UNIVERSITÀ DEGLI STUDI DI BARI ALDO MORO

UNIVERSITÀ DEGLI STUDI DI NAPOLI FEDERICO II

MULTIPARTITE ENTANGLEMENT AND QUANTUM FRUSTRATION

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BARI THEORY XMAS WORKSHOP 2024

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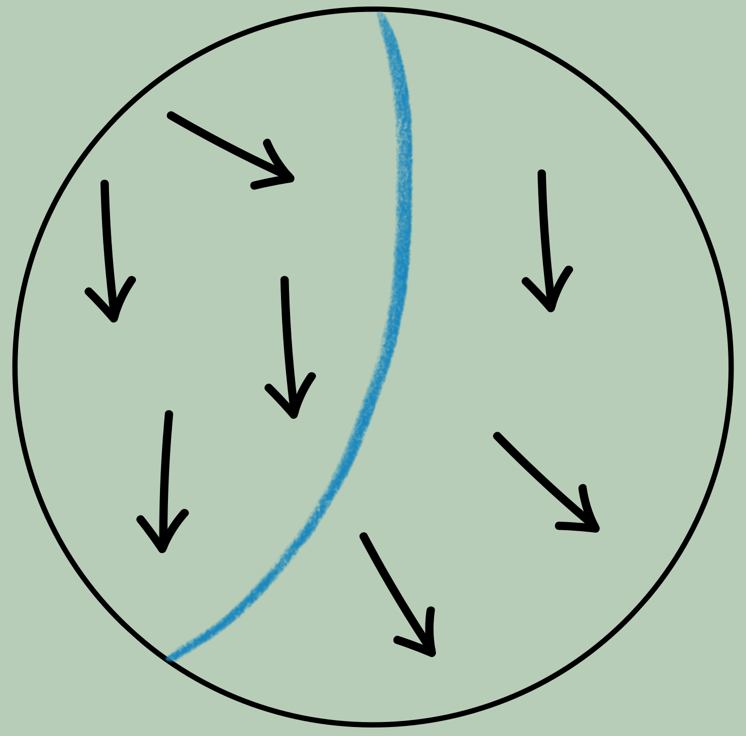
Overview

1. Introduction
2. Emerging of frustration
3. Numerical approach
4. Multipartite entanglement analysis
5. Optimiization: Annealing

MULTIPARTITE ENTANGLEMENT

A complex phenomenon

balanced bipartition



- **More than one** bipartition possible & **exponentially growing** of bipartitions
- information can be extracted by analyzing the **bipartite** entanglement associated to **each bipartition**
- **statistical approach** based on the Random Matrix Theory

MMES AND THE EMERGE OF FRUSTRATION

perfect MMES: maximum bipartite entanglement with respect to all the balanced bipartitions

$$\pi_{ME}(|\psi\rangle) = \binom{n}{[n/2]}^{-1} \sum_{|A|=[n/2]} \pi_A(|\psi\rangle)$$

searching for a perfect MMES results in the **emergence of frustration**

P Facchi et al. "Multipartite entanglement and frustration". In: New Journal of Physics 12.2 (Feb. 2010), p. 025015. issn: 1367-2630. doi: 10.1088/1367 2630/12/2/025015. url: http://dx.doi.org/10.1088/1367-2630/12/ 2/025015.

EMERGE OF QUANTUM FRUSTRATION

existence of **perfect MMES** is not garanteed for every qubits system

number of qubits	Existence of perfect MMES
3	Yes
4	No
5	Yes
6	Yes
≥ 7	No

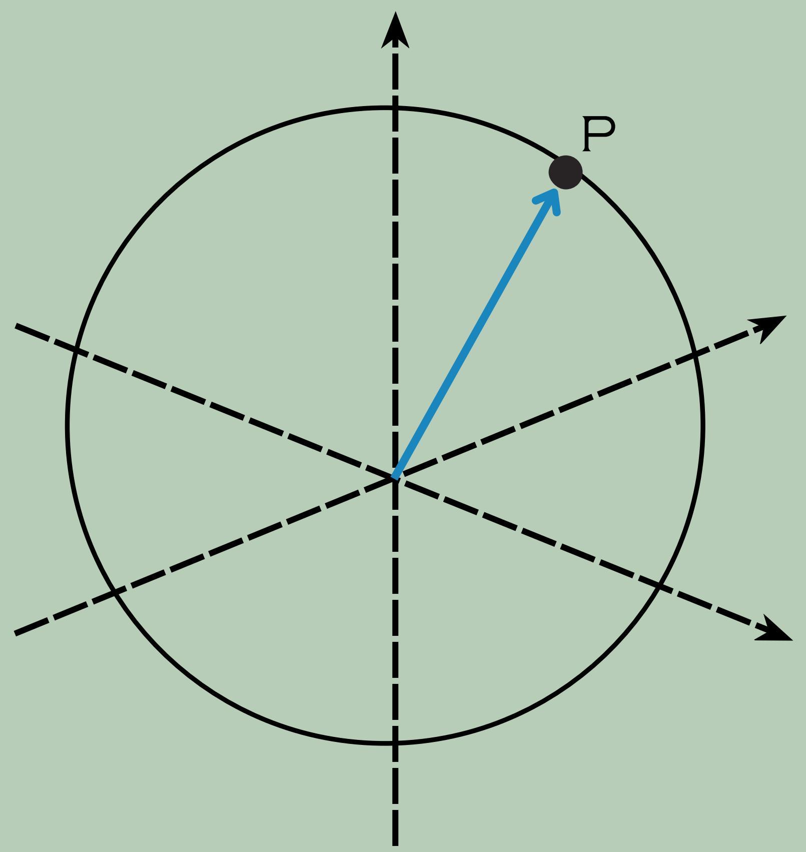
The components of the states do not satisfy all the conditions in order to have all balanced bipartition maximal entangled when: $n \neq 3, 5, 6$

$$\pi_{ME} > \frac{1}{2^{n_A}}$$

- Random

$$|\psi\rangle = \sum_{k=0}^{2^n-1} z_k|k\rangle, \quad \text{with} \quad z_k \in \mathbb{C}$$

Unit hypersphere

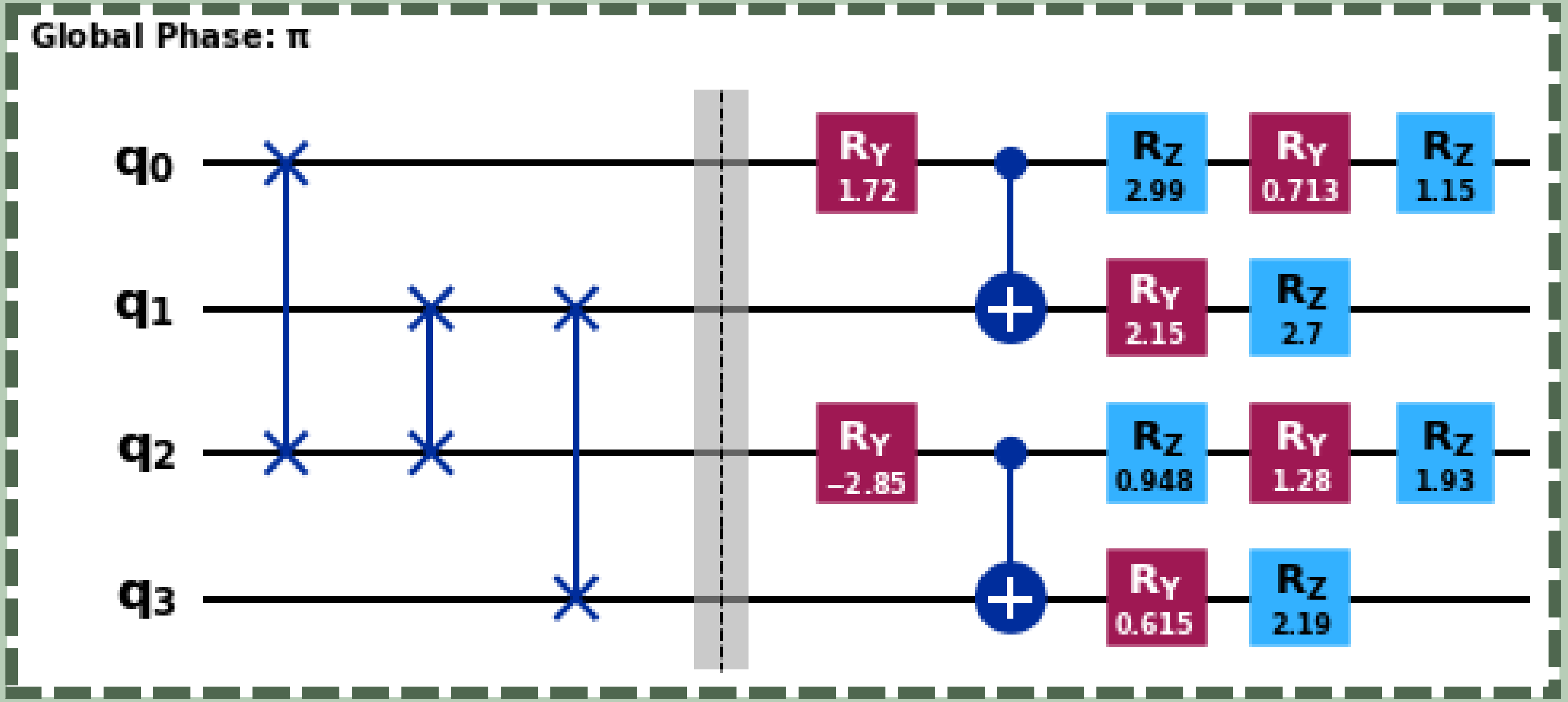


- Uniform real-phased:

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{2^n-1} s_k|k\rangle \quad \text{with} \quad s_k = \{+1, -1\}$$

example $k = 27$

$$|k\rangle = |27\rangle = |11011\rangle$$



SIMULATIONS

Gabriele Cenedese et al. "Generation of Pseudo-Random Quantum States on Actual Quantum Processors". In: Entropy 25.4 (Apr. 2023), p. 607. issn: 1099-4300. doi: 10.3390/e25040607. url: http://dx.doi.org/10.3390/e25040607

Creation of python vectors with uniform random value +1 and -1

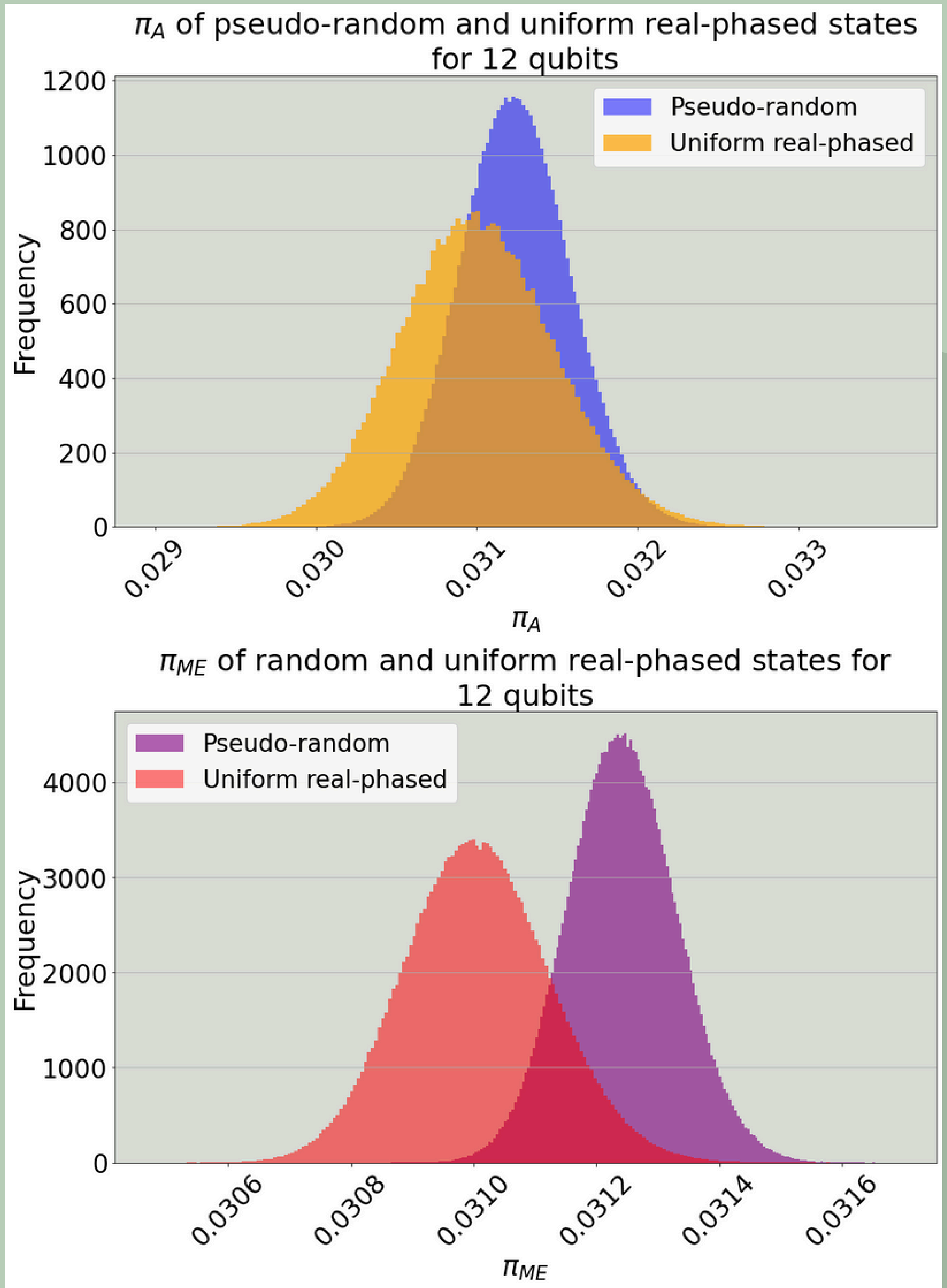
```
vettore = [1] + [random.choice(values) for _ in range(dim)]
initial_state = [1/(np.sqrt(2**n_qubits)) * vettore[i] for i in range(2**n_qubits)]
circuit = QuantumCircuit(n_qubits)
circuit.initialize(initial_state)
```

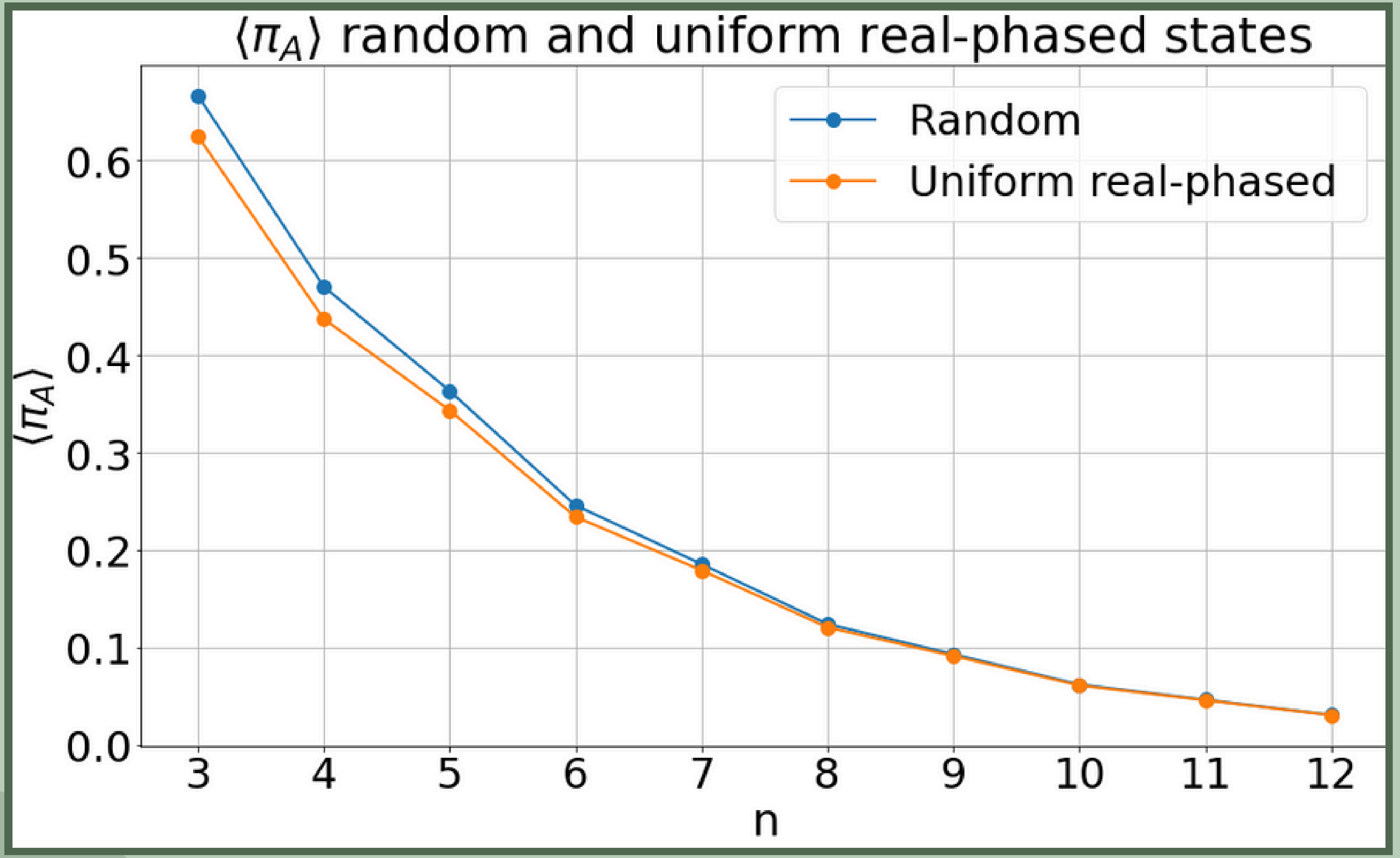
Initialization of these vectors in quantum states using a proper quantum gate of the Qiskit library n = 12

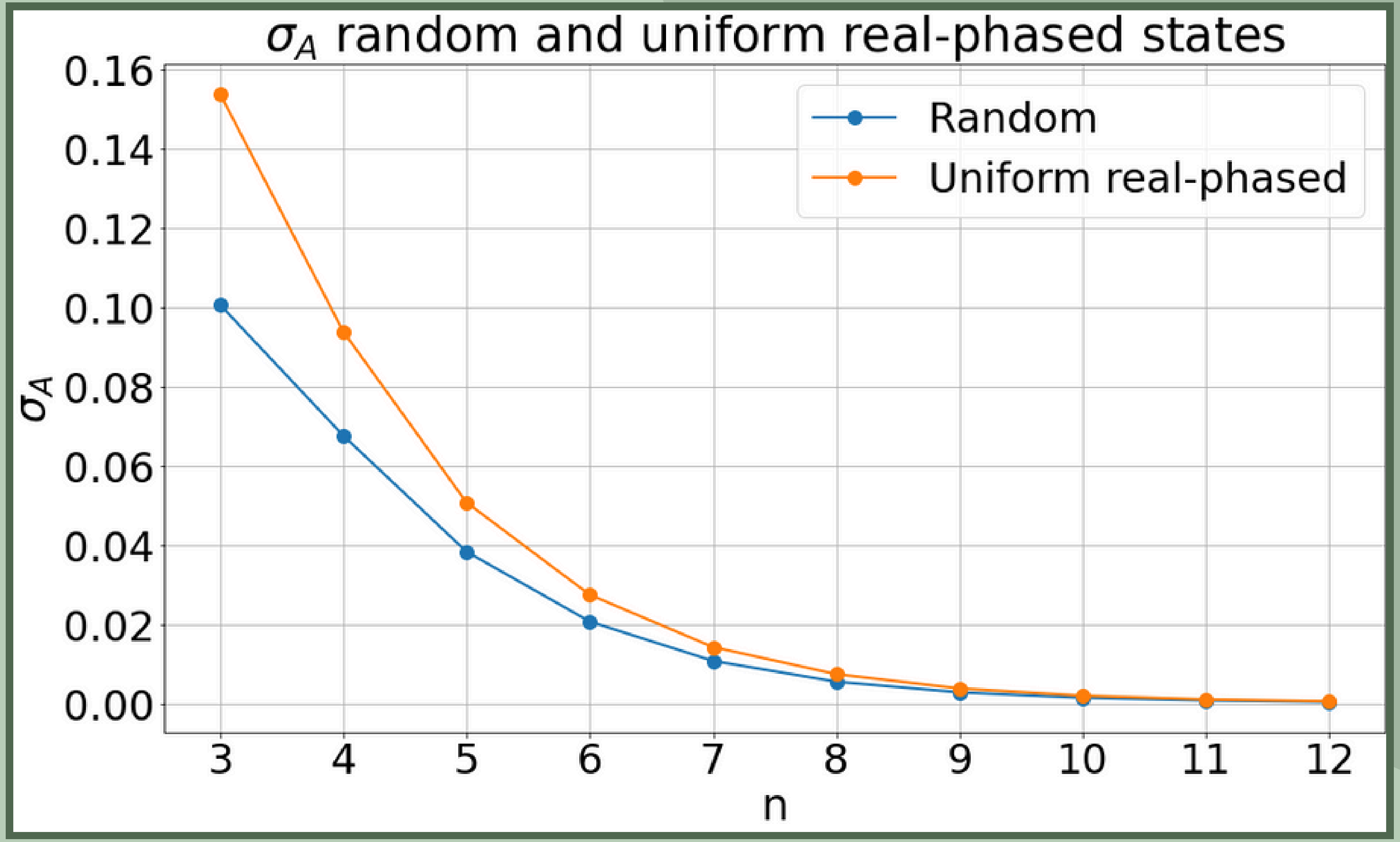
Histograms

one bipartition purities π_A

averaged purities π_{ME}




⟨π_A⟩ random and uniform real-phased states
⟨π_A⟩
0.0
0.1
0.2
0.3
0.4
0.5
0.6
3
4
5
6
7
8
9
10
11
12
n
Random
Uniform real-phased


σ_A random and uniform real-phased states
σ_A
0.00
0.02
0.04
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0.08
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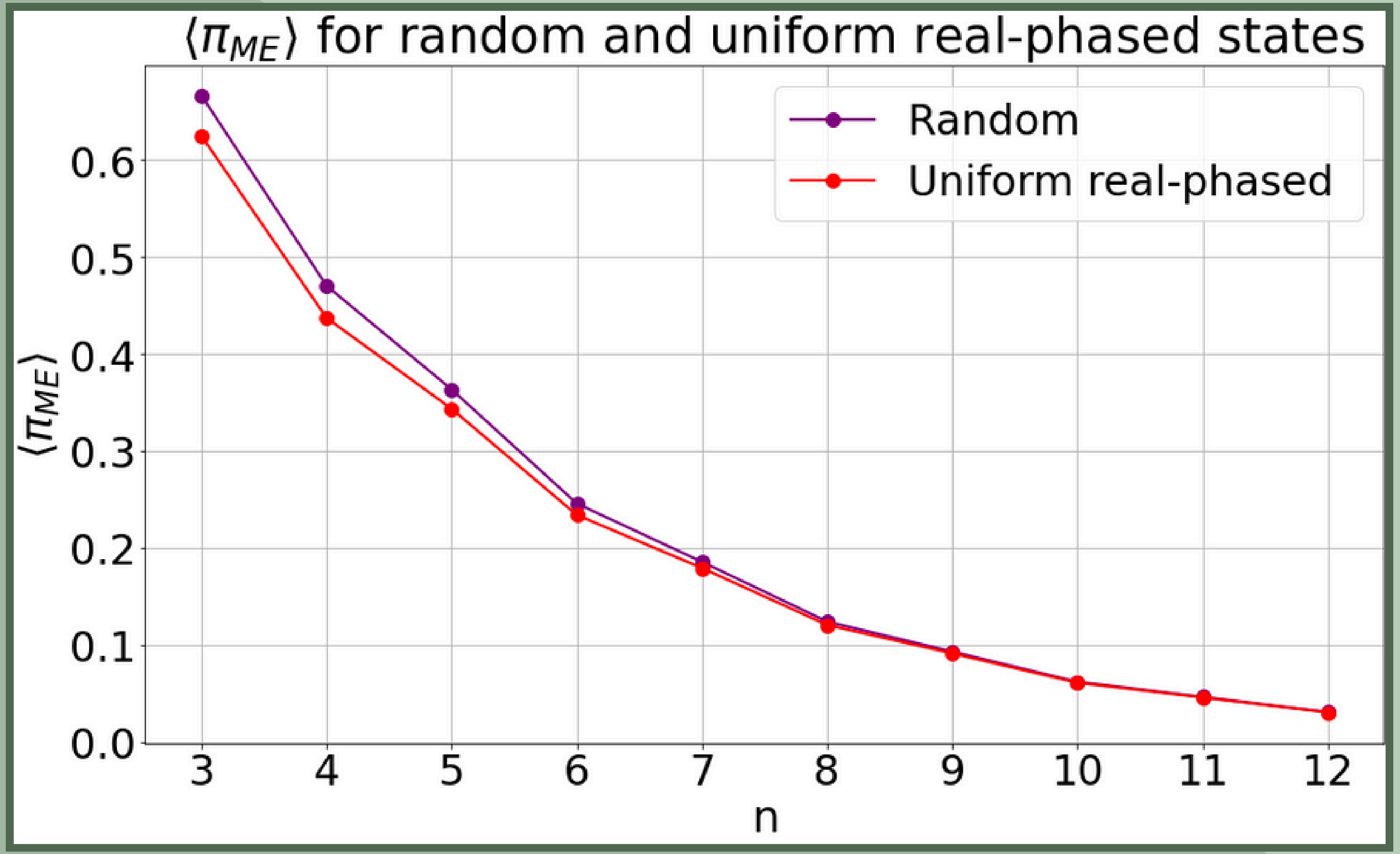
12
n
Random
Uniform real-phased


⟨π_{ME}⟩ for random and uniform real-phased states
⟨π_{ME}⟩
0.0
0.1
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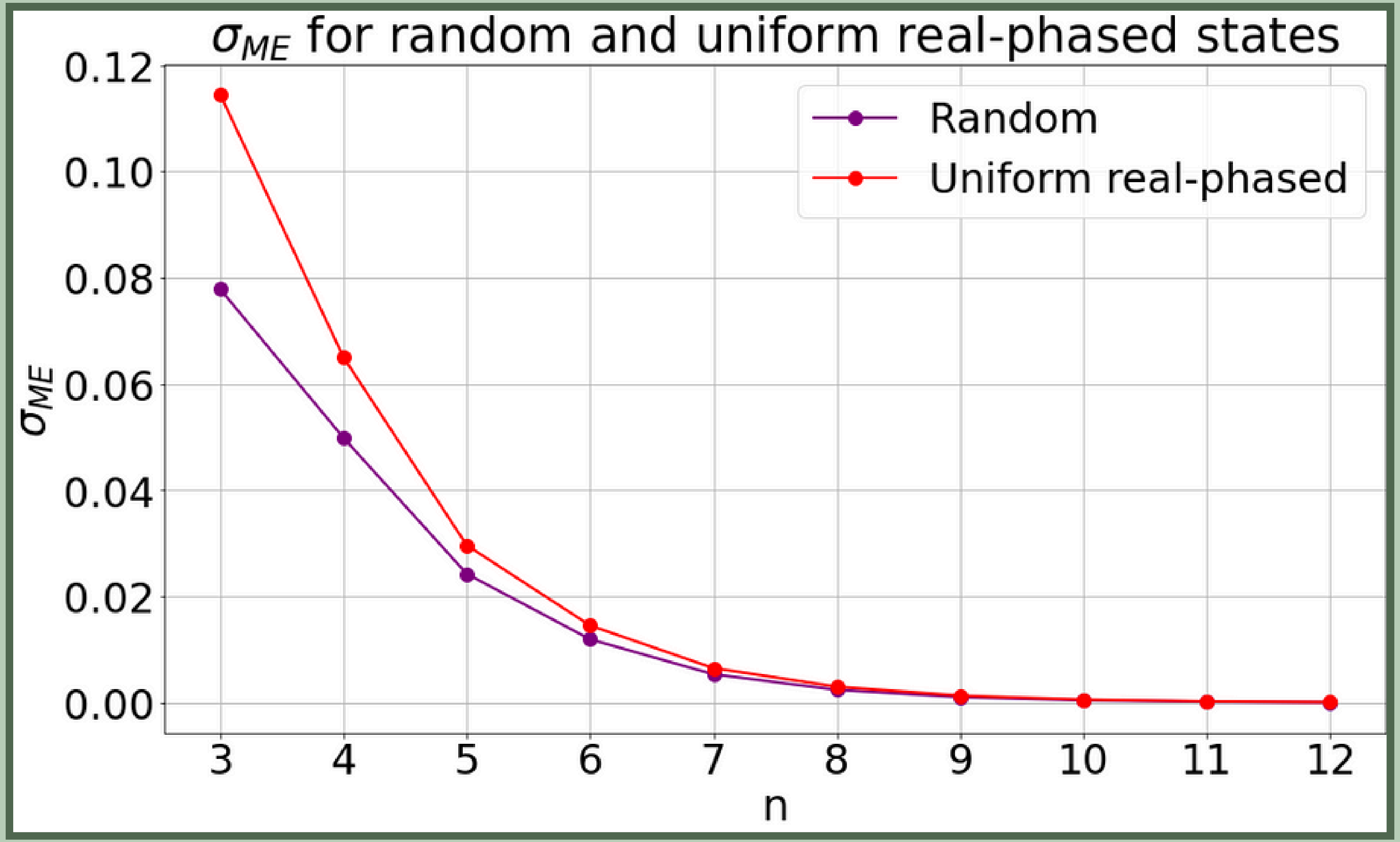
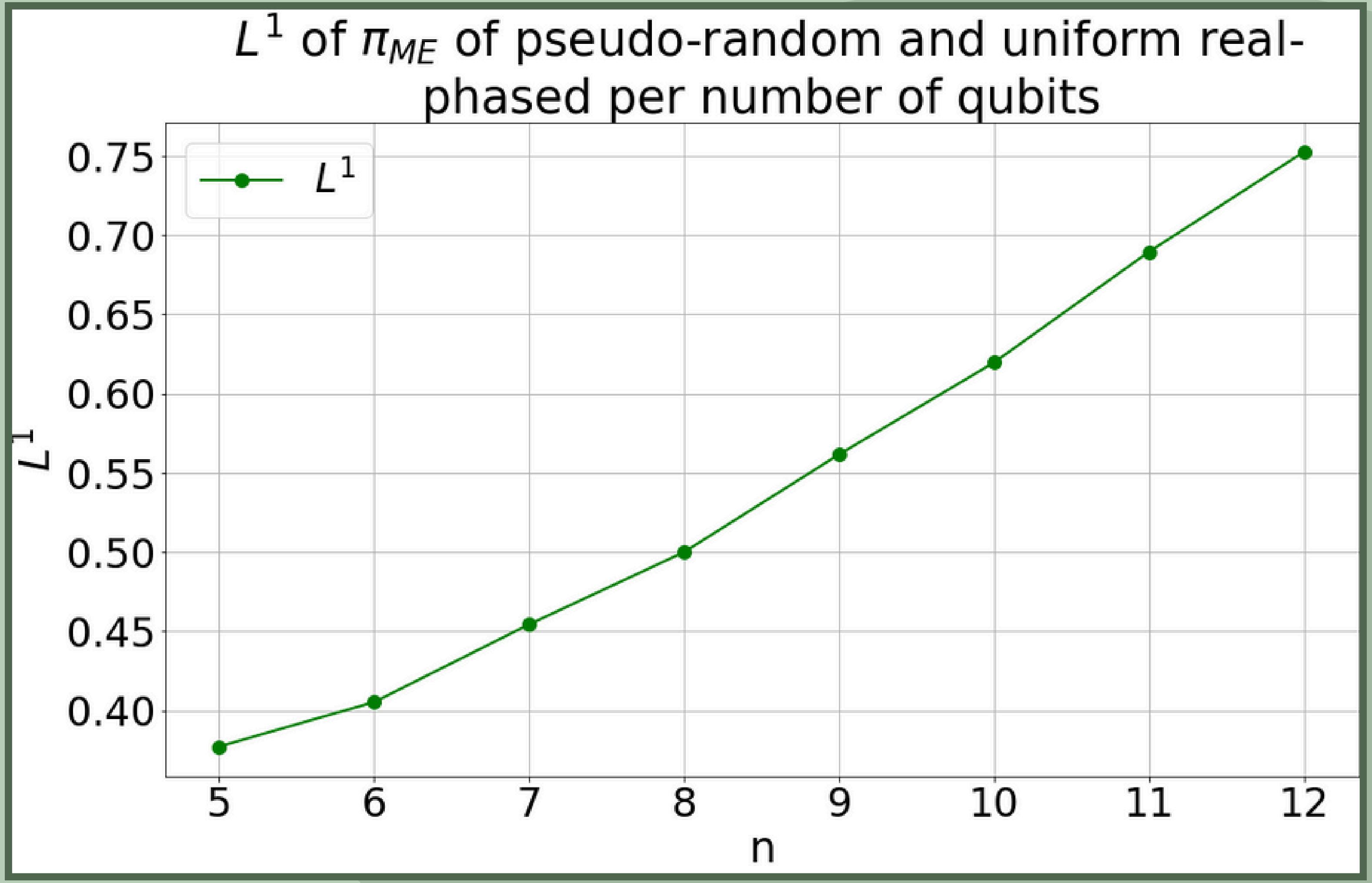
Random
Uniform real-phased


σ_{ME} for random and uniform real-phased states
σ_{ME}
0.00
0.02
0.04
0.06
0.08
0.10
0.12
3
4
5
6
7
8
9
10
11
12
n
Random
Uniform real-phased

Absolute distance behavior

$$d_1 = \frac{1}{2}\sum_{i=1}^{n} |p(x_i) - q(x_i)| \cdot \delta x \quad \text{with} \quad 0 \leq d_1 \leq 1$$

π_A



π_{ME}

Optimization of the potential π_{ME}

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{2^n-1} s_k |k\rangle \quad \text{with} \quad s_k = \{+1, -1\}$$

It is a discrete submanifold with a **finite(very large)** number of states: 2^{2^n-1}

$$n = 5 : \text{states} \sim 2,12 \times 10^9 \quad \longrightarrow \quad n = 10 : \text{states} \sim 9 \times 10^{307}$$

$$\pi_{\text{ME}}(s) = \frac{1}{N^2} \sum_{l,l',k,k' \in \mathbb{Z}_2^n} \Delta(l, l'; k, k') s_l s_l' s_k s_k'$$

we want to **minimize** $\pi_{ME}(s)$

Simulated Annealing and tempering(Parisi and Marinari)

$$E = \pi_{ME}(s)$$

each states a different set of coefficient

$$s = \{s_k\}$$

→

different value of energies

$$E$$

Flipping **one** component of the state we end up with a new states with a new value of energy

Procedure

Flipping one component

$$|\psi_1\rangle \rightarrow |\psi_2\rangle \iff E_1 \rightarrow E_2$$

→ **Is the move accettable?**

CONDITION

$$x < e^{(E_1 - E_2)\beta}$$

YES: Keep $|\psi_2\rangle$ and E_2

NO: Come back to $|\psi_1\rangle$ and E_1

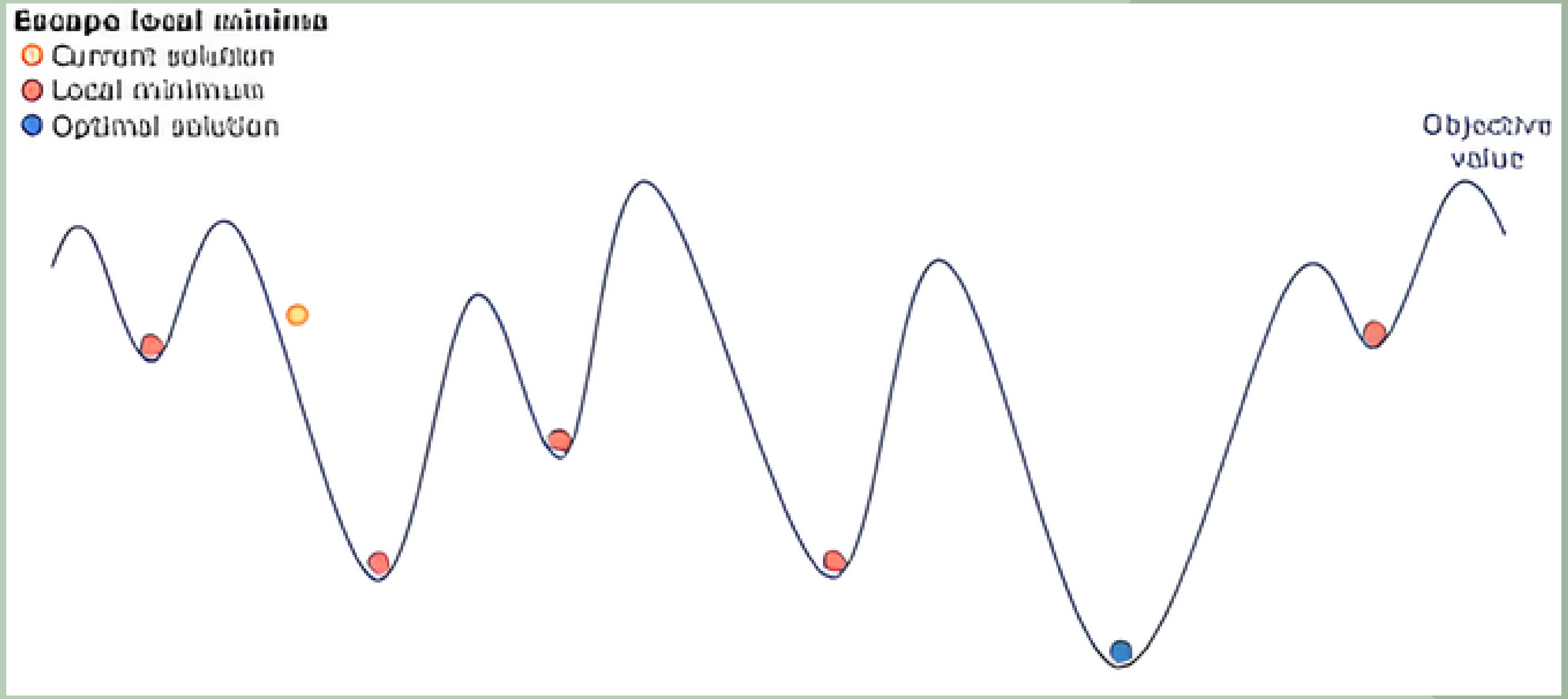
Why $x < e^{(E_1 - E_2)\beta}$?

Also move to **higher** energies can be accepted

↓

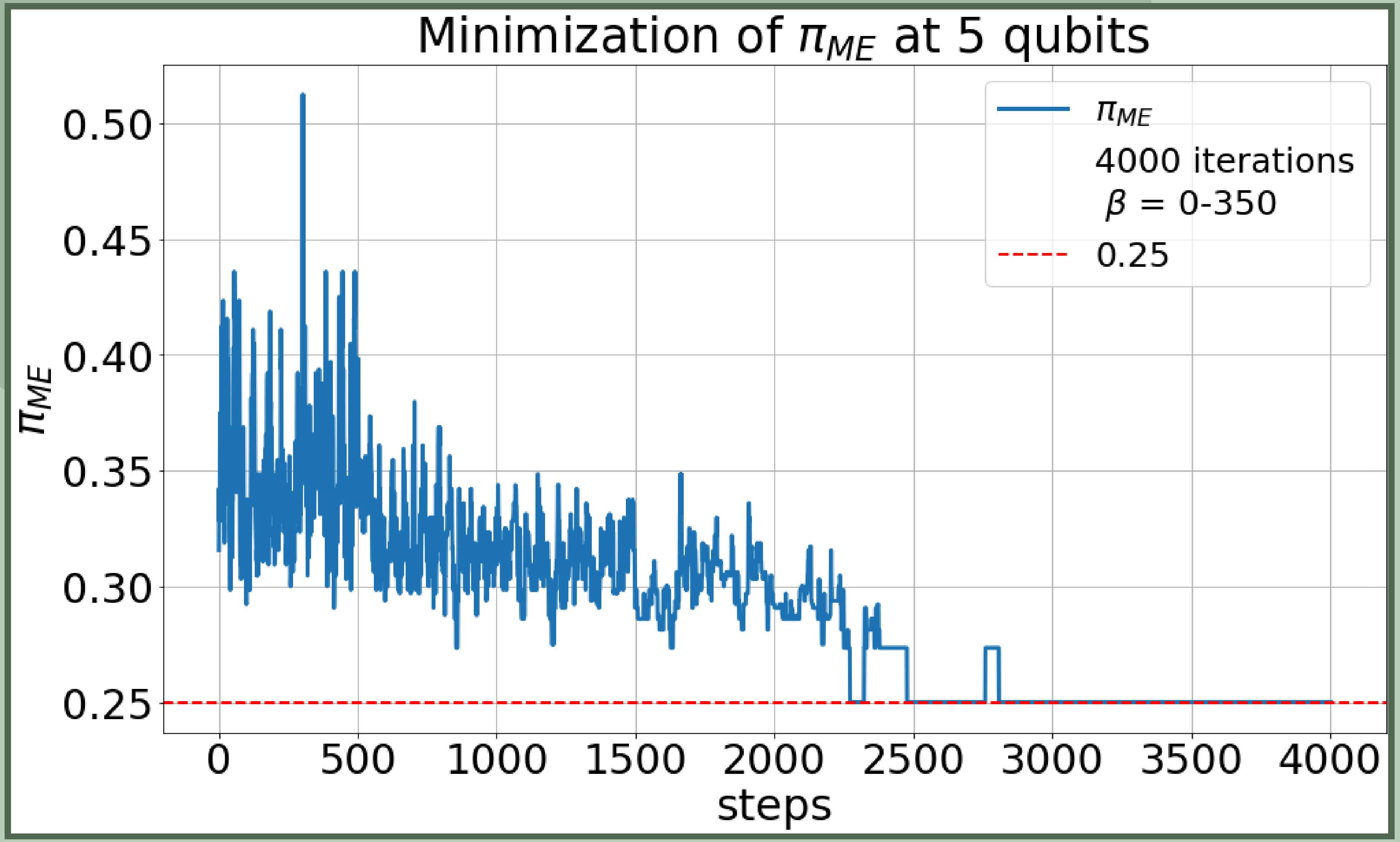
You can escape from the **relative minima** of

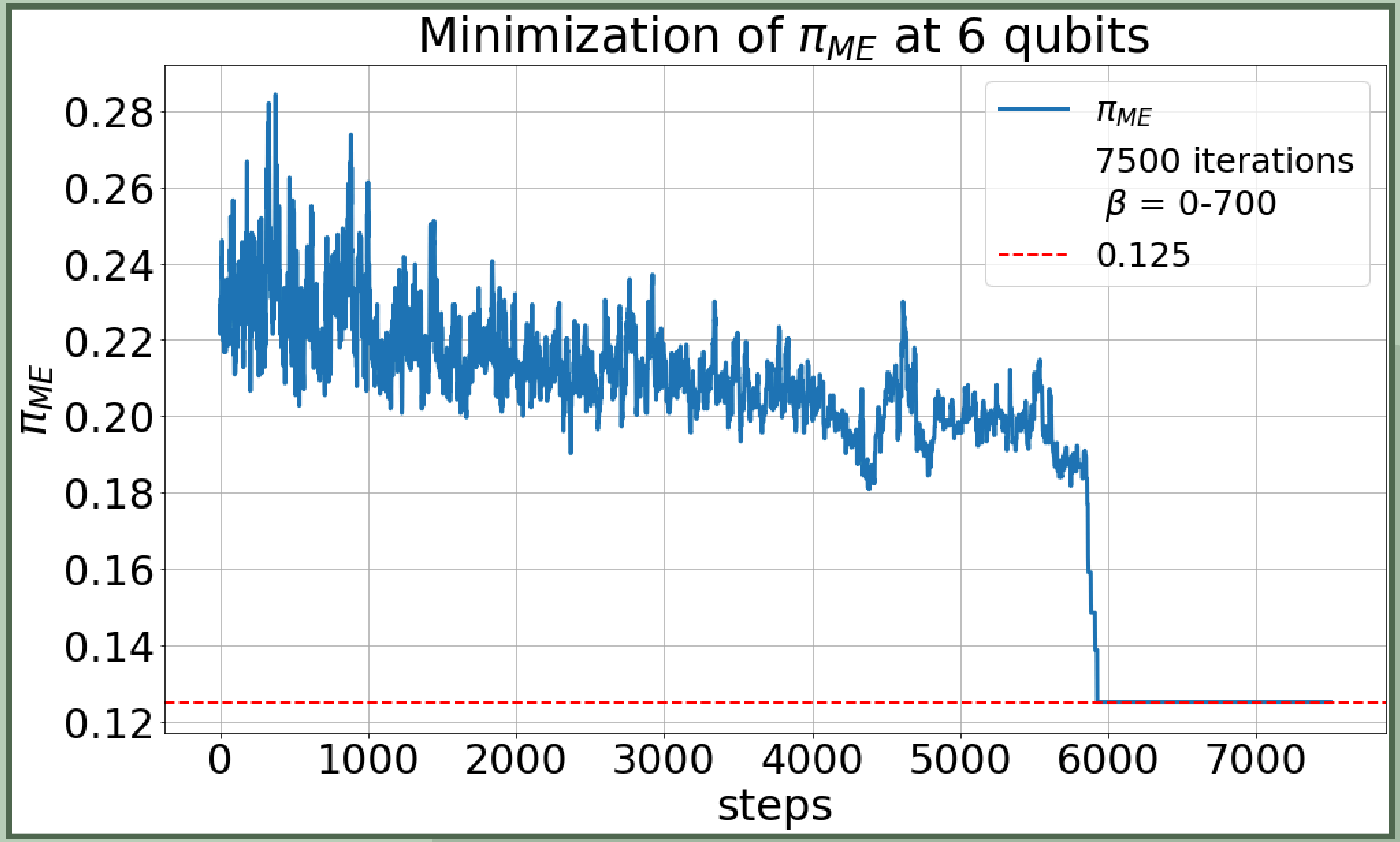
$$\pi_{ME}(s)$$



This can not be accomplished using the condition: $E_2 < E_1$

Annealing plot



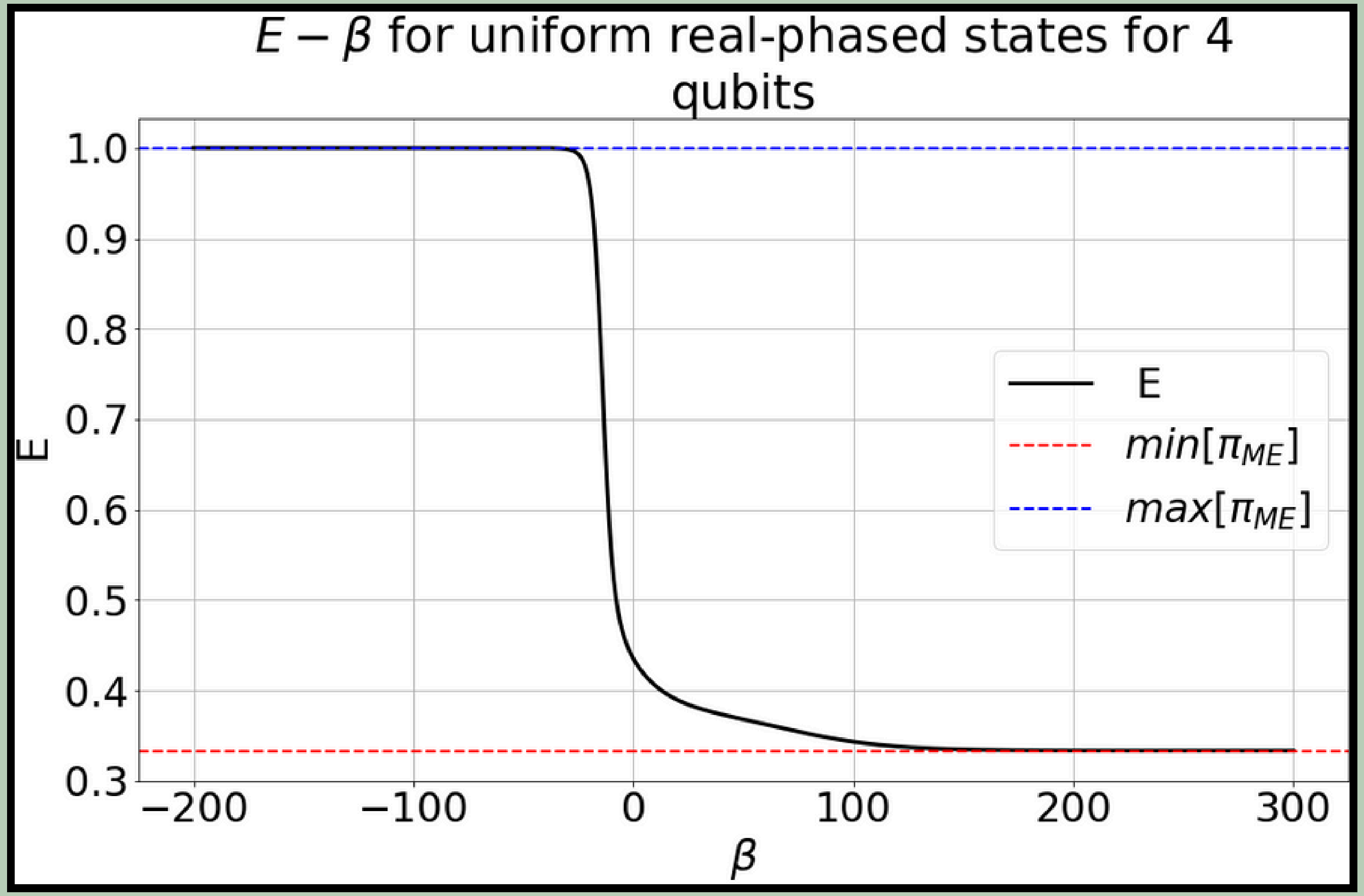


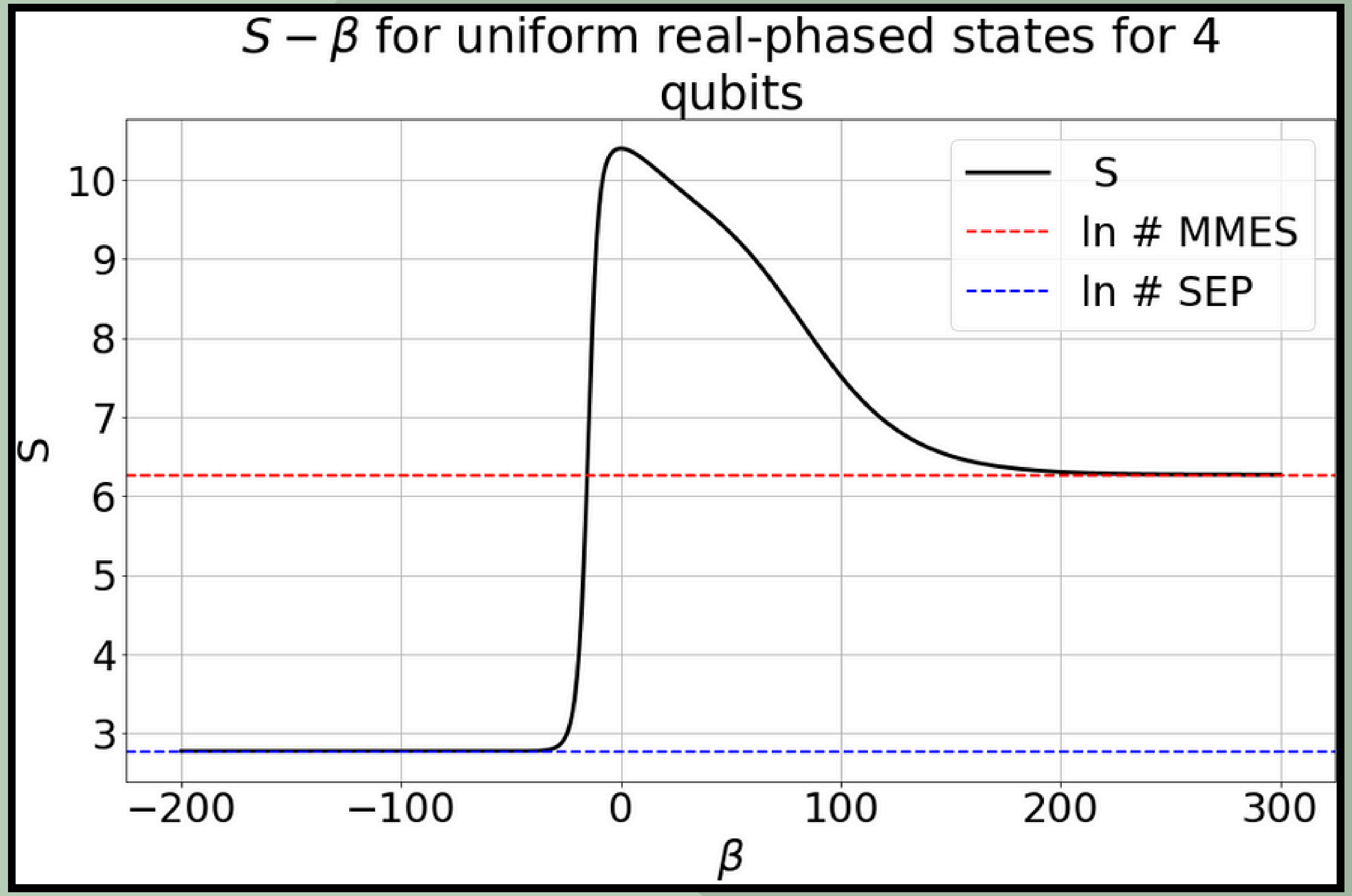
Future implementations

- Concluding the optimizatin of the 7 qubits systems. At the moment:

$$min[\pi_{ME}] = 0.13504 \quad \text{over to} \quad \pi_{min} = 0.125$$

- Analysis of the thermodynamics behavouir for 5 qubits on:





BACK-UP SLIDES

Quantifier of entanglement: **Purity** $\pi_A = \mathrm{tr}(\rho_A^2) = \sum_k p_k^2$

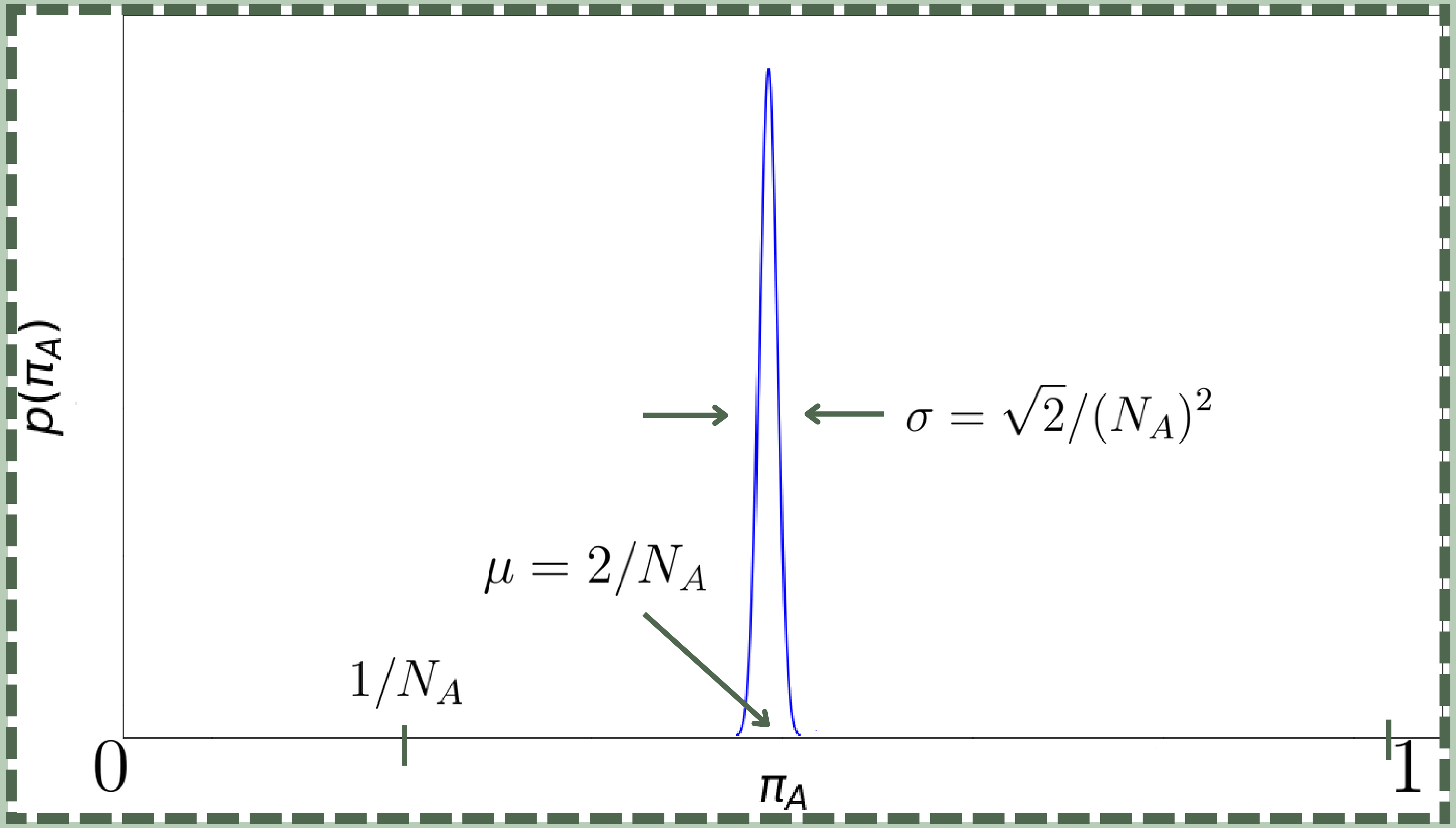
Purity lives in a well-defined intervall: $\mathbb{I} = \left[\frac{1}{N_A}, 1\right]$ where:

- $\pi_A = \frac{1}{N_A}$ ⇒ maximal entangled state
- $\pi_A = 1$ ⇒ separable state

where $N_A = 2^{n_A}$ and where n_A is the number of qubits in A

DISTRIBUTION OF BIPARTITE ENTANGLEMENT

case of n-qubits system uniformly distributed on the unitary hypersphere (Haar)



P.Facchi et al. “Statistical mechanics of multipartite entanglement”. In: Jour nal of Physics A: Mathematical and Theoretical 42.5 (Jan. 2009), p. 055304. issn: 1751-8121. doi: 10.1088/1751-8113/42/5/055304. url: http: //dx.doi.org/10.1088/1751-8113/42/5/055304

The parameters of the distributions can be approximate in the limit of large number of qubits

$$\mu = \langle \pi_A \rangle = \frac{N_A + N_{\overline{A}}}{N+1}$$

$$\sigma^2 = \langle (\pi_A - \mu)^2 \rangle = \frac{2(N_A^2 - 1)(N_{\overline{A}}^2 - 1)}{(N+1)^2(N+2)(N+3)}$$

$$\mu = 2/N_A$$

$$\sigma = \sqrt{2}/(N_A)^2$$

where:

$N_A = 2^{n_A}$ and where n_A is the number of qubits in A

$N_{\overline{A}} = 2^{n_{\overline{A}}}$ and where $n_{\overline{A}}$ is the number of qubits in $\overline{A}$

$N = N_A N_{\overline{A}} = 2^{n_A + n_{\overline{A}}} = 2^n$ and where n is the number of qubits of the system

P.Facchi et al. “Statistical mechanics of multipartite entanglement”. In: Jour nal of Physics A: Mathematical and Theoretical 42.5 (Jan. 2009), p. 055304. issn: 1751-8121. doi: 10.1088/1751-8113/42/5/055304. url: http: //dx.doi.org/10.1088/1751-8113/42/5/055304

MAXIMAL MULTIPARTITE ENTANGLED STATES: MMES

perfect MMES: maximum bipartite entanglement with respect to all the balanced bipartitions

↓

How to identify them ?

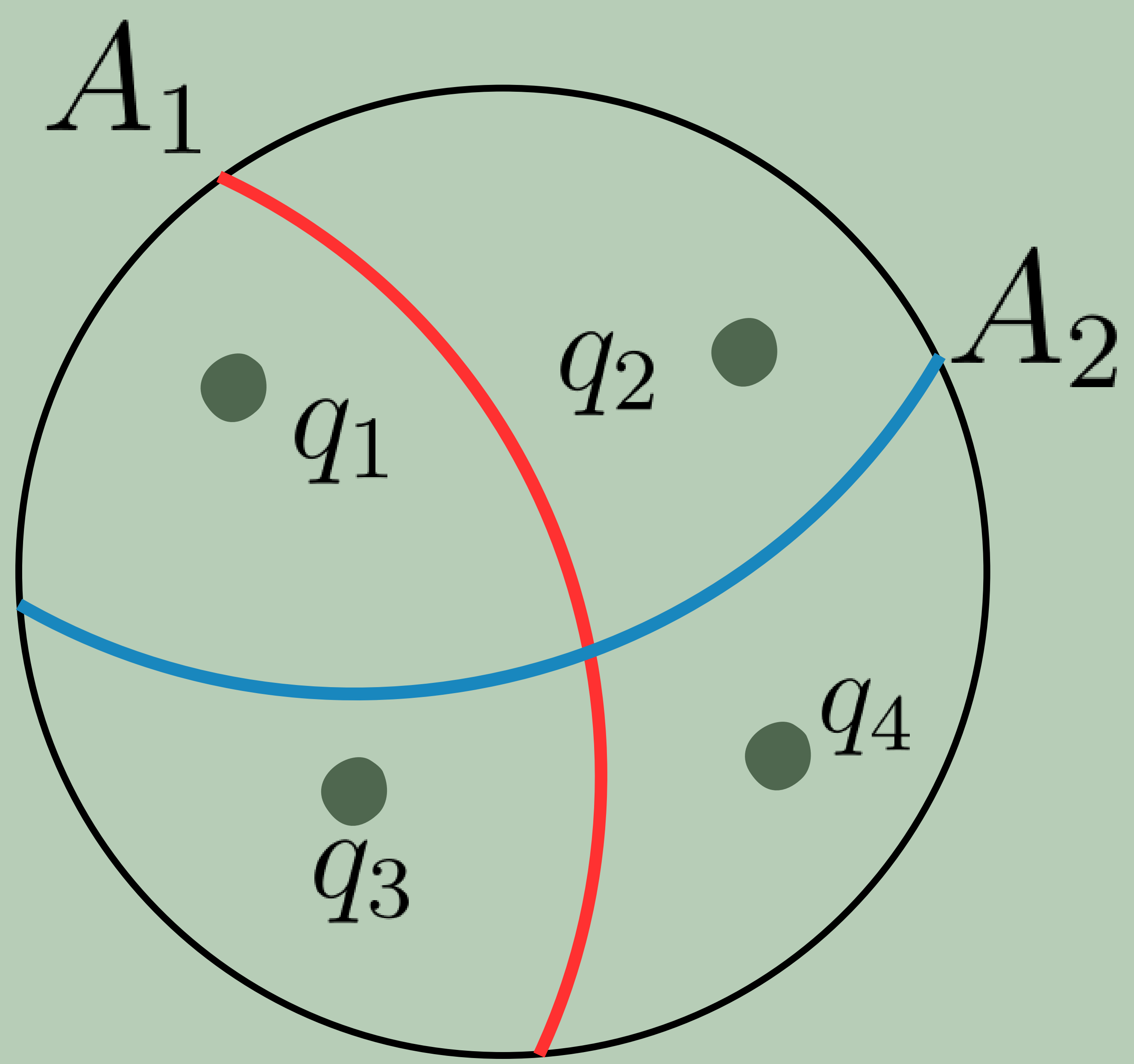
P Facchi et al. "Multipartite entanglement and frustration". In: New Journal of Physics 12.2 (Feb. 2010), p. 025015. issn: 1367-2630. doi: 10.1088/1367 2630/12/2/025015. url: http://dx.doi.org/10.1088/1367-2630/12/ 2/025015.

n qubits state

$$|\psi\rangle = \sum_{k=0}^{2^n-1} z_k|k\rangle$$

bipartition

$$(A, \overline{A})$$



$$\rho_A = Tr_{\overline{A}}(|\psi\rangle\langle\psi|)$$

$$\pi_A = Tr(\rho_A^2) = \sum_{k=0}^{2^{n_A}-1} z_k|k\rangle$$

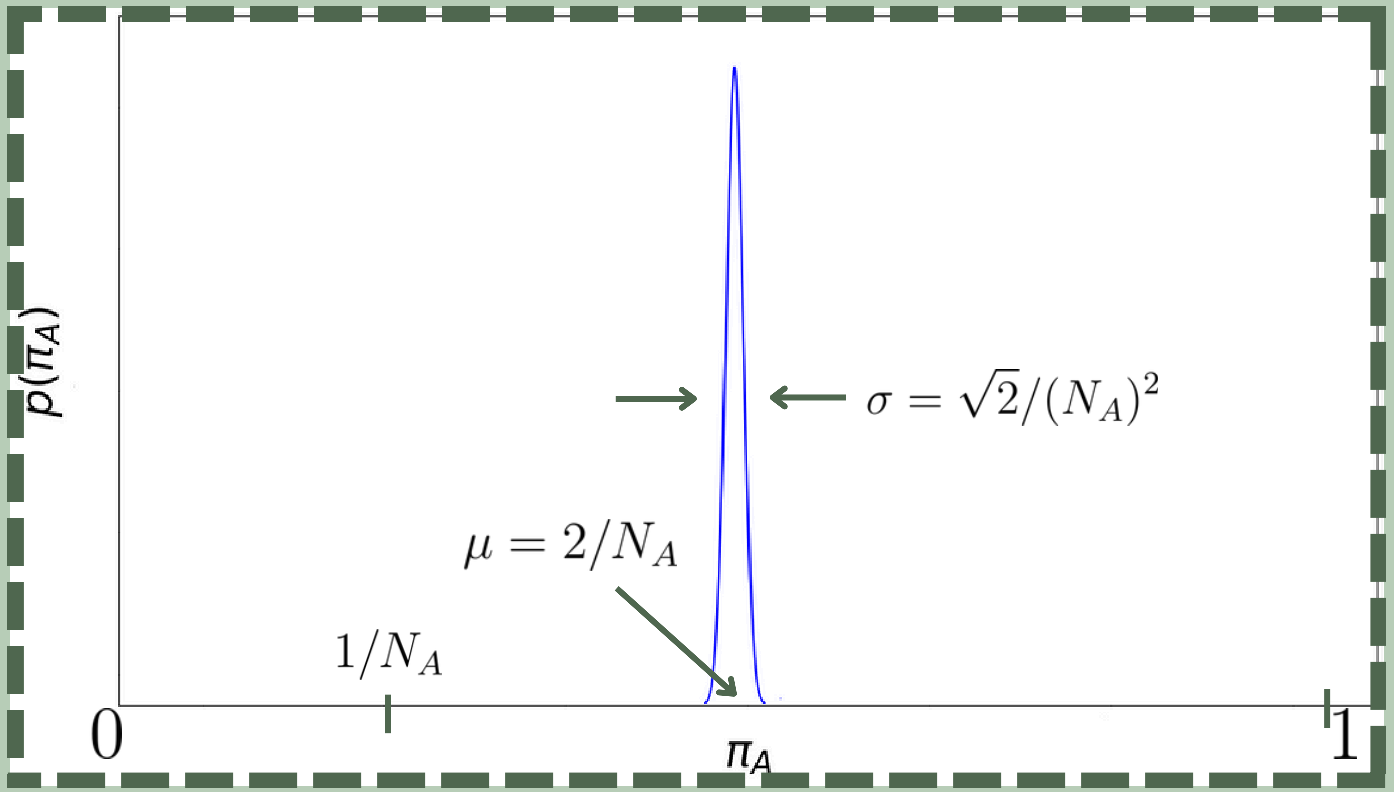
- π_A has to be $\frac{1}{N_A}$ for any bipartition
- ρ_A has to be propotional to the identy operator

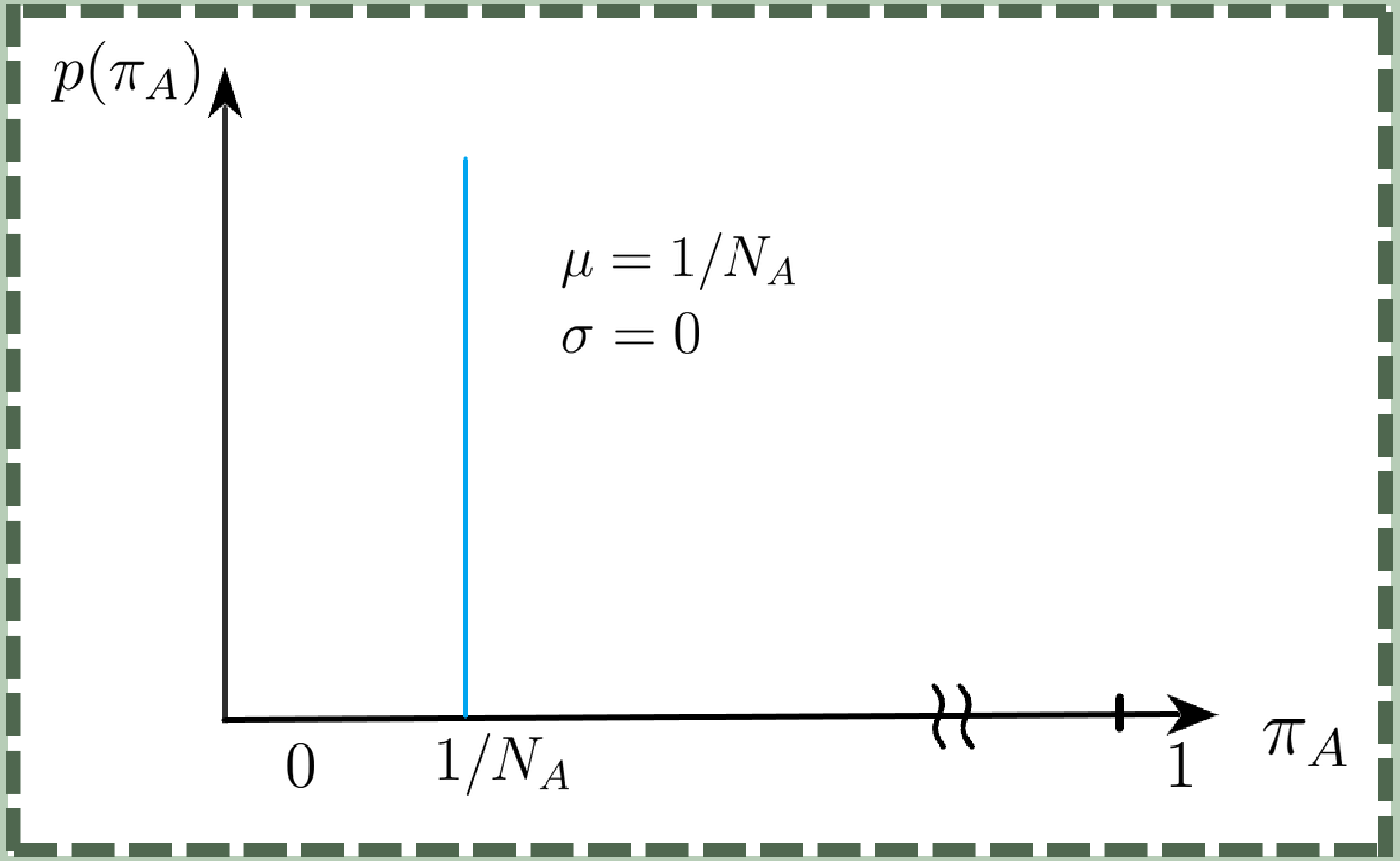
π_A has to be $\dfrac{1}{N_A}$ for any bipartition

$$\pi_{ME}(|\psi\rangle) = \binom{n}{[n/2]}^{-1} \sum_{|A|=[n/2]} \pi_A(|\psi\rangle)$$

The state is a **perfect MMES** ⟺ the potential of the multipartite entanglement has to be $\frac{1}{N_A}$

- The generic bipartite entanglement distribution becomes a **delta dirac** function





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Number of uniform states per number of qubits

The number of elements is equal to $2^{2^n - 1}$

number of qubits	uniform real-phased states
3	128
4	32768
5	$2,12 \times 10^{9}$
6	$9,22 \times 10^{18}$
7	$1,07 \times 10^{38}$
8	$5,69 \times 10^{76}$
9	$6,70 \times 10^{153}$
10	$8,99 \times 10^{307}$
11	$> 10^{308}$

The pseudo-random sub-manifold is continuous

Number of states simulated

number of qubits	Pseudo-Random	Uniform Real-Phased
3	2×10^6	128
4	2×10^6	32678
5	2×10^6	2×10^6
6	2×10^6	2×10^6
7	2×10^6	2×10^6
8	2×10^6	2×10^6
9	2×10^6	2×10^6
10	2×10^6	2×10^6
11	10^6	10^6
12	10^6	10^6

Collection of data

choose a balanced bipartition $(A, \overline{A})$

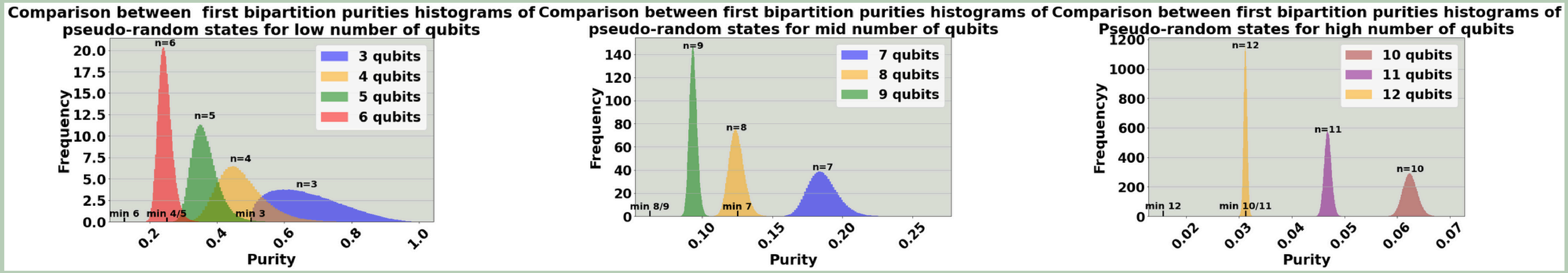
$$|\psi\rangle \xrightarrow{\text{Partial Trace}} \rho_A \xrightarrow{\text{sum over the squared eigenvalues}} \pi_A$$

For each state until 10 qubits, **all purities** relative to each balanced bipartition have been estimated.

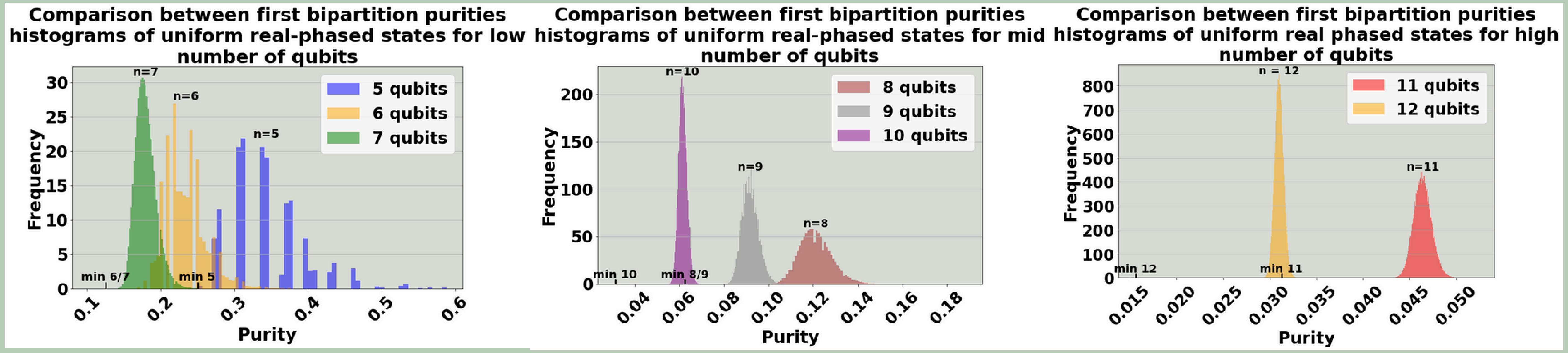
for 11 and 12 qubits high **computational cost**

Histograms of π_A

Pseudo-random

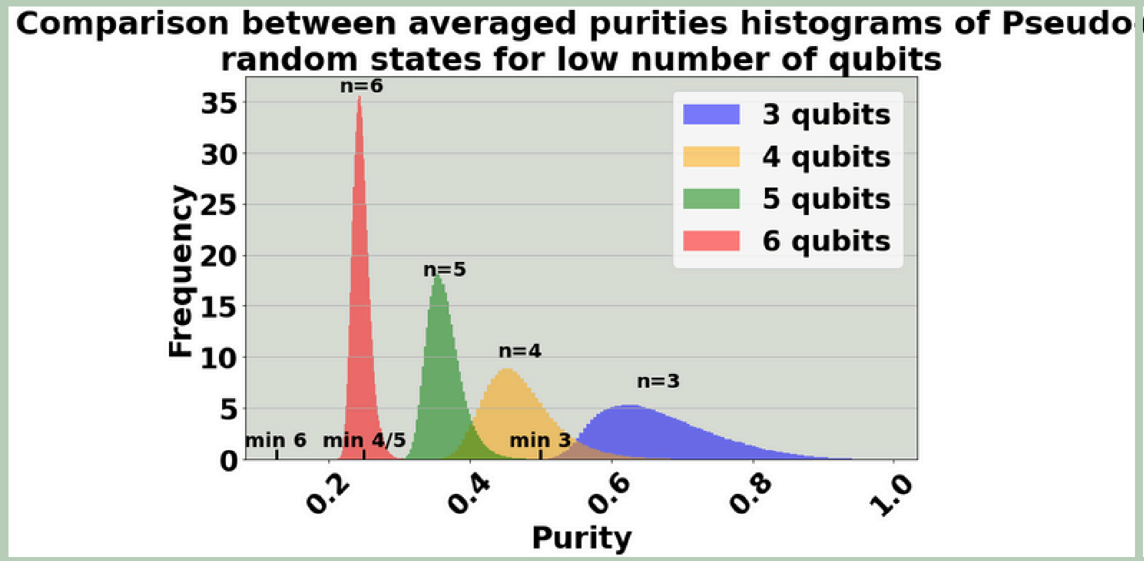


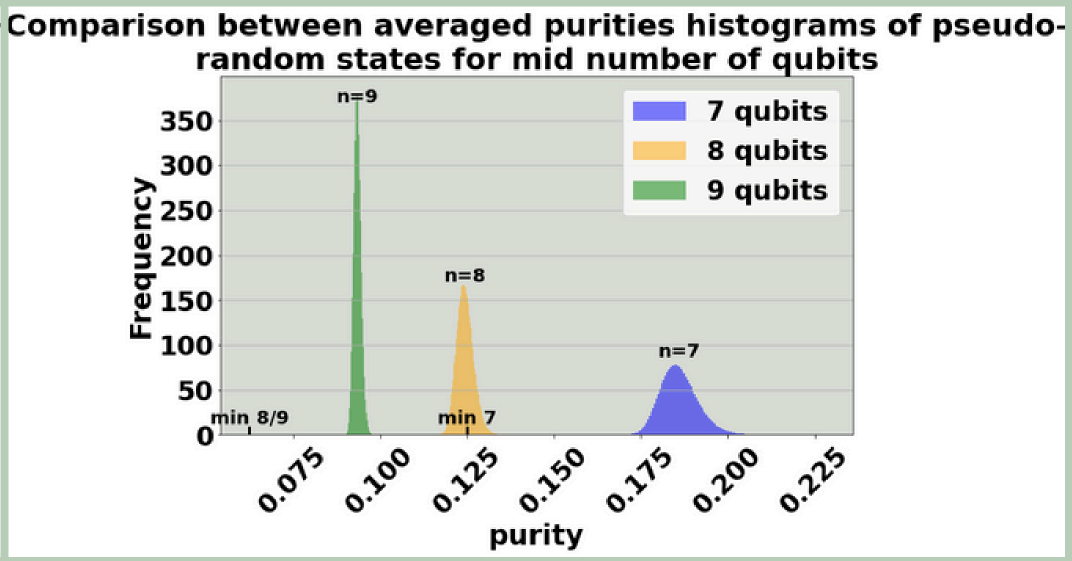
Uniform real-phased

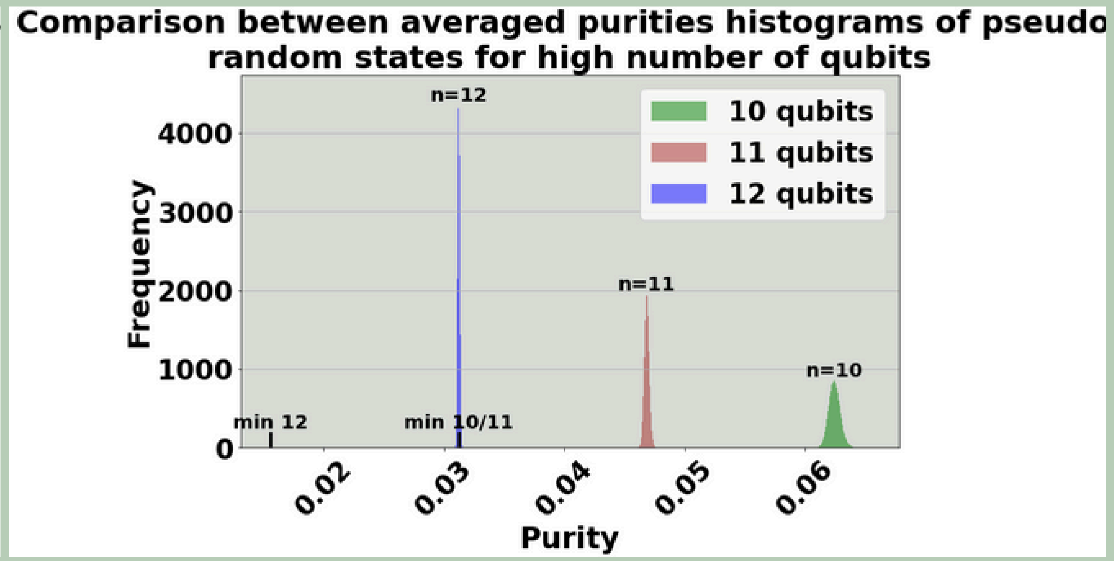


Histograms of π_{ME}

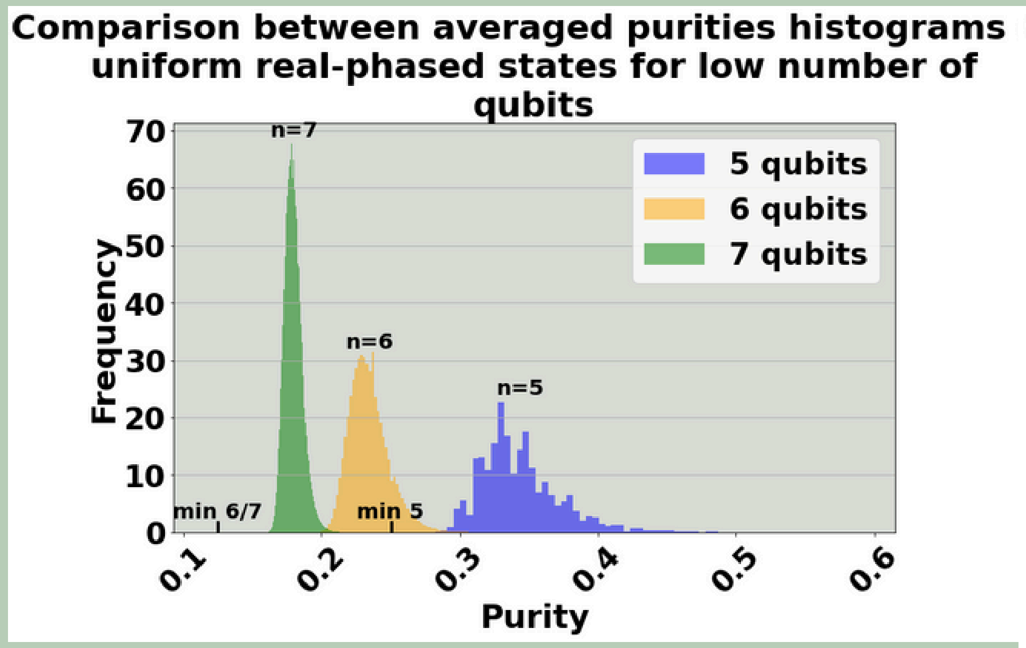
Pseudo-random

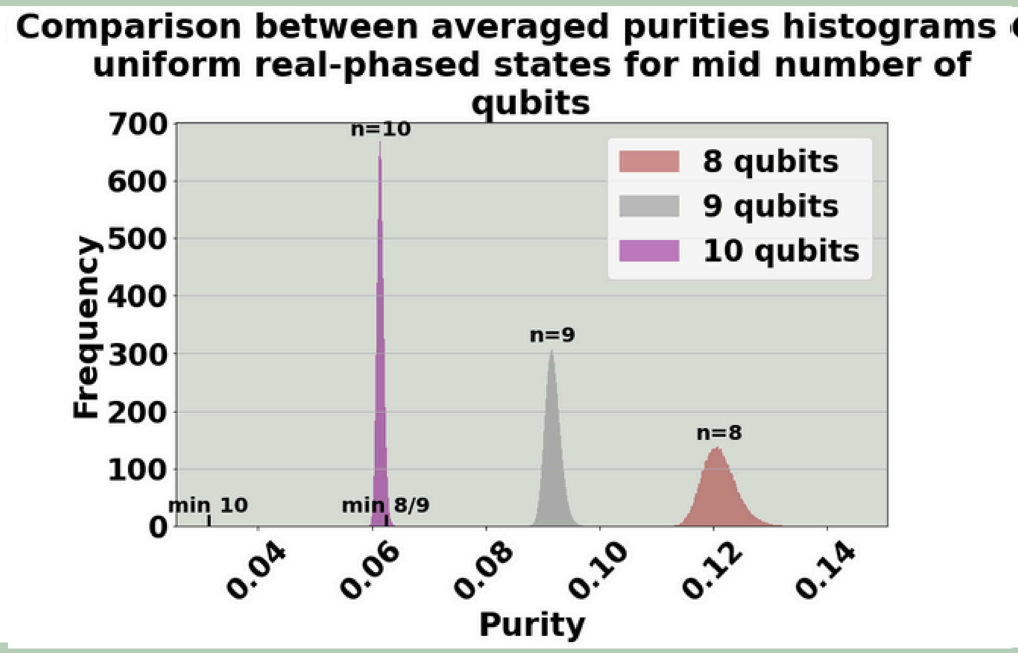


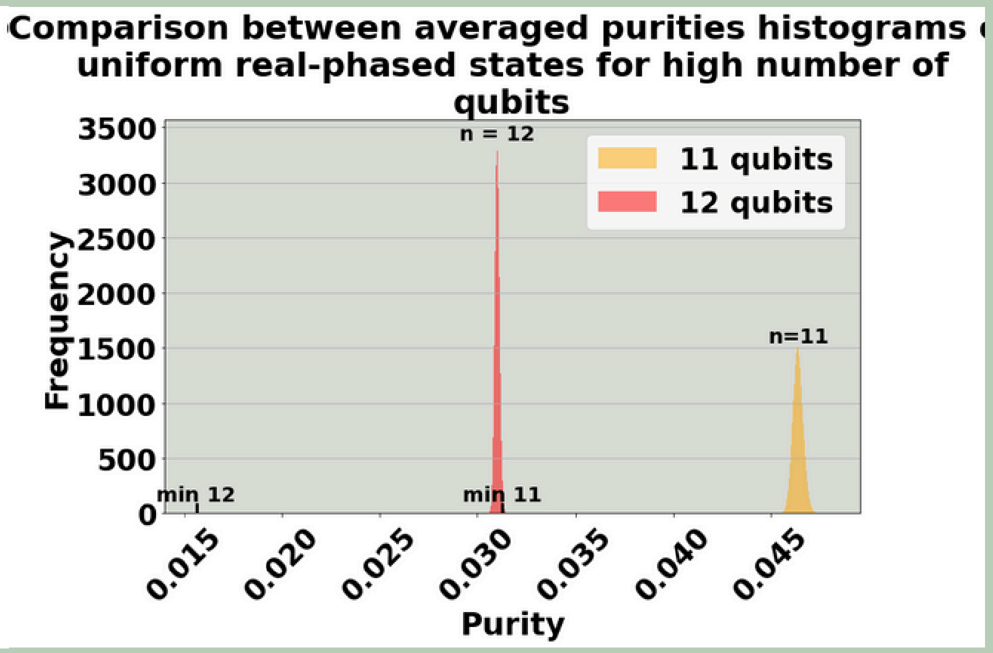




Uniform real-phased







Analytical expectations of the purity distribution relative to the generic bipartition for **Random state**

$$\mu = \langle \pi_A \rangle = \frac{N_A + N_{\overline{A}}}{N+1}$$

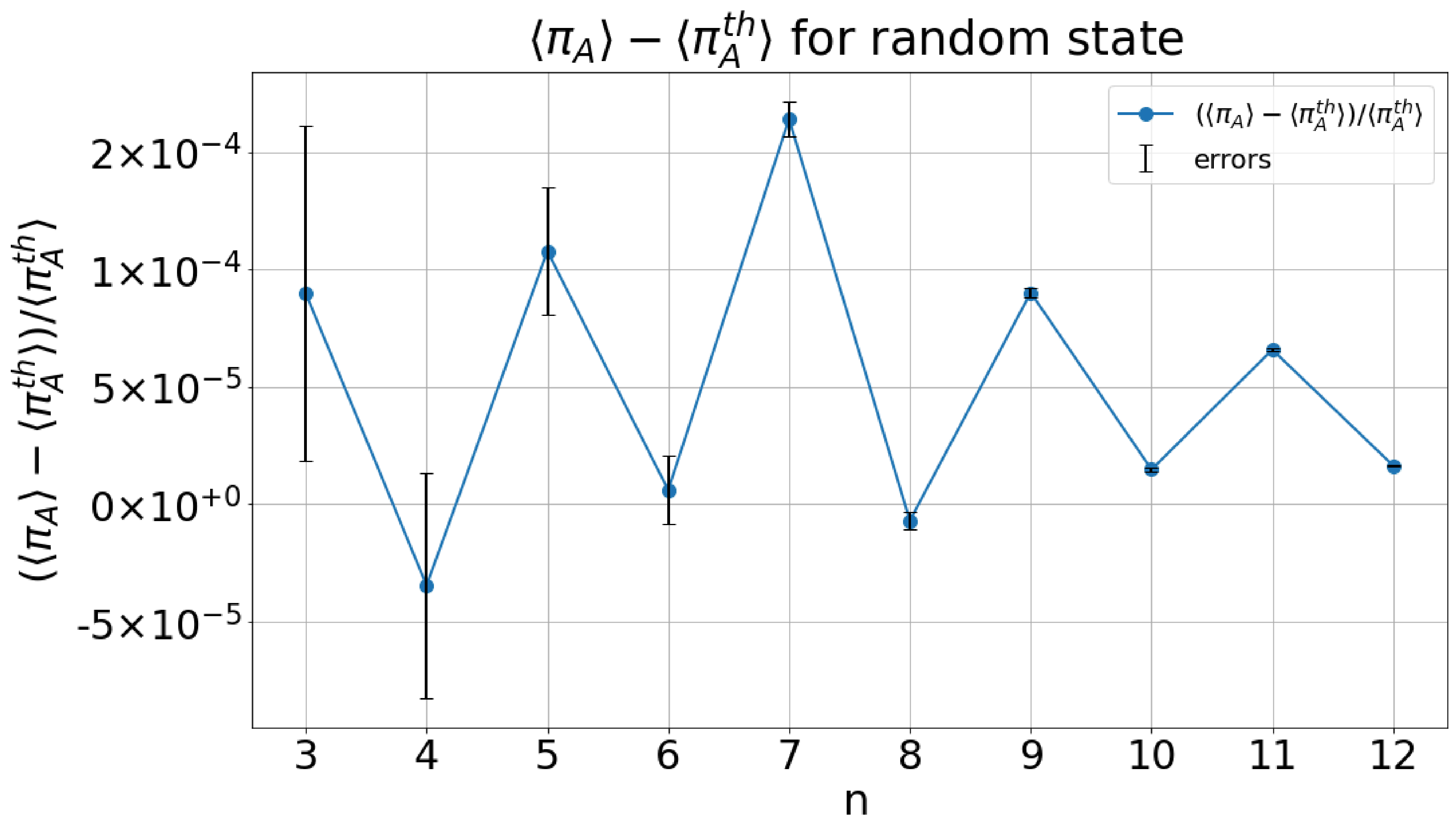
$$\sigma_A^2 = \langle (\pi_A - \mu)^2 \rangle = \frac{2(N_A^2 - 1)(N_{\overline{A}}^2 - 1)}{(N+1)^2(N+2)(N+3)}$$

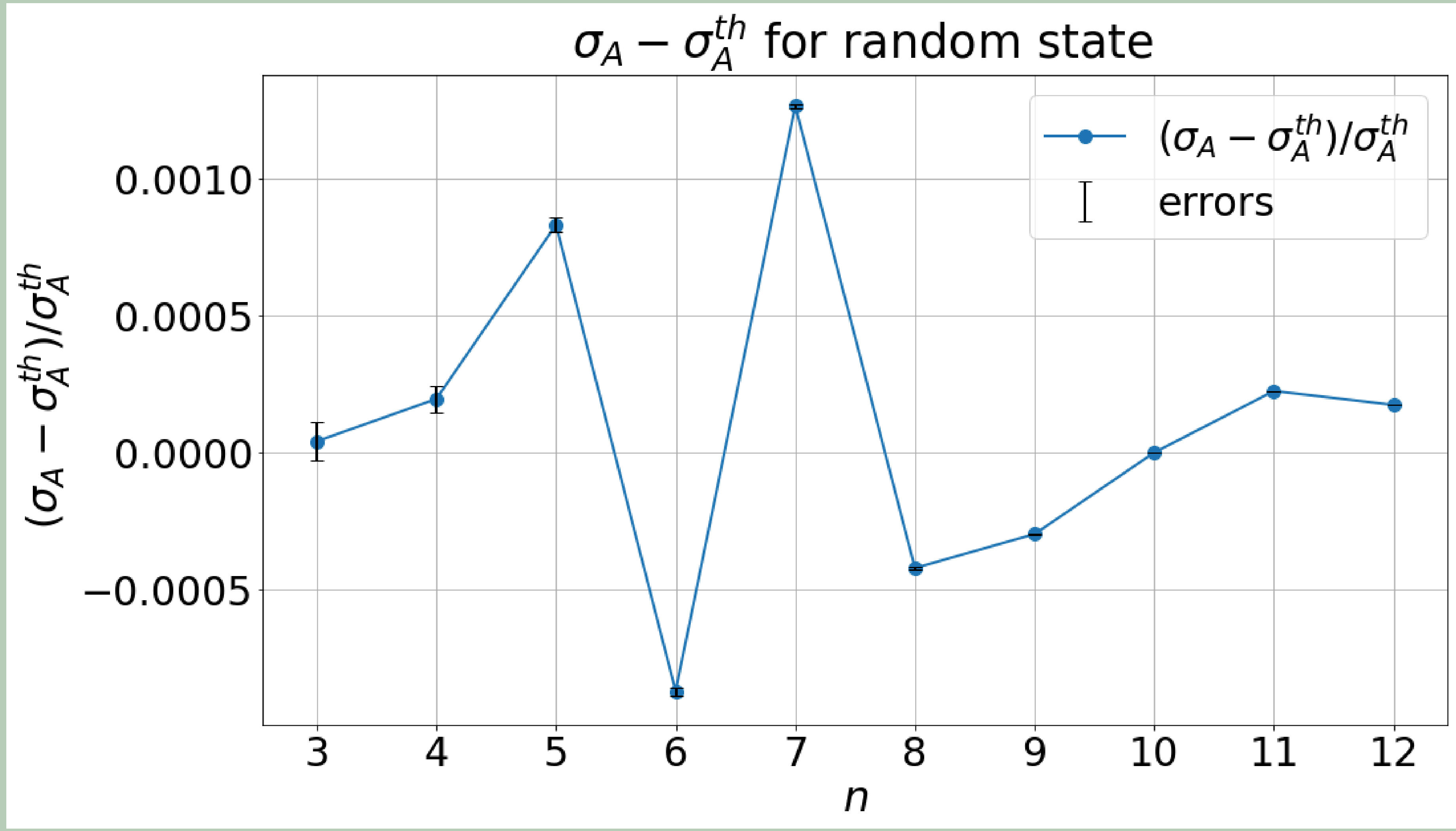
$N_A = 2^{n_A}$ and where n_A is the number of qubits in

$N_{\overline{A}} = 2^{n_{\overline{A}}}$ and where $n_{\overline{A}}$ is the number of qubits in

$N = N_A N_{\overline{A}} = 2^{n_A + n_{\overline{A}}} = 2^n$ and where n is the number of qubits of the system

Comparison numerial vs theoretical



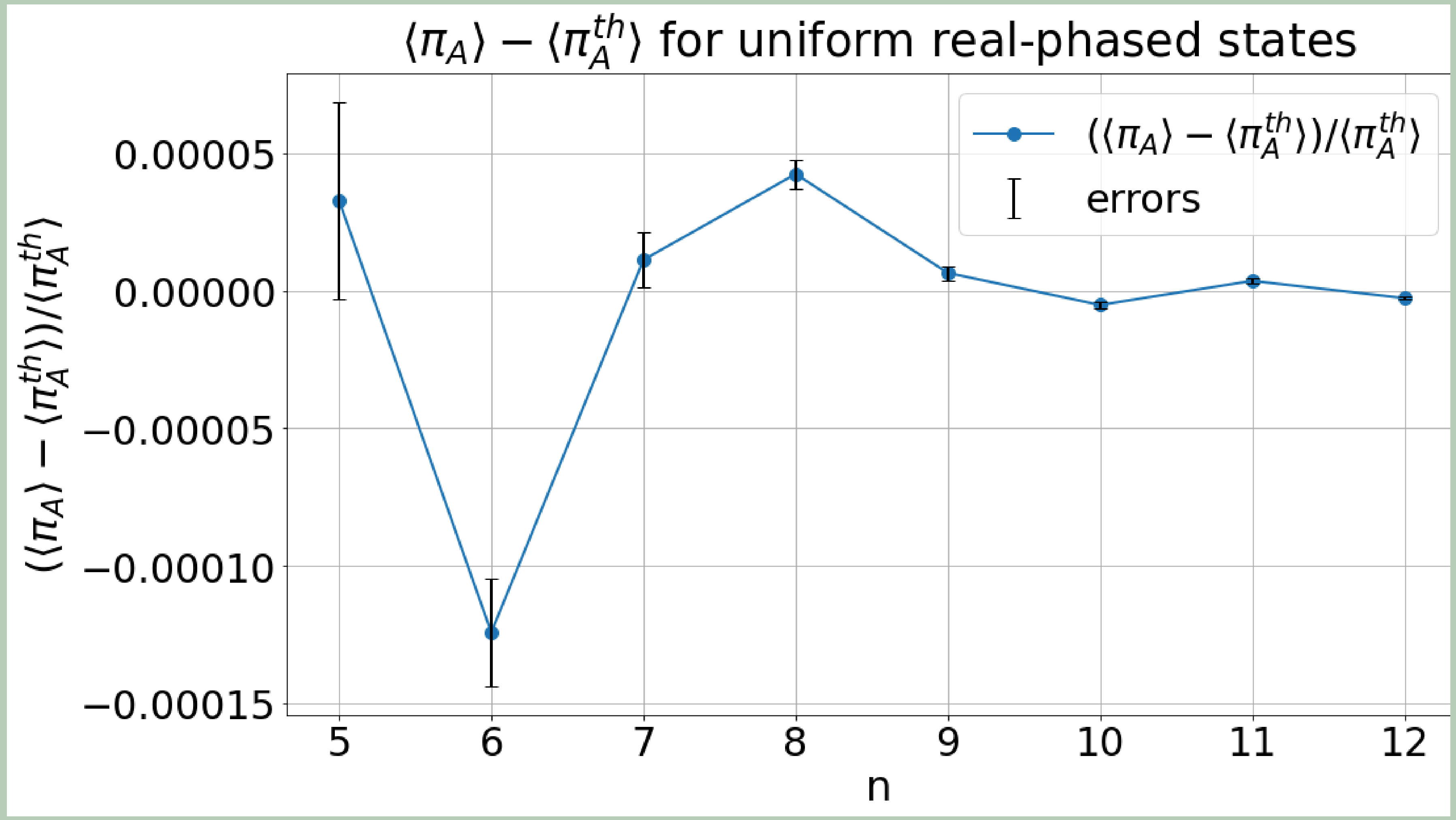


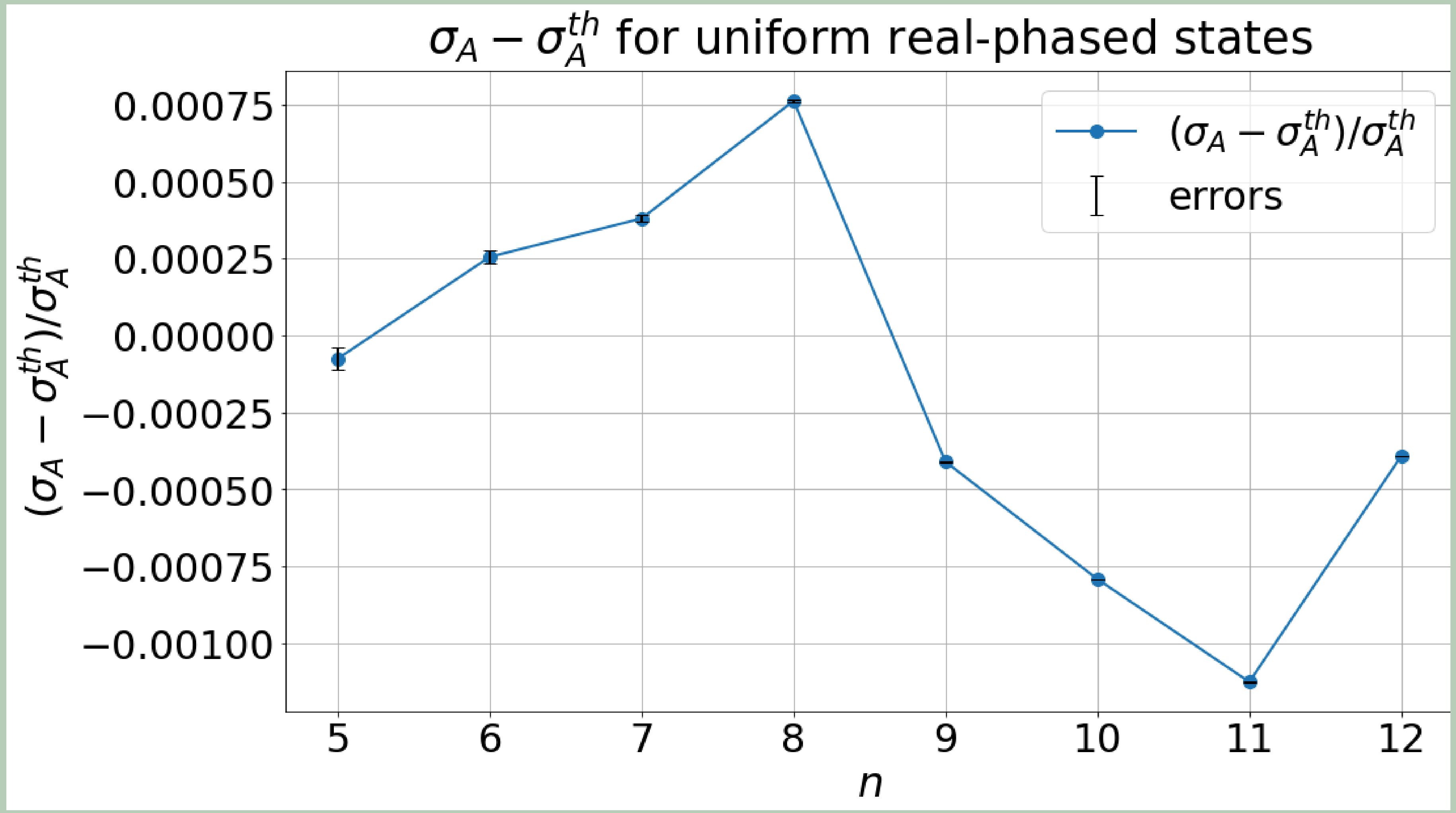
Analytical expectations of the purity distribution relative to the generic bipartition for **Uniform real-phased**

$$\mu = \langle \pi_A \rangle = \frac{N_A + N_{\overline{A}} - 1}{N}$$

$$\sigma_A^2 = \langle (\pi_A - \mu)^2 \rangle = \frac{N_A N_{\overline{A}} - N_A - N_{\overline{A}} + 1}{N^3}$$

Comparison numerial vs theoretical

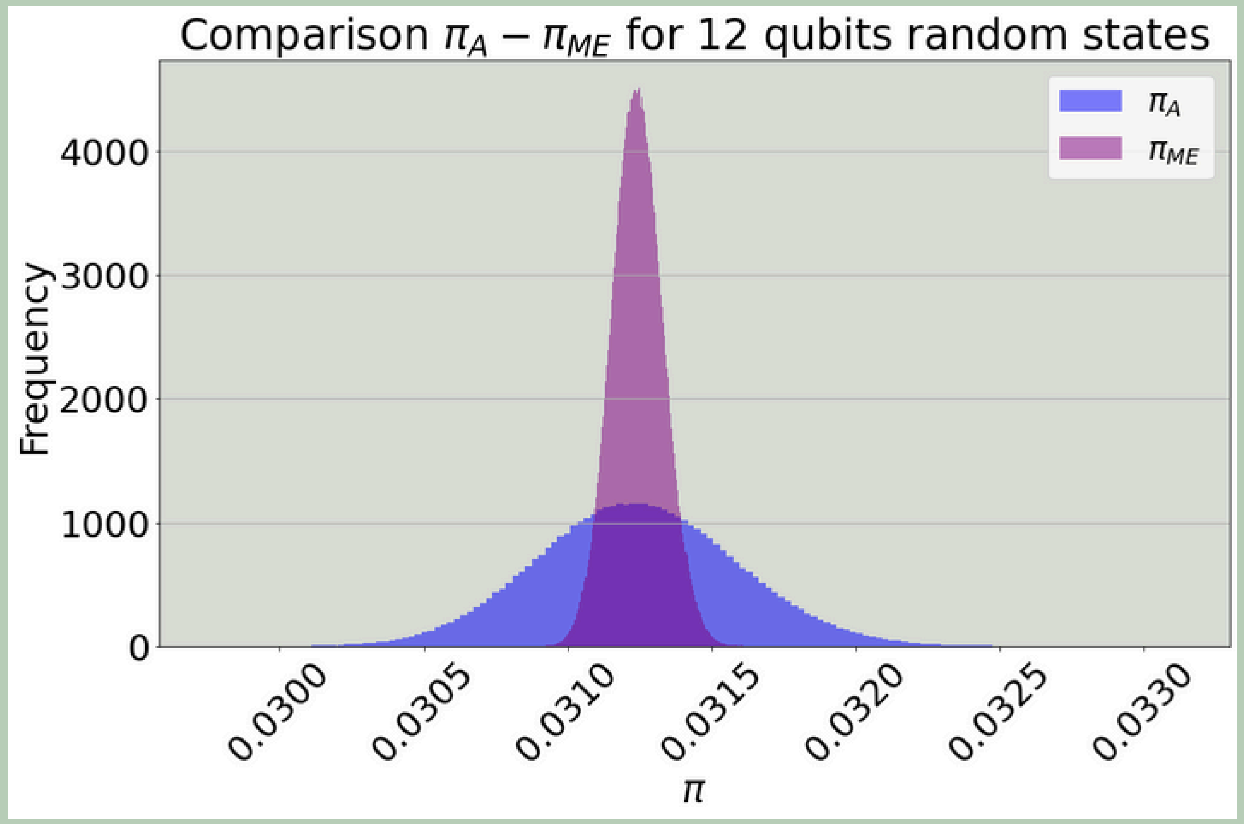


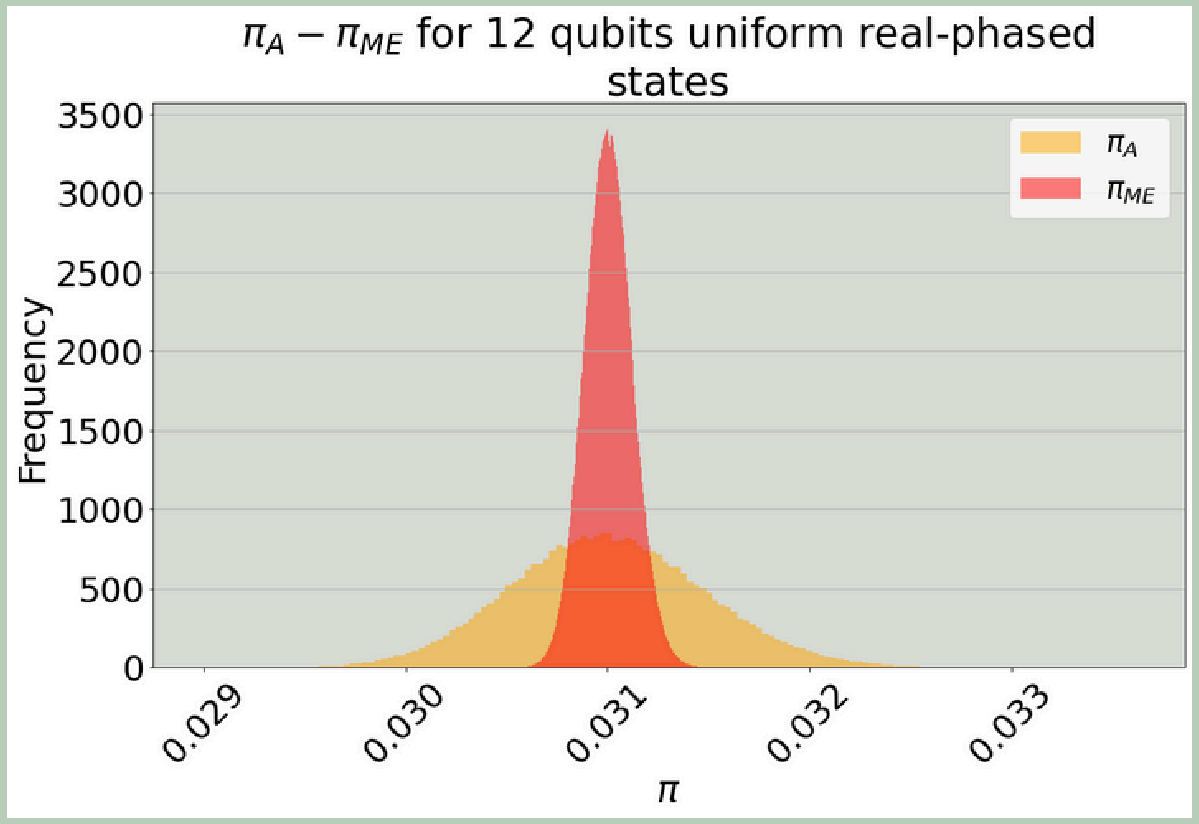


n = 12

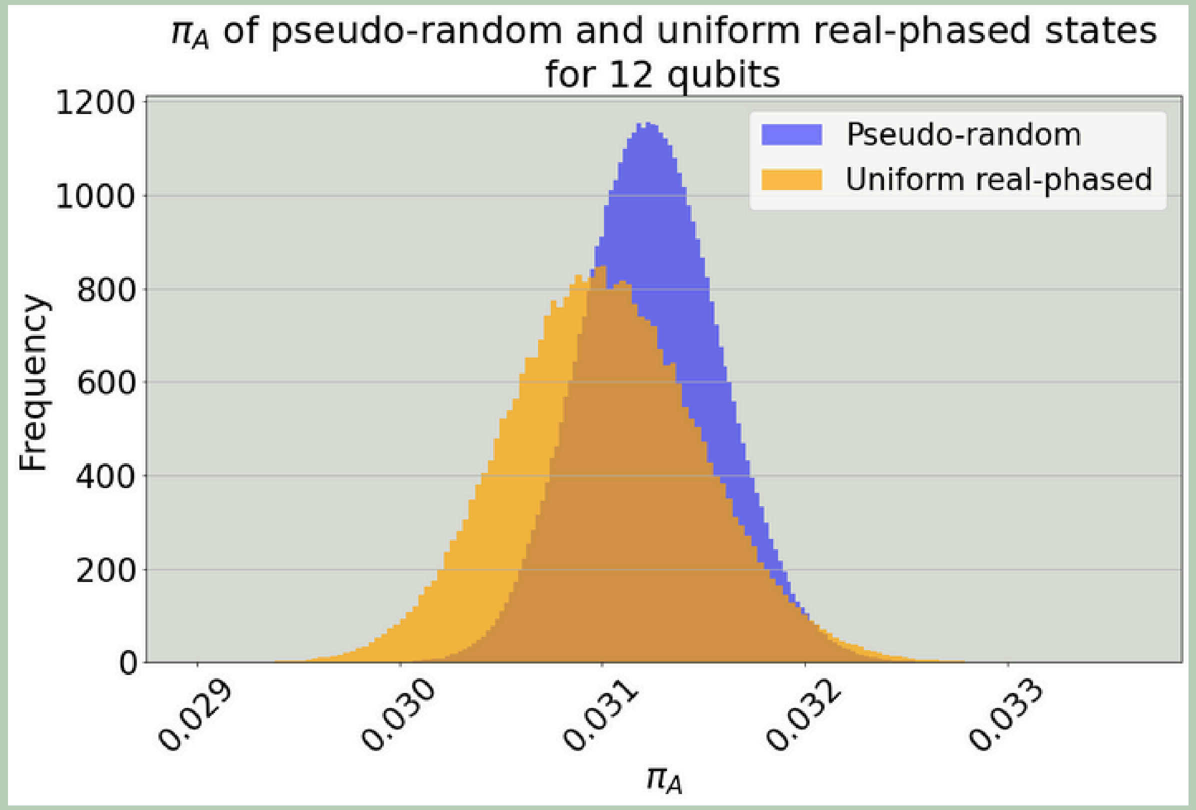
Comparisons

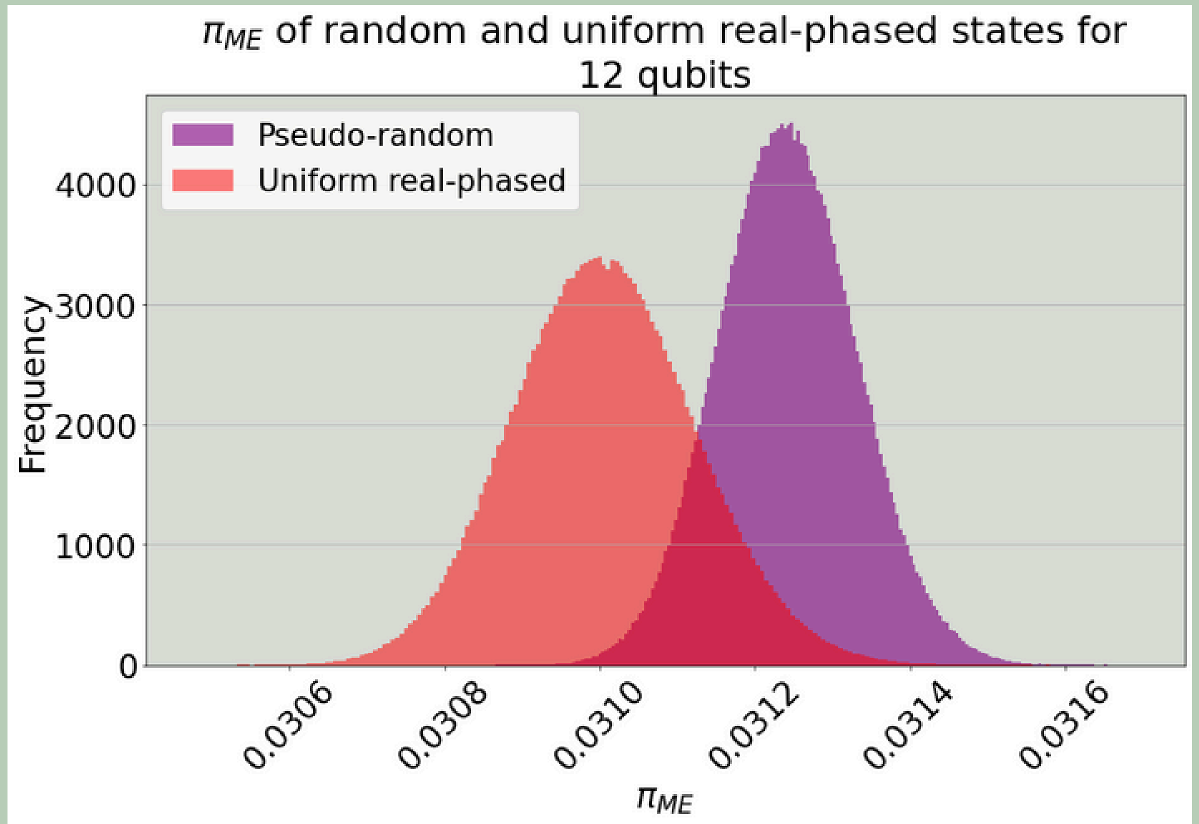
between different data



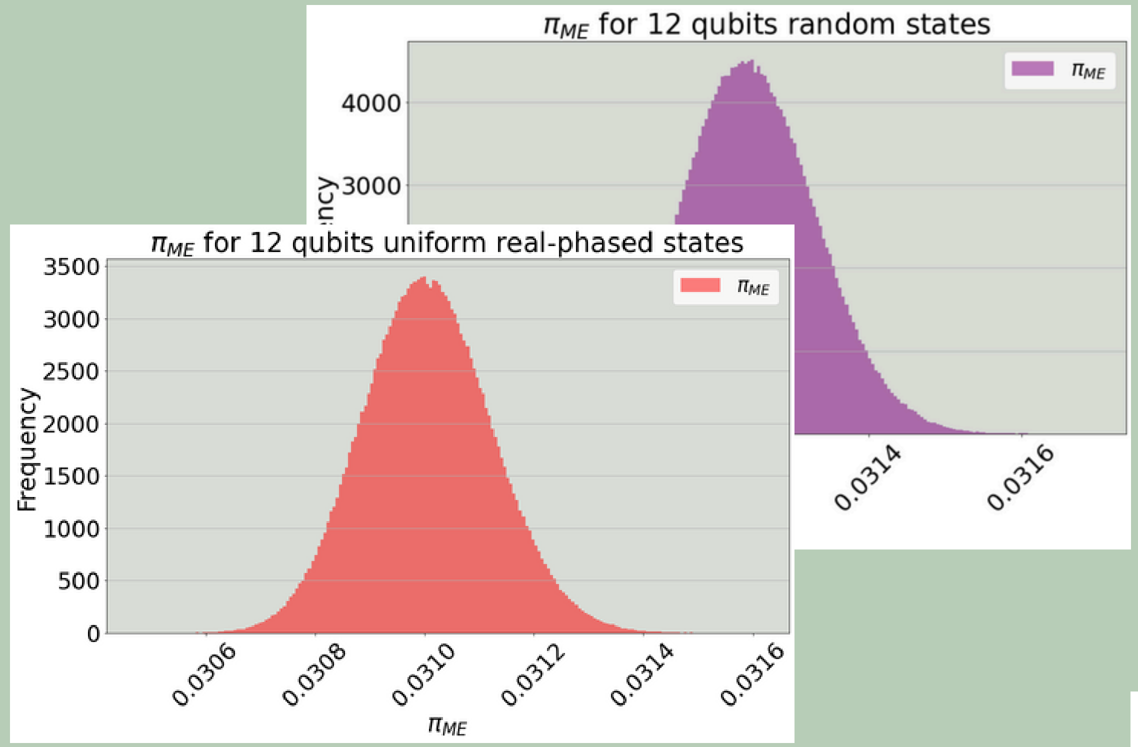


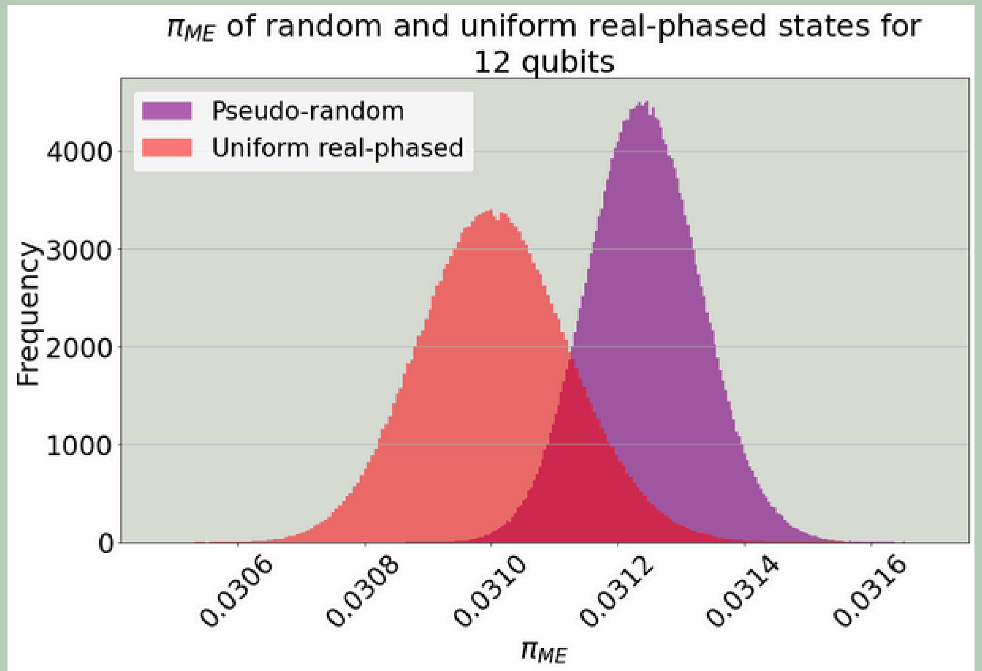
between different sub-manifold

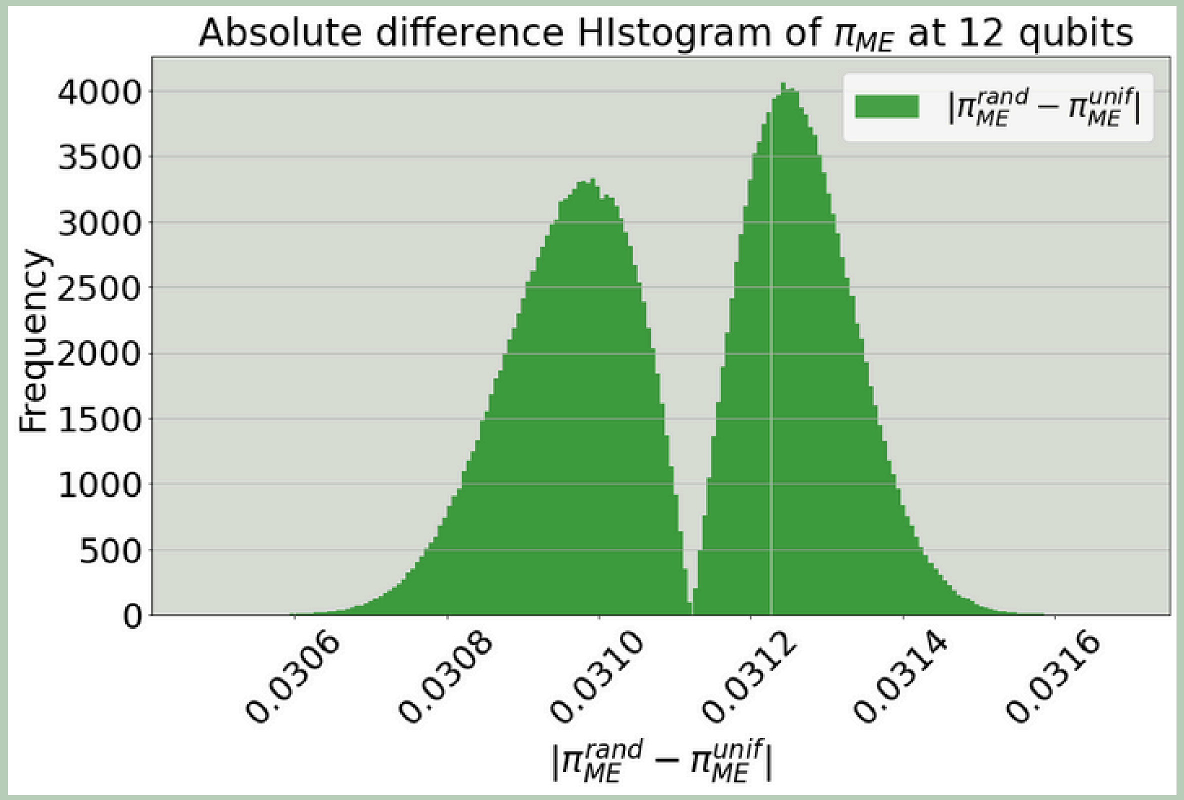




L^1 distance compuation





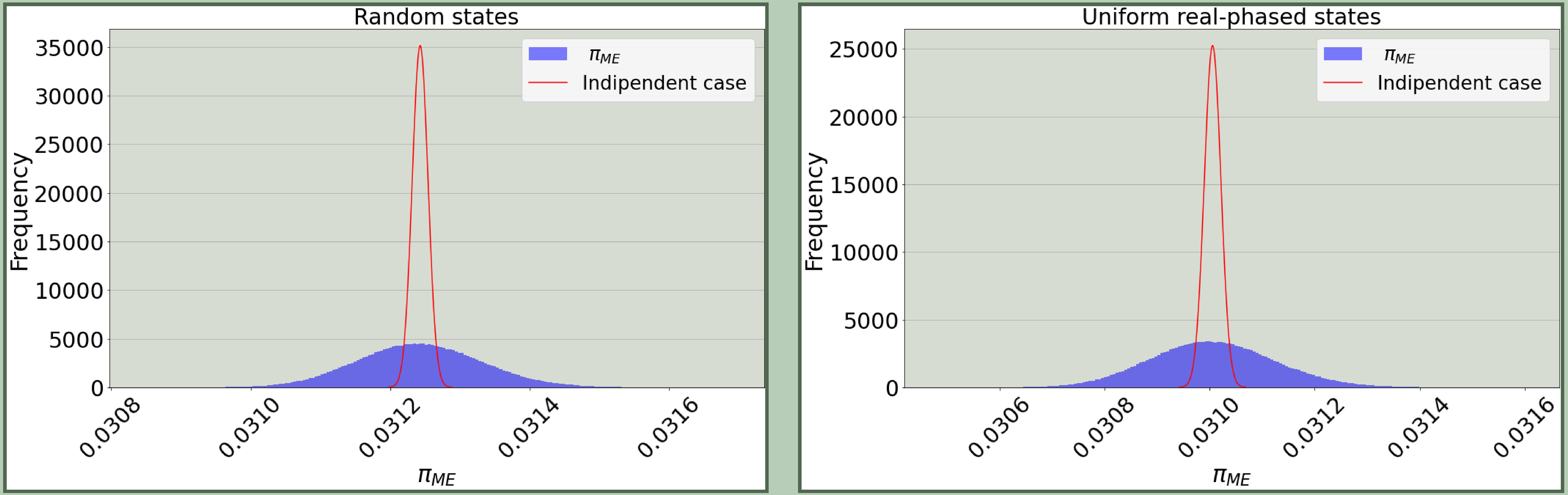


Absolute distance values

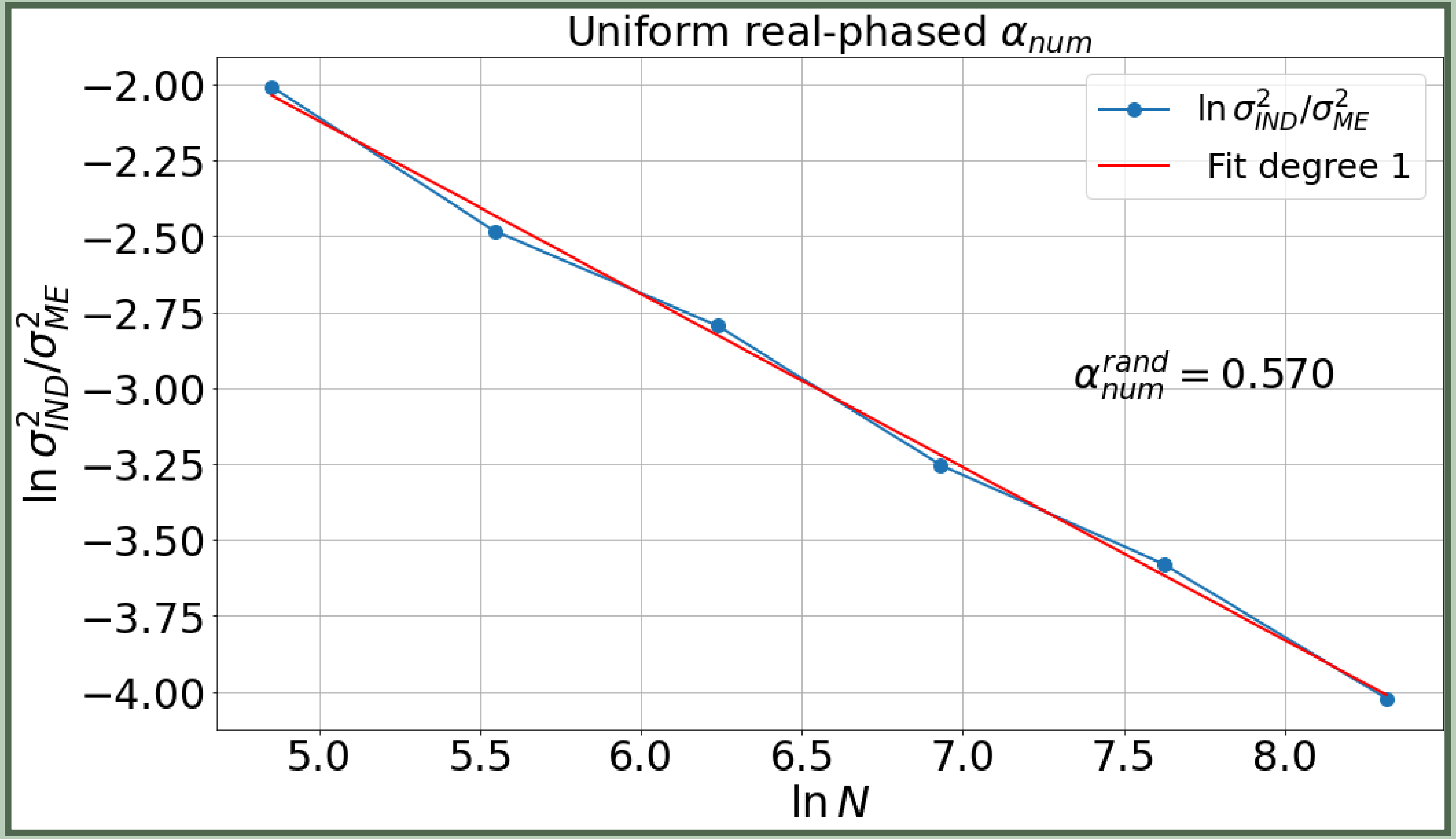
Qubits	One bipartition	Mean over all bipartitions
3	1.70661478	1.74463625
4	1.78527711	1.53627322
5	1.05875891	0.7543195
6	0.67784909	0.8103450
7	0.55854826	0.9085205
8	0.54466821	0.9997795
9	0.54290989	1.1231185
10	0.5374433	1.2391315
11	0.540405	1.37901017
12	0.5377885	1.50515837

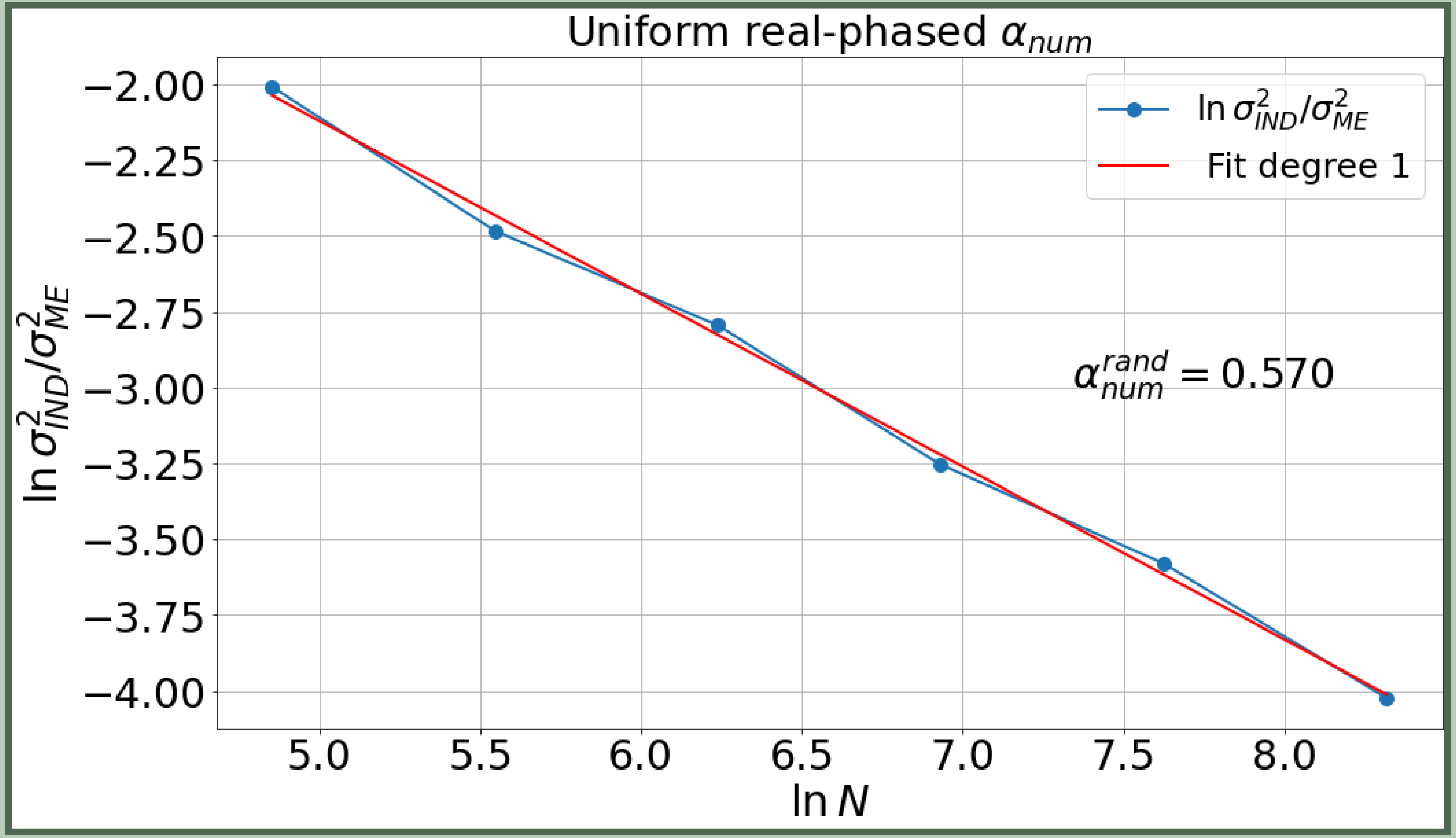
Identification of a frustration quantifier

- Non indipendence of π_{ME} from the bipartitions



Numerical estimation





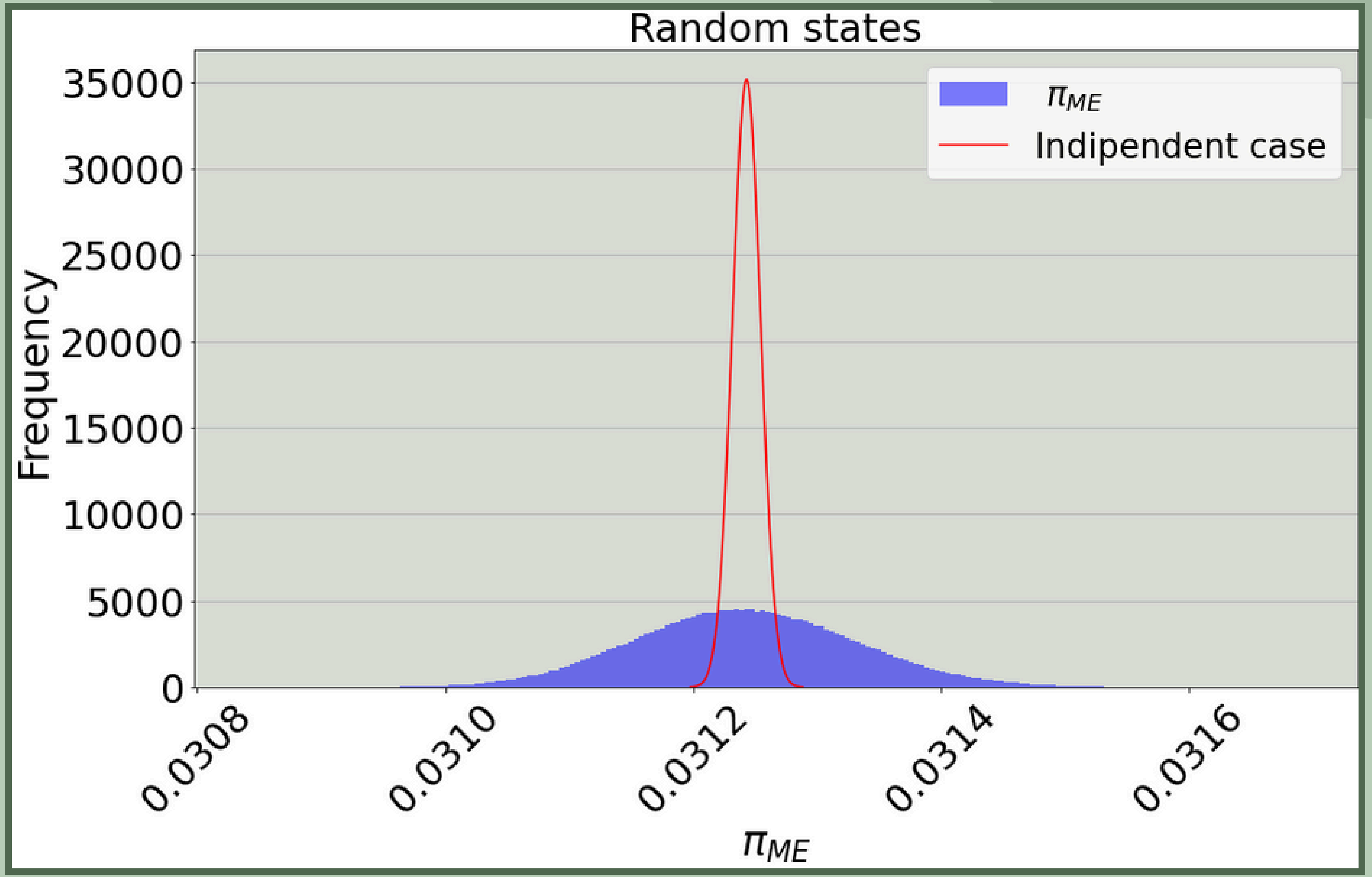
$$\alpha_r = 0.548$$

$$\alpha_u = 0.570$$

Major problem

non independece $\not\Longleftrightarrow$ quantum frustration

Why? → **Simmetry** of the distribution



- $$\frac{\sigma^2_{\mathrm{IND}}}{\sigma^2_{\mathrm{ME}}} = N^{-\alpha}$$ with $\alpha < 1$

The bipartitions interfere each other, enlarging the variance of the distribution of

$$\pi_{ME}$$

- **Random case**

$$\sigma^2_{ME} \sim N^{-2.415}$$

$$\sigma^2_{IND} \sim N^{-3}$$

$$\alpha^{asym} = 0.585$$

P Facchi et al. “Classical statistical mechanics approach to multipartite entanglement”. In: Journal of Physics A: Mathematical and Theoretical 43.22 (May 2010), p. 225303. issn: 1751-8121. doi: 10.1088/1751-8113/43/22/ 225303. url: http://dx.doi.org/10.1088/1751-8113/43/22/225303.

Procedure for the evaluation of α

$$\sigma_{\mathrm{ME}}^2 = \frac{\sigma_1^2}{\mathbb{P}^\gamma}$$

$$\sigma_{\mathrm{IND}}^2 = \frac{\sigma_1^2}{\mathbb{P}}$$

$$\frac{\sigma_{\mathrm{IND}}^2}{\sigma_{\mathrm{ME}}^2} = \mathbb{P}^{\gamma-1}$$

$$\Rightarrow$$

$$\ln \frac{\sigma_{\mathrm{IND}}^2}{\sigma_{\mathrm{ME}}^2} = (\gamma - 1) \ln \mathbb{P}$$

at 12 qubits: $\ln \mathbb{P} = \beta \ln N$ with $\beta = 0.937$

$$\ln \frac{\sigma_{\mathrm{IND}}^2}{\sigma_{\mathrm{ME}}^2} \sim (\gamma - 1)\beta \ln N$$

$$\alpha = (1 - \gamma)\beta$$

NOT IN THE LIMIT OF LARGE n

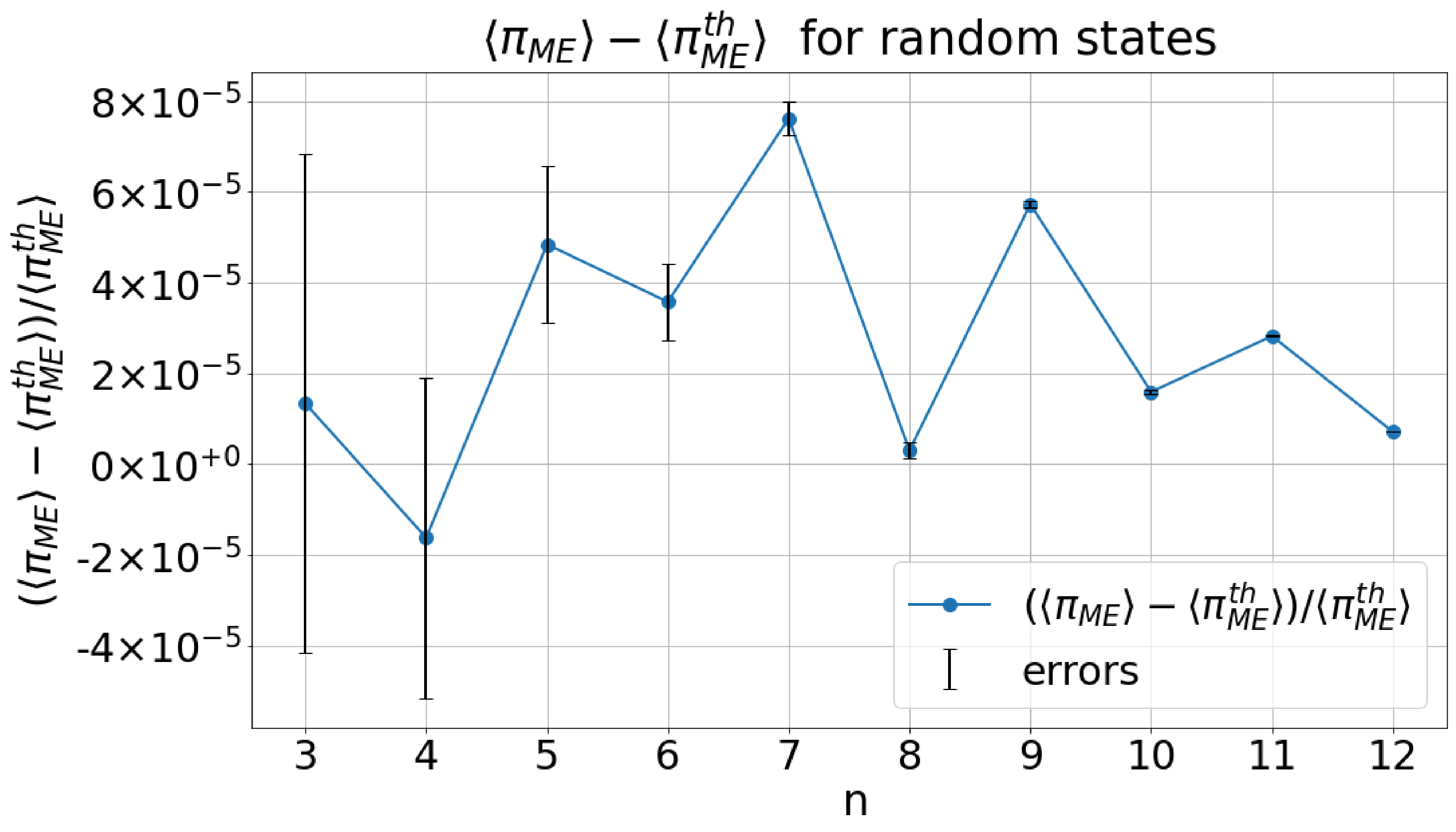
Analytical expectation of π_{ME} for random states

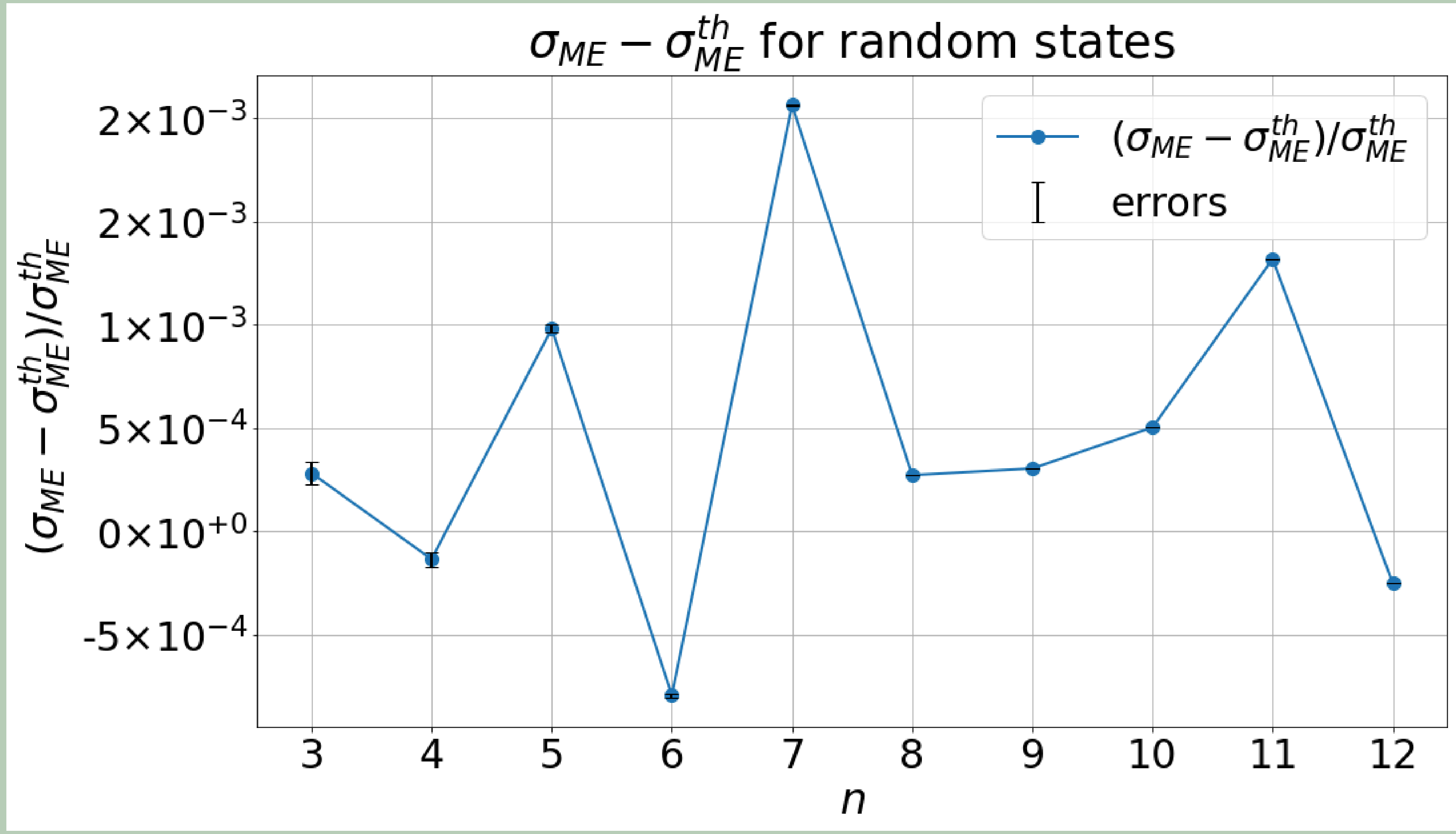
$$\sigma_{th}^2 = \frac{(N+1)f_2(N) - 2(N_A + N_{\overline{A}})^2}{(N+1)^2(N+2)(N+3)}$$

Qubits	$f_2(N)$
3	14
4	22
5	31.8
6	49
7	71.83
8	109.77
9	161.95
10	246.43
11	364.87
12	553.70

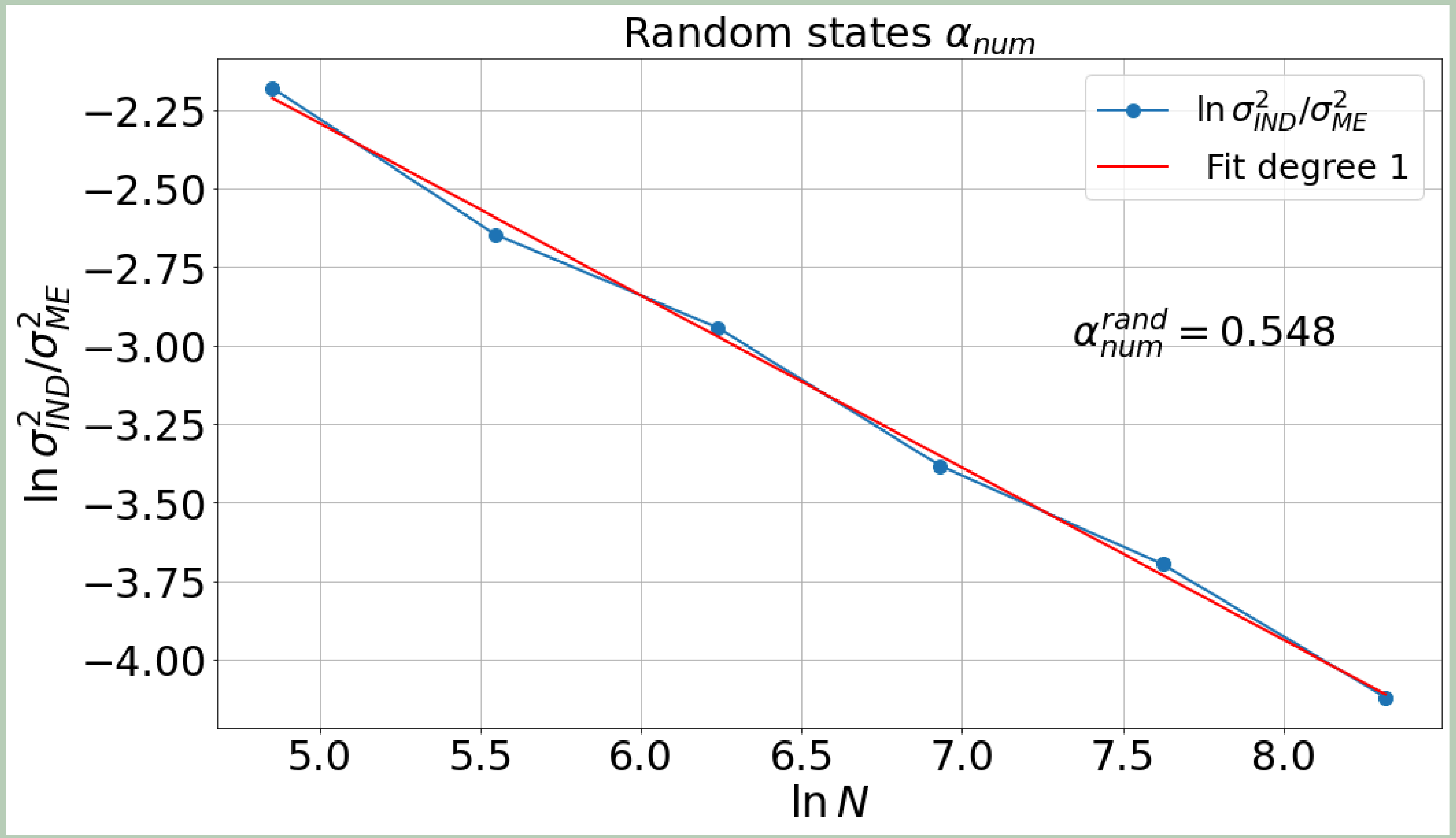
P Facchi et al. "Classical statistical mechanics approach to multipartite entanglement". In: Journal of Physics A: Mathematical and Theoretical 43.22 (May 2010), p. 225303. issn: 1751-8121. doi: 10.1088/1751-8113/43/22/ 225303. url: http://dx.doi.org/10.1088/1751-8113/43/22/225303.

Comparison numerial vs theoretical





Numerical



Theoretical