Asymptotics of open quantum systems: a (quite) gentle introduction

Daniele Amato

University of Bari & INFN
daniele.amato@ba.infn.it

Joint work with Paolo Facchi (UNIBA & INFN),
and Arturo Konderak (CFT-Warsaw)

Bari Theory Xmas Workshop 2024, Bari, 17th December 2024

Closed quantum systems

Closed quantum system: **No interaction** with the surroundings

$\mathcal{H}$ Hilbert space $d = \dim(\mathcal{H}) < \infty$

State of the system: $|\psi\rangle \in \mathcal{H}$, $\| \, |\psi\rangle \, \| = 1$

Closed quantum systems

Closed quantum system: **No interaction** with the surroundings

$\mathcal{H}$ Hilbert space $d = \dim(\mathcal{H}) < \infty$

State of the system: $|\psi\rangle \in \mathcal{H}$, $\| \, |\psi\rangle \, \| = 1$

Evolution: $|\psi(0)\rangle \mapsto |\psi(t)\rangle = U(t)\,|\psi(0)\rangle \;\; t \in \mathbb{R}^+$

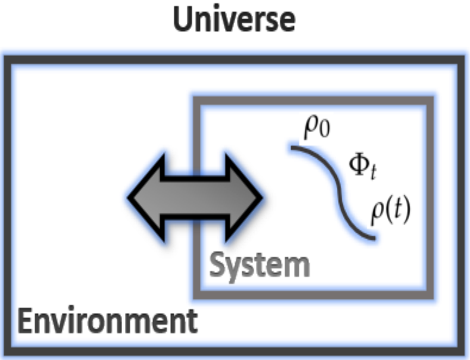
1. $U(t)$ **unitary** for any $t \in \mathbb{R}^+$
2. $U(t_1 + t_2) = U(t_1)U(t_2)$ for all $t_1, t_2 \in \mathbb{R}^+$ (**group** property)
3. **Time continuity**

Open quantum systems: Definition

Open quantum system: Interaction with the surroundings (**environment**)

Open quantum systems: Definition

Open quantum system: Interaction with the surroundings (**environment**)



System-environment flow of **energy/matter**

Open quantum systems I

$\mathcal{H}$ Hilbert space $d = \dim(\mathcal{H}) < \infty$

State of the system: **density operator** ρ on $\mathcal{H}$

1. $\rho \geqslant 0$ **Positivity**
2. $\mathrm{Tr}(\rho) = 1$ **Unit trace**

Open quantum systems I

$\mathcal{H}$ Hilbert space $d = \dim(\mathcal{H}) < \infty$

State of the system: **density operator** ρ on $\mathcal{H}$

1. $\rho \geqslant 0$ **Positivity**
2. $\mathrm{Tr}(\rho) = 1$ **Unit trace**

$$\rho = \sum_{k=1}^{d} p_k \, |\psi_k\rangle\langle\psi_k| \;\; \{\psi_k\}_{k=1}^{d} \text{ basis of } \mathcal{H}$$

Positivity $\Rightarrow p_k \geqslant 0$ for all $k = 1, \ldots, d$

$$\text{Unit-trace} \Rightarrow \sum_{k=1}^{d} p_k = 1$$

Conclusion

$\{p_k\}_{k=1}^{d}$ probability vector $\Rightarrow$ **Ensemble** interpretation of ρ

Open quantum systems II

$\mathcal{S}(\mathcal{H})$: states on $\mathcal{H}$

Evolution: $(\Phi_t)_{t\in\mathbb{R}^+}$ family of maps on $\mathcal{S}(\mathcal{H})$ (**quantum dynamical map**)

1. Φ_t **trace-preserving** (TP) $\mathrm{Tr}(\Phi_t(\rho)) = 1$ for all $\rho \in \mathcal{S}(\mathcal{H})$
2. Φ_t **completely positive** (CP) $\Rightarrow \Phi_t(\rho) \geqslant 0$ for all $\rho \in \mathcal{S}(\mathcal{H})$
3. **Time continuity**

Open quantum systems II

$\mathcal{S}(\mathcal{H})$: states on $\mathcal{H}$

Evolution: $(\Phi_t)_{t\in\mathbb{R}^+}$ family of maps on $\mathcal{S}(\mathcal{H})$ (**quantum dynamical map**)

1. Φ_t **trace-preserving** (TP) $\mathrm{Tr}(\Phi_t(\rho)) = 1$ for all $\rho \in \mathcal{S}(\mathcal{H})$
2. Φ_t **completely positive** (CP) $\Rightarrow \Phi_t(\rho) \geqslant 0$ for all $\rho \in \mathcal{S}(\mathcal{H})$
3. **Time continuity**

Φ_t CPTP $\Rightarrow \rho$ density operator $\mapsto \Phi_t(\rho)$ density operator

Main message

Entanglement $\Rightarrow$ complete positivity

Closed vs open quantum dynamics

Closed-system case: $|\psi(0)\rangle \mapsto |\psi(t)\rangle = U(t)\,|\psi(0)\rangle \;\; U^\dagger(t)U(t) = \mathbb{I}$

$$\rho \in \mathcal{S}(\mathcal{H}) \mapsto \Phi_t(\rho) = U(t)\rho U^\dagger(t) \in \mathcal{S}(\mathcal{H})$$

K. Kraus, Ann. Phys. **64**, 311 (1970).

Closed vs open quantum dynamics

Closed-system case: $|\psi(0)\rangle \mapsto |\psi(t)\rangle = U(t)\,|\psi(0)\rangle \;\; U^\dagger(t)U(t) = \mathbb{I}$

$$\rho \in \mathcal{S}(\mathcal{H}) \mapsto \Phi_t(\rho) = U(t)\rho U^\dagger(t) \in \mathcal{S}(\mathcal{H})$$

Open-system case: **Kraus representation**

$$\rho \in \mathcal{S}(\mathcal{H}) \mapsto \Phi_t(\rho) = \sum_{k=1}^{N} A_k(t)\rho A_k^\dagger(t) \in \mathcal{S}(\mathcal{H})$$

$$\{A_k(t)\}_{t\in\mathbb{R}^+} \;\; \sum_{k=1}^{N} A_k^\dagger(t)A_k(t) = \mathbb{I}$$

K. Kraus, Ann. Phys. **64**, 311 (1970).

Closed vs open quantum dynamics

Closed-system case: $|\psi(0)\rangle \mapsto |\psi(t)\rangle = U(t)\,|\psi(0)\rangle \;\; U^\dagger(t)U(t) = \mathbb{I}$

$$\rho \in \mathcal{S}(\mathcal{H}) \mapsto \Phi_t(\rho) = U(t)\rho U^\dagger(t) \in \mathcal{S}(\mathcal{H})$$

Open-system case: **Kraus representation**

$$\rho \in \mathcal{S}(\mathcal{H}) \mapsto \Phi_t(\rho) = \sum_{k=1}^{N} A_k(t)\rho A_k^\dagger(t) \in \mathcal{S}(\mathcal{H})$$

$$\{A_k(t)\}_{t\in\mathbb{R}^+} \;\; \sum_{k=1}^{N} A_k^\dagger(t)A_k(t) = \mathbb{I}$$

- $A_k(t)$ not unitary for some k and $t \in \mathbb{R}^+$
- $A_k(t_1 + t_2) \neq A_k(t_1)A_k(t_2)$ for some k and $t_1, t_2 \in \mathbb{R}^+$

Conclusion

Open-system dynamics is **much more** general.

K. Kraus, Ann. Phys. **64**, 311 (1970).

Asymptotics of open quantum systems: motivation

- Unitary gates inside **noiseless** and **decoherence-free** subspaces
- Quantum **reservoir engineering**
- Quantum **associative memories**

Hopfield: Nobel Prize for Physics 2024

J. J. Hopfield, Proc. Natl. Acad. Sci. U.S.A. **79**, 2554–2558 (1982).
P. Zanardi, and M. Rasetti, Phys. Rev. Lett. **79**, 3306 (1997).
F. Verstraete, M. M. Wolf, and J. I. Cirac, Nature Phys. **5**, 633 (2009).
M. Lewenstein, *et al.*, Quantum Sci. Technol. **6**, 045002 (2021).

Asymptotics of open quantum systems: motivation

- Unitary gates inside **noiseless** and **decoherence-free** subspaces
- Quantum **reservoir engineering**
- Quantum **associative memories**

Hopfield: Nobel Prize for Physics 2024

Main message

The asymptotic dynamics is also interesting for various applications!

J. J. Hopfield, Proc. Natl. Acad. Sci. U.S.A. **79**, 2554–2558 (1982).
P. Zanardi, and M. Rasetti, Phys. Rev. Lett. **79**, 3306 (1997).
F. Verstraete, M. M. Wolf, and J. I. Cirac, Nature Phys. **5**, 633 (2009).
M. Lewenstein, *et al.*, Quantum Sci. Technol. **6**, 045002 (2021).

Basic notions of open-quantum-system asymptotics

$(\Phi_t)_{t\in\mathbb{R}^+}$ quantum dynamical map

Stationary states: $\Phi_t(\rho_i) = \rho_i$ for all $t \in \mathbb{R}^+$

$\mathrm{Fix}((\Phi_t)) = \mathsf{span}\{\rho_i\}$ **Fixed-point subspace**

Basic notions of open-quantum-system asymptotics

$(\Phi_t)_{t\in\mathbb{R}^+}$ quantum dynamical map

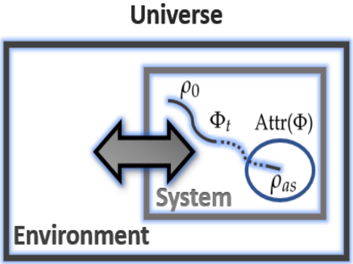
Stationary states: $\Phi_t(\rho_i) = \rho_i$ for all $t \in \mathbb{R}^+$

$\mathrm{Fix}((\Phi_t)) = \mathrm{span}\{\rho_i\}$ **Fixed-point subspace**

Asymptotic subspace:

$$\mathrm{Attr}((\Phi_t)) = \mathrm{span}\left\{\rho_{\mathrm{as}} \in \mathcal{S}(\mathcal{H}) \,\middle|\, \lim_{t\to\infty} \Phi_t(\rho_0) = \rho_{\mathrm{as}} \text{ for some } \rho_0 \in \mathcal{S}(\mathcal{H})\right\}$$

Asymptotic dynamics: $\rho_{\mathrm{as}} \in \mathrm{Attr}((\Phi_t)) \mapsto \Phi_t(\rho_{\mathrm{as}}) \in \mathrm{Attr}((\Phi_t))$



What we did so far....

- D. A., P. Facchi, and A. Konderak, J. Phys. A: Math. Theor. **56**, 265304 (2023);
- D. A., P. Facchi, and A. Konderak, In: M. Correggi, and M. Falconi (eds), Quantum Mathematics II. INdAM 2022. Springer INdAM Series **58**, 169-181 (2023);
- D. A., P. Facchi, and A. Konderak, arXiv:2403.12926 [quant-ph] (2024);
- D. A., and P. Facchi, Sci. Rep. **14**, 14366 (2024).

Paolo Facchi's home page: https://home.ba.infn.it/∼facchi/Home.html

What we did so far....

- D. A., P. Facchi, and A. Konderak, J. Phys. A: Math. Theor. **56**, 265304 (2023);
- D. A., P. Facchi, and A. Konderak, In: M. Correggi, and M. Falconi (eds), Quantum Mathematics II. INdAM 2022. Springer INdAM Series **58**, 169-181 (2023);
- D. A., P. Facchi, and A. Konderak, arXiv:2403.12926 [quant-ph] (2024);
- D. A., and P. Facchi, Sci. Rep. **14**, 14366 (2024).

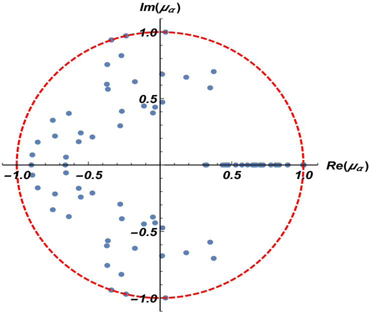
Paolo Facchi's home page: https://home.ba.infn.it/∼facchi/Home.html

Thanks for your attention!

Spectrum of CPTP maps

Φ CPTP map $\quad \mathrm{spec}(\Phi) =$ spectrum of Φ

1. $\mu_0 = 1 \in \mathrm{spec}(\Phi)$
2. $\mathrm{spec}(\Phi) \subseteq \{\mu \in \mathbb{C} \,|\, |\mu| \leqslant 1\}$
3. $\mu \in \mathrm{spec}(\Phi) \Rightarrow \mu^* \in \mathrm{spec}(\Phi)$



A recap on Markovian evolutions

$(\Phi_t)_{t\in\mathbb{R}^+}$ **Quantum dynamical semigroup**

$$\Updownarrow$$

a) $\Phi_0 = 1$

b) $\Phi_{t+s} = \Phi_t \circ \Phi_s$ for any $t, s \geqslant 0$ (**Semigroup** property)

c) **Time continuity**

$$\Phi_t = e^{t\mathcal{L}} \quad \mathcal{L}(\rho) = -i[H,\rho] + \mathcal{L}_D(\rho) \text{ GKLS generator}$$

$$\mathcal{L}_D(\rho) = \sum_{k=1}^{N}\left(A_k\rho A_k^\dagger - \frac{1}{2}\{A_k^\dagger A_k, \rho\}\right) \text{ Dissipative part}$$

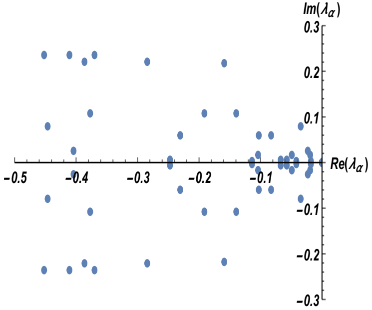
- $H = H^\dagger \in \mathcal{B}(\mathcal{H})$ effective **Hamiltonian**
- $A_k \in \mathcal{B}(\mathcal{H})$ jump or **noise operators**

V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, J. Math. Phys. **17**, 821 (1976).
G. Lindblad, Commun. Math. Phys. **48**, 119 (1976).

Spectrum of GKLS generators

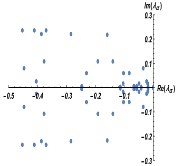
$\mathcal{L}$ GKLS generator $\quad \mathrm{spec}(\mathcal{L})$ = spectrum of $\mathcal{L}$

- $\lambda_0 = 0 \in \mathrm{spec}(\mathcal{L})$
- $\lambda_\alpha \in \mathbb{D} = \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}(\lambda) \leqslant 0\}$
- $\lambda_\alpha \in \mathrm{spec}(\mathcal{L}) \Rightarrow \lambda_\alpha^* \in \mathrm{spec}(\mathcal{L})$



Interplay between spectrum and Markovian dynamics

$\mathcal{L}$ GKLS generator



$$\mathrm{spec}(\mathcal{L}) = \mathrm{spec}_P(\mathcal{L}) \cup \mathrm{spec}_B(\mathcal{L})$$

Peripheral spectrum: $\mathrm{spec}_P(\mathcal{L}) = \{\lambda \in \mathrm{spec}(\mathcal{L}) \,|\, \mathrm{Re}(\lambda) = 0\}$

Bulk spectrum: $\mathrm{spec}_B(\mathcal{L}) = \{\lambda \in \mathrm{spec}(\mathcal{L}) \,|\, \mathrm{Re}(\lambda) < 0\}$

$$\mathcal{L}(X) = \lambda X \quad \lambda \in \mathrm{spec}_B(\mathcal{L}) \Rightarrow \|e^{t\mathcal{L}}(X)\| = |e^{\lambda t}|\|X\| \xrightarrow{t\to\infty} 0$$

Key message

Bulk eigenvectors are related to **transient** dynamics

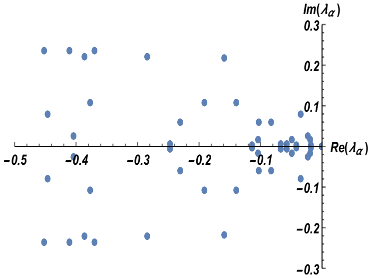
Asymptotic dynamics of Markovian evolutions

$(\Phi_t)_{t\in\mathbb{R}^+}$ quantum dynamical semigroup with $\Phi_t = e^{t\mathcal{L}}$

Fixed point subspace: $\mathrm{Fix}((\Phi_t)) = \{X \in \mathcal{B}(\mathcal{H}) \,|\, \mathcal{L}(X) = 0\}$

Attractor subspace:

$$\mathrm{Attr}((\Phi_t)) = \mathrm{span}\{X \in \mathcal{B}(\mathcal{H}) \,|\, \mathcal{L}(X) = \lambda X \text{ for some } \lambda \text{ with } \mathrm{Re}(\lambda) = 0\}$$



Asymptotic dynamics of Markovian evolutions: structure theorem

$$\mathrm{Attr}(\mathcal{L}) \ni X = 0_{d_0} \oplus \bigoplus_{k=1}^{M} X_k \otimes \rho_k$$

- $X_k \in \mathcal{B}(\mathcal{H}_{k,1})$
- $\rho_k > 0$ density operator on $\mathcal{H}_{k,2}$

$$\Phi_t(X) = 0_{d_0} \oplus \bigoplus_{k=1}^{M} U_k(t) X_k U_k^\dagger(t) \otimes \rho_k$$

- $U_k(t) = e^{-iH_k t}$ continuous group of unitaries on $\mathcal{H}_{k,1}$ ($H_k = H_k^\dagger$)

D. Amato, Spectrum and dynamics of open quantum systems, PhD thesis, University of Bari (2024).

Number of steady and asymptotic states

$(\Phi_t)_{t\in\mathbb{R}^+}$ quantum dynamical semigroup

$m_0 = \dim(\mathrm{Fix}((\Phi_t)))$: # linearly independent stationary states

$m_P = \dim(\mathrm{Attr}((\Phi_t)))$: # linearly independent asymptotic states

$m_P \geqslant m_0 \geqslant 1$: **at least one** stationary state of $(\Phi_t)_{t\in\mathbb{R}^+}$

Goal

Sharp upper **bounds** for m_0 and m_P

Upper bounds for m_0 and m_P

d-level quantum system

Hamiltonian generator: $\mathcal{L}(\rho) = -i[H, \rho]$ $H = H^\dagger$ Hamiltonian

Theorem 1

$\mathcal{L} \neq 0$ *Hamiltonian GKLS generator* $\Rightarrow m_0 \leqslant d^2 - 2d + 2 \quad m_P = d^2$

Theorem 2

$\mathcal{L}$ *non-Hamiltonian generator* $\Rightarrow m_0 \leqslant m_P \leqslant d^2 - 2d + 2$

- The bounds depend **only** on d
- The bounds are **sharp**

Sharpness of the bounds

$\{e_i\}_{i=1}^d$ basis of $\mathcal{H}$

- $H = \rho_1 |e_1\rangle\langle e_1| + \rho_2 \sum_{i=2}^d |e_i\rangle\langle e_i| \quad \rho_1 \neq \rho_2 \in \mathbb{R}$

$$\mathcal{L}_H(\rho) = -i[H, \rho] \text{ satisfies } m_0 = d^2 - 2d + 2$$

$$\Downarrow$$

Hamiltonian bound is sharp

- $P_1 = |e_1\rangle\langle e_1|, P_2 = \mathbb{I} - P_1$

$$\mathcal{L}(\rho) = \sum_{i=1}^{2} P_i \rho P_i^\dagger - \rho \text{ satisfies } m_0 = m_P = d^2 - 2d + 2$$

$$\Downarrow$$

Non-Hamiltonian bound is sharp