Lectures :

1) Particle acceleration up to ultra high energies

2) GRBs (an example of relativistic shocks)

3) Cosmic ray propagation in extragalactic radiation fields

4) Cosmic ray propagation in extragalactic magnetic fields

Andrew Taylor

DESY.

Lecture 1 Plan:

1) An intro to the world of non-thermal particles (ie. “Cosmic Rays”)

2) Shocks- what are they? What do they do to the gas passing through them?

3) Cosmic Ray Acceleration at Shocks

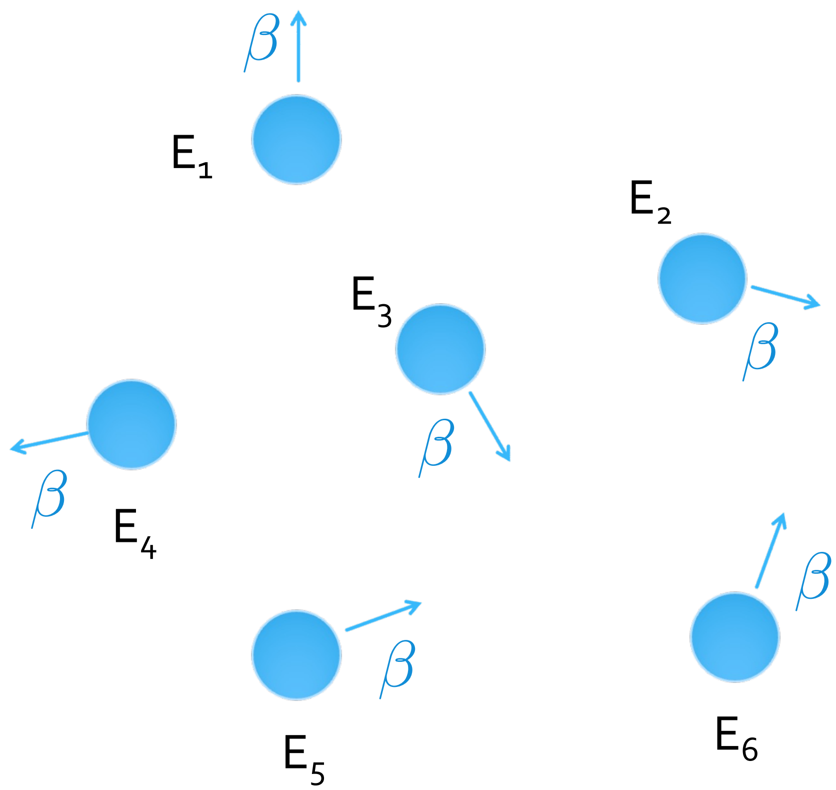
The World of Non-Thermal Particles

Andrew Taylor

DESY.

Thermal Particles

Thought experiment- imagine an ensemble of particles all with the same energy bouncing around in a box.....

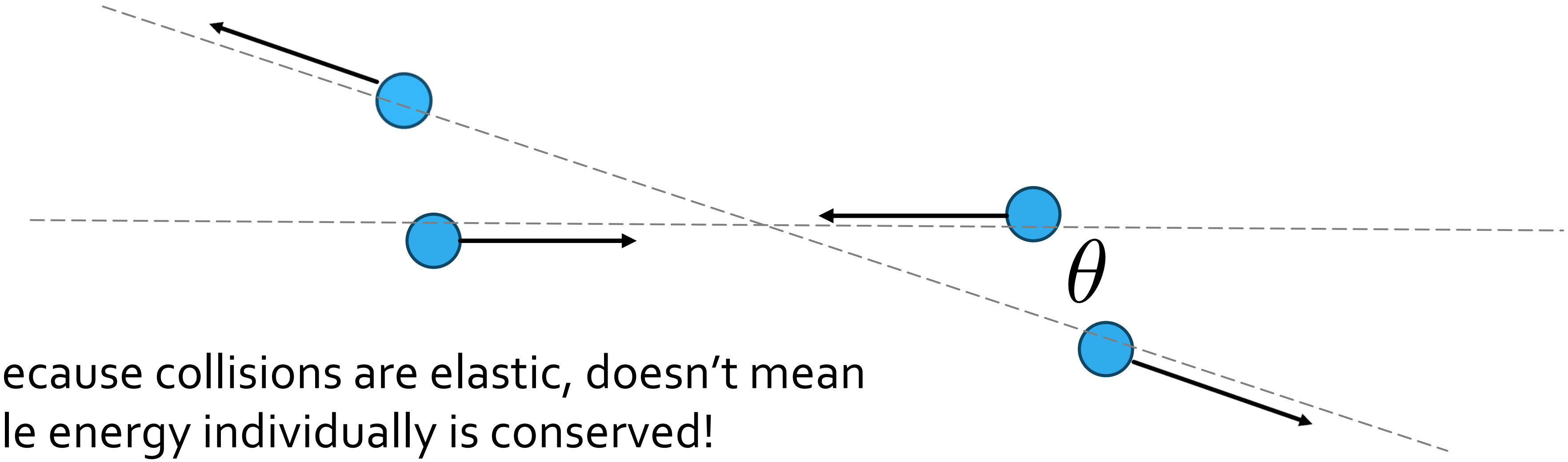


Andrew Taylor

DESY.

Elastic Collisions

In the com frame:



Just because collisions are elastic, doesn't mean particle energy individually is conserved!

$$\mathbf{p}_1' = \frac{1}{2}\left[\begin{pmatrix} cos\theta & \sin\theta \\ -sin\theta & \cos\theta \end{pmatrix}(\mathbf{p_1} - \mathbf{p_2}) + (\mathbf{p_1} + \mathbf{p_2})\right]$$

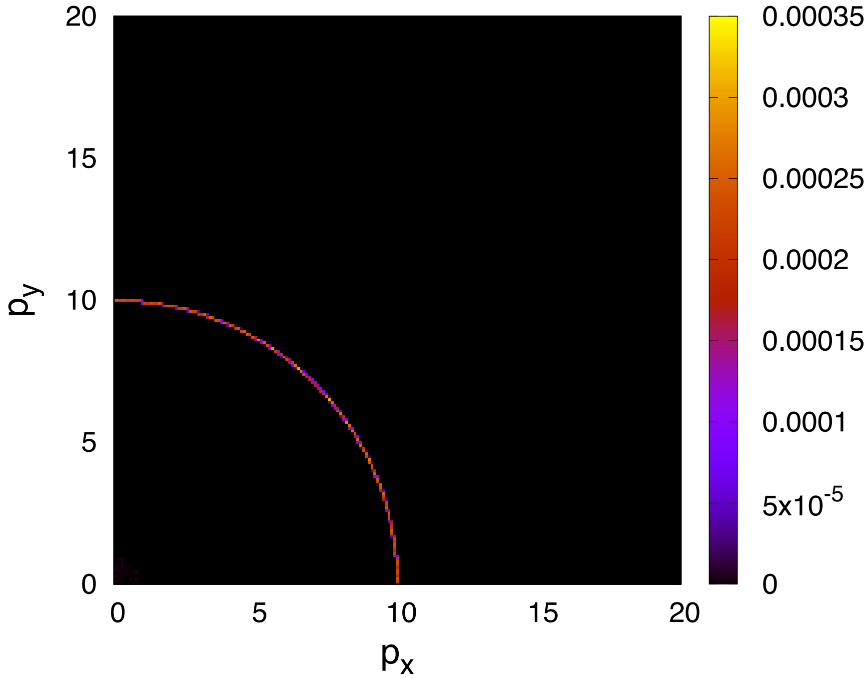
$$\mathbf{p}_2' = \frac{1}{2}\left[\begin{pmatrix} cos\theta & \sin\theta \\ -sin\theta & \cos\theta \end{pmatrix}(\mathbf{p_2} - \mathbf{p_1}) + (\mathbf{p_1} + \mathbf{p_2})\right]$$

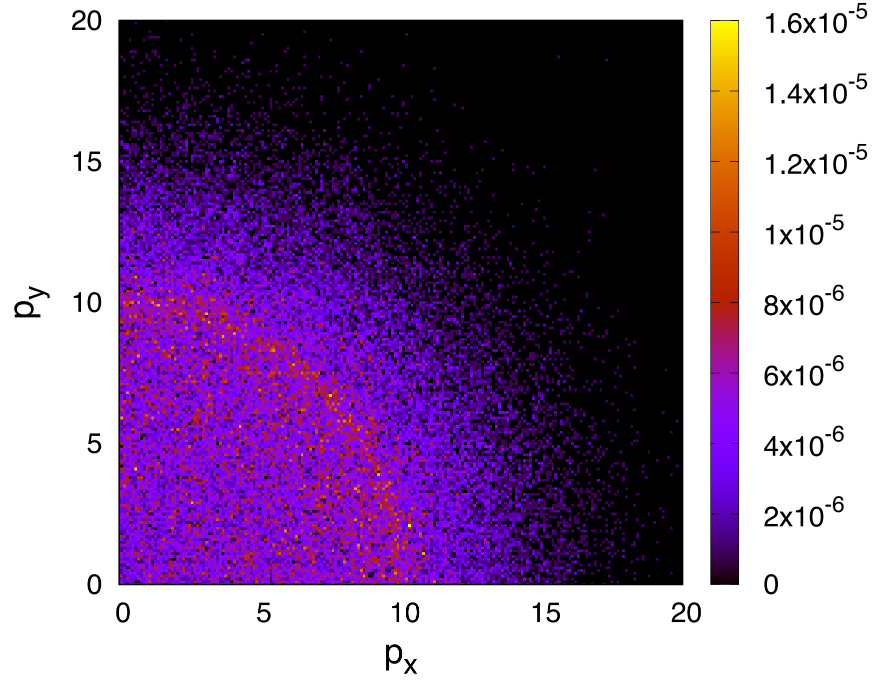
Andrew Taylor

DESY.

Elastic Collisions

p_x and p_y of each particle start off correlated, and through scattering become decorrelated, appreciated by looking at phase space distribution





each scattering p_x and p_y are individually randomised

$$\mathbf{dp^2 = dp_x^2 + dp_y^2}$$

Andrew Taylor

DESY.

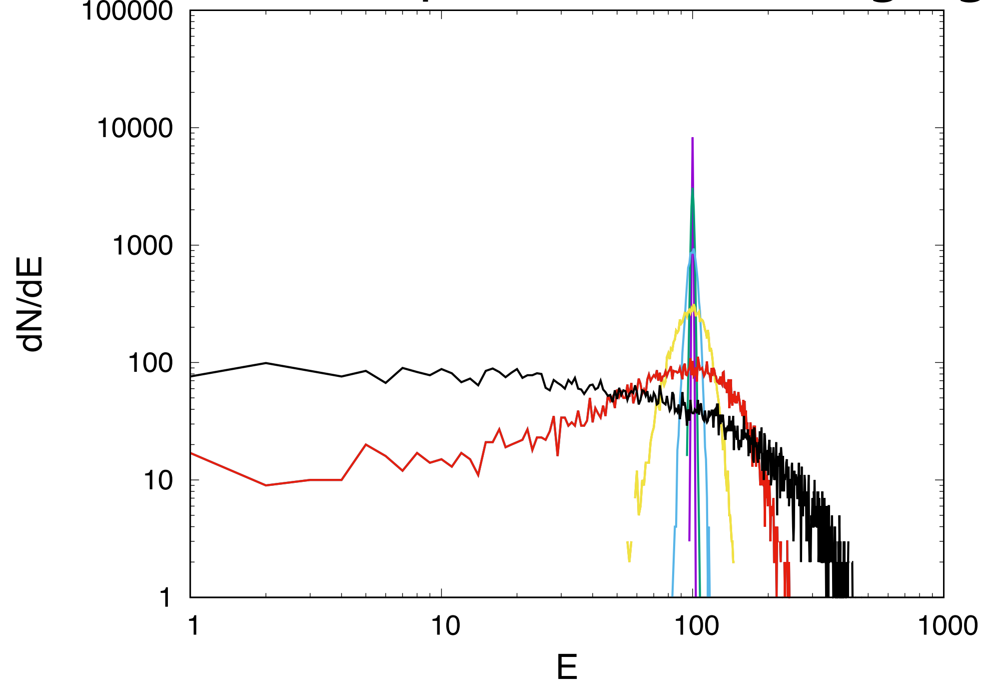
Origin of Thermalised Particle Distribution Function

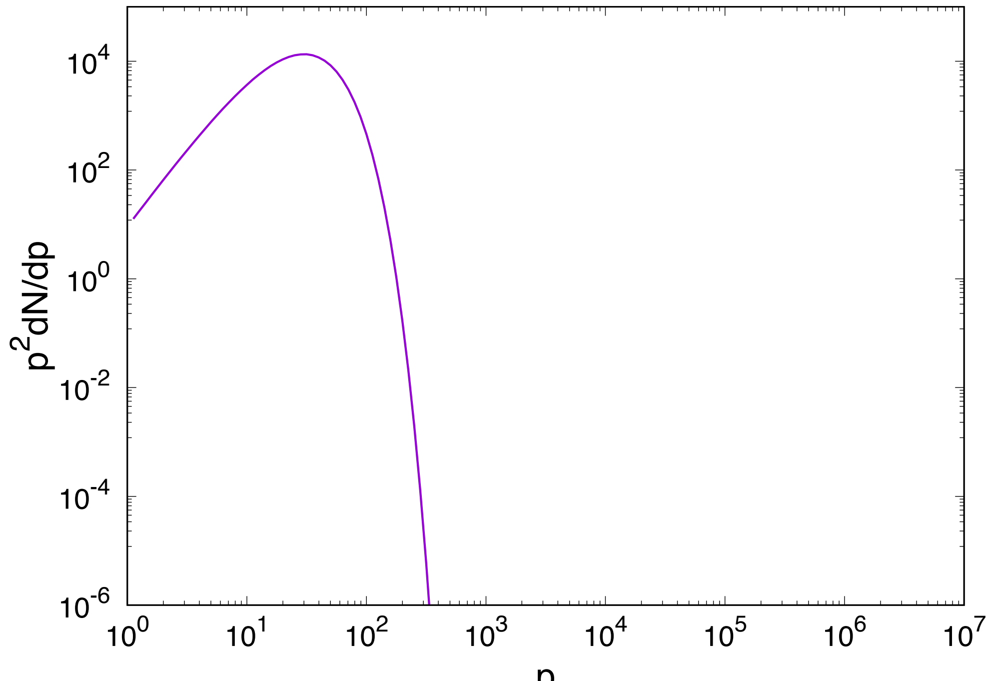
Ensemble of particles exchanging energies:

E_1	E_2	E_3	E_4	E_5	E_6
100	100	100	100	100	100
101	99	100	100	100	100
101	99	100	100	99	101
100	99	101	100	99	101
100	98	101	100	99	102
99	99	101	100	99	102

Relaxing to a Thermal Distribution

Ensemble of particles exchanging energies:



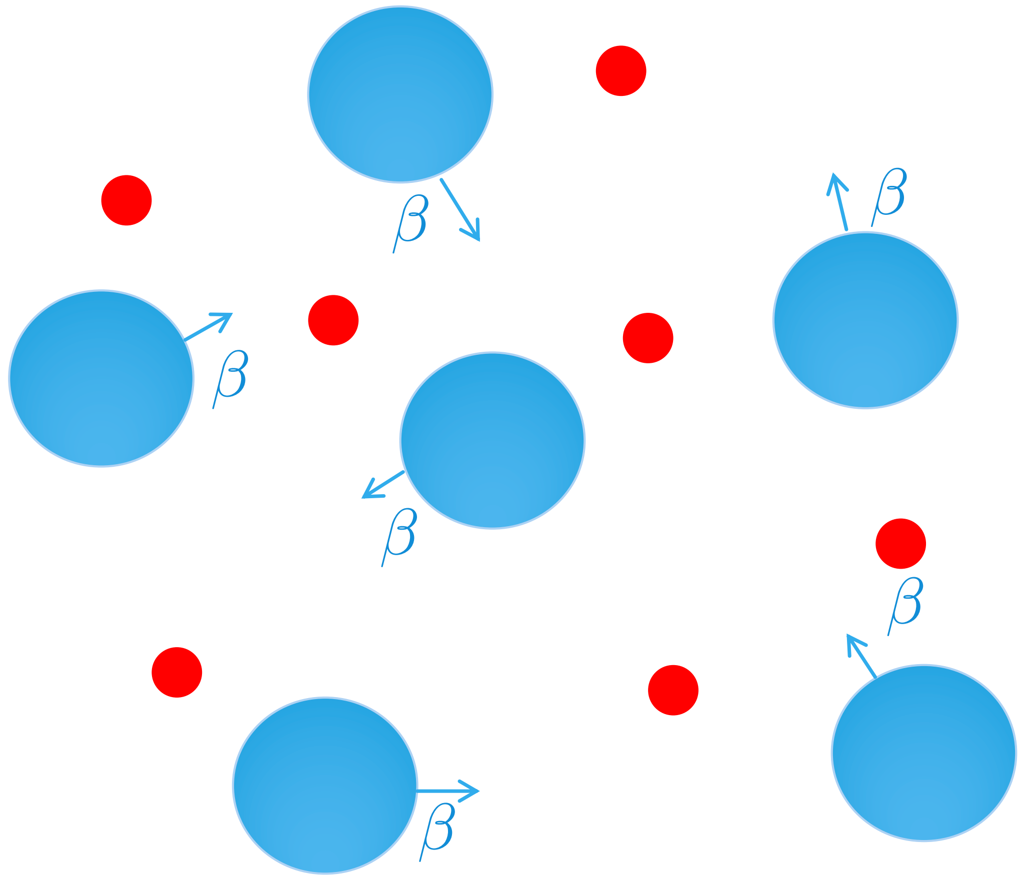


Andrew Taylor

DESY.

Microscopic Particles Thermalising off Macroscopic Objects

Another thought experiment- imagine an ensemble of thermalized macro particles (blue) with a cold set of micro particles injected (red).....

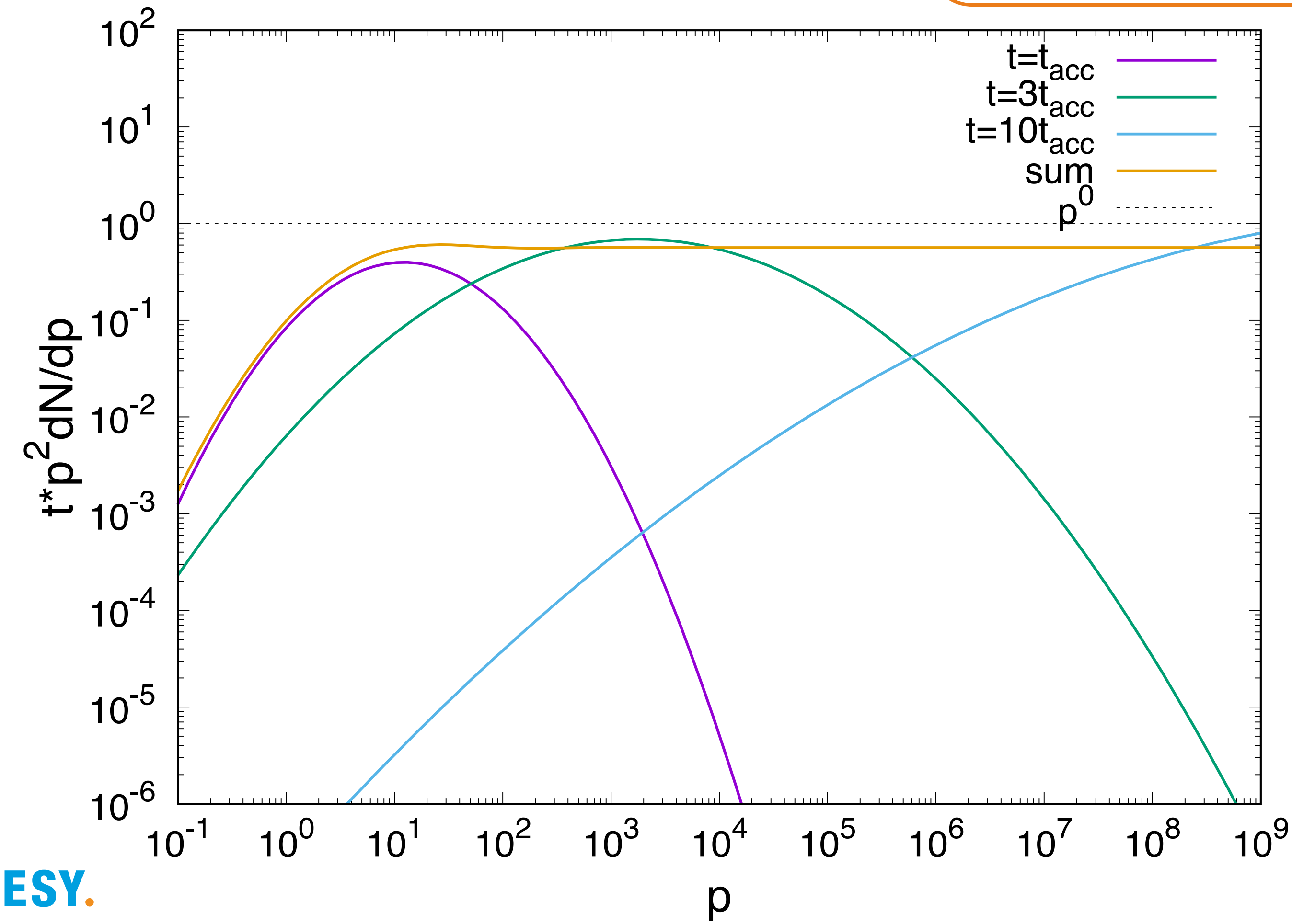


Andrew Taylor

DESY.

Green's Function for Stochastic Acceleration

$$\frac{dN}{dp} \propto e^{-(\ln p)^2/4(t/t_{acc})}$$



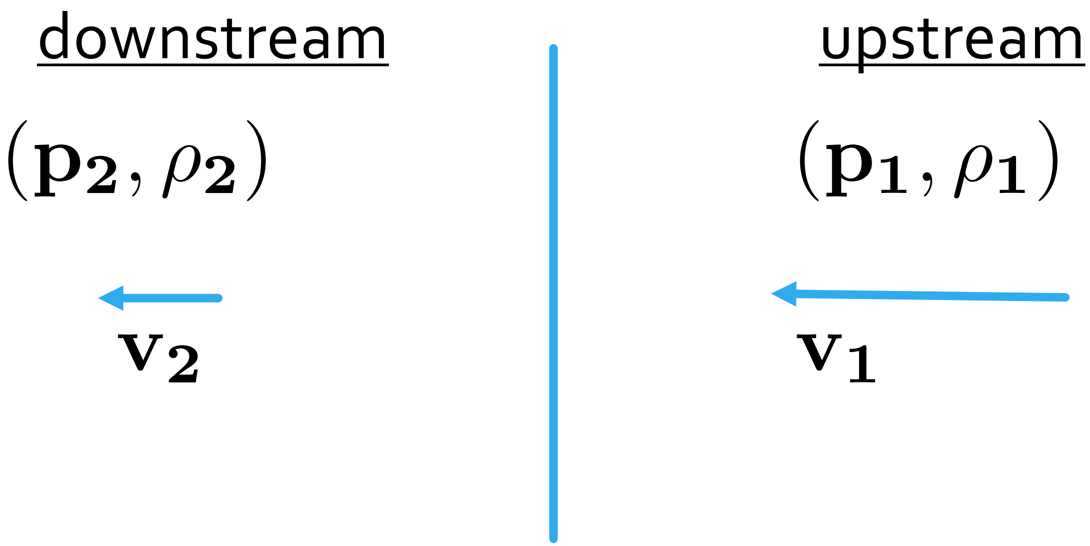
Andrew Taylor

DESY.

Shocks…….a Surprise!

Collisional Shock- Conservation Conditions

viewed in shock frame



Number Flux: $\rho_1 \mathbf{v_1} = \rho_2 \mathbf{v_2}$

Momentum Flux: $\mathbf{p_1} + \rho_1 \mathbf{v_1^2} = \mathbf{p_2} + \rho_2 \mathbf{v_2^2}$

Energy Flux: $$\frac{\gamma}{\gamma - 1}\mathbf{p_1 v_1} + \frac{1}{2}\rho_1 \mathbf{v_1^3} = \frac{\gamma}{\gamma - 1}\mathbf{p_2 v_2} + \frac{1}{2}\rho_2 \mathbf{v_2^3}$$

Collisional Shock- Cold Shock Case

Momentum Flux:

$$\rho_1 \mathbf{v_1^2} = \mathbf{p_2} + \rho_2 \mathbf{v_2^2}$$

$$\frac{\mathbf{p_2}}{\rho_1 \mathbf{v_1^2}} = \left(\mathbf{1} - \frac{\mathbf{v_2}}{\mathbf{v_1}}\right)$$

Energy Flux:

$$\frac{\mathbf{1}}{\mathbf{2}}\rho_1 \mathbf{v_1^3} = \left(\frac{\gamma}{\gamma - \mathbf{1}}\right)\mathbf{p_2 v_2} + \frac{\mathbf{1}}{\mathbf{2}}\rho_2 \mathbf{v_2^3}$$

$$\frac{\mathbf{2}\gamma}{\gamma - \mathbf{1}}\frac{\mathbf{p_2 v_2}}{\rho_1 \mathbf{v_1^3}} = \left(\mathbf{1} - \left(\frac{\mathbf{v_2}}{\mathbf{v_1}}\right)^{\mathbf{2}}\right) = \left(\mathbf{1} - \frac{\mathbf{v_2}}{\mathbf{v_1}}\right)\left(\mathbf{1} + \frac{\mathbf{v_2}}{\mathbf{v_1}}\right)$$

$$\frac{\mathbf{v_2}}{\mathbf{v_1}}\left(1-\frac{\mathbf{v_2}}{\mathbf{v_1}}\right)=\left(\frac{\gamma-1}{2\gamma}\right)\left(1-\left(\frac{\mathbf{v_2}}{\mathbf{v_1}}\right)^2\right)$$

PAUSE

Why not have a go at solving this

Collisional Shock- Cold Shock Case

$$\frac{\mathbf{v_2}}{\mathbf{v_1}}\left(\mathbf{1}-\frac{\mathbf{v_2}}{\mathbf{v_1}}\right)=\left(\frac{\gamma-\mathbf{1}}{\mathbf{2}\gamma}\right)\left(\mathbf{1}-\left(\frac{\mathbf{v_2}}{\mathbf{v_1}}\right)^{\mathbf{2}}\right)$$

$$\left(\frac{\mathbf{v_2}}{\mathbf{v_1}}-\mathbf{1}\right)\left(\frac{\mathbf{v_2}}{\mathbf{v_1}}-\left(\frac{\gamma-\mathbf{1}}{\gamma+\mathbf{1}}\right)\right)=\mathbf{0}$$

So what are collisional shocks good for?
Stimulating the unstimulated degrees of freedom in the system where energy can be stored

Collisional Shock- Partition of Momentum and Energy

Downstream Momentum Partition:

$$\mathbf{p_2} = \frac{3}{4}\rho_1 \mathbf{v_1^2}$$

Downstream Energy Partition:

$$\frac{\gamma}{\gamma - 1}\mathbf{p_2 v_2} = \frac{15}{16}\left[\frac{1}{2}\rho_1 \mathbf{v_1^3}\right]$$

Andrew Taylor

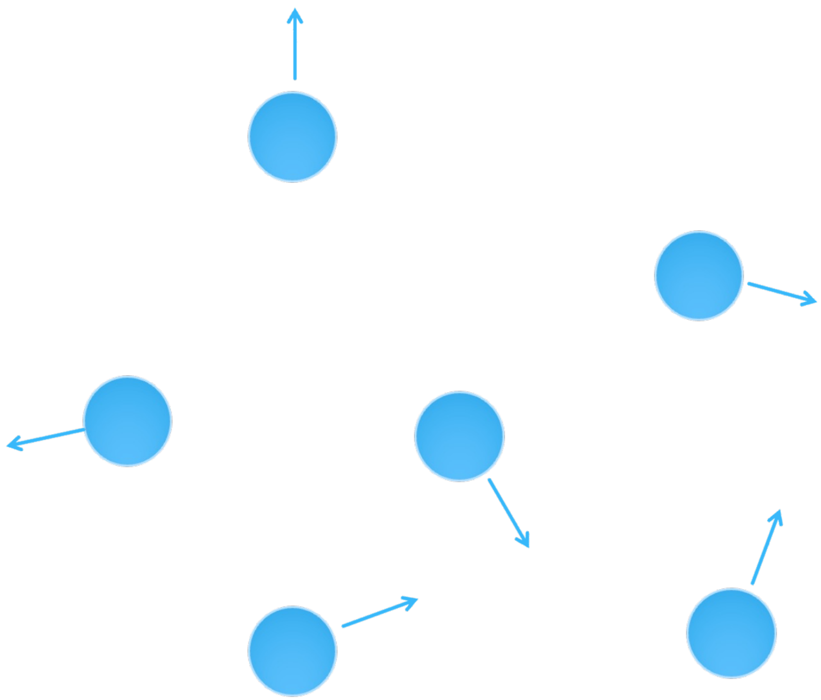
Collision Time

$$\mathbf{t = \frac{1}{n_e \sigma_T c}}$$

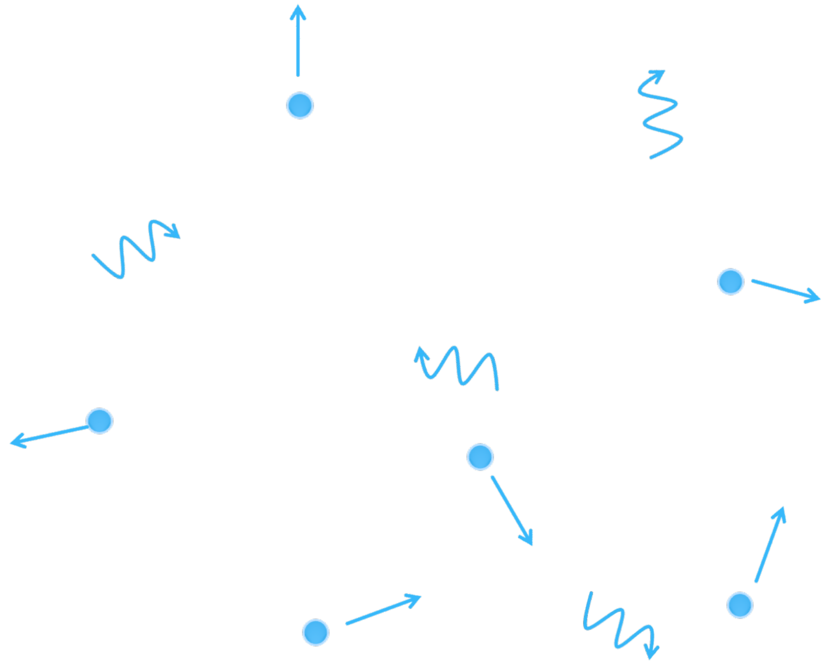
$$\mathbf{\approx \left(\frac{1\ cm^{-3}}{n_e} \right)\ Myr}$$

Energy Exchange at Shocks

Collisional Shock

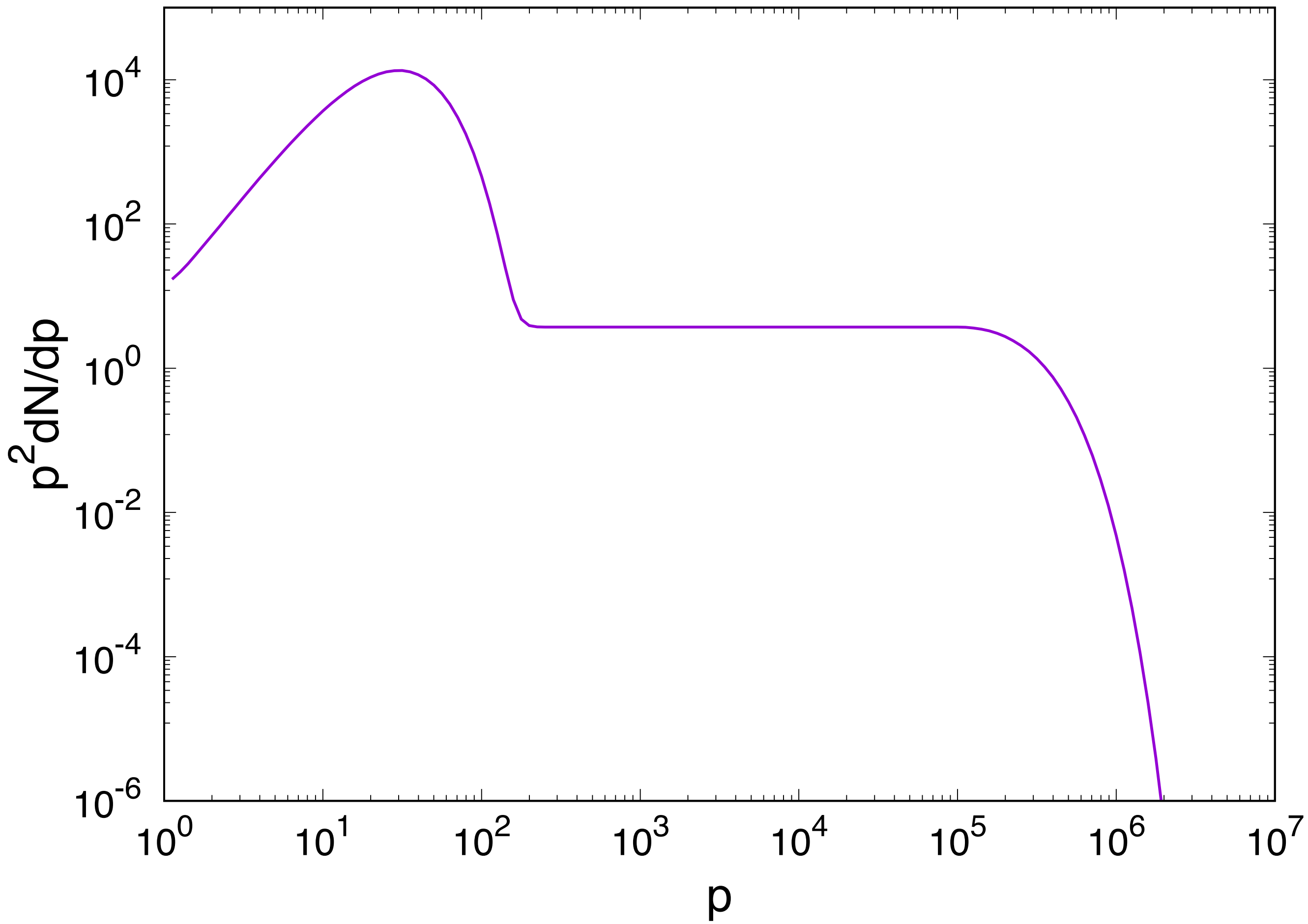


Collisionless Shock



Andrew Taylor

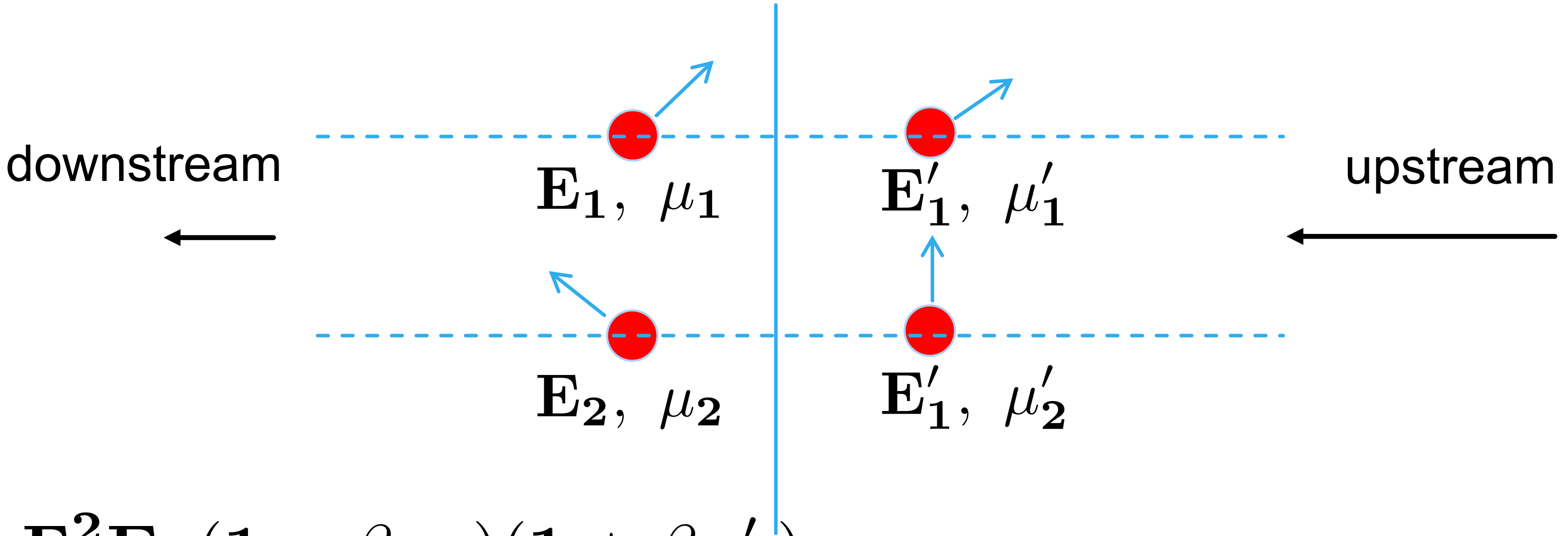
Collisionless Shock- the Injection Problem



Andrew Taylor

DESY.

Particle Acceleration at Collisionless Shocks



$$\mathbf{E_2} = \Gamma^2 \mathbf{E_1}(1-\beta\mu_1)(1+\beta\mu_2')$$

$$\mu' = \frac{\mu-\beta}{1-\beta\mu}$$

$$\mathbf{E_2} = \Gamma^2 \mathbf{E_1}(1-\beta\mu_1)\left(1+\beta\left(\frac{\mu_2-\beta}{1-\beta\mu_2}\right)\right)$$

$$\mathbf{E_2} = \mathbf{E_1}\left(\frac{1+\beta\mu_1}{1+\beta\mu_2}\right)$$

Fermi Shock Acceleration

Energy

$$\frac{\Delta E}{E} = \frac{4v}{3c} = \frac{4}{3}\beta \text{(energy gain)}$$

$$E_1 = \left(1 + \frac{4}{3}\beta\right) E_0$$

$$E_n = \left(1 + \frac{4}{3}\beta\right)^n E_0$$

Number

$$\frac{\Delta N}{N} = -\frac{4v}{3c} = -\frac{4}{3}\beta \text{ (advection downstream)}$$

$$N_1 = \left(1 - \frac{4}{3}\beta\right) N_0$$

$$N_n = \left(1 - \frac{4}{3}\beta\right)^n N_0$$

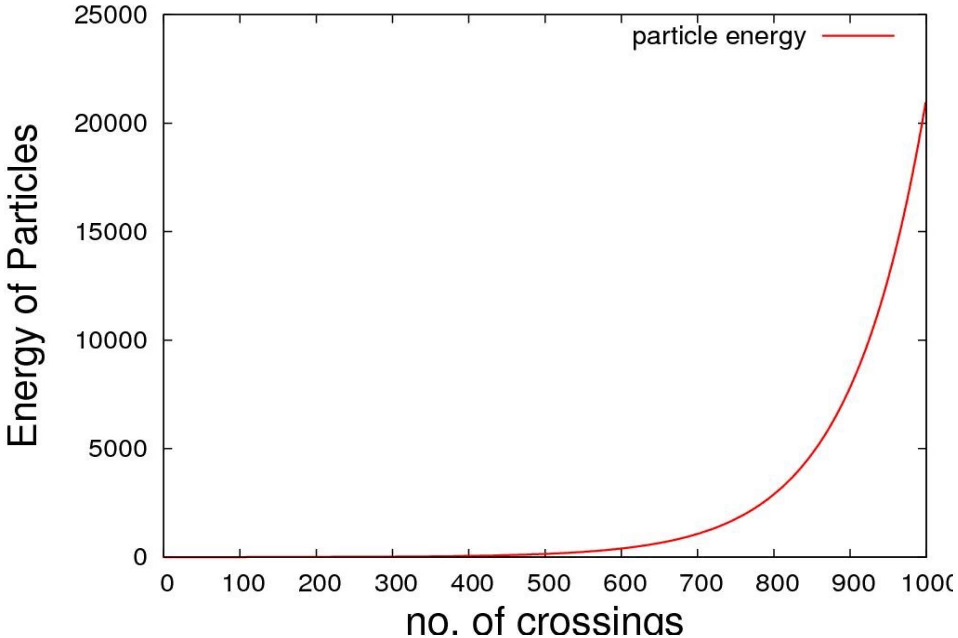
So $n \sim 1/\beta$ crossings are needed before the particle population is significantly altered → SNRs have $v_{sh} \sim 10^3$ km s^{-1} so $\beta \sim 10^{-2}$

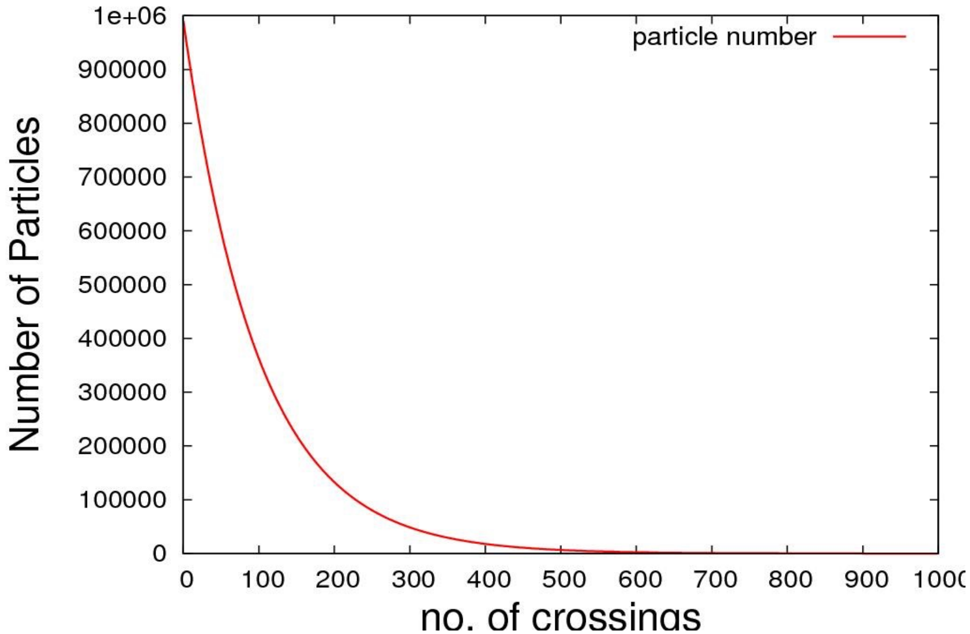
Fermi Shock Acceleration

<u>Energy</u>

<u>Number</u>

$\beta \sim 10^{-2}$





PAUSE

Why not have a go at determining the resultant spectrum

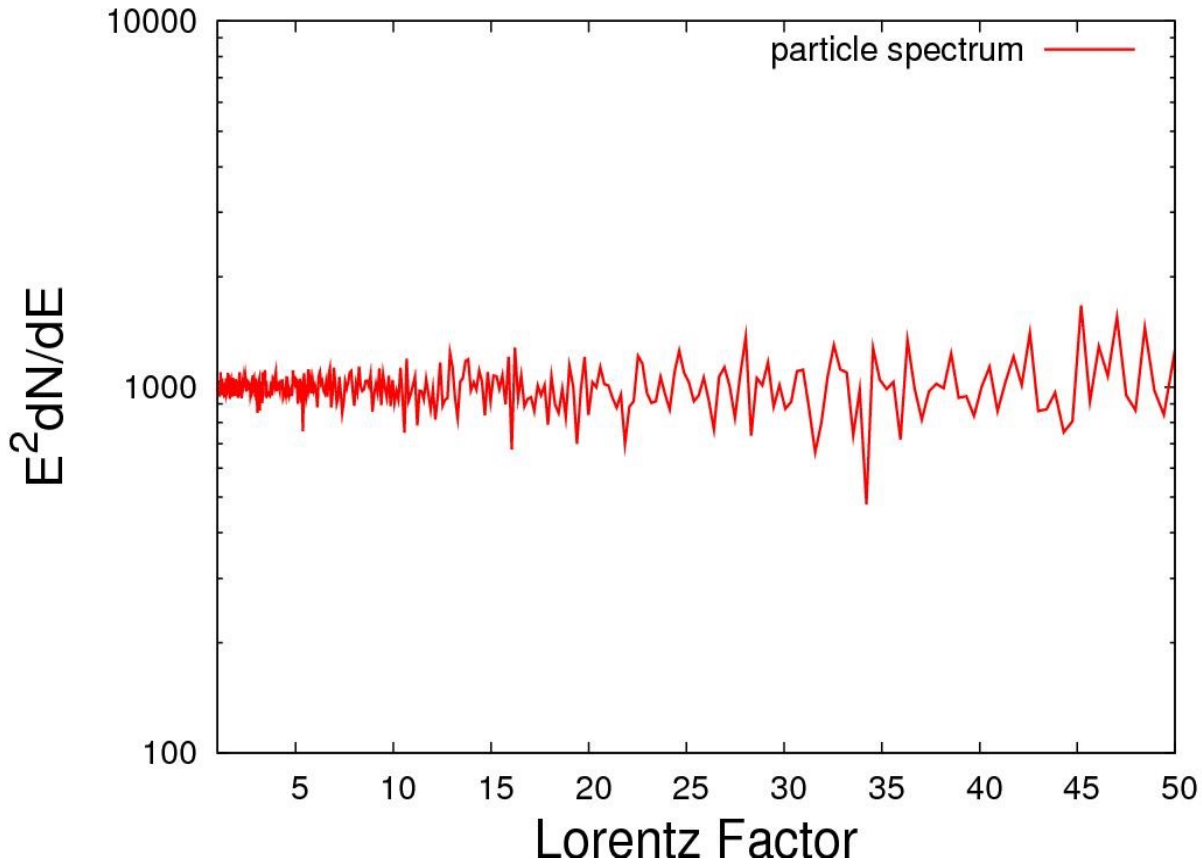
Fermi Shock Acceleration

So,

$$\frac{\mathbf{\Delta N}}{\mathbf{\Delta E}} = \frac{\mathbf{N_0}}{\mathbf{E_0}} \left(\frac{\mathbf{1 - 4\beta/3}}{\mathbf{1 + 4\beta/3}} \right)^{\mathbf{n}}$$

$$\approx \frac{\mathbf{N_0}}{\mathbf{E_0}} \left(\mathbf{1 + 4\beta/3}\right)^{\mathbf{-2n}}$$

$$\approx \mathbf{N_0 E_0 E^{-2}}$$



The flat spectrum, with dN/dE ~ E^{-2}, is produced when the
acceleration time and the escape time are equal (and have the same energy dependence)

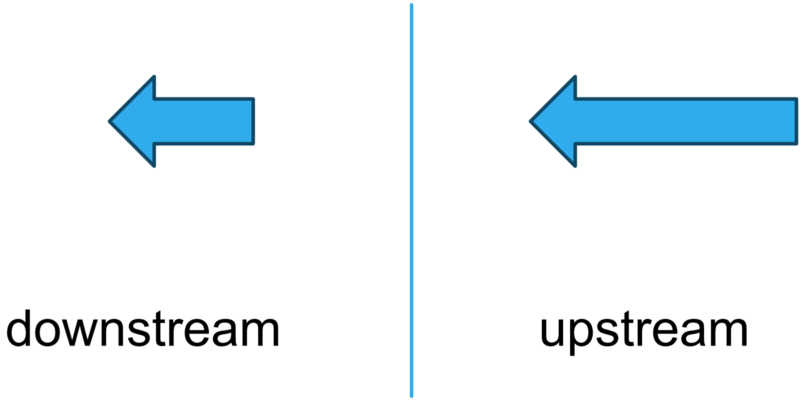
Andrew Taylor

DESY.

Diffusive Shock Acceleration (Fermi First Order)

Fermi (First Order) Acceleration Time

[viewed in shock rest frame]



Spatial Transport Equation (Continuity Equation)

$$\frac{\partial \mathbf{f}}{\partial \mathbf{t}} = -\nabla_{\mathbf{x}} \cdot [-(\mathbf{D_{xx}} \nabla_{\mathbf{x}} \mathbf{f}) + \mathbf{vf}] + \mathbf{Q}$$

Diffusion

Advection

Source term

Andrew Taylor

DESY.

Fermi (First Order) Acceleration Time

$$\mathbf{t_{acc} = E\frac{\Delta t_{cycle}}{\Delta E_{cycle}}}$$

Transport of particles in each region is dictated by competition between diffusion and advection

downstream | upstream

$$\mathbf{t_{diff} = \frac{R^2}{D_{xx}}}$$

$$\mathbf{t_{adv} = \frac{R}{v_{adv}}}$$

Balancing these timescales

$$\mathbf{t_{resid} = \frac{D_{xx}}{(c\beta_{sh})^2}}$$

Fermi (First Order) Acceleration Time

$$\mathbf{t_{acc} = E\frac{\Delta t_{cycle}}{\Delta E_{cycle}}}$$

$$\mathbf{t_{resid} = \frac{D_{xx}}{(c\beta_{sh})^2}}$$

However, during the time it takes advection to dominate over diffusion, the particle will have crossed the shock $\mathbf{1/\beta}$ times

$$\mathbf{\Delta t_{cycle} = \frac{D_{xx}}{(c^2\beta_{sh})}}$$

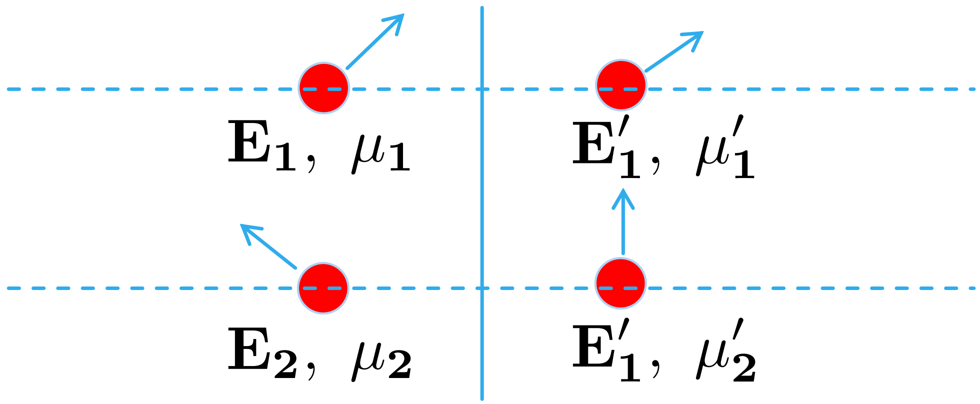
Fermi (First Order) Acceleration Time

$$\mathbf{t_{acc} = E\frac{\Delta t_{cycle}}{\Delta E_{cycle}}}$$

$$\mathbf{\Delta t_{cycle} = \frac{D_{xx}}{(c^2\beta_{sh})}}$$

$$\mathbf{\Delta E_{cycle} = E\beta_{sh}}$$

$$\mathbf{t_{acc} = \frac{D_{xx}}{(c\beta_{sh})^2} = \frac{t_{scat}}{\beta_{sh}^2}}$$



$$\mathbf{E_2 = E_1\left(\frac{1+\beta\mu_1}{1+\beta\mu_2}\right)}$$

Andrew Taylor

DESY.

The Need for Efficient Accelerators
….what means efficient?

Andrew Taylor

Particle Acceleration in AGN

$$\mathbf{t_{acc}} = \eta \frac{\mathbf{R_{lar}}}{\mathbf{c}\beta^{\mathbf{2}}}$$

$$\mathbf{t_{esc.}} = \frac{\mathbf{R^2}}{\eta \mathbf{c} \mathbf{R_{lar}}}$$

Maximum energy
(Hillas criterion)

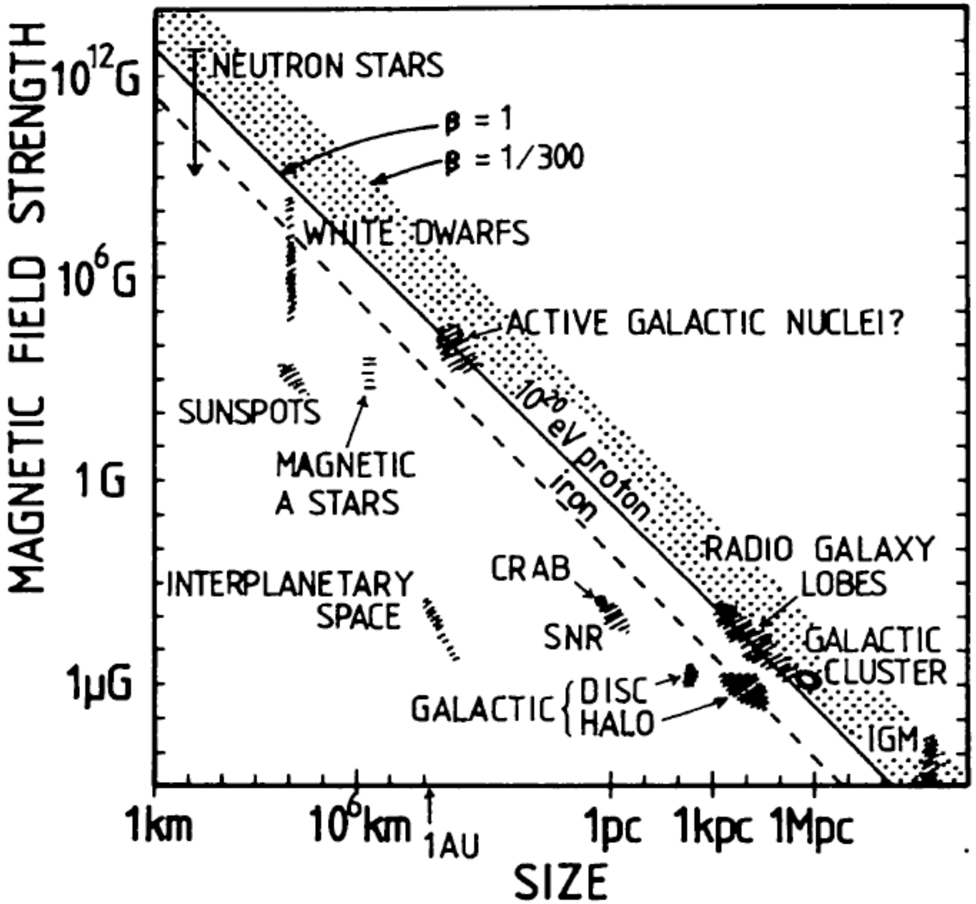
$$\mathbf{R_{lar}} = \frac{\beta}{\eta}\mathbf{R}$$

AM Hillas (1984)

$$\mathbf{R_{lar}(E, B)} = \left(\frac{\mathbf{E}}{\mathbf{10\ EeV}}\right)\left(\frac{\mathbf{1\ mG}}{\mathbf{B}}\right)\mathbf{10\ pc}$$

The Hillas Criterion (Implicitly Assumes Accelerator is Efficient)

AM Hillas (1984)



$\eta \approx 1$ assumed in above plot

Andrew Taylor

DESY.

The End of the Accelerated Spectra: Cutoffs

Particle Transport Equation in Momentum Space

$$\frac{\partial \mathbf{f}}{\partial \mathbf{t}} + \nabla_{\mathbf{p}} \cdot \mathbf{J}_{\mathbf{p}} = \frac{\mathbf{Q}}{\mathbf{p}^2}$$

$$\frac{\partial \mathbf{f}}{\partial \mathbf{t}} = \nabla_{\mathbf{p}} \cdot \left[(\mathbf{D}_{\mathbf{pp}} \nabla_{\mathbf{p}} \mathbf{f}) + \frac{\mathbf{p}}{\mathbf{3}} (\nabla \cdot \mathbf{v}) - \frac{\mathbf{p}}{\tau_{\mathbf{loss}}(\mathbf{p})} \mathbf{f} \right] - \frac{\mathbf{f}}{\tau_{\mathbf{esc}}(\mathbf{p})} + \frac{\mathbf{Q}}{\mathbf{p}^2}$$

Andrew Taylor

DESY.

Particle Transport Equation

- Cut-offs arise naturally in the general solution of the transport equation for particles

$$\frac{\partial \mathbf{f}}{\partial \mathbf{t}} = \nabla_{\mathbf{p}} \cdot \left[(\mathbf{D}_{\mathbf{pp}} \nabla_{\mathbf{p}} \mathbf{f}) - \frac{\mathbf{p}}{\tau_{\mathbf{loss}}(\mathbf{p})} \mathbf{f} \right] - \frac{\mathbf{f}}{\tau_{\mathbf{esc}}(\mathbf{p})} + \frac{\mathbf{Q}}{\mathbf{p}^{\mathbf{2}}}$$

Acceleration

Radiative Losses

Escape

Source term

Andrew Taylor

DESY.

Cut-off Shape

- Interplay of acceleration and cooling defines the value of the cut-off of the primary particles:

$$\frac{\mathbf{dN}}{\mathbf{dE_e}} \propto \mathbf{E_e^{-\Gamma} e^{-(E_e/E_{\max})^{\beta_e}}} \qquad \beta_\mathbf{e} = \mathbf{2 - q - r}$$

- In the following, demonstrations for this result will be shown for the case of stochastic acceleration scenarios. However, in reality, this result is more general, holding also for shock acceleration scenarios.

[see Schlickeisser et al. 1985, Zirakashvili et al. 2007, Stawarz et al. 2008]

A Simple Case- q=1, only escape

- Bohm diffusion (**q=1**) + only escape results in simple exponential cutoff.
- Some simplifications to the transport equation:

$$\frac{\partial \mathbf{f}}{\partial \mathbf{t}} = \nabla_{\mathbf{p}} \cdot \left[(\mathbf{D}_{\mathbf{pp}} \nabla_{\mathbf{p}} \mathbf{f}) - \frac{\mathbf{p}}{\tau_{\mathbf{loss}}(\mathbf{p})} \mathbf{f} \right] - \frac{\mathbf{f}}{\tau_{\mathbf{esc}}(\mathbf{p})} + \frac{\mathbf{Q}}{\mathbf{p}^2}$$

Steady state

No losses

Delta injection

Andrew Taylor

DESY.

A Simple Case (II)- q=1, only escape

- Rearranging the terms (and explicitly stating the dependences from p of the parameters):

$$\frac{1}{p^2}\frac{\partial}{\partial p}\left(p^2 D_0 \frac{p}{p_0}\frac{\partial f}{\partial p}\right) - \frac{f}{\tau_{esc}(p)} = \delta(p), \qquad \tau_{esc}(p) \propto p^{-1}$$

$$\frac{\partial^2 f}{\partial p^2} + \frac{3}{p}\frac{\partial f}{\partial p} - \left(\frac{1}{D_0 \tau_0}\right) f = \delta(p)$$

Cutoff comes from balancing 1st and 3rd term → $f \propto A e^{-p/p_\tau}$

Recall generally, $\beta_e = 2 - q - r$

$$q = 1,\ r = 0,\ \rightarrow\ \beta_e = 1$$

Andrew Taylor

DESY.

(Note- energy losses for the $r = 0$ case will not alter this result)

Particle Acceleration with Cooling

$$\mathbf{t_{cool} = \frac{m_e}{(4/3)\Gamma_e \sigma_T U_B}}$$

$$\sigma_T \mathbf{U_{Bcrit} \frac{hc}{(m_e c^2)^2} = (2}\pi\mathbf{/3)}\alpha$$

$$\mathbf{t_{cool} = \frac{9}{8\pi\alpha} \frac{h}{E_e} \frac{U_{B_{crit.}}}{U_B}}$$

Andrew Taylor

DESY.

Particle Acceleration with Cooling

1 $$\mathbf{t_{cool}} = \frac{9}{8\pi\alpha}\frac{\mathbf{h}}{\mathbf{E_e}}\frac{\mathbf{U_{B_{crit.}}}}{\mathbf{U_B}}$$

2 $$\mathbf{t_{lar}} = \frac{2\pi\mathbf{E_e}}{\mathbf{eBc}} = \mathbf{\Gamma_e}\left(\frac{\mathbf{B_{crit}}}{\mathbf{B}}\right)\frac{\mathbf{h}}{\mathbf{m_e}}$$

3 $$\mathbf{E_\gamma^{sync}} = \mathbf{\Gamma_e^2}\left(\frac{\mathbf{B}}{\mathbf{B_{crit}}}\right)\mathbf{m_e}$$

$$\mathbf{t_{cool}} = \frac{9}{8\pi\alpha}\left(\frac{\mathbf{m_e}}{\mathbf{E_\gamma^{sync}}}\right)\mathbf{t_{lar}}$$

DESY.

Andrew Taylor

$$\mathbf{t_{cool} = \frac{9}{8\pi\alpha}\left(\frac{m_e}{E_\gamma^{sync}}\right) t_{lar}}$$

PAUSE

Try using relations 1-3 from previous slide to obtain the above result

Particle Acceleration with Cooling

$$\mathbf{t_{acc}} = \eta \frac{\mathbf{R_{lar}}}{\mathbf{c}\beta^{\mathbf{2}}}$$

$$\mathbf{t_{cool}} = \frac{\mathbf{9}}{\mathbf{8}\pi\alpha}\left(\frac{\mathbf{m_e}}{\mathbf{E_\gamma^{sync}}}\right)\mathbf{t_{lar}}$$

$$\mathbf{E_\gamma^{sync}} \approx \frac{\mathbf{9}}{\mathbf{4}}\eta^{-\mathbf{1}}\beta^{\mathbf{2}}\frac{\mathbf{m_e}}{\alpha}$$

Maximum synchrotron energy tells us how efficient accelerator is!

Where is $\mathbf{E_\gamma^{sync}}$?

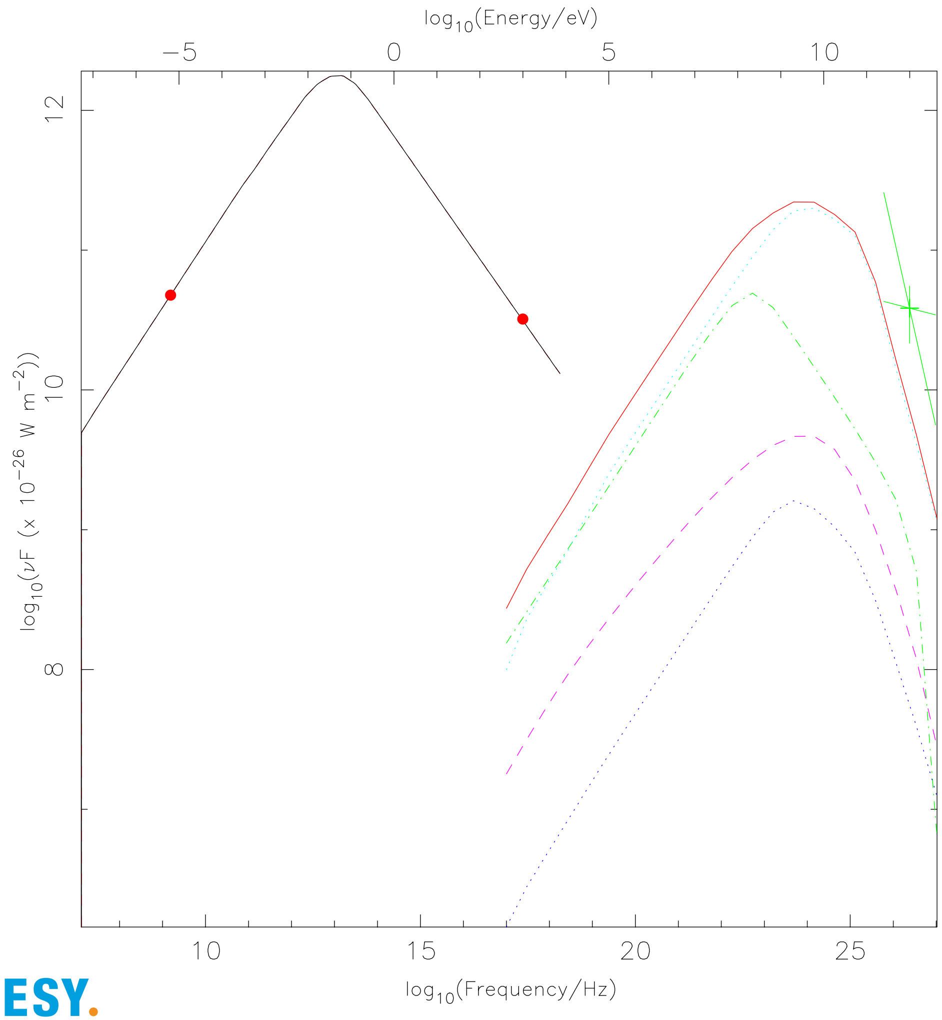
Andrew Taylor

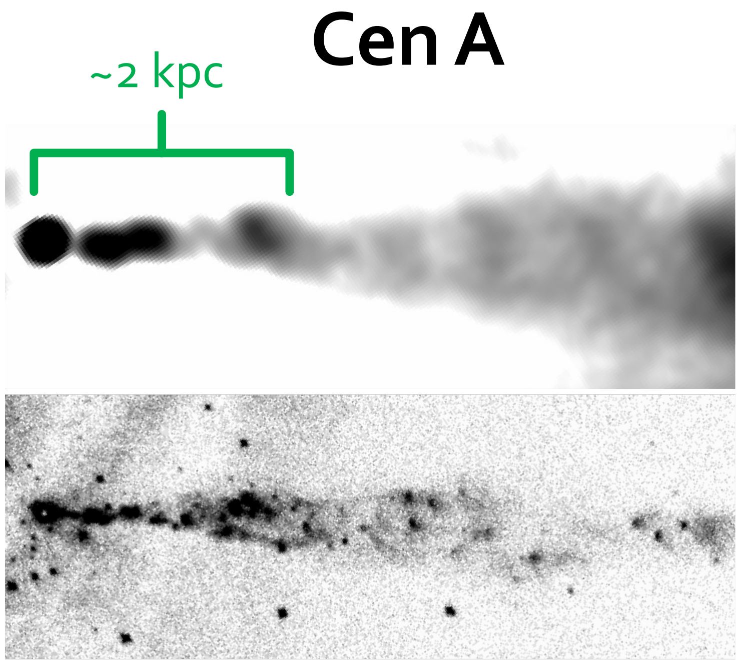
DESY.

Synchroton Cutoff for AGN?

Evidence of a candidate source (Cen A) operating as an efficient accelerator?

Hardcastle et al. (1103.1744)



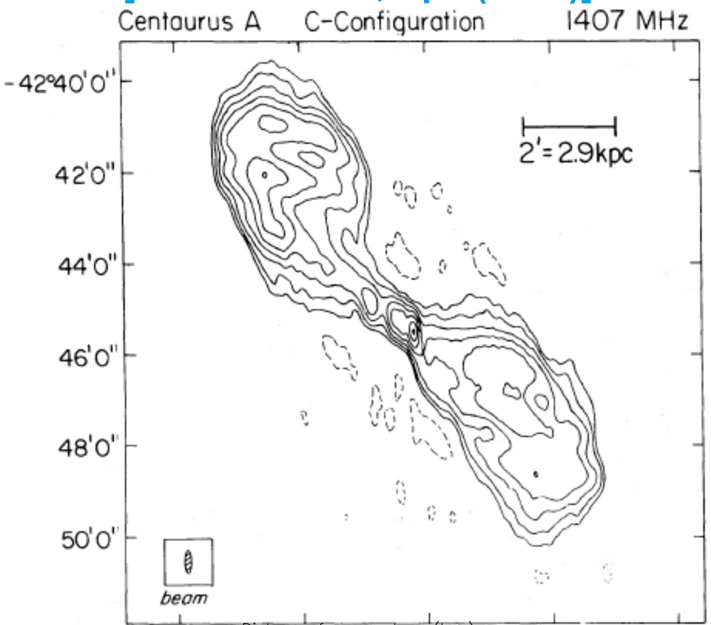


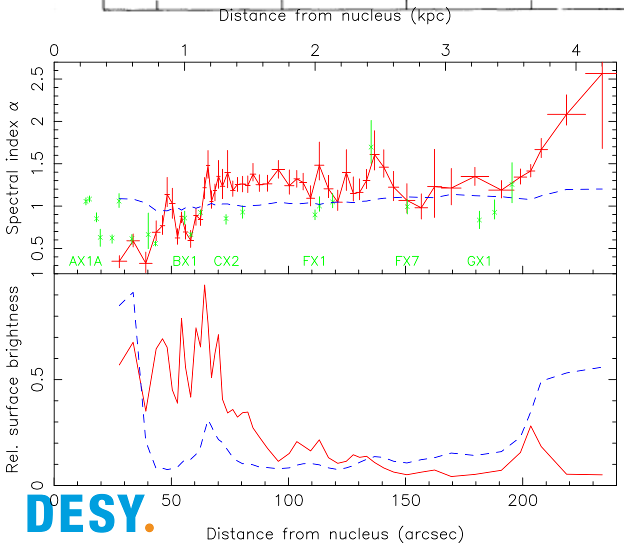
$$\eta < 10^4$$

Andrew Taylor

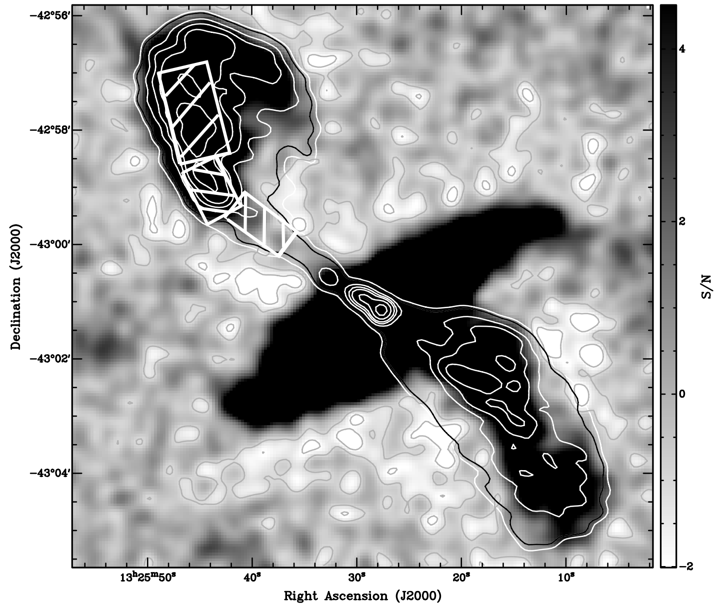
Centaurus A's Inner Jet-
A Cosmic Lab

[J. Burns et al., ApJ (1983)]





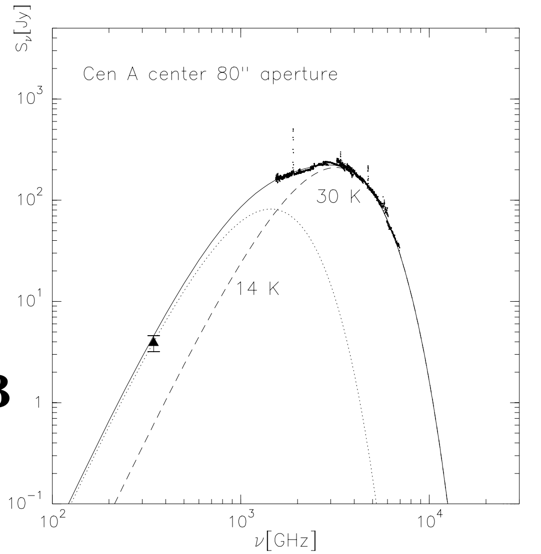
[A. Weiss et al., A&A (2008)]



$$B_{eq} = 60\ \mu G$$

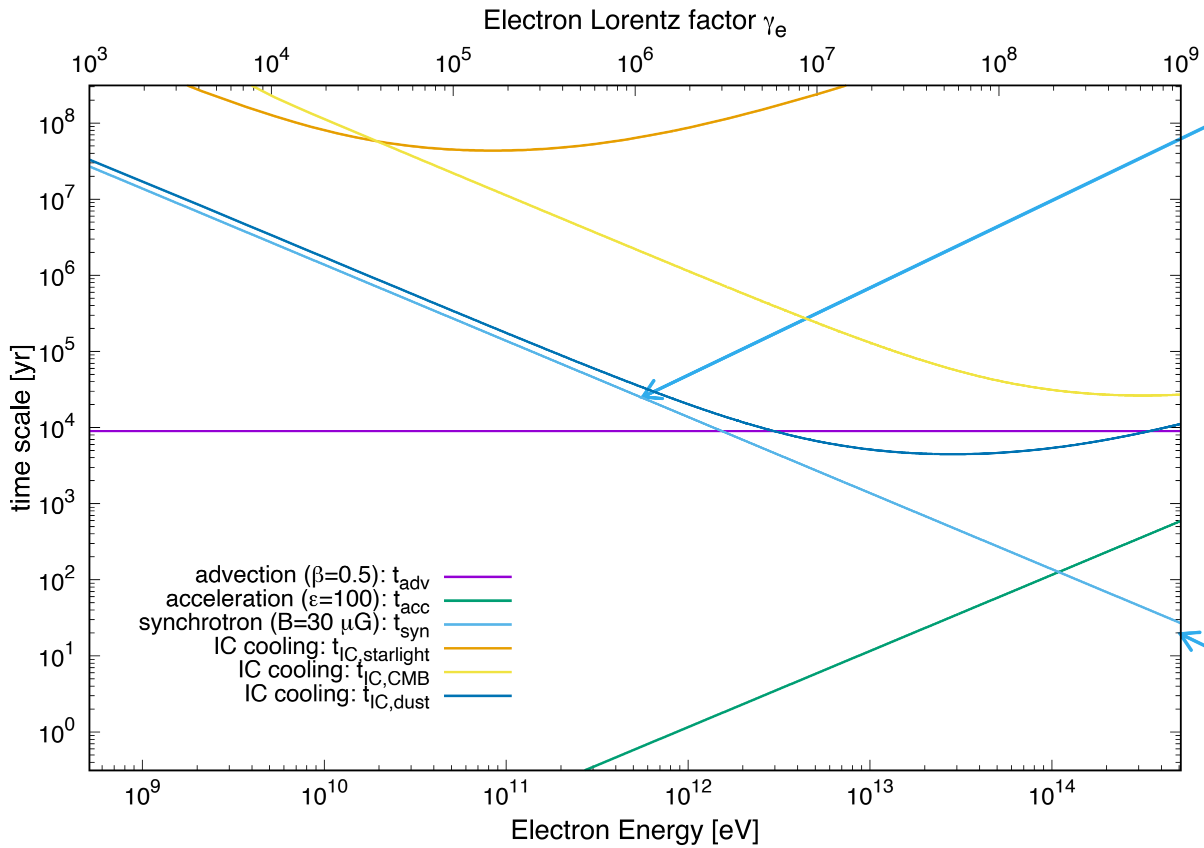
$$U_B \approx 10\ eV\ cm^{-3}$$

$$U_{IR} \approx 10\ eV\ cm^{-3}$$



DESY.

Transport & Cooling Times of Electrons in Cen A's Jets



Synchrotron cooling dominates

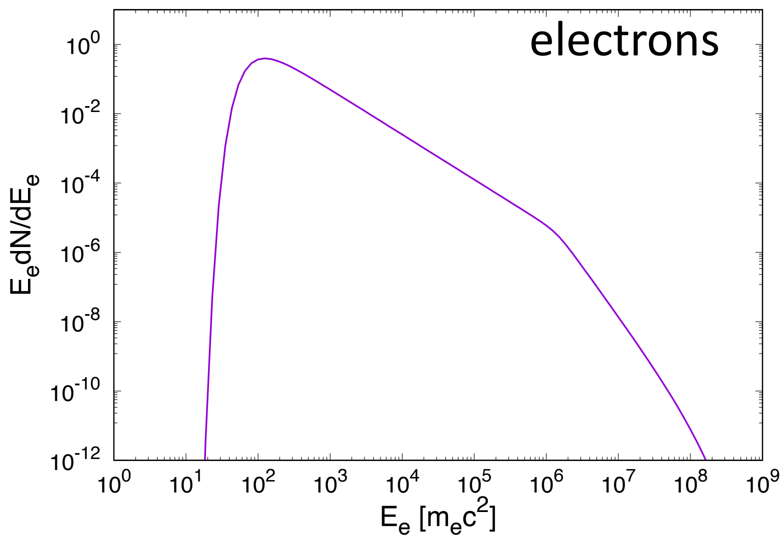
Cooling time becomes shorter than advection time

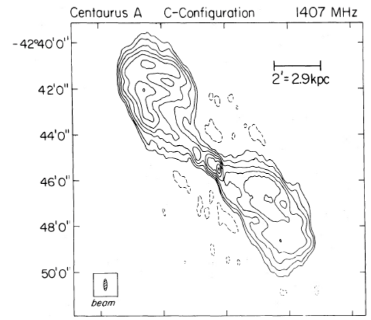
Electrons accelerated to 0.1 PeV energies

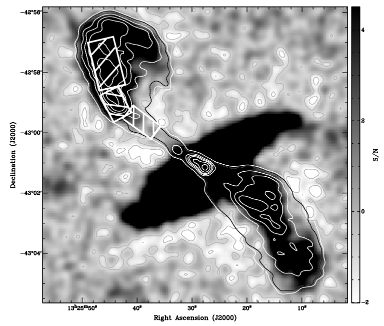
$$\frac{\partial \mathbf{n}}{\partial \mathbf{t}} = -\nabla_{\mathbf{p}} \cdot \left[\frac{\mathbf{p}}{\tau_{\mathbf{acc}}(\mathbf{p})}\mathbf{n} - \frac{\mathbf{p}}{\tau_{\mathbf{loss}}(\mathbf{p})}\mathbf{n} \right] - \frac{\mathbf{n}}{\tau_{\mathbf{esc}}(\mathbf{p})} + \mathbf{Q}$$

drev

Distinguishing Cen A's Nucleus and Inner Jet SED

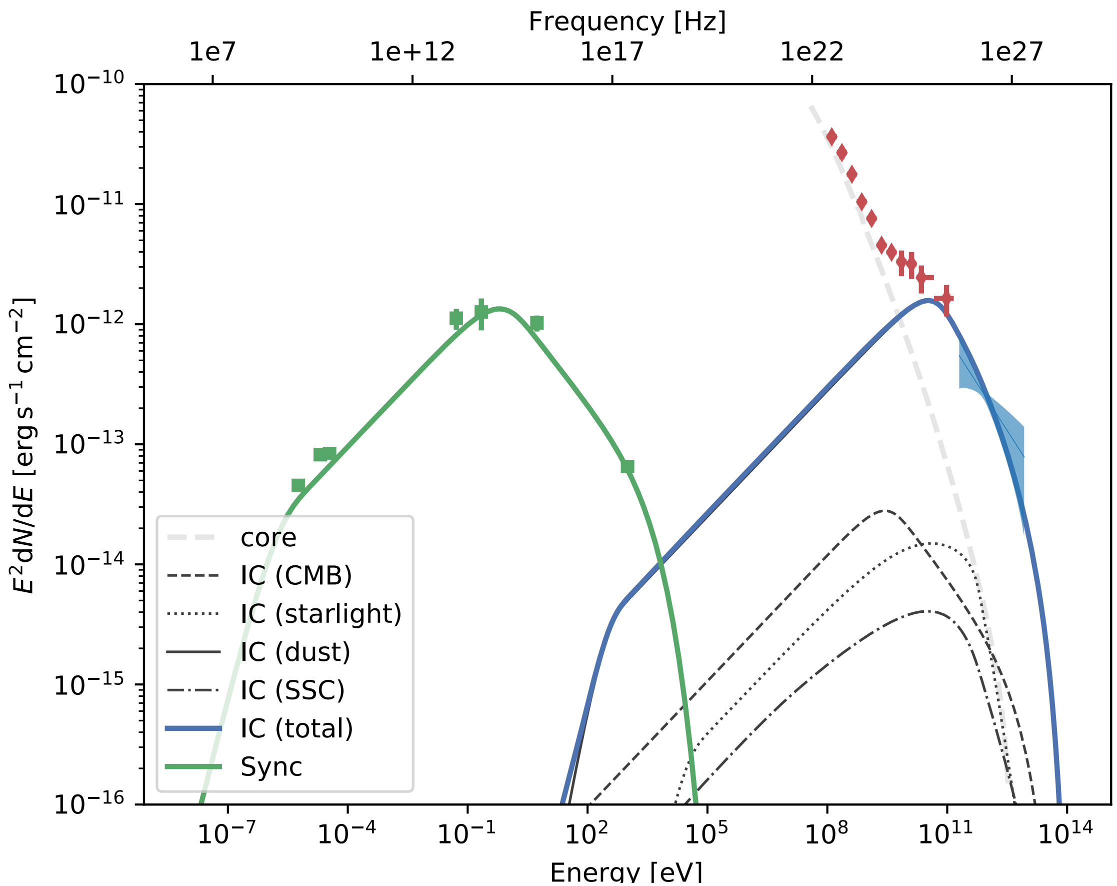






$$\eta = \frac{1}{\Gamma_e^{\max 2} (B/B_{crit})\alpha}$$

$$\approx 10^4 \left(\frac{10^8}{\Gamma_e^{\max}}\right)^2 \left(\frac{20\mu G}{B}\right)$$

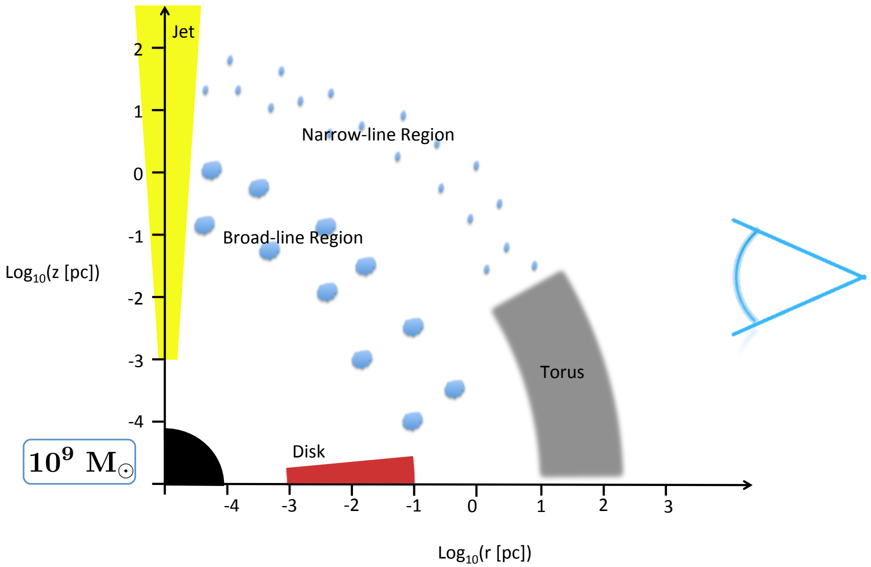


[HESS- F. Rieger, A. Taylor, et al., Nature]

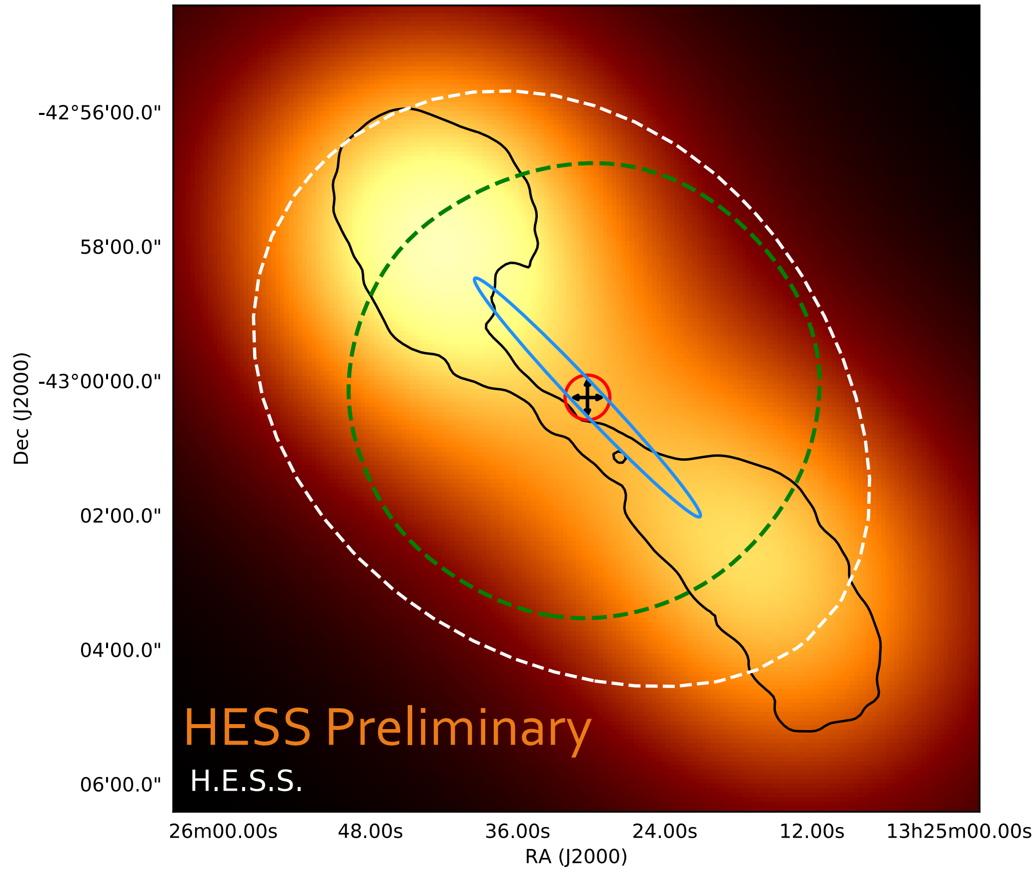
Conclusions

- Thermalisation occurs through collisional interactions
- Non-thermal particle production is possible in non-collisional systems in which free energy is available in the form of a converging flow
- Astrophysical shocks are a prime example of such systems
- To accelerate particle up to ultra high energies requires fast (ie. mildly relativistic or relativistic) shocks
- One telltale sign that a shock can accelerate up to ultra high energies would be that it produces synchrotron emission beyond MeV energies.

Centaurus A - VHE Extension



HESS Detected Extension on ~2kpc scale

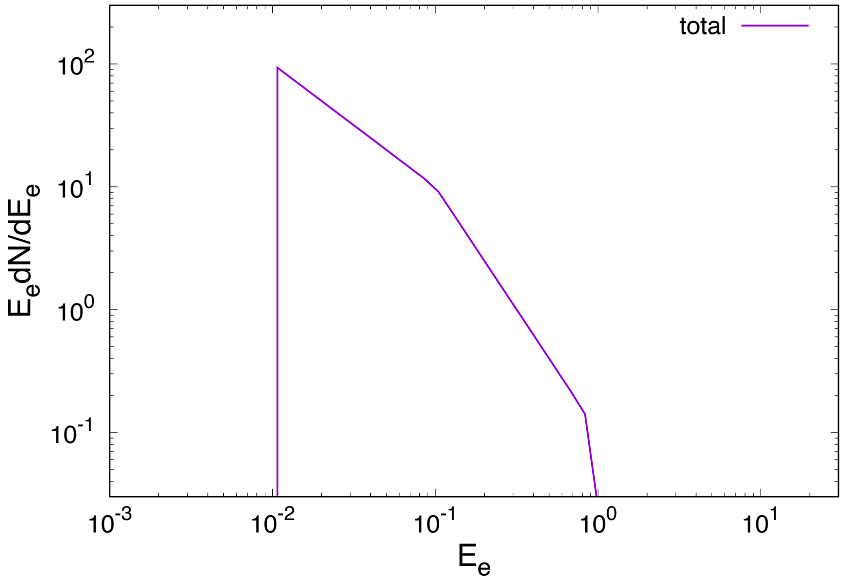


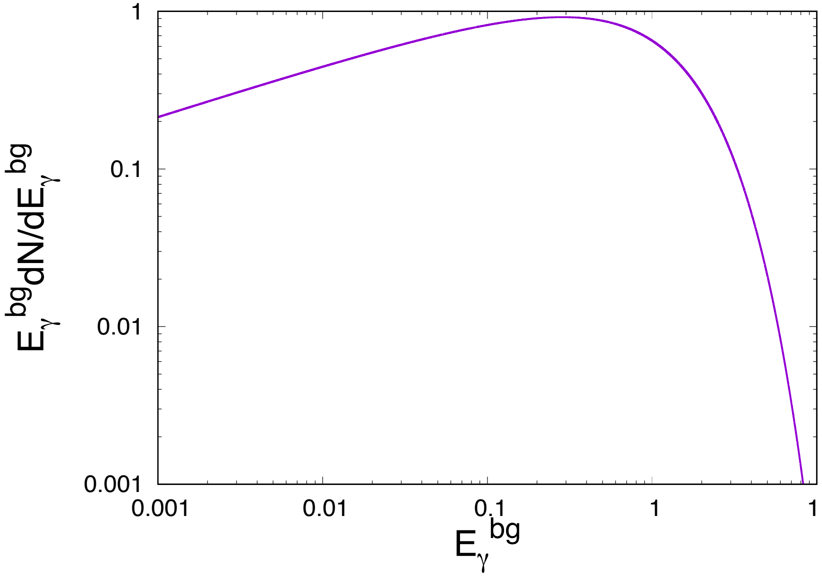
[HESS- F. Rieger, A. Taylor, et al., Nature- accepted today!]

Andrew Taylor

DESY.

Future Probes- Cutoff Region in Synchrotron Spectrum

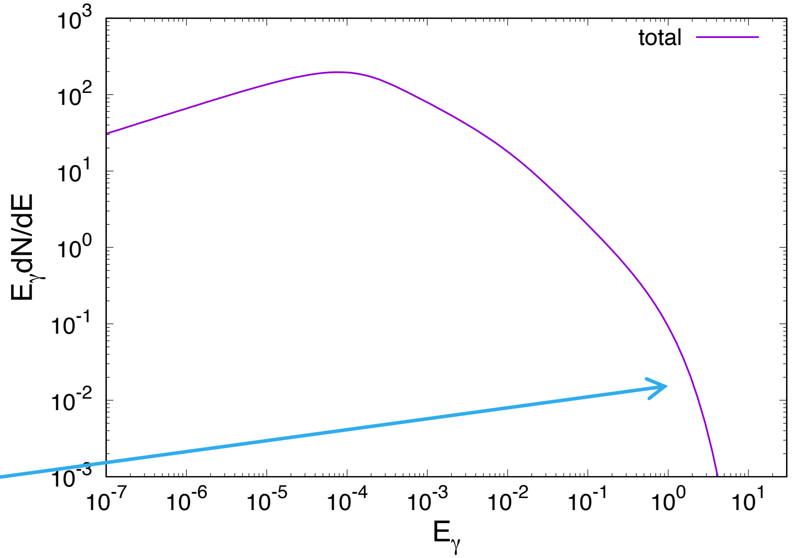




$$\mathbf{E_\gamma^{sync} = \Gamma_e^2 \left(\frac{B}{B_{crit}}\right) m_e}$$

$$\mathbf{B_{crit} = 4 \times 10^{13}\ G}$$

$$E_\gamma \frac{dN}{dE_\gamma}_{tot} = \int \left(\frac{E_\gamma}{E_e^2}\right) \frac{dN}{dE_\gamma} \left(\frac{E_\gamma}{E_e^2}\right) E_e \frac{dN}{dE_e} dE_e$$



Possibility to probe cutoff region

DESY.

Collisional Shock- Enthalpy

$$\gamma = \frac{\mathbf{w_{nonrel.}}}{\mathbf{e}}$$

$$= \frac{\mathbf{e} + \mathbf{p}}{\mathbf{e}}$$

$$\mathbf{e} = \frac{\mathbf{p}}{\gamma - \mathbf{1}}$$

$$\mathbf{w_{nonrel.}} = \frac{\gamma}{\gamma - \mathbf{1}}\mathbf{p}$$

$$\mathbf{w_{rel.}} = \frac{\gamma}{\gamma - \mathbf{1}}\mathbf{p} + \rho$$

$$= \mathbf{w_{nonrel.}} + \rho$$

Andrew Taylor

Intuitive Insights into Cut-off Shape Origin

Consider the steady-state case of diffusion (constant diffusion coefficient) of particles into an absorbing medium

$$\nabla \cdot (\mathbf{D_{xx}} \nabla \mathbf{f}) - \frac{\mathbf{f}}{\tau(\mathbf{x})} = \delta(\mathbf{r})$$

For $\tau(\mathbf{x}) = \tau_*(\mathbf{x}/\mathbf{x}_*)^{\mathbf{2}}$ $\quad \mathbf{f} \propto \text{const.}$

For $\tau(\mathbf{x}) = \tau_*$ $\quad \mathbf{f} \propto \mathbf{e}^{-\mathbf{x}/\mathbf{x}_\tau}$

For $\tau(\mathbf{x}) = \tau_*(\mathbf{x}/\mathbf{x}_*)^{-\mathbf{2}}$ $\quad \mathbf{f} \propto \mathbf{e}^{-(\mathbf{x}/\mathbf{x}_\tau)^{\mathbf{2}}}$

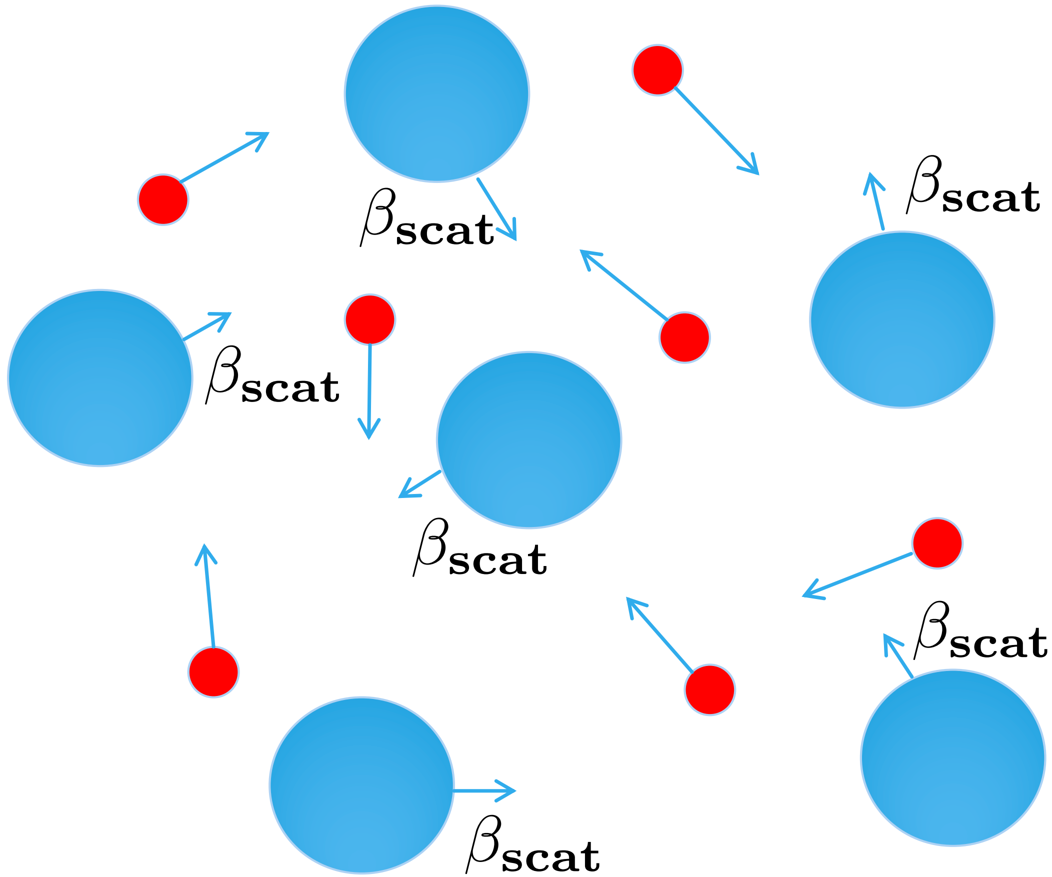
Stochastic Acceleration (Fermi Second Order)

Andrew Taylor

DESY.

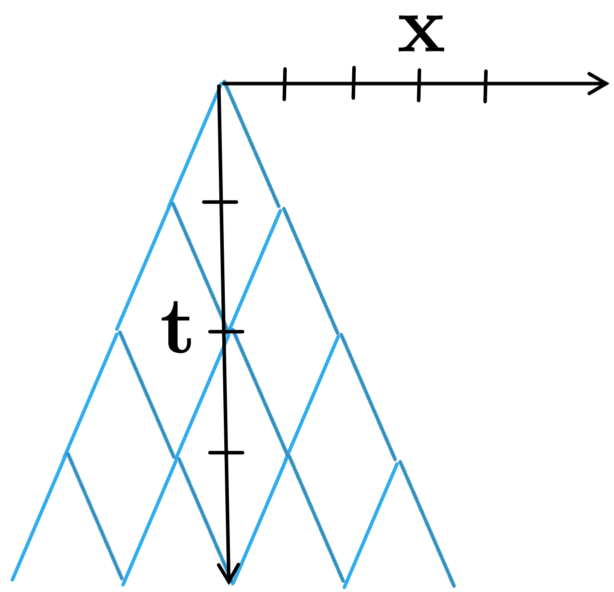
Stochastic Acceleration/Propagation

$$D_{xx}D_{pp} \approx \beta_{scat}^2 p^2$$



Andrew Taylor

Random Walks



$$\gamma(\mathbf{t}+\mathbf{1}) = \mathbf{t}!$$

$$\gamma(\mathbf{t}+\mathbf{1}) = \int_0^\infty \mathbf{x}^{\mathbf{t}} \mathbf{e}^{-\mathbf{x}} \mathbf{dx}$$

$$\mathbf{f}(\mathbf{x},\mathbf{t}) = \frac{\gamma(\mathbf{t}+\mathbf{1})}{[\gamma([\mathbf{t}-\mathbf{x}]/\mathbf{2}+\mathbf{1})\gamma([\mathbf{x}+\mathbf{t}]/\mathbf{2}+\mathbf{1})](\mathbf{2}^{\mathbf{t}})}$$

$$\mathbf{f}(\mathbf{x},\mathbf{t}) \approx \frac{\mathbf{e}^{-\mathbf{x}^2/(\mathbf{2t})}}{[\pi/(\mathbf{t}/\mathbf{2})]^{\mathbf{1}/\mathbf{2}}}$$

To be returned to in a later lecture.....
no harm at having a go at deriving
this from above

Random Walks in Physical Space and Momentum Space

Spatial spread:

$$\mathbf{\Delta x = D_{xx}/c} \qquad \mathbf{\frac{dN}{dx} \propto e^{-x^2/4D_{xx}t}}$$

$$\mathbf{\frac{dN}{dx} \propto e^{-x^2/4c^2 t_{scat} t}}$$

Momentum spread:

$$\mathbf{\frac{\Delta E}{E}} \propto \beta$$

$$\mathbf{\frac{dN}{dp} \propto e^{-(\ln p)^2/4(D_{pp}/p^2)t}}$$

$$\mathbf{t_{acc} = \frac{p^2}{D_{pp}}}$$

$$\mathbf{\frac{dN}{dp} \propto e^{-(\ln p)^2/4(t/t_{acc})}}$$

Andrew Taylor

DESY.

Momentum Continuity Equation (Boltzman Equation)

$$\frac{\partial \mathbf{f}}{\partial \mathbf{t}} = \nabla_{\mathbf{p}} \cdot \left[(\mathbf{D}_{\mathbf{pp}} \nabla_{\mathbf{p}} \mathbf{f}) - \frac{\mathbf{p}}{\tau_{\mathbf{loss}}(\mathbf{p})} \mathbf{f} \right] - \frac{\mathbf{f}}{\tau_{\mathbf{esc}}(\mathbf{p})} + \frac{\mathbf{Q}}{\mathbf{p}^2}$$

Acceleration

Radiative Losses

Escape

Source term

Where f is the phase space density:

$$\mathbf{f} = \frac{\mathbf{dN}}{\mathbf{d^3xd^3p}}$$

DESY.

Andrew Taylor

Stochastic Particle Acceleration- Random Walk Result (Momentum)

$$\frac{\partial \mathbf{f}}{\partial \mathbf{t}} = \nabla_{\mathbf{p}} \cdot \left[(\mathbf{D_{pp}} \nabla_{\mathbf{p}} \mathbf{f}) - \frac{\mathbf{p}}{\tau_{\mathbf{loss}}(\mathbf{p})} \mathbf{f} \right] - \frac{\mathbf{f}}{\tau_{\mathbf{esc}}(\mathbf{p})} + \frac{\mathbf{Q}}{\mathbf{p^2}}$$

Steady state

No losses

Delta injection

Assuming $\mathbf{D_{pp}} \propto \mathbf{p^q}$

$$\frac{\partial^2 \mathbf{f}}{\partial \mathbf{p^2}} + \frac{(\mathbf{2}+\mathbf{q})}{\mathbf{p}} \frac{\partial \mathbf{f}}{\partial \mathbf{p}} - \frac{\mathbf{4}\tau_{\mathbf{acc}}}{\tau_{\mathbf{esc}}} \frac{\mathbf{f}}{\mathbf{p^2}} = \delta(\mathbf{p})$$

For $\mathbf{f} = \mathbf{p}^{-\alpha}$ and $\mathbf{q} = \mathbf{2}$

$$\alpha^{\mathbf{2}} - \mathbf{3}\alpha - \frac{\mathbf{4}\tau_{\mathbf{acc}}}{\tau_{\mathbf{esc}}} = \mathbf{0}$$

$$\alpha^2 - 3\alpha - \frac{4\tau_{\mathrm{acc}}}{\tau_{\mathrm{esc}}} = 0$$

PAUSE

Why not have a go at solving this

Particle Acceleration- When Are E^{-2} Type Spectra Expected?

$$\alpha^2 - 3\alpha - \frac{4\tau_{acc}}{\tau_{esc}} = 0$$

$$\alpha = \frac{3}{2} \pm \left(\frac{4\tau_{acc}}{\tau_{esc}} + \frac{9}{4}\right)^{1/2}$$

$$\frac{\tau_{acc}}{\tau_{esc}} = 1$$

$$f = \frac{dN}{d^3p} = p^{-4}$$

Andrew Taylor

DESY.

Fermi (Second Order) Acceleration Time

$$\mathbf{t_{acc} = E\frac{\Delta t_{scat}}{\Delta E_{scat}}}$$

$$\mathbf{\Delta E_{scat} = E}\beta^{\mathbf{2}}_{\mathbf{scat}}$$

$$\mathbf{t_{acc}} = \frac{\mathbf{t_{scat}}}{\beta^{\mathbf{2}}_{\mathbf{scat}}}$$