \frac{1}{\tau}\frac{F_a}{k_B T}\int_0^\tau \hat{n}_i(s)\dot{x}_i(s)\ ds$$

- Peculiar tail structures → • Associated to particles close to topological defects





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- **Peculiar take structures** of Rate Functions signal peculiar dynamical behaviours
  - → singularities and linear tails → **big jumps** (general mechanism)
  - → anomalous tail structure → **motion close to defects**

Free AOUP: saddle point

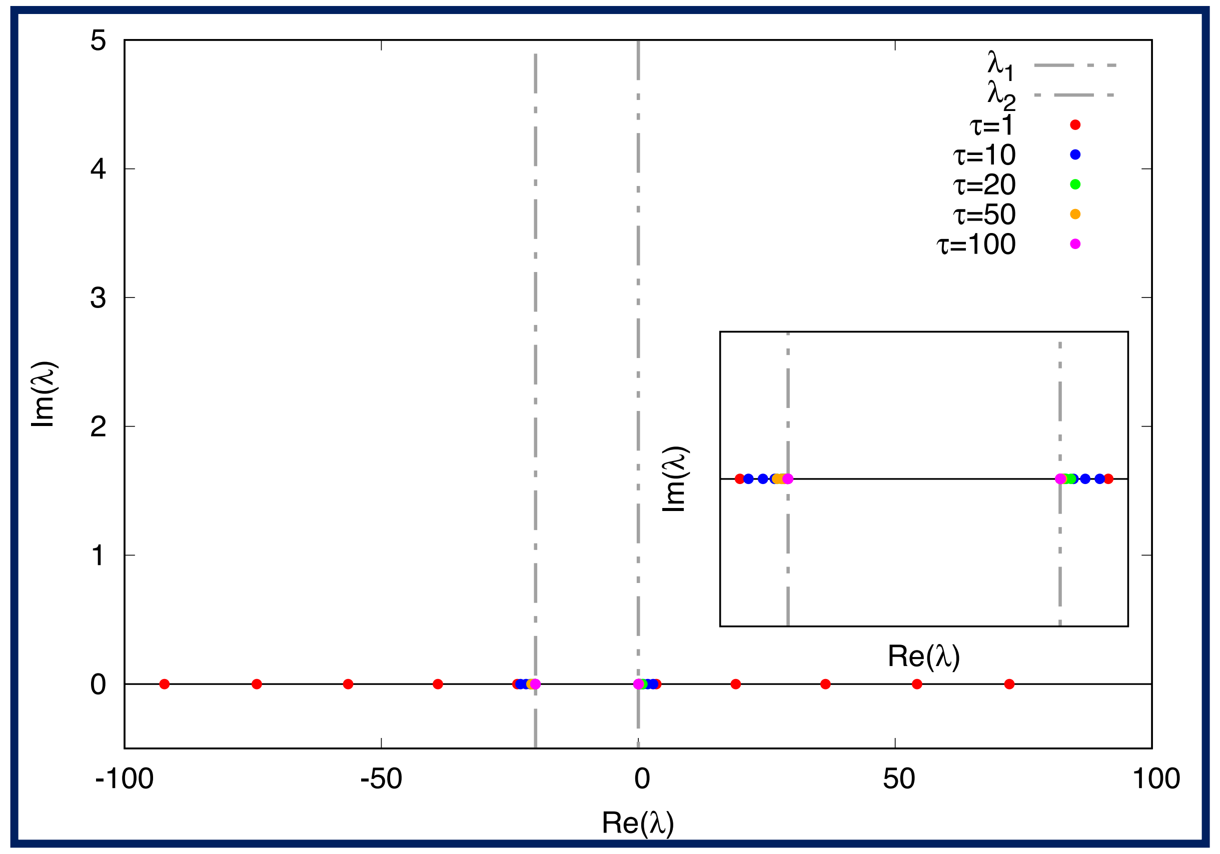
J Stat Mech 2021,
Semeraro, Suma, Petrelli,
Cagnetta and Gonnella

- Cumulant Generating Function

$$\langle e^{\mu \mathscr{W}_a} \rangle = \frac{e^{\tau \frac{d}{2}(\gamma_R - \alpha)}}{\left(\frac{1+e^{-2\tau\alpha}}{2}\right)^{d/2} \left(2 + \frac{\gamma_R^2 + \alpha^2}{\gamma_R \alpha} \tanh(\tau_\alpha)\right)^{d/2}} = F(\mu) e^{\tau \frac{d}{2}(\gamma_R - \alpha)} \qquad \alpha = \sqrt{\gamma_R^2 - 4D_R \mu \gamma (1 + D_T \gamma \mu)}$$

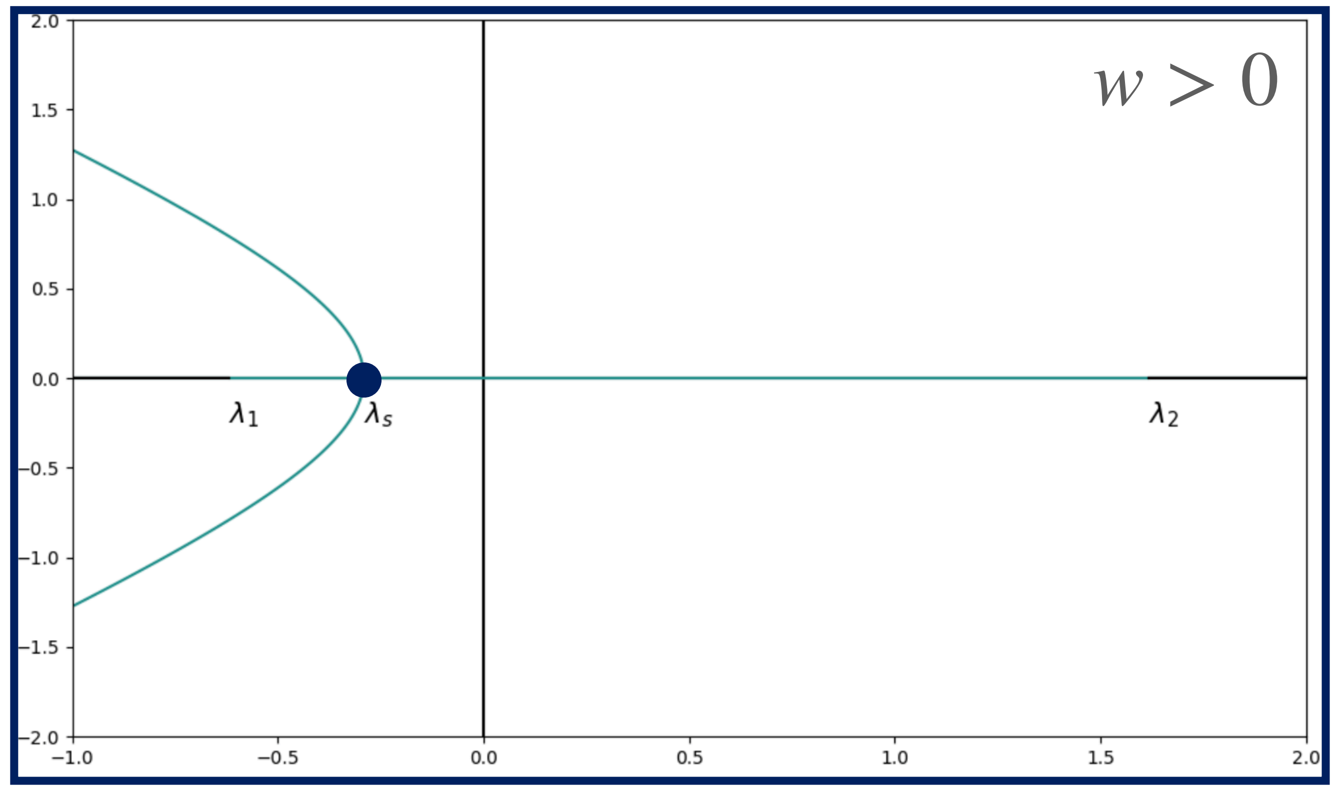
Introduction to Path-Integral Methods in Physics and Polymer Science, Wiegel 1986, Work Scientific

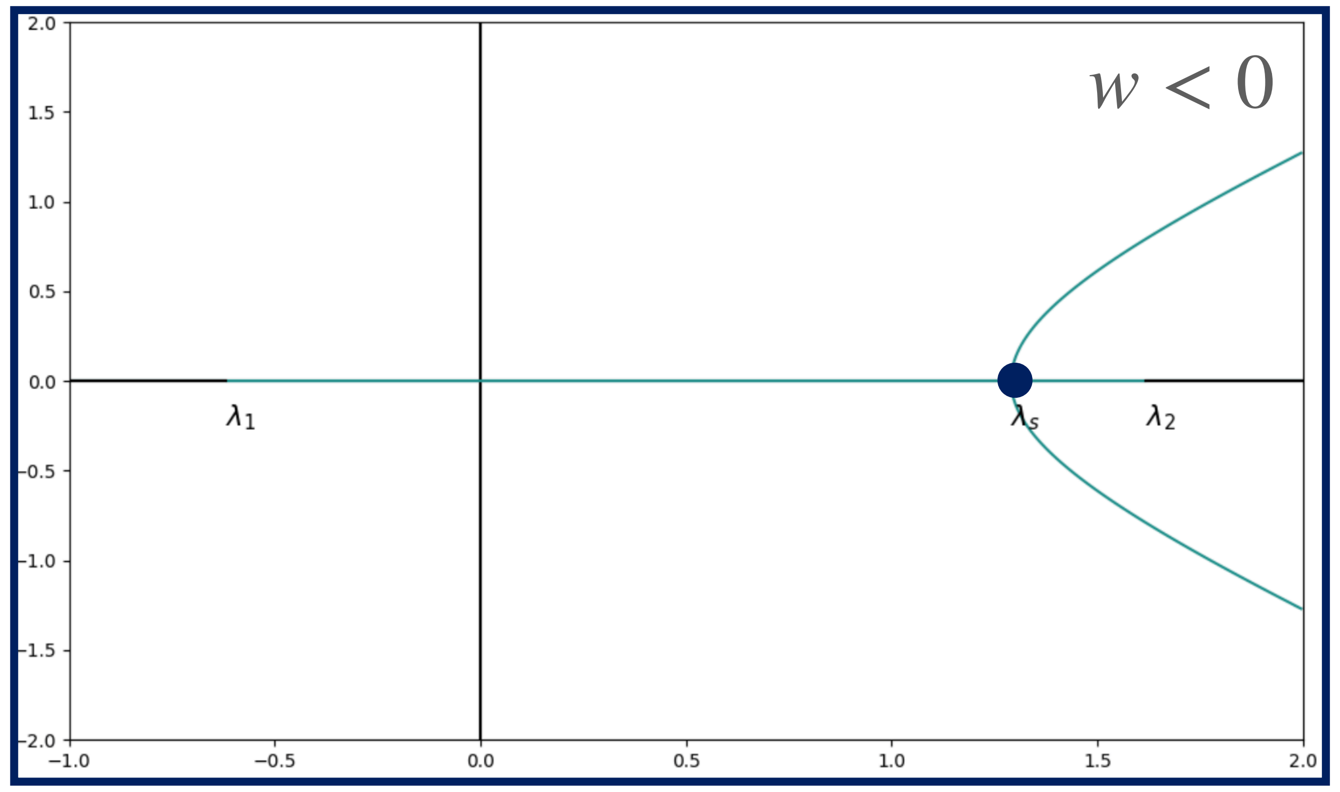
- Sources of singularities
  - Branch points $\tilde{\mu}_{1/2} = \frac{\gamma_R^2}{4D_R\gamma}\left(-\frac{1 \pm \sqrt{1+4A}}{2A}\right)$ $\quad A = \frac{\gamma_R^2 D_T}{4D_R}$
  - Poles of $F(\mu)$ $\quad \tilde{\mu}_{1/2}(y) = \frac{\gamma_R^2}{4D_R\gamma}\left(-\frac{1 \pm \sqrt{1+4A(1+y^2)}}{2A}\right)$



- Saddle-point estimation of $p(w) = \frac{1}{2\pi\imath}\int_{-\imath\infty}^{+\imath\infty} d\mu \; F(\mu) e^{\tau \frac{d}{2}(\gamma_R - \alpha)}$
  - Steepest descent paths

    $Im[\mu w - \phi(\mu)] = 0$
  - Saddle-points $\quad \mu w - \phi(\mu) = 0 \rightarrow \tilde{\mu}_{\pm}^{(s)} = \frac{\gamma_R^2}{4D_R\gamma}\left(-\frac{1 \pm \sqrt{1 + 4A\left(\frac{4\tilde{w}^2 - 1}{4(A+\tilde{w})^2}\right)}}{2A}\right)$





- Integration along steepest dissent paths deformed to pass by $\tilde{\mu}_{\pm}^{(s)}$ and avoid non-analicities of the integrand

$$p(w) \asymp \frac{F(\tilde{\mu}^{(s)})}{2\pi}\left(\frac{2\pi}{\mathscr{C}}\right)^{1/2} e^{-\tau I(w)} \longrightarrow \text{Extraction of the Rate Function } I(w)$$

LDT for quadratic functionals of Gauss-Markov chains

- Continuous model $(\gamma, T, d = 1)$

$$\begin{cases} \dot{x}(t) = a(t) - \kappa x(t) + \sqrt{2}\ \xi(t) \\ \dot{a}(t) = -a(t) + Pe\sqrt{2}\ \eta(t) \end{cases} \longrightarrow$$

- Discrete model as a Gauss-Markov chain
  $\tau = N \cdot dt,\ x_n, a_x \equiv x(n \cdot dt), a(n \cdot dt), \{\xi_n\}, \{\eta_n\}$ sequence of normal rv

$$\begin{cases} r_{n+1} = (1 - \kappa dt)\ r_n + a_n\ dt + \sqrt{2dt}\ \xi_n \\ a_{n+1} = (1 - dt)\ a_n + Pe\sqrt{2dt}\ \eta_n \end{cases} \longrightarrow \quad X_{n+1} = SX_n + D\zeta_n \qquad X_n = (x_n, a_n)^T$$

initial conditions covariance matrix

$$\Sigma_0 = \begin{pmatrix} \frac{1+\kappa+Pe^2}{\kappa(1+\kappa)} & \frac{Pe^2}{1+\kappa} \\ \frac{Pe^2}{1+\kappa} & Pe^2 \end{pmatrix} \qquad \Sigma_0 = \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_a^2 \end{pmatrix}$$

$$S = \begin{pmatrix} 1 - \kappa dt & dt \\ 0 & 1 - dt \end{pmatrix} \qquad D = \begin{pmatrix} \sqrt{2dt}dt & dt \\ 0 & Pe\sqrt{2dt} \end{pmatrix} \qquad \zeta_n = \begin{pmatrix} \xi_n & 0 \\ 0 & \eta_n \end{pmatrix}$$

- Entire trajectory $\{(x_0, a_0), \ldots, (a_N, x_N)\}$ is Gaussian distributed with zero mean and covariance matrix

$$\Sigma_N = \begin{pmatrix} \Sigma_0^{-1} + S^\top D^{-2} S & -S^\top D^{-2} & & & \\ -D^{-2}S & D^{-2} + S^\top D^{-2} S & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & D^{-2} + S^\top D^{-2} S & -S^\top D^{-2} \\ & & & -D^{-2}S & D^{-2} \end{pmatrix}^{-1}$$

- Discretisation of Active Work as a quadratic functional

$$\mathscr{W}_a \cdot \tau = \int_0^\tau a(t)\dot{r}(t)\ dt \longrightarrow \quad W_N = \frac{1}{2}\sum_{n=1}^{N}(a_n + a_{n-1})(r_n - r_{n-1}) = \frac{1}{2}(r_0 \quad a_0 \quad \ldots \quad r_N \quad a_N)\,\mathsf{M}_N \begin{pmatrix} r_0 \\ a_0 \\ \vdots \\ r_N \\ a_N \end{pmatrix}$$

quasi-Toeplitz block matrix $\longrightarrow$

$$\mathsf{M}_N \equiv \begin{pmatrix} -E_+ & E_-^\top & & & \\ E_- & 0 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 0 & E_-^\top \\ & & & E_- & E_+ \end{pmatrix} \qquad E_\pm \equiv \frac{1}{2}\begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}$$

LDT for quadratic functionals of Gauss-Markov chains

- Evaluation of the SCGF (generalization of Szegö theorem)

$$\varphi(\lambda) = \lim_{N\to\infty} \frac{1}{N} \boxed{\log < e^{\lambda W_N} >} = -\frac{1}{4\pi}\int_0^{2\pi} \ln\det F_\lambda(\theta)\ d\theta$$

Symbol matrix

$$F_\lambda(\theta) \equiv Ve^{-i\theta} + U + V^\top e^{i\theta}$$
$$-(D^{-2}S+\lambda E_-)e^{-i\theta} + D^{-2} + S^\top D^{-2}S - (D^{-2}S+\lambda E_-)^\top e^{i\theta}$$

- Positive definiteness

$$\log < e^{\lambda W_N} > = -\frac{1}{2}\ln\det\boxed{(\Sigma_N^{-1} - \lambda M_N)} - N\ln(2\ dt\ Pe) - \frac{1}{2}\ln\det\Sigma_0$$

↑ Gaussian integral

bulk block Toeplitz

$$\Sigma_N^{-1} - \lambda M_N = \begin{pmatrix} L & V^\top & & \mathsf{T}_N & \\ V & U & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & U & V^\top \\ & & & V & R \end{pmatrix}$$

$$L = \Sigma_0^{-1} + S^\top D^{-2}S + \lambda E_+$$
$$R = D^{-2} - \lambda E_+$$
$$U = D^{-2} + S^\top D^{-2}S$$
$$V = -D^{-2}S - \lambda E_-$$

- bulk block Toeplitz matrix T_N is positive definite ⟶ symbol matrix $F_\lambda(\theta)$ positive definite for all $(0,2\pi)$ ⟶ **Primary domain** $P = (\tilde{\lambda}_-, \tilde{\lambda}_+)$

- Schur complement $\mathsf{S}_N \equiv \begin{pmatrix} L - V^\top(\mathsf{T}_N^{-1})_{11}V & -V^\top(\mathsf{T}_N^{-1})_{1N}V^\top \\ -V(\mathsf{T}_N^{-1})_{N1}V & R - V(\mathsf{T}_N^{-1})_{NN}V^\top \end{pmatrix} \xrightarrow{N\to\infty} \begin{pmatrix} \mathscr{L}_\lambda & 0 \\ 0 & \mathscr{R}_\lambda \end{pmatrix}$ positive definite

Hermitian

positive definiteness of ⟶

$$\mathscr{L}_\lambda \equiv \Sigma_0^{-1} + S^\top D^{-2}S + \lambda E_+ - (D^{-2}S+\lambda E_-)^\top \Phi_\lambda(0) H_\lambda^{-1}(D^{-2}S+\lambda E_-)$$
$$\mathscr{R}_\lambda \equiv D^{-2} - \lambda E_+ - (D^{-2}S+\lambda E_-)K_\lambda^{-1}\Phi_\lambda(0)(D^{-2}S+\lambda E_-)^\top$$

⟶ **Effective domain** $E = (\lambda_-, \lambda_+)$

$$\Phi_\lambda(n) \equiv \frac{1}{2\pi}\int_0^{2\pi} F_\lambda^{-1}(\theta)e^{-in\theta}d\theta$$

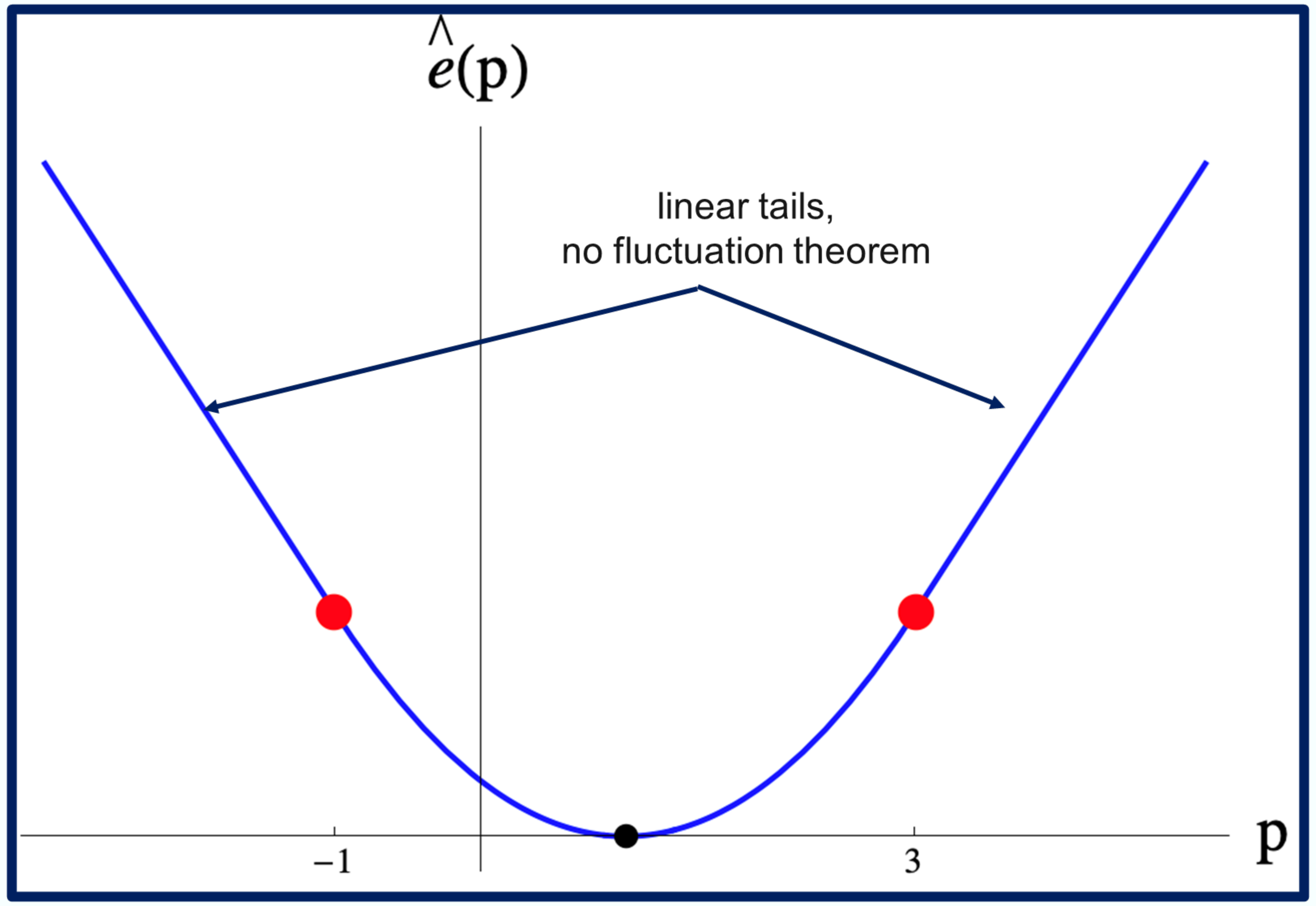
$$H_\lambda \equiv I + (D^{-2}S+\lambda E_-)\Phi_\lambda(1) \qquad K_\lambda \equiv I + \Phi_\lambda(1)(D^{-2}S+\lambda E_-)$$

invertible

Examples of Singular Rate Functions

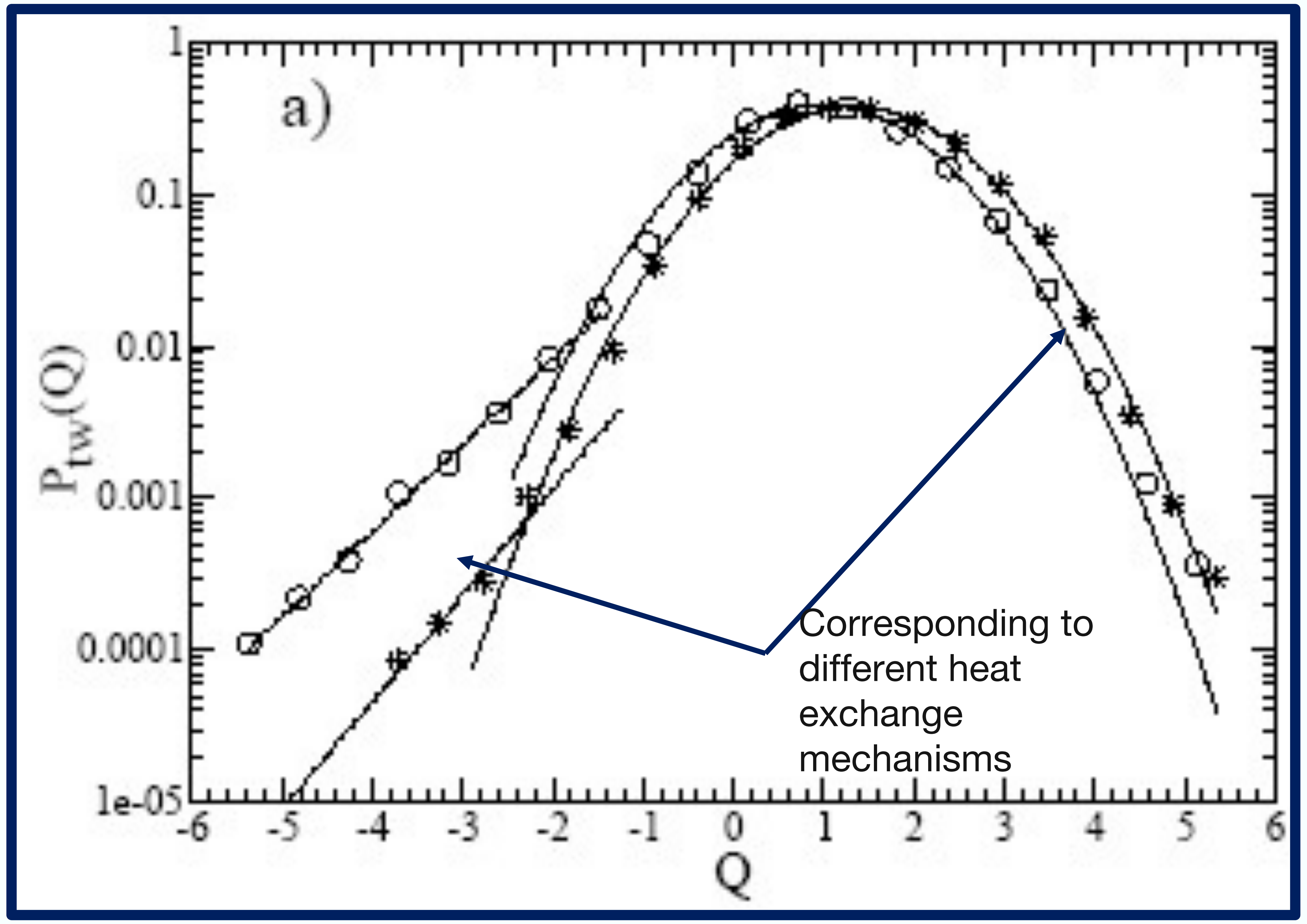
- Heat exchanged by an overdamped Brownian particle dragged by a moving harmonic potential

PRL 2003, Cohen, van Zon



- Heat exchanged between non-equilibrium aging glassy systems and the thermal bath

EPL 2004, Crisanti, Ritort



- Other examples for single particle models:

J Stat Mech 2006, Visco

J Stat Mech 2012, Seifert, Speck et al

4, Burioni

J Stat Phys 2022, Farago

Phys Rep 2009, Touchette

J Phys A 2013, Gradenigo et al.

PRE 2018, Nyawo et al

- Other examples for interacting particle systems, urn models, etc:

....l.

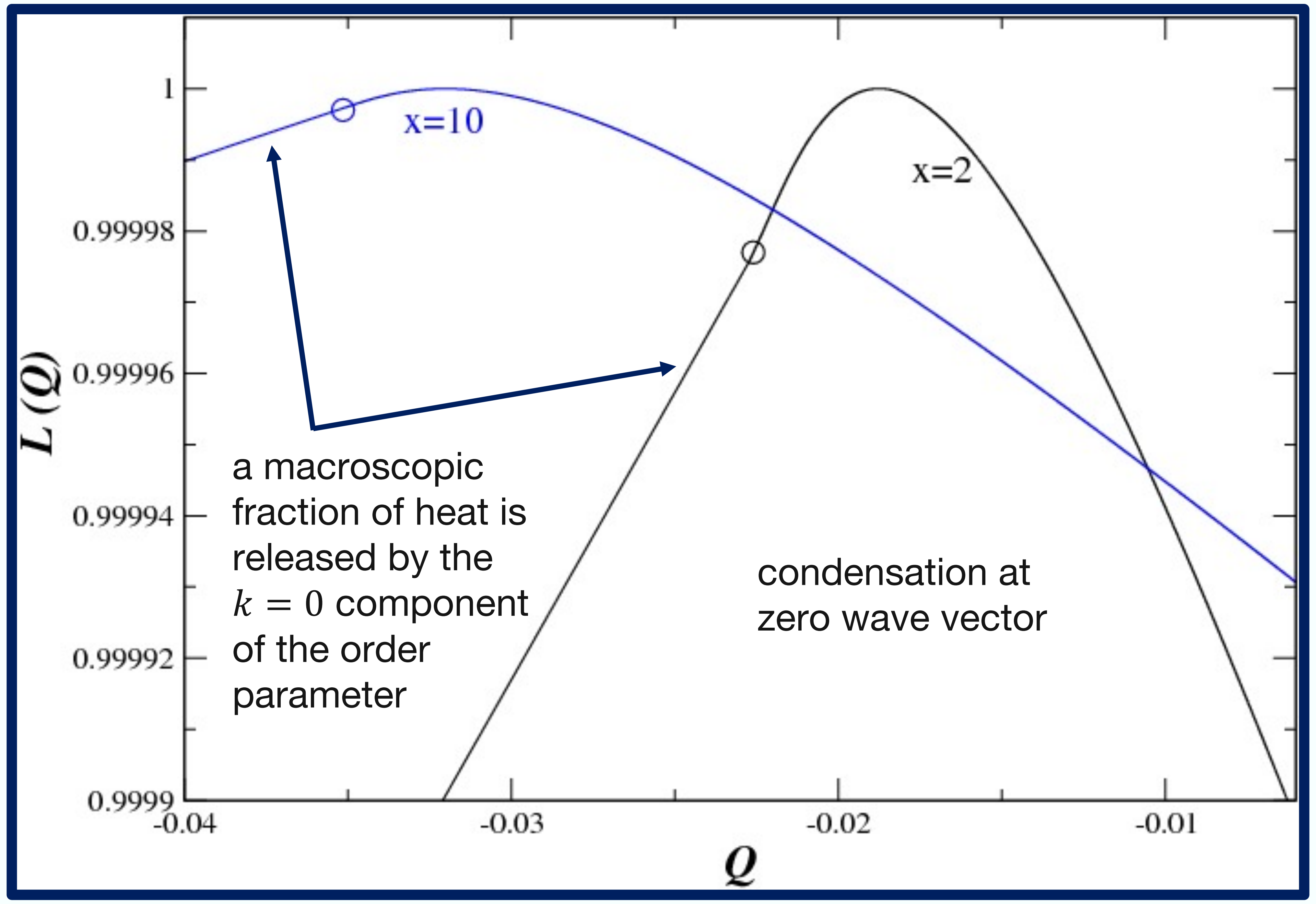
PRL 2014, Nossan, Evans, Majumdar

PRE 2014, Zannetti, Corberi, Gonnella

Examples of Singular Rate Functions

- Heat released by a ferromagnet after a quench below the critical point

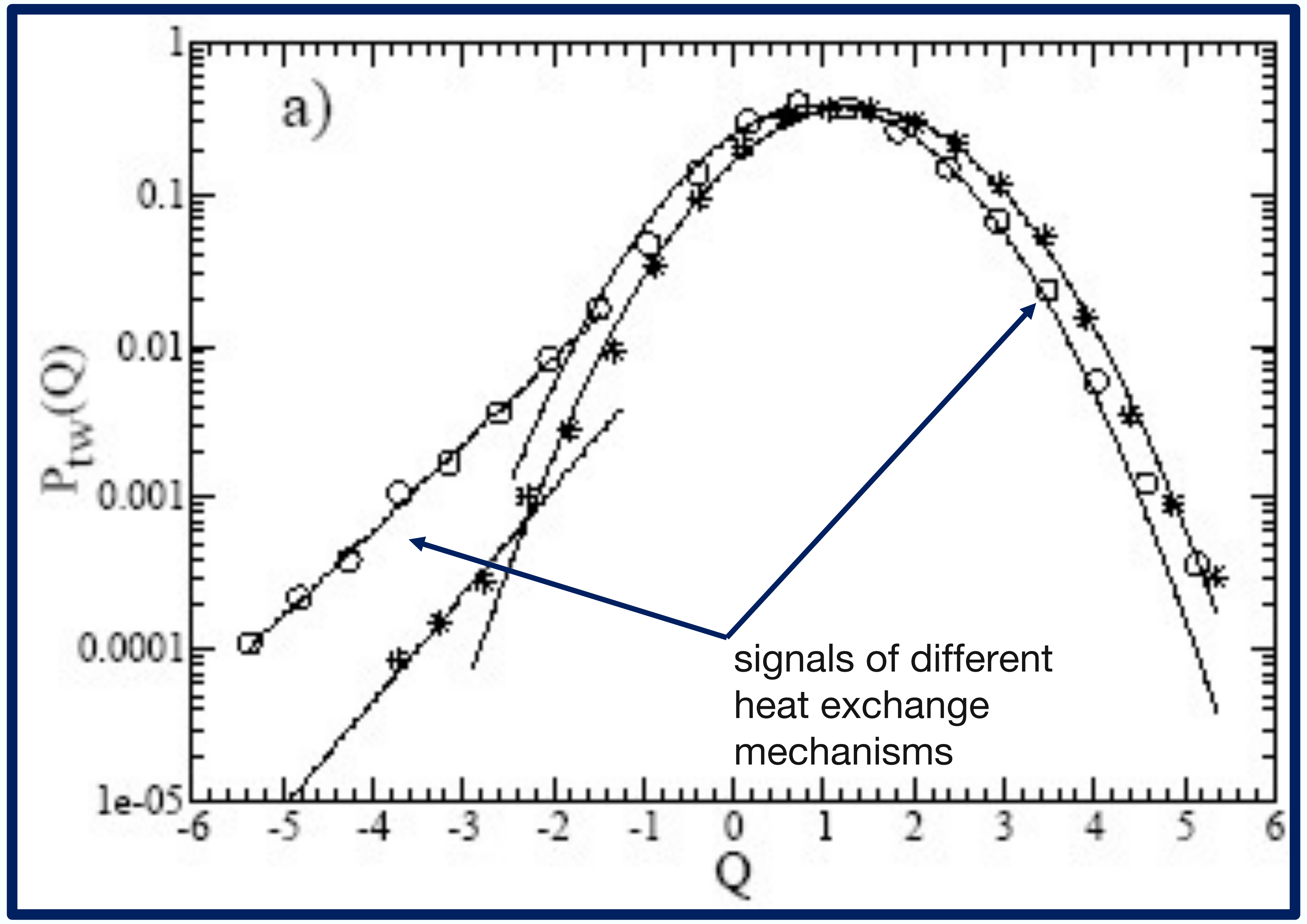
J Phys A 2013, Piscitelli, Corberi, Gonnella.



Presented in Venice, this conference, october 2012

- Heat exchanged between non-equilibrium aging glassy systems and the thermal bath in contact

EPL 2004, Crisanti et al



- Other examples:

PRL 2014, Gambassi