Particles acceleration at astrophysical shocks

Exercises

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Exercise n.1

Particle spectrum from the distribution function

- We want to derive the spectrum of particle accelerated at a plane parallel shock using the distribution function in momentum

Definition $$f(t, \vec{x}, \vec{p}) \equiv \frac{dN}{dV\, d^3p}$$

- Derivation of the transport equation

We can use the Liuville's theorem because we can neglect collisions (otherwise we should use the Boltzman equation)

$$\begin{cases} \dfrac{Df}{Dt} = \dfrac{\partial f}{\partial t} + \vec{u} \cdot \nabla_x f + \dot{\vec{p}}\, \nabla_p f = 0 & \rightarrow \text{Vlasov equation} \\ \dot{\vec{p}} = q\, \dfrac{\vec{u}}{c} \times \vec{B} & \text{Lorentz force} \end{cases}$$

$$\Rightarrow \boxed{\frac{\partial f}{\partial t} + \vec{u}\, \dot{\nabla}_x f = \nabla \left[D_{xx} \nabla f \right] + \frac{p}{3} \nabla \cdot \vec{u}\, \frac{\partial f}{\partial p} + Q(x, p)}$$

See e.g. Vietri's book sec. 4.3

D_{xx} is the spatial diffusion coefficient

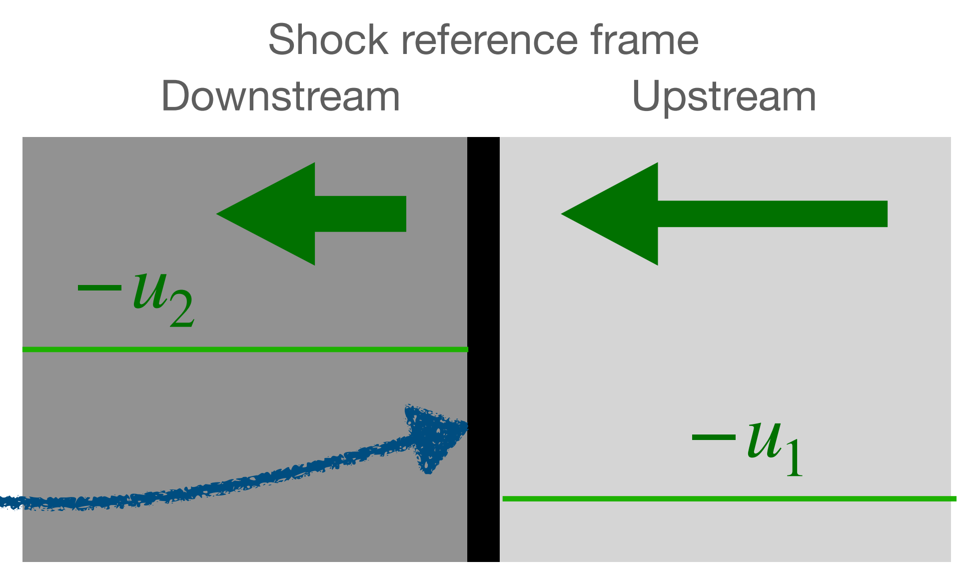
Exercise n.1

Particle spectrum from the distribution function

- Step 1: plane parallel (1D) stationary system: $\cancel{\frac{\partial}{\partial t}}$, $\nabla_x \to \partial_x$

$$u \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[D \frac{\partial f}{\partial x}\right] + \frac{p}{3}\frac{du}{dx}\frac{\partial f}{\partial p} + Q(x,p)$$

- Note: assume D spatially constant upstream and downstream



- Velocity profile:

$$u(x) = -u_2 + (u_2 - u_1)\theta(x)$$

- Injection occurs only at the shock discontinuity:

$$\begin{cases} Q(x,p) = q_0\, \delta(p - p_0)\, \delta(x) \\ q_0 = \dfrac{\eta_{\rm inj} n_1 u_i}{4\pi p_0^2} \end{cases}$$

- Questions: derive the distribution function in the whole region (upstream, downstream and at the shock)
  - use the boundary condition: $f(x = +\infty) = 0$
  - assume that a fraction $\xi_{\rm cr} \approx 0.1$ of the shock bulk pressure is converted into CR energy

Exercise n.1

Particle spectrum from the distribution function

- Hint 1: start solving the equation upstream/downstream using the boundary condition $f(x=0,p)=f_0(p)$

$$\frac{\partial}{\partial x}\left[D\frac{\partial f}{\partial x}\right] - u\frac{\partial f}{\partial x} = 0$$

- Hint 2: solve equation for $f_0(p)$ by integrating across the shock discontinuity

$$\int_{0^-}^{0^+}\left\{\frac{\partial}{\partial x}\left[D\frac{\partial f}{\partial x}\right] - u\frac{\partial f}{\partial x} + \frac{p}{3}\frac{du}{dx}\frac{\partial f}{\partial p} + Q(x,p)\right\} = 0$$

- Hint 3: the normalisation is obtained from $P_{\rm CR} = \xi_{\rm CR}\rho_1 u_1^2$

$$P_{\rm CR} = \frac{1}{3}\int 4\pi\, p^2 dp\, f_0(p) pc \;\Rightarrow\; f_0(p) = \frac{3\xi_{\rm CR}\rho_1 u_1^2}{4\pi c(m_p c)^4\Lambda}\left(\frac{p}{m_p c}\right)^{-s}$$

$$\Lambda' = \int_1^{p_{\rm max}/p_0} y^{3-s} dy$$

Exercise n.1

Particle spectrum from the distribution function

- Solution:

$$f_0(p) = \frac{3\,\xi_{\rm cr}\,\rho_0\,u_{\rm sh}^2}{4\pi\,(mc)^4 c\,\Lambda}\left(\frac{p}{mc}\right)^{-s}$$

$$s = \frac{3u_1}{u_1 - u_2}$$

$$f_1(x,p) = f_0(p)\,e^{-u_1 x/D_1}$$

$$f_2(x,p) = f_0(p)$$

Exercise n.2

CR spectrum escaping from a SNR

Determine the CR spectrum escaping from a SNR shock during the Sedov-Taylor phase, assuming that, during this phase, the shock converts a constant fraction of energy in escaping CR and the maximum energy decreases as a power low in time.

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CR spectrum escaping from a SNR

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Solution

- Energy flux escaping the SNR shock: $$4\pi p^2 dp\, Q_{\rm esc}(p)\, pc = \xi_{\rm esc}(t)\frac{1}{2}\rho_0 v_{\rm sh}^3\, 4\pi R_{\rm sh}^2\, dt$$
- Maximum energy in time: $p_{\max}(t) = p_M \left(t/t_{\rm ST}\right)^{-\delta}$ $\Rightarrow \dfrac{dp_{\max}}{dt} = -\,\delta\,\dfrac{p_{\max}}{t}$
- Evolution during the Sedov-Taylor phase $$\begin{cases} R_{\rm sh} = R_{\rm ST}\left(\dfrac{t}{t_{\rm ST}}\right)^{2/5} \\ v_{\rm sh} = \dfrac{dR_{\rm sh}}{dt} = \dfrac{2}{5} v_{\rm ST}\left(\dfrac{t}{t_{\rm ST}}\right)^{-3/5} \end{cases} \Rightarrow v_{\rm sh}^3 R_{\rm sh}^2 \propto t^{-1}$$
- Total released spectrum: $$Q_{\rm esc}(p) = \delta\left(\frac{2}{5}\right)^3 \frac{\rho_0}{c} R_{\rm ST}^3\, v_{\rm ST}^2\, t\, \xi_{\rm esc}(t)\, p^{-4} \propto \xi_{\rm esc}(t)\, p^{-4}$$

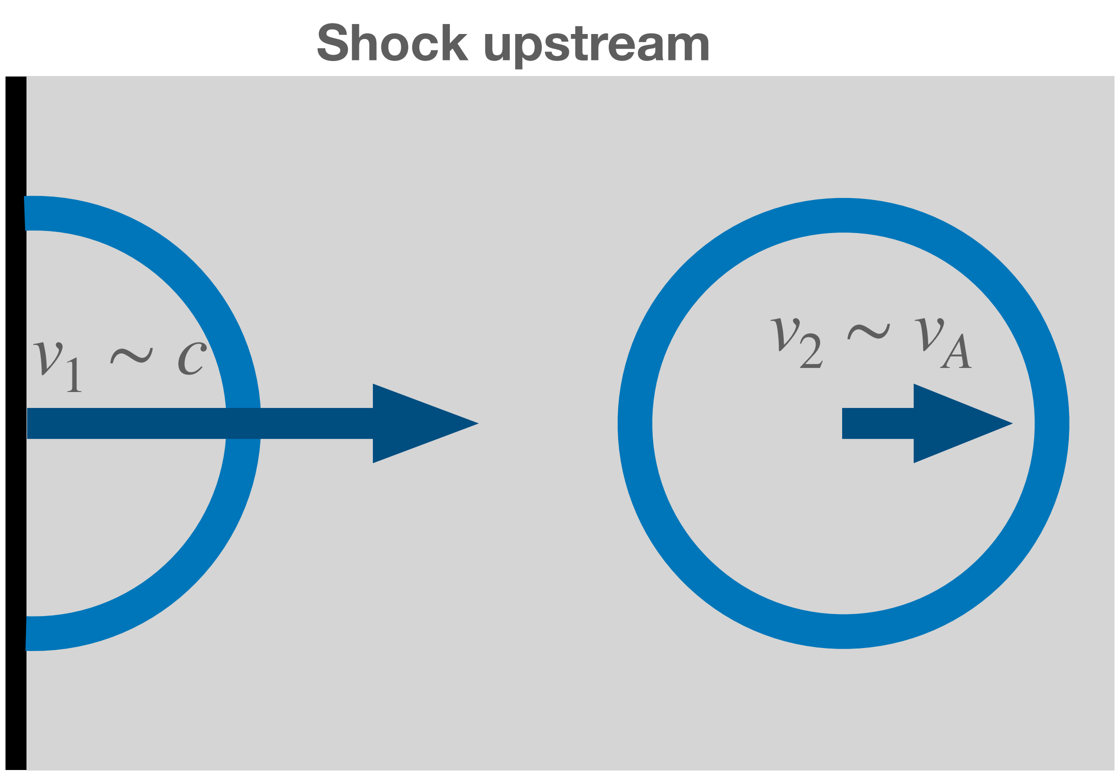
NOT RELATED TO FERMI ACCELERATION!!!

Additional question: how we can get a spectrum steeper that p^{-4}?

Exercise n.3

Magnetic field amplification

- Estimate the amplification of magnetic field due to the streaming of CR particles upstream of a plane parallel shock assuming that $\xi_{\rm cr}$ is the fraction of shock kinetic energy converted into CR energy
- Hint: use the idea that particles upstream are fully isotropized after travelling a diffusion length



Alfvén speed: $v_A = \dfrac{B_0}{\sqrt{4\pi\rho}}$

Momentum of Alfvén waves $p_m \simeq \dfrac{1}{v_A}\dfrac{\delta B^2}{4\pi}$

Exercise n.3 - solution

Magnetic field amplification

- Assume that the momentum lost by particles goes into magnetic field

Initial momentum $p_1 \simeq n_{\rm cr}\gamma m v_1$; final momentum $p_2 \simeq n_{\rm cr}\gamma m v_2$

Momentum of Alfvén waves $p_m \simeq \frac{1}{v_A}\frac{\delta B^2}{4\pi}$

$$\frac{\delta B^2}{4\pi v_A} = p_1 - p_2 = n_{\rm cr} m\gamma(c/3 - v_A) \Rightarrow \delta B^2 \simeq 4\pi\, n_{\rm cr} m\gamma\, v_A\, c/3$$

$$\Rightarrow \frac{\delta B^2}{B_0^2} \simeq \frac{4\pi n_{\rm cr} m\gamma c v_A}{B_0^2} = \frac{4\pi n_{\rm cr} m\gamma c/3\, v_A}{v_A^2 4\pi\rho} = \frac{n_{\rm cr}}{n_{\rm ism}}\frac{c/3}{v_A}$$

Using the distribution function from Ex. 1 $\quad n_{\rm cr} = \int f_0(p) d^3p \;\rightarrow\; \frac{n_{\rm cr}}{n_{\rm ism}} = 3\xi_{\rm cr}\left(\frac{v_{\rm sh}}{c}\right)^2 \frac{\Lambda'}{\Lambda}$

$$\Rightarrow \frac{\delta B}{B_0} \simeq \left(\xi_{\rm cr}\frac{\Lambda'}{\Lambda}\frac{v_{\rm sh}}{c}\frac{v_{\rm sh}}{v_A}\right)^{1/2} \sim 1$$

$$\Lambda' = \int_1^{p_{\max}/p_0} y^{2-s} dy$$