Modified (?) Lednicky-Lyuboshitz model for p-p femtoscopy

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ALICE

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Plan

- **Overview of the Lednicky's model**
- **Solution to the Schrodinger's equation with Coulomb potential**
- **Introducing a short-range potential (non-asymptotic and asymptotic regions)**
- **Proposing a model**
- **Results if consider the strong potential in s- and p-waves**
- **Comparisons and discussion**

WF according to Lednicky

(as I understand it) People usually refer to this when they use Lednicky model for p-p:

$$\psi(\vec{k}, \vec{r}) = e^{i\sigma_0}\left[\sqrt{A_c(\eta)}e^{i\vec{k}\vec{r}}\ {}_1F_1\Big(-i\eta, 1, i(kr - \vec{k}\vec{r})\Big) + f(k)\,\frac{G_0(\eta, kr) + i\,F_0(\eta, kr)}{r}\right]$$

$$\eta = \frac{1}{ka}, \quad a - Bohr\ radius \qquad A_c(\eta) = \frac{2\pi\eta}{e^{2\pi\eta} - 1} \quad - Gamov\ factor\ (Coulomb\ penetration\ factor)$$

$$\sigma_0 = arg[\Gamma(1 + i\eta)] \quad - Coulomb\ s - wave\ phase\ shift$$

$$f(k) \quad - scattering\ amplitude$$

$$F_0(kr, \eta) \quad - regular\ s - wave\ Coulomb\ function$$

$$G_0(kr, \eta) \quad - singular\ s - wave\ Coulomb\ function$$

This formalism (for example) was used in ALICE paper for p-p femtoscopy (Run1):

PHYSICAL REVIEW C 92, 054908 (2015)

Closer look at the Lednicky's WF

The WF: $$\psi(\vec{k},\vec{r}) = e^{i\sigma_0}\left[\sqrt{A_c(\eta)}e^{i\vec{k}\vec{r}}\,{}_1F_1\Big(-i\eta,\,1,\,i(kr-\vec{k}\vec{r})\Big) + f(k)\,\frac{G_0(\eta,kr)+i\,F_0(\eta,kr)}{r}\right]$$

is an asymptotical solution to the Schrodinger's equation and can be used only outside the strong potential range!!!

(Ledndicky also noted that (for example): https://arxiv.org/abs/nucl-th/0501065v3)

Behaviour at r->0 (k != 0):

$$\left.\begin{array}{l} e^{i\vec{k}\vec{r}} \underset{r\to 0}{\to} 1 \\ {}_1F_1\Big(-i\eta,\,1,\,i(kr-\vec{k}\vec{r})\Big) \underset{r\to 0}{\to} 0 \\ G_l(\eta,\rho) \underset{r\to 0}{\sim} r^{-l} \underset{l=0,\,r\to 0}{\to} 1 \\ F_l(\eta,\rho) \underset{r\to 0}{\sim} r^{l+1} \underset{l=0,\,r\to 0}{\to} 0 \end{array}\right\} \Longrightarrow \psi(r) \underset{r\to 0}{\sim} \frac{1}{r}$$

WF is singular at 0 !!!

Application to the femtoscopic CF

General expression (for p-p):

$$C_{pp}(k) = \frac{1}{2}\sum_{S=0}^{1}\frac{2S+1}{(2s_p+1)^2}\int d^3r \cdot S(\vec{r}) \cdot |\psi(-\vec{k},\vec{r}) + (-1)^S\psi(\vec{k},\vec{r})|^2$$

$$S(r) \sim exp(-\frac{r^2}{4R_{inv}^2})$$ — **assuming Gaussian source**

Integration over the coordinate space — [0, inf) (?)

But if so, we integrate the WF also in the region where it has singular behaviour!!!

Why don't we see this problem in the final CF ?

Application to the femtoscopic CF

Let's take a look at the CF again:

$$C_{pp}(k) = \frac{1}{2}\sum_{S=0}^{1}\frac{2S+1}{(2s_p+1)^2}\int d^3r \cdot S(\vec{r}) \cdot |\psi(-\vec{k},\vec{r}) + (-1)^S\psi(\vec{k},\vec{r})|^2$$

$$d^3r = r^2 \, sin\,\theta \; dr \; d\varphi \; d\theta$$

$$S(\vec{r}) \underset{r\to 0}{\sim} 1$$

$$\psi(r) \underset{r\to 0}{\sim} \frac{1}{r} \implies |\psi(r)|^2 \underset{r\to 0}{\sim} \frac{1}{r^2}$$

cancel each other

But if we don't see the problem it doesn't make it right!

Also if you would try to expand the WF to higher partial waves you'll get a singularity in CF.

How to deal with it?

The WF proposed by Lednicky is an asymptotical solution => can't be integrated over all the coordinate space

⇩

Use of a cutoff?

- **At which distance r?**
- **What to do in the region [0, cutoff]? -> how to sufficiently define the WF there?**

We would like to propose a solution.

Schrodinger's equation with Coulomb potential

Dividing angular and radial parts

Schrodinger's equation for a pair of charged particles in CMS:

$$\left[-\frac{\hbar^2}{2\mu}\nabla^2 + \frac{Z_1 Z_2 e^2}{r} - \frac{\hbar^2 k^2}{2\mu}\right]\psi = 0 \qquad \mu = \frac{m_1 m_2}{m_1 + m_2} \qquad \vec{r} = \vec{r}_1 - \vec{r}_2 \qquad \vec{k} = \frac{m_2 \vec{p}_1 - m_1 \vec{p}_2}{m_1 + m_2}$$

Since the potential is spherically symmetric, the solution can be expressed as a product of two separate solutions: to the angular part and to the radial. Let's write the expected partial wave expansion:

$$\psi = \sum_{l=0}^{\infty} (2l+1) i^l \frac{R_l(r)}{r} P_l(\cos\theta)$$

Where $R_l(r)$ is the solution to the radial Schrodinger's equation:

$$\frac{d^2 R_l(r)}{dr^2} + \left[k^2 - \frac{l(l+1)}{r^2} - \frac{2}{a_B r}\right] R_l(r) = 0 \qquad a_B = \frac{\hbar^2}{\mu Z_1 Z_2 e^2}$$

Albert Messiah, Quantum Mechanics, v1

Radial part solution

After doing some math one can obtain these two solutions:

$$R_l(\rho) = \tilde{C}_l \rho^{l+1} e^{i\rho} \begin{cases} {}_1F_1(l+1+i\eta,\, 2l+2,\, -2i\rho) \\ U(l+1+i\eta,\, 2l+2,\, -2i\rho) \end{cases} \qquad \eta = \frac{1}{ka_B} \qquad \rho = k \cdot r$$

${}_1F_1(\ldots)$ — confluent (Kummer's) hypergeometric function of the first kind — regular at 0

$U(\ldots)$ — confluent (Tricomi's) hypergeometric function of the second kind — singular at 0

- **we want our solution to be regular at 0 => choose the Kummer's function**
- **it also must behave as a standing wave at the infinity => define the C_l coeff.**

$$R_l^{reg}\Big|_{r\to\infty} \sim sin(\rho - \eta ln(2\rho) - l\frac{\pi}{2} + \sigma_l)$$

Albert Messiah, Quantum Mechanics, v1

Final solution for Coulomb

Our final regular solution in terms of partial waves:

$$\psi^{reg} = \sum_{l=0}^{\infty} (2l+1) i^l e^{i\sigma_l} \frac{F_l(\eta, \rho)}{\rho} P_l(\cos\theta)$$

$$F_l(\eta, \rho) = C_l \rho^{l+1} e^{i\rho} {}_1F_1(l+1+i\eta, 2l+2, -2i\rho)$$ — regular Coulomb WF; $$C_l = \frac{2^l e^{-\frac{\pi\eta}{2}} |\Gamma(l+1+i\eta)|}{(2l+1)!}$$

Let's introduce also irregular Coulomb WFs:

$$u_l^{(\pm)} = e^{\mp i\sigma_l}\left(G_l \pm i F_l\right)$$

$$F_l = \frac{1}{2i}\left(u_l^{(+)} e^{i\sigma_l} - u_l^{(-)} e^{-i\sigma_l}\right)$$

$$G_l = \frac{1}{2}\left(u_l^{(+)} e^{i\sigma_l} + u_l^{(-)} e^{-i\sigma_l}\right)$$

$$u_l^{(\pm)} \underset{r\to\infty}{\sim} e^{\pm i(\rho - \eta \ln 2\rho - \frac{l\pi}{2})}$$

$$G_l \underset{r\to\infty}{\sim} \sin\left(\rho - \eta \ln(2\rho) - l\frac{\pi}{2} + \sigma_l\right)$$

$$\sigma_l = \arg\Gamma(l+1+i\eta)$$

We can rewrite our regular solution:

$$\psi^{reg} = \frac{1}{2\rho} \sum_{l=0}^{\infty} (2l+1)\, i^{l+1}\left(u_l^{-}(\eta,\rho) - e^{2i\sigma_l} u_l^{+}(\eta,\rho)\right) P_l(\cos\theta)$$

Albert Messiah, Quantum Mechanics, v1

Introducing short-range potential (asymptotic solution)

Coulomb + short-range (asymptotical)

If an additional (to Coulomb) short-range potential "fades" not slower than r^{-2}, then outside its range (d, inf) we can implement it only by an additional phase shift:

$$\psi_{c+s} \underset{r>d}{=} \frac{1}{2\rho} \sum_{l=0}^{\infty} (2l+1)\, i^{l+1} \left(u_l^-(\eta,\rho) - e^{2i\sigma_l} e^{2i\delta_l} u_l^+(\eta,\rho) \right) P_l(cos\,\theta)$$

δ_l — phase shift corresponding to the short-range potential

We can rewrite:

$$\psi_{c+s} \underset{r>d}{=} \frac{1}{r} \sum_{l=0}^{\infty} (2l+1)\, i^l e^{i\sigma_l} \left(\frac{F_l(\eta,\rho)}{k} + f_l(k) \left(G_l(\eta,\rho) + i\, F_l(\eta,\rho) \right) \right) P_l(cos\,\theta)$$

If we consider a short-range potential only in s-wave:

$$\psi_{c+s}^{l=0} \underset{r>d}{=} \sqrt{A_c(\eta)}\; e^{i\sigma_0} e^{i\vec{k}\vec{r}}\; {}_1F_1\left(-i\eta,\, 1,\, i(kr - \vec{k}\vec{r}\,) \right) + e^{i\sigma_0} f_0(k) \frac{G_0(\eta,\rho) + i\, F_0(\eta,\rho)}{r}$$

We used:

$$\frac{1}{kr} \sum_{l=0}^{\infty} (2l+1)\, i^l\, e^{i\sigma_l} F_l(\eta,\rho)\, P_l(cos\,\theta) = \sqrt{A_c(\eta)}\; e^{i\sigma_0} e^{i\vec{k}\vec{r}}\; {}_1F_1\left(-i\eta,\, 1,\, i(kr - \vec{k}\vec{r}\,) \right)$$

$$f_l(k) = \frac{e^{2i\delta_l} - 1}{2ik}$$

Albert Messiah, Quantum Mechanics, v1

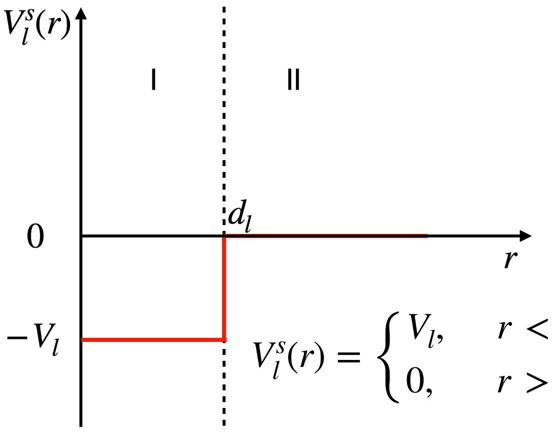
Square-well as a short-range potential

Schrodinger's equation (Coulomb+sq.-well)

Let's introduce a short-range potential as a square-well for each value of l.

Radial Schrodinger's equation in (I) sector:

$$\frac{\mathrm{d}^2 R_l^{(I)}}{\mathrm{d}r^2} + \left[k^2 - \frac{l(l+1)}{r^2} - \frac{2}{a_B r} - \frac{2\mu}{\hbar^2} V_l\right] R_l^{(I)} = 0$$



$$V_l^s(r) = \begin{cases} V_l, & r < d_l \quad (I) \\ 0, & r > d_l \quad (II) \end{cases}$$

Substituting the sq.-well potential:

$$\frac{\mathrm{d}^2 R_l^{(I)}}{\mathrm{d}r^2} + \left[\tilde{k}^2 - \frac{l(l+1)}{r^2} - \frac{2}{a_B r}\right] R_l^{(I)} = 0$$

$$\tilde{k}_l = \sqrt{k^2 - \frac{2\mu}{\hbar^2} V_l}$$

But we already know the solution!

$$R_l^{(I)} = \frac{e^{i\tilde{\sigma}_l} F_l(\tilde{\eta}_l, \tilde{\rho}_l)}{\tilde{k}_l}$$

$$\tilde{\rho}_l = \tilde{k}_l \cdot r$$

$$\tilde{\eta}_l = \frac{1}{\tilde{k}_l a_B}$$

Matching conditions

We also already know the solution for the sector (II). It's the asymptotical one (slide 13). Thus, our radial solutions for both sectors and the total:

$$R_l^{(I)} = \frac{1}{\tilde{k}_l}\, e^{i\tilde{\sigma}_l} F_l(\tilde{\eta}_l, \tilde{\rho}_l)$$

$$R_l^{(II)} = e^{i\sigma_l}\left(\frac{F_l(\eta,\rho)}{k} + f_l(k)\,\left(G_l(\eta,\rho) + i\,F_l(\eta,\rho)\right)\right)$$

$$\Longrightarrow\quad R_l^{tot} = \begin{cases} A_l\, R_l^{(I)}, & r < d_l \\ R_l^{(II)}, & r \geq d_l \end{cases}$$

From the matching conditions

$$A_l\, R_l^{(I)}\Big|_{r=d} = R_l^{(II)}\Big|_{r=d}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(ln\,(A_l\, R_l^{(I)})\right)\Big|_{r=d} = \frac{\mathrm{d}}{\mathrm{d}r}\left(ln\,(R_l^{(II)})\right)\Big|_{r=d}$$

One can find

$$A_l = \frac{\tilde{k}_l}{F_l(\tilde{\eta}_l, \tilde{k}_l d)}\, e^{i(\sigma_l - \tilde{\sigma}_l)}\left(\frac{F_l(\eta, kd)}{k} + f_l(k)\,\left(G_l(\eta, kd) + i\,F_l(\eta, kd)\right)\right)$$

$$ctg\,\delta_l = \frac{G_l(\eta, kd)}{F_l(\eta, kd)}\,\frac{\tilde{k}\, f_l(\tilde{\eta}, \tilde{k}d) - k\, g_l(\eta, kd)}{k\, f_l(\eta, kd) - \tilde{k}\, f_l(\tilde{\eta}, \tilde{k}d)}$$

where

$$f_l(\eta,\rho) = \frac{\mathrm{d}}{\mathrm{d}r}\left(ln\,(F_l(\eta,\rho))\right)\Big|_{r=d}$$

$$g_l(\eta,\rho) = \frac{\mathrm{d}}{\mathrm{d}r}\left(ln\,(G_l(\eta,\rho))\right)\Big|_{r=d}$$

Final solution

Knowing the total radial solution for each partial wave, we can write our final WF:

$$\psi(k,\, r) = \frac{1}{r} \sum_{l=0}^{\infty} (2l+1)\; i^l R_l(k,\, r)\; P_l(cos\; \theta)$$

$$R_l(k,\, r) = \begin{cases} \frac{F_l(\tilde{\eta}_l,\, \tilde{k}_l r)}{F_l(\tilde{\eta}_l,\, \tilde{k}_l d)}\; e^{i\sigma_l} \left(\frac{F_l(\eta, kd)}{k} + f_l(k)\, \left(G_l(\eta, kd) + i\; F_l(\eta, kd) \right) \right), & r < d \\ e^{i\sigma_l} \left(\frac{F_l(\eta, \rho)}{k} + f_l(k)\, \left(G_l(\eta, \rho) + i\; F_l(\eta, \rho) \right) \right), & r \geq d \end{cases}$$

Examples for s- and p-waves

s- and p-waves

If we consider presence of the short-range potential only in the s-wave, our WF:

$$\psi(k,r) = e^{i\sigma_0} \begin{cases} \dfrac{F_0(\tilde{\eta}_0, \tilde{k}_0 r)}{F_0(\tilde{\eta}_0, \tilde{k}_0 d)} \left(\dfrac{F_0(\eta, kd)}{kr} + f_0(k) \dfrac{G_0(\eta, kd) + i\, F_0(\eta, kd)}{r} \right) + & \\ \quad + \sqrt{A_c(\eta)}\; e^{i\vec{k}\vec{r}}\; {}_1F_1\left(-i\eta, 1, i(kr - \vec{k}\vec{r})\right) - \dfrac{F_0(\eta,\rho)}{kr} & r < d \\ f_0(k) \dfrac{G_0(\eta,\rho) + i\, F_0(\eta,\rho)}{r} + \sqrt{A_c(\eta)}\; e^{i\vec{k}\vec{r}}\; {}_1F_1\left(-i\eta, 1, i(kr - \vec{k}\vec{r})\right) & r \geq d \end{cases}$$

The same as in **Gmitro, M., Kvasil, J., Lednický, R. *et al.*, *Czech J Phys* 36, 1281–1287 (1986) !**

Final WF in case of s- and p-waves:

$$\psi(k,r) = \begin{cases} \sqrt{A_c(\eta)}\; e^{i\sigma_0} e^{i\vec{k}\vec{r}}\; {}_1F_1\left(-i\eta, 1, i(kr - \vec{k}\vec{r})\right) + & \\ + \displaystyle\sum_{l=0}^{1} (2l+1)\, i^l\, e^{i\sigma_l} \left[\dfrac{F_l(\tilde{\eta}_l, \tilde{k}_l r)}{F_l(\tilde{\eta}_l, \tilde{k}_l d)} \left(\dfrac{F_l(\eta, kd)}{kr} + f_l(k) \dfrac{G_l(\eta, kd) + i\, F_l(\eta, kd)}{r} \right) - \dfrac{F_l(\eta,\rho)}{kr} \right] P_l(cos\theta) & r < d \\ \sqrt{A_c(\eta)}\; e^{i\sigma_0} e^{i\vec{k}\vec{r}}\; {}_1F_1\left(-i\eta, 1, i(kr - \vec{k}\vec{r})\right) + \sum_{l=0}^{1} (2l+1)\, i^l\, e^{i\sigma_l}\, f_l(k) \dfrac{G_l(\eta,\rho) + i\, F_l(\eta,\rho)}{r} P_l(cos\theta) & r \geq d \end{cases}$$

Try on the theoretical femtoscopic CF

Can we use a square-well?

Square-well is indeed a very rough approximation. But is it so bad?

It was shown in phase shifts analyses that for small energies we are not sensitive to the shape of the strong potential (generally for a short-range one). Which means that we cannot extract any particular information about the shape from experimental data (at least from phase shifts).
(for example: H. A. Bethe, Phys. Rev. 76, 38 – Published 1 July 1949)

But we see femtoscopic peak in small energy (momentum) region!

So, could be an option. Let’s see.

Defining the potential parameters

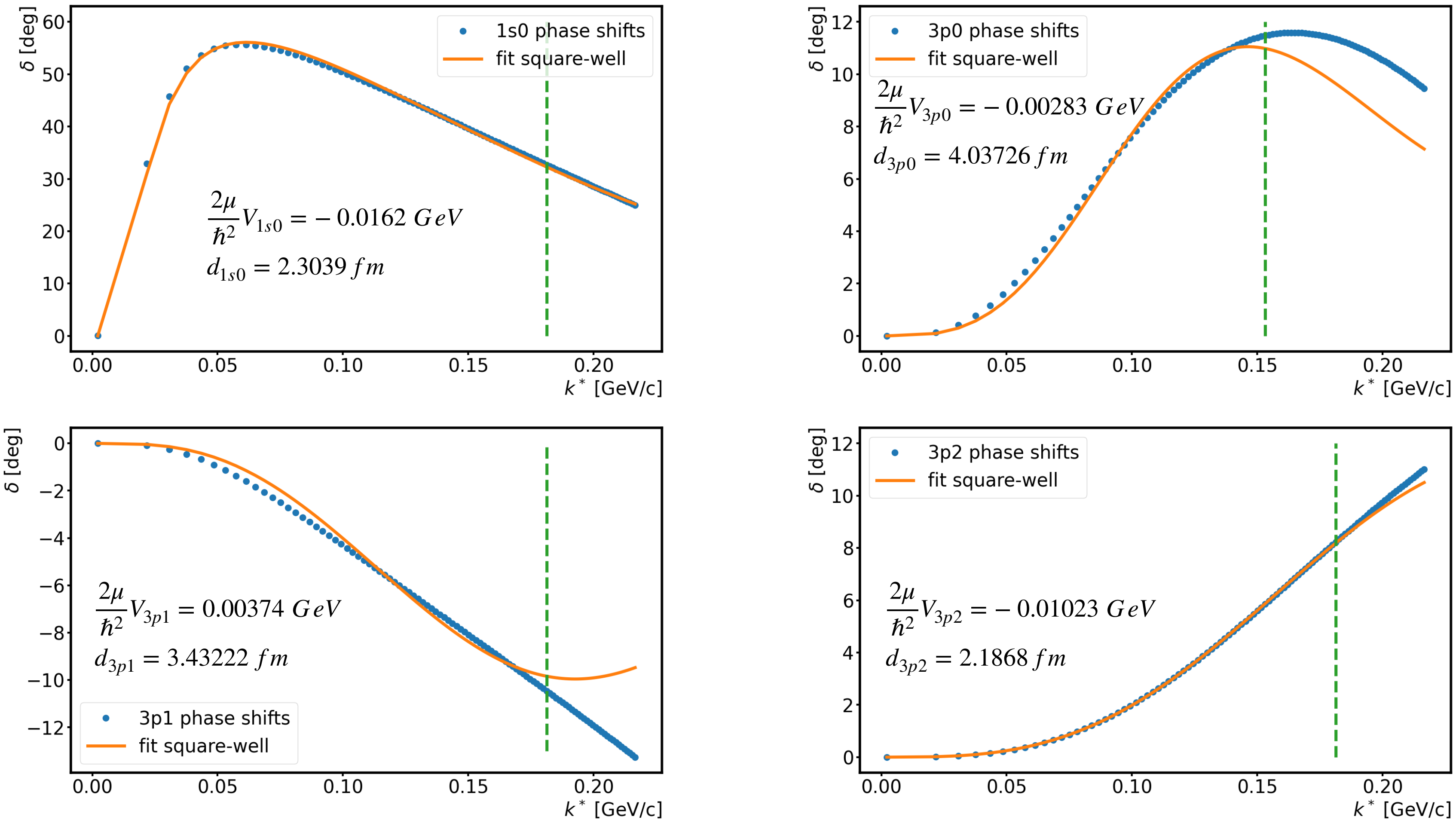
OK, we have an analytical WF for a square-well potential, but what are its parameters (depth and width)?

One can obtain them by fitting momenta(energy)-dependent phase shifts with our matching condition:

$$ctg\,\delta_l = \frac{G_l(\eta, kd)}{F_l(\eta, kd)}\,\frac{\tilde{k}\,f_l(\tilde{\eta}, \tilde{k}d) - k\,g_l(\eta, kd)}{k\,f_l(\eta, kd) - \tilde{k}\,f_l(\tilde{\eta}, \tilde{k}d)}$$

Phase shifts fit: square-well

Phase shifts data are taken from: https://nn-online.org/



Green line — chosen range of the fit

Theoretical correlation function

Theoretical expression:

$$C_{pp}(k^*) = \frac{1}{2}\sum_{S=0}^{1}\frac{2S+1}{(2s_p+1)^2}\sum_{L,\,J}\omega_{LJ}\int d^3\vec{r}S(\vec{r})\,|\,\psi^S_{-\vec{k}}(\vec{r}) + (-1)^S\psi^S_{\vec{k}}(\vec{r})\,|^2$$

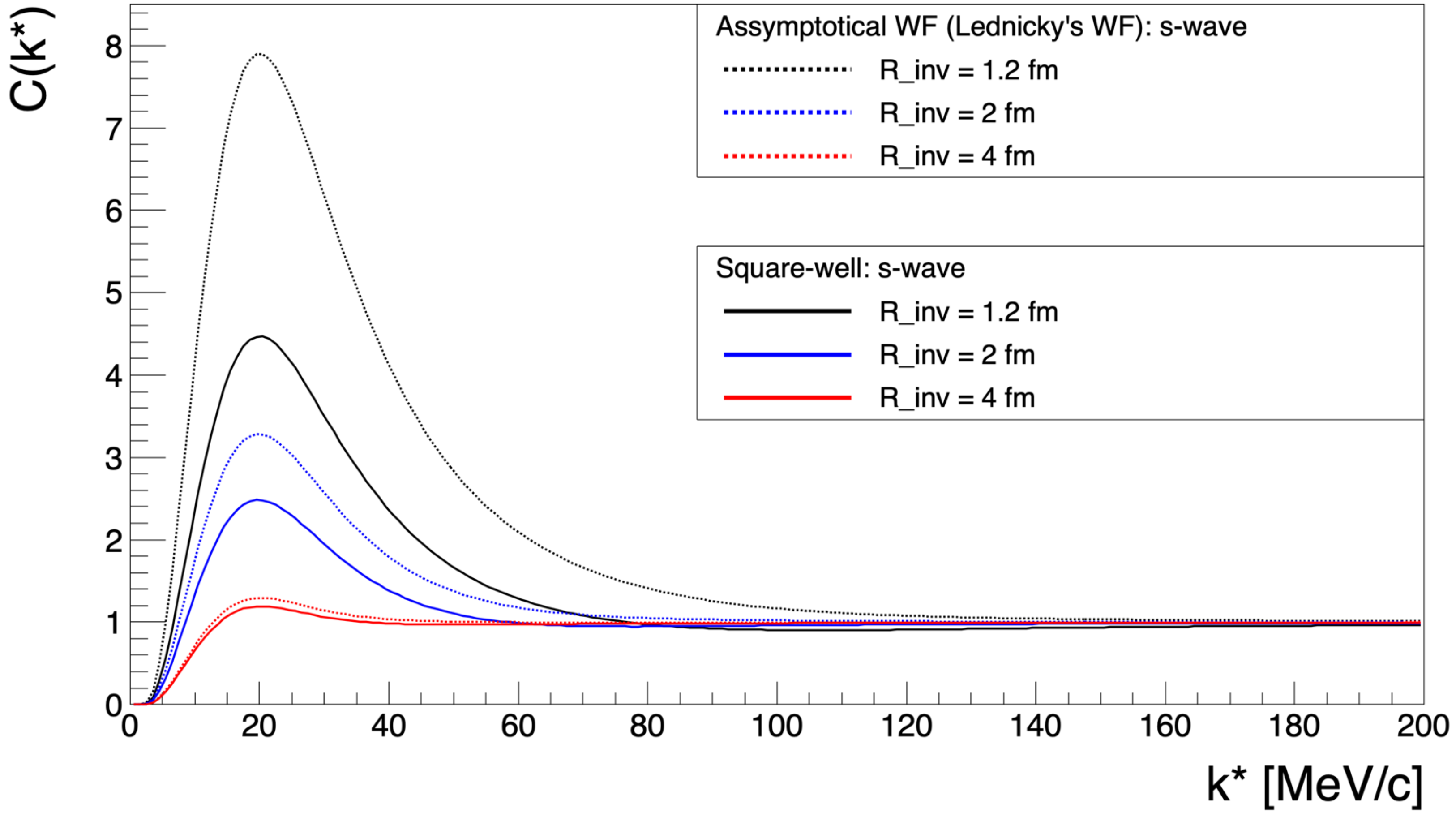
$$\omega_{LJ} = \frac{2J+1}{(2L+1)(2S+1)}$$

We will work with L=[0, 1]. Corresponding states: ${}^{2S+1}L_J: \quad {}^1s_0,\ {}^3p_0,\ {}^3p_1,\ {}^3p_2$

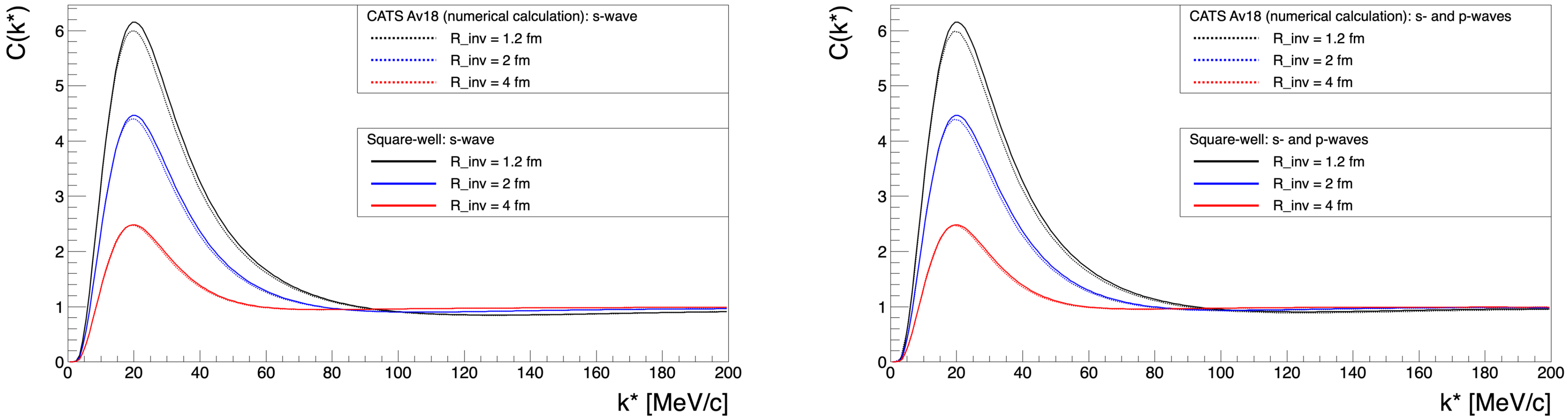
$$S(r) = \frac{1}{8\pi^{\frac{3}{2}}R^3_{inv}}exp(-\frac{r^2}{4R^2_{inv}})$$

— assuming Gaussian source

Result: with or without square-well potential



Result: comparison with CATS



CATS is a framework that provides a numerical solution to the Schrodinger's equation for a given strong potential model (e.g. Argonne v18).

Mihaylov, D.L., Mantovani Sarti, V., Arnold, O.W. *et al.*, *Eur. Phys. J. C* 78, 394 (2018).

GitHub repo: https://github.com/dimihayl/DLM

Summary

- **Lednicky's WF — cannot be used for all distances r**
- **Our model takes care of the WF at small distances**
- **Noticeable difference between CF with our WF and Lednicky's (especially for small source sizes)**
- **Are we sensitive to the shape of a short range potential in femtoscopy?**
- **The model has been used by us for the past year to extract proton source size in pp 900 GeV and Pb—Pb 5.36 TeV collisions at ALICE**
- **We will try to plug in the Levy SF and fit the proton CF in ALICE pp collisions**

Backup slides

Effective range parameters (Bethe's formula)

By fitting the momenta-dependent phase shifts:

$$f(k) = \frac{1}{k\ ctg(\delta_l) - ik}$$

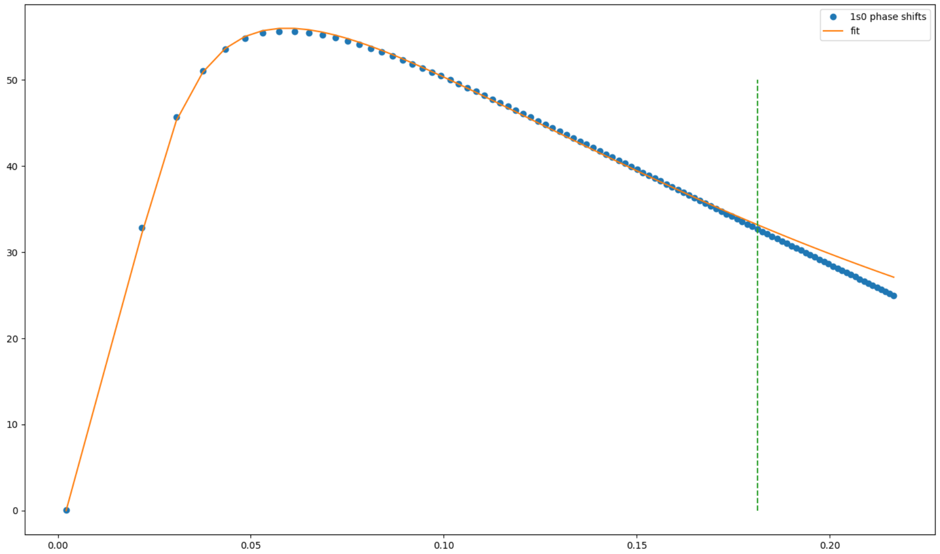
$$k\ ctg(\delta_l) \sim -\frac{1}{f_0} + \frac{1}{2} d_0 k^2 + (-P d_0^3 k^4) + O(k^6)$$

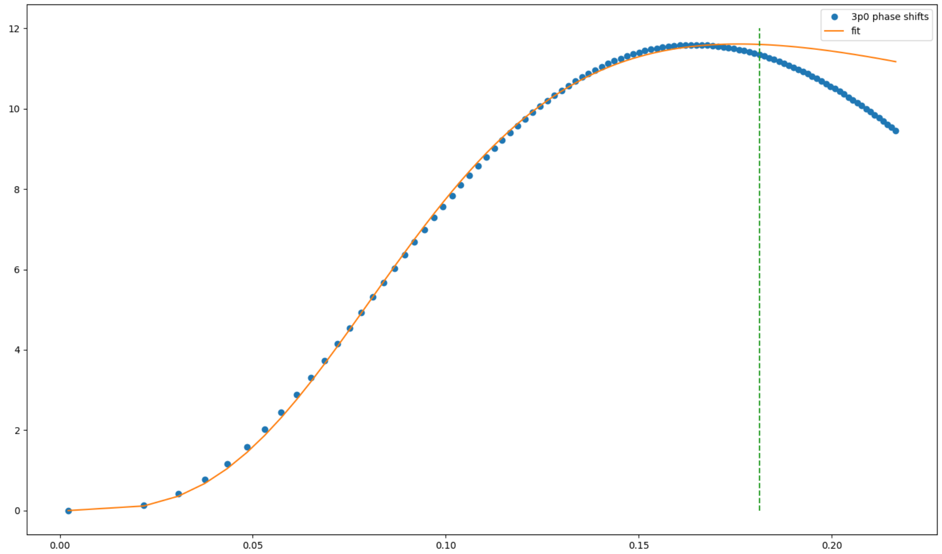
H. A. Bethe, Phys. Rev. 76, 38 – Published 1 July 1949

Extract effective range parameters.

I used coefficient P != 0 only for 1s0 and 3p0 states.

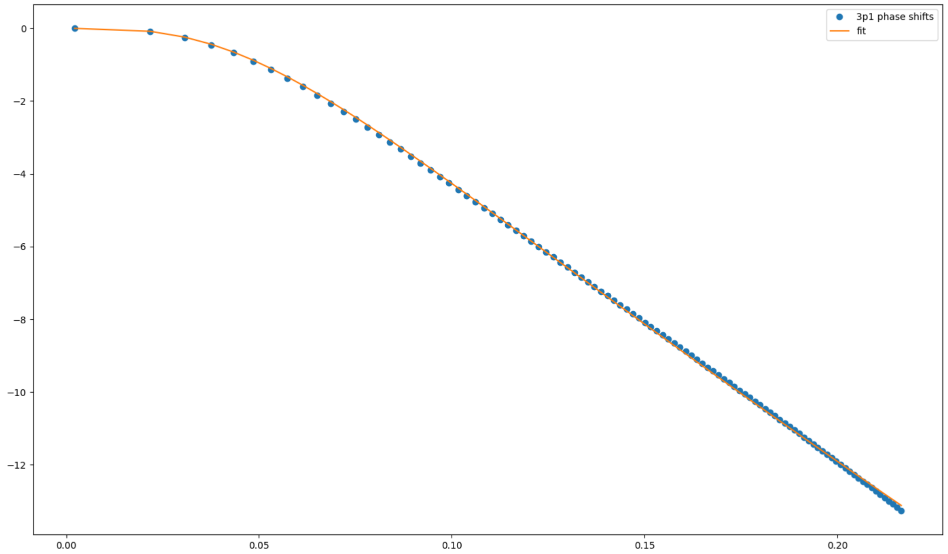
Phase shifts fit: eff. range parameters

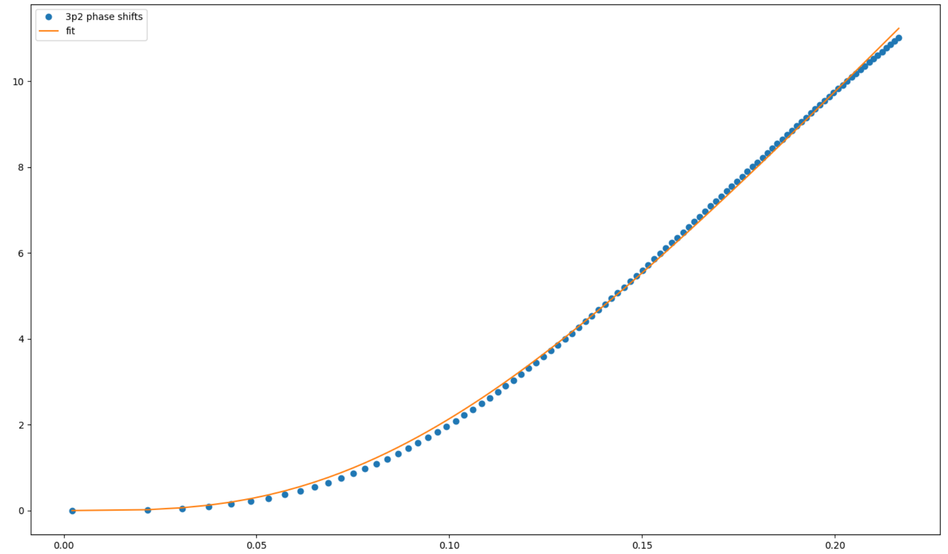




Phase shifts data are taken from: https://nn-online.org/

Phase shifts fit: eff. range parameters





Phase shifts data are taken from: https://nn-online.org/