Factorization (or not?)

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DIPARTIMENTO DI FISICA
"ALDO PONTREMOLI"

Resummation, Evolution, Factorization 2025

Milan, October 2025

Summary

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- Relevance of Factorization in general

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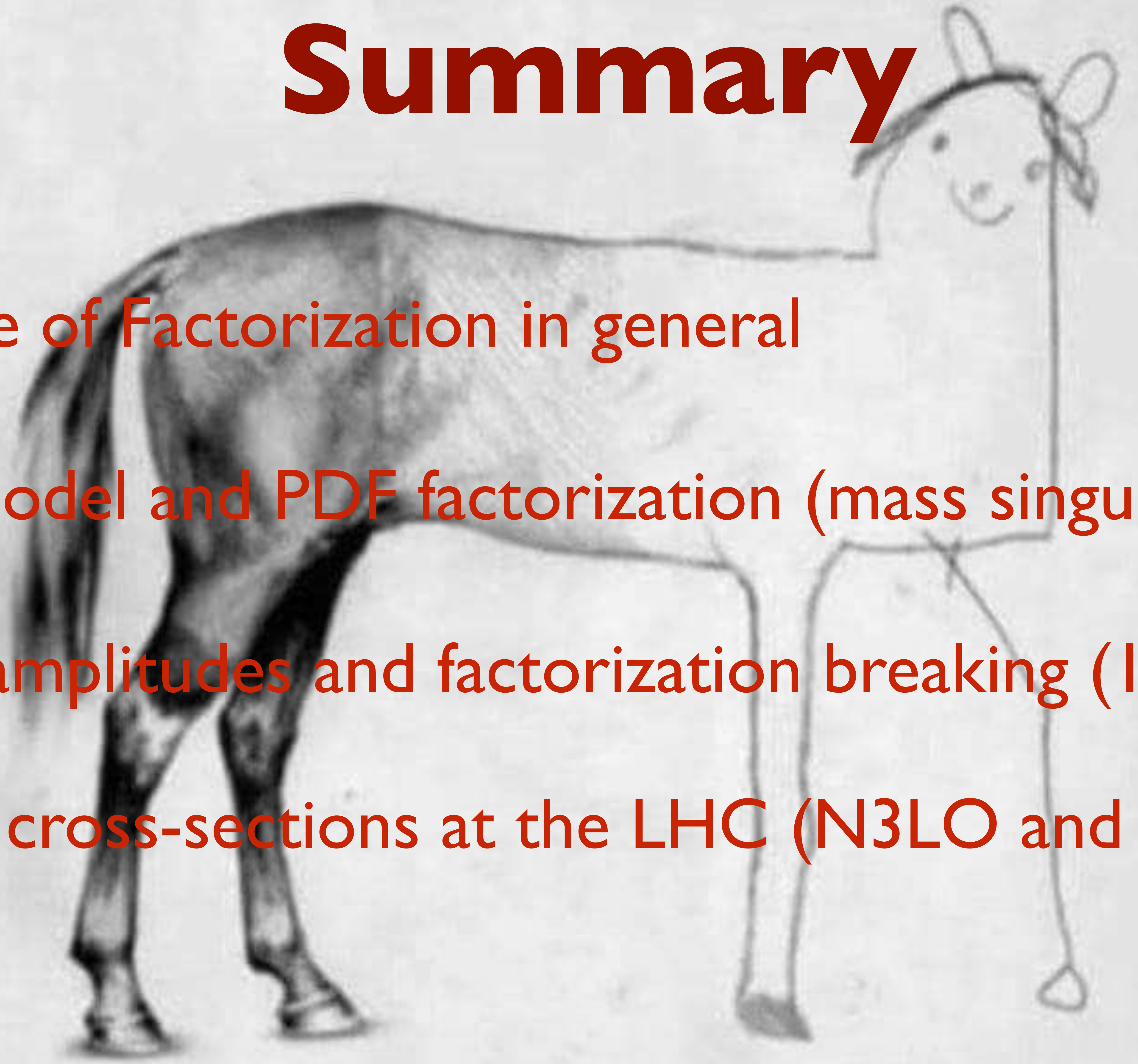

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- Effect on cross-sections at the LHC (N3LO and N4LO…)

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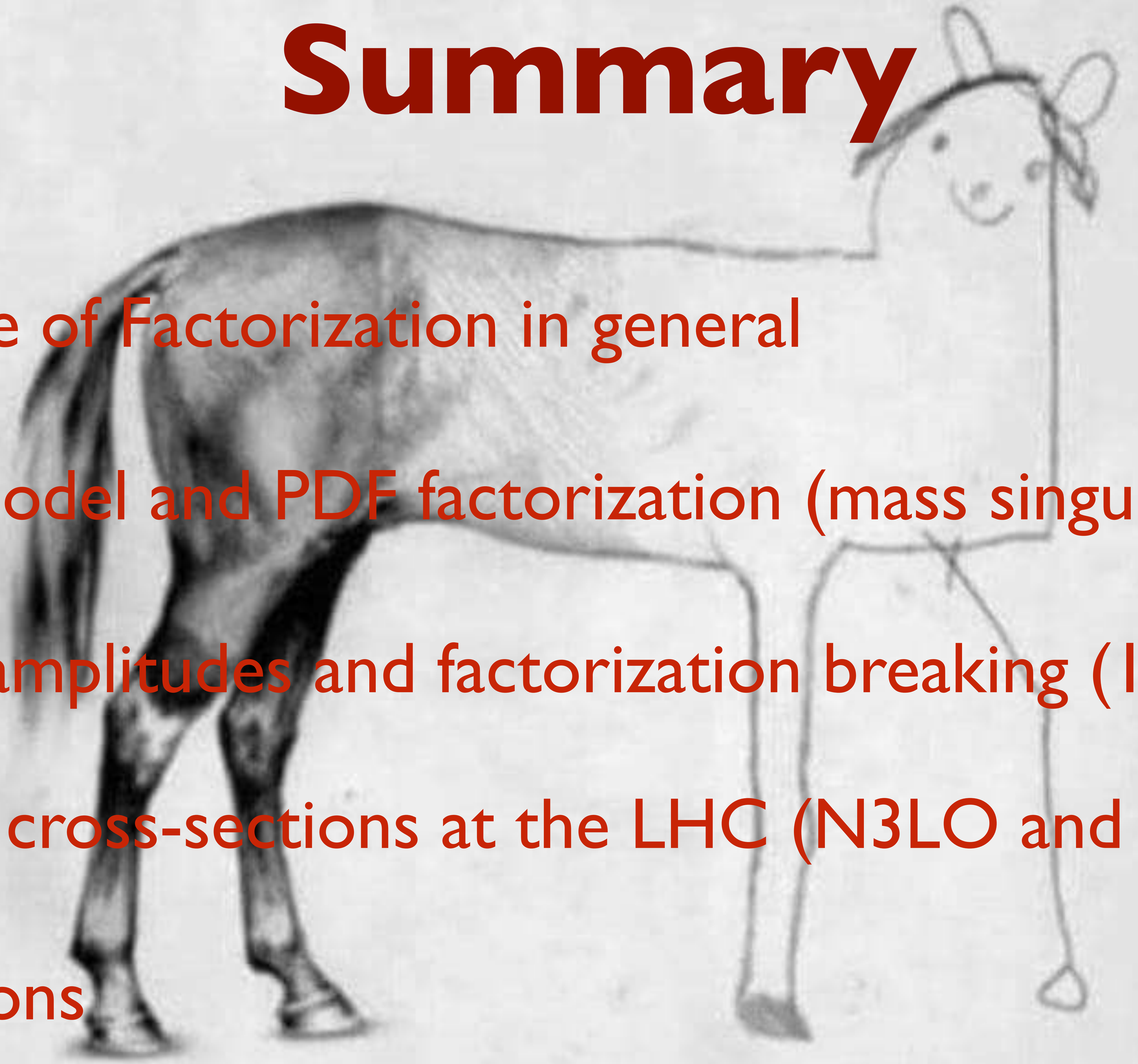

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- Parton Model and PDF factorization (mass singularities)
- Splitting amplitudes and factorization breaking (1 and 2 loops)
- Effect on cross-sections at the LHC (N3LO and N4LO…)
- Conclusions

Factorization in high school

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Main Classes of Polynomial Factorization

1. **Common Factor Extraction**
   $ax + ay = a(x + y)$

2. **Difference of Squares**
   $a^2 - b^2 = (a - b)(a + b)$

3. **Perfect Square Trinomials**
   $a^2 + 2ab + b^2 = (a + b)^2$
   $a^2 - 2ab + b^2 = (a - b)^2$

4. **Sum and Difference of Cubes**
   $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
   $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

5. **Quadratic Trinomials**
   $ax^2 + bx + c = (px + q)(rx + s)$

6. **Grouping Method**
   $ax + ay + bx + by = (a + b)(x + y)$

Nicer, simpler, more elegant results 😀

Factorization in life

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- Work and Holidays : need strict factorization!

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- Really relevant : Church and State, keep them separate

Factorization in Science (general)

Identify relevant scale, degrees of freedom and effective theory to provide the best description of the process in the simplest way. Also allows to simplify complex problems into manageable pieces that can later be recombined.

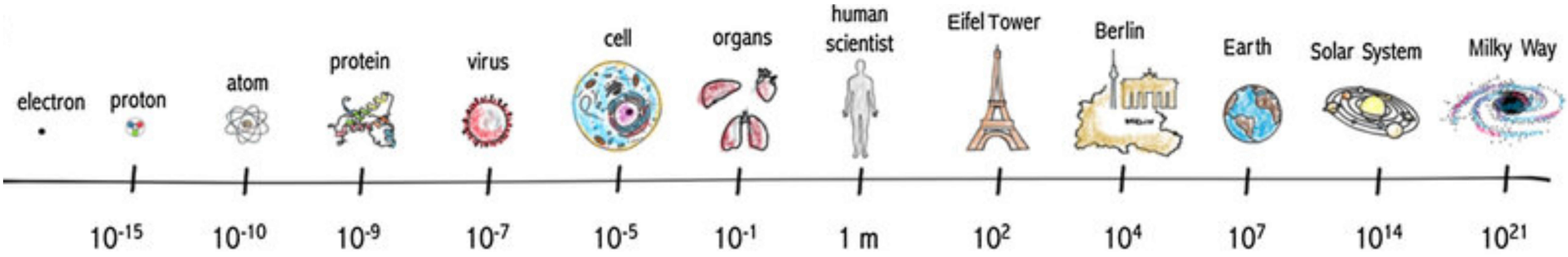
science relies on factorization

nuclear physics

chemistry

medicine

geology



particle physics

quantum physics

biology

engineering

cosmology

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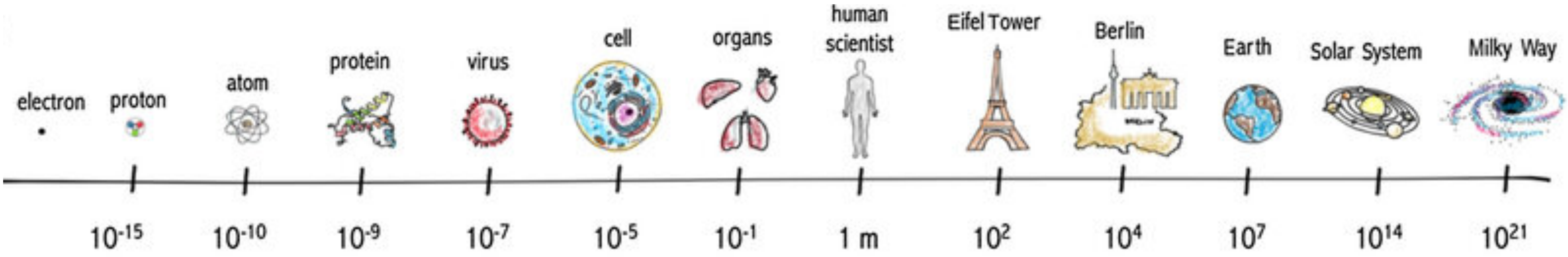

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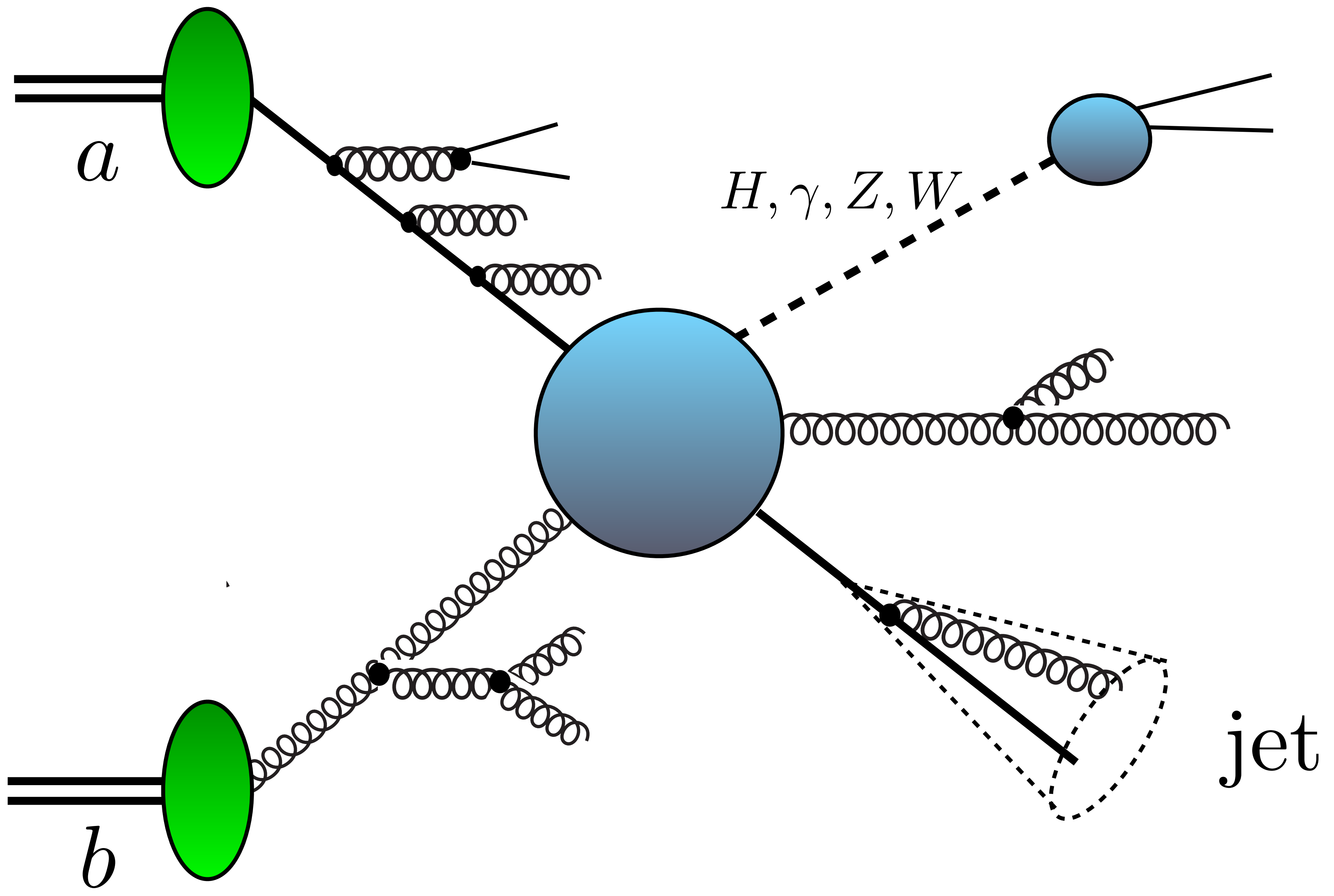
When the car breaks, goes to the mechanic, not to the quantum mechanic… relevance

(some) Factorization in QCD

- Not a full discussion about factorization in QCD
- PDF factorization (a very rough sketch of factorization of mass singularities)
- “Origin" of PDF factorization from collinear factorization of amplitudes
- Strict collinear factorization and breaking at higher orders (most of the talk)
- What other people did on the subject and I still don’t understand…

- In the LHC era, QCD is everywhere and factorization essential!



non-perturbative parton distributions

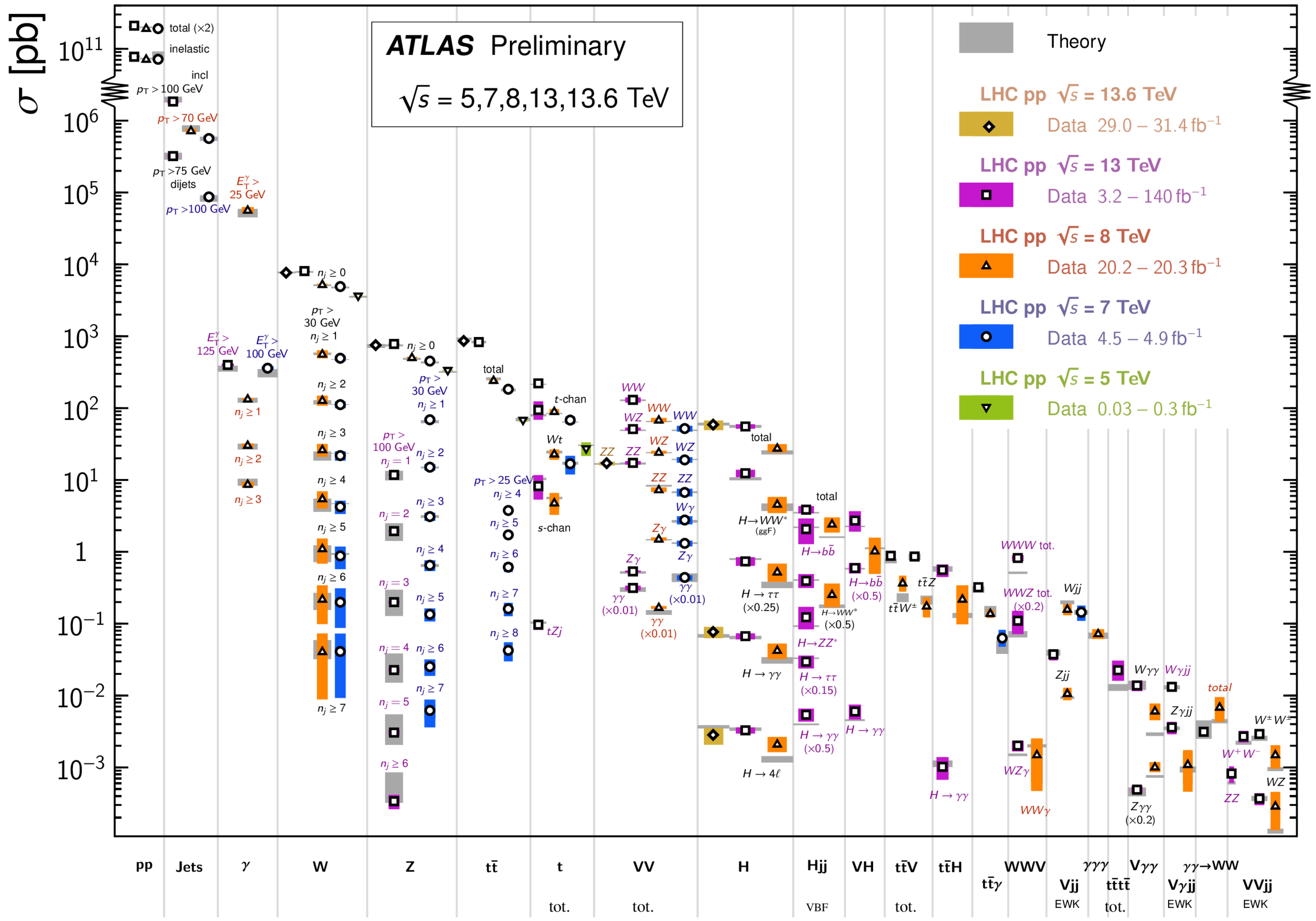
$$d\sigma = \sum_{ab} \int dx_a \int dx_b\, f_a(x_a, \mu_F^2) f_b(x_b, \mu_F^2) \times d\hat{\sigma}_{ab}(x_a, x_b, Q^2, \alpha_s(\mu_R^2)) + \mathcal{O}\left(\left(\frac{\Lambda}{Q}\right)^m\right)$$

perturbative partonic cross-section

Partonic cross-section: expansion in $\alpha_s(\mu_R^2) \ll 1$ $\quad d\hat{\sigma} = \alpha_s^n\, d\hat{\sigma}^{(0)} + \alpha_s^{n+1}\, d\hat{\sigma}^{(1)} + \ldots$

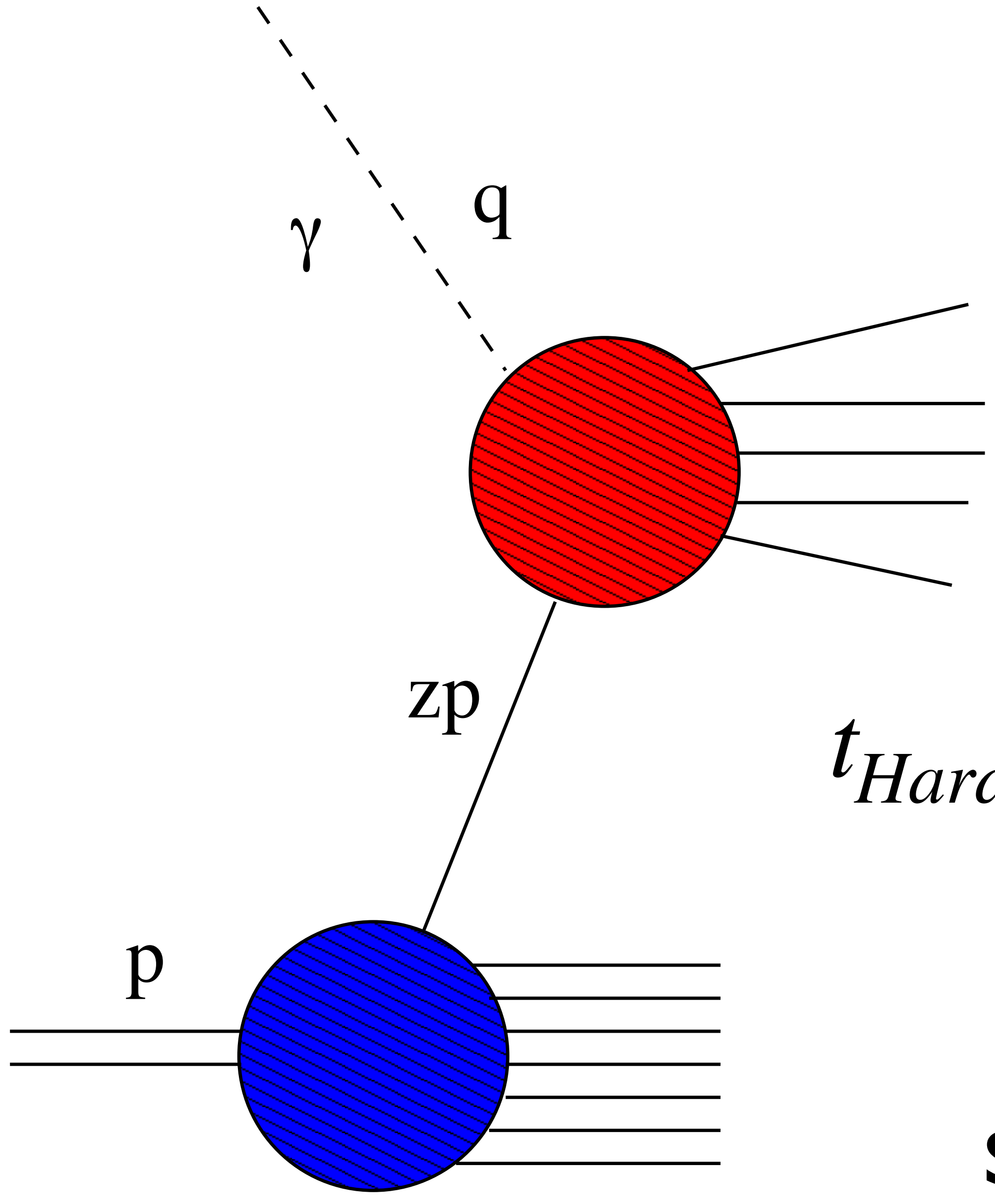
- parton distributions obtained from global fits thanks to UNIVERSALITY (FACTORIZATION)

Standard Model Production Cross Section Measurements



Parton Model

Factorization



hard interaction time

$$t_{Hard} \sim \frac{1}{Q} \ll t_{soft} \sim \frac{1}{\Lambda_{QCD}}$$

soft interaction time

large distances

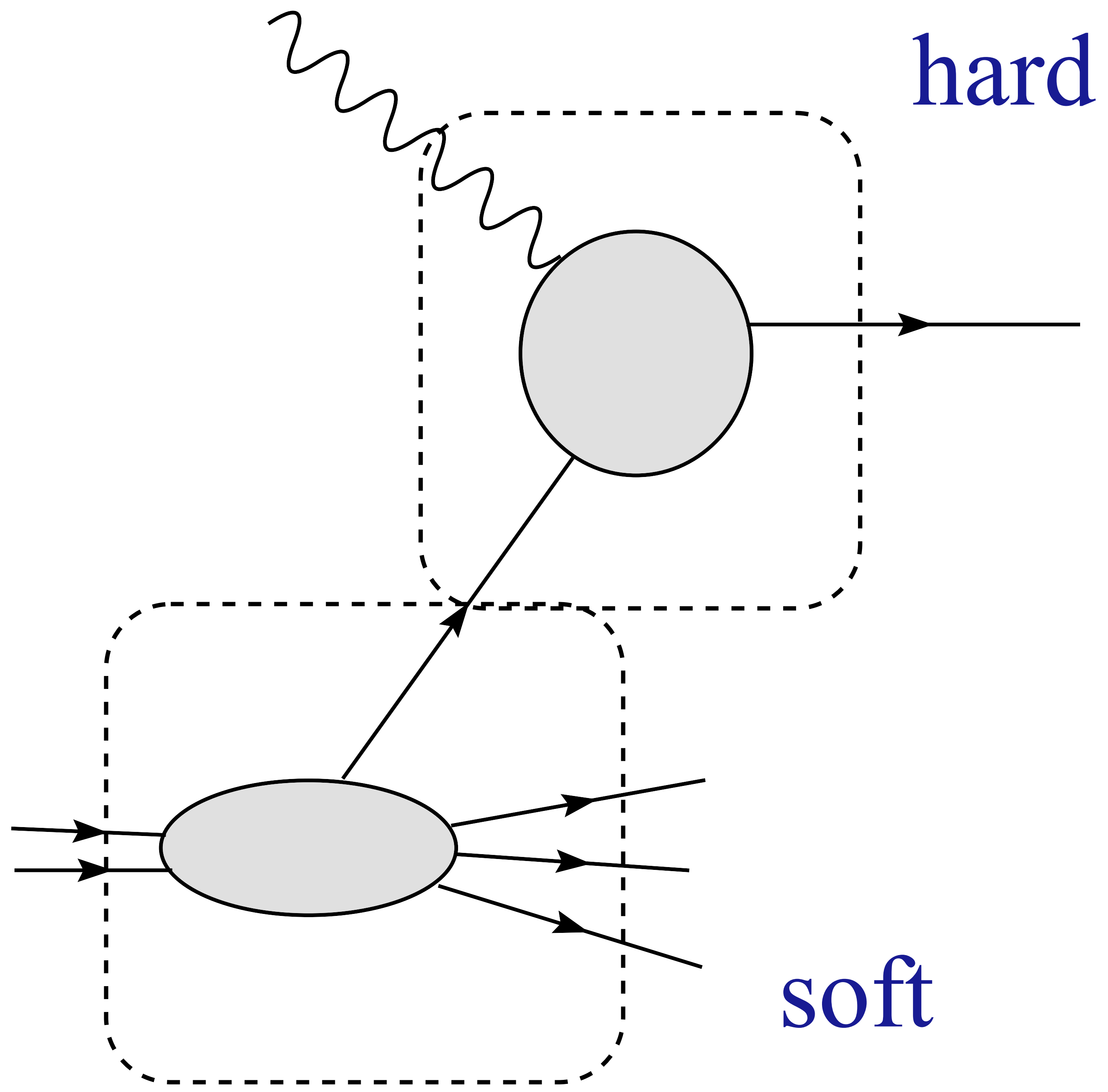
$$\sigma(ep \to eX) = \int_0^1 dz \sum_i f_i(z)\, \hat{\sigma}(eq_i \to eX)$$

small distances

- Probability to find parton "i" with momentum fraction z in proton
- $\hat{\sigma}$ ="partonic" cross section → computed perturbatively
- Parton distributions (PDF) from experiments : ***universal***

$$F_2(x, \cancel{Q^2}) = \sum_q e_q^2\, x\, f_q(x)$$ Bjorken scaling

Parton model with QCD corrections: separation between soft and hard physics



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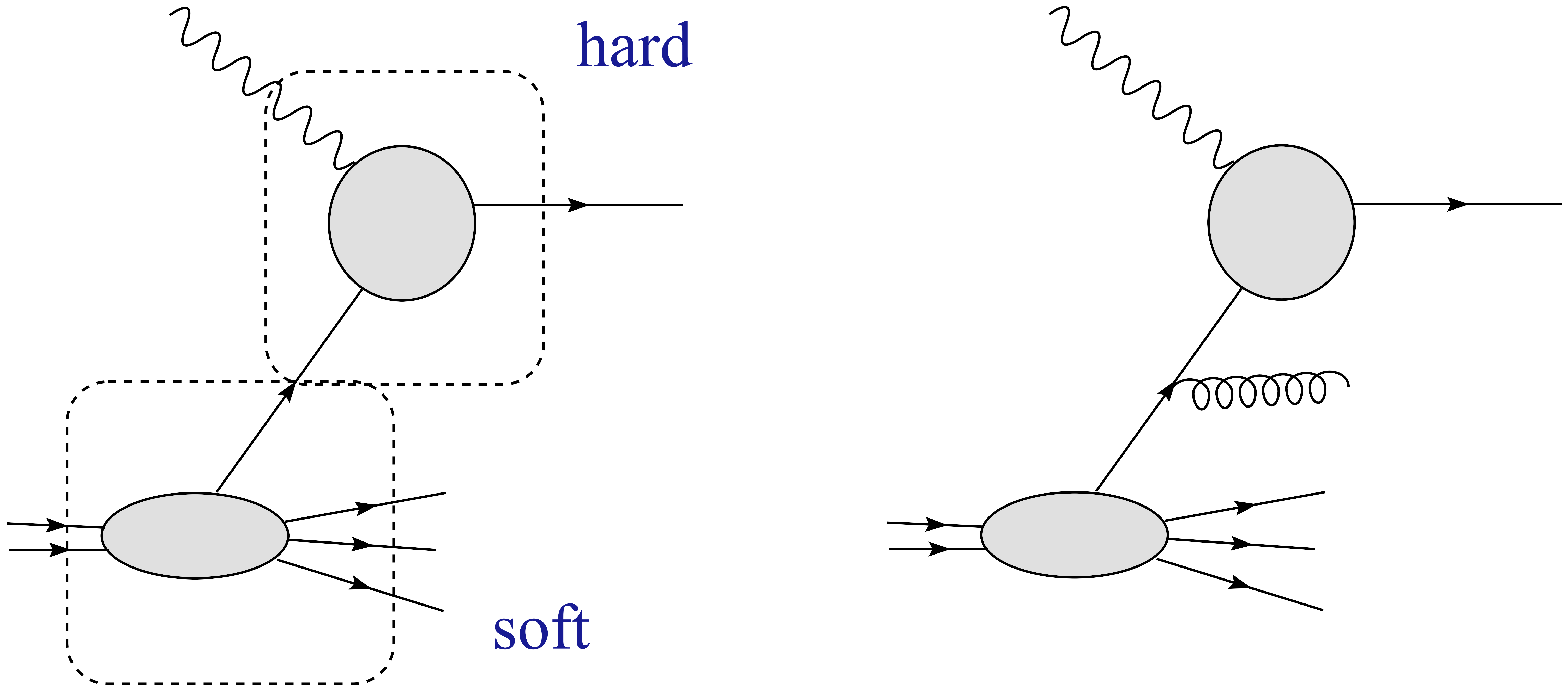

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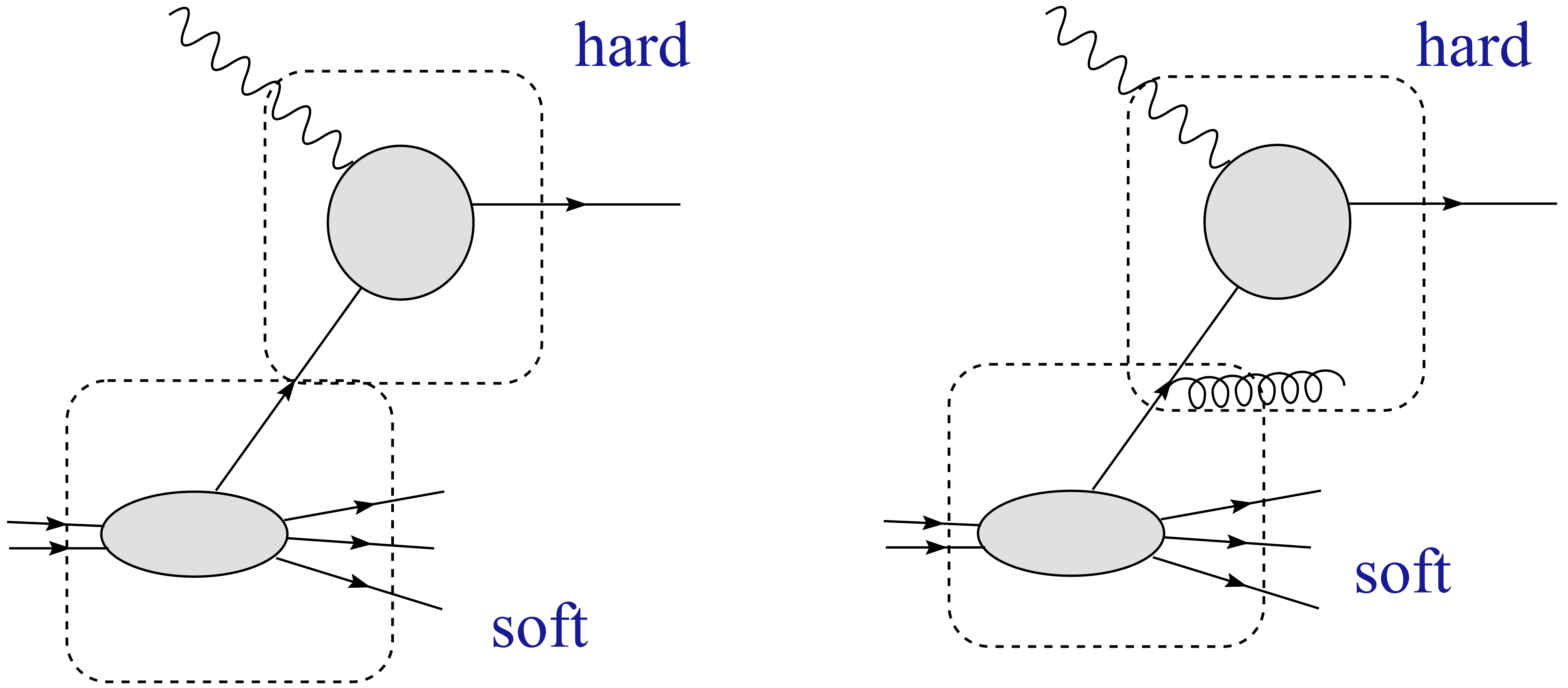

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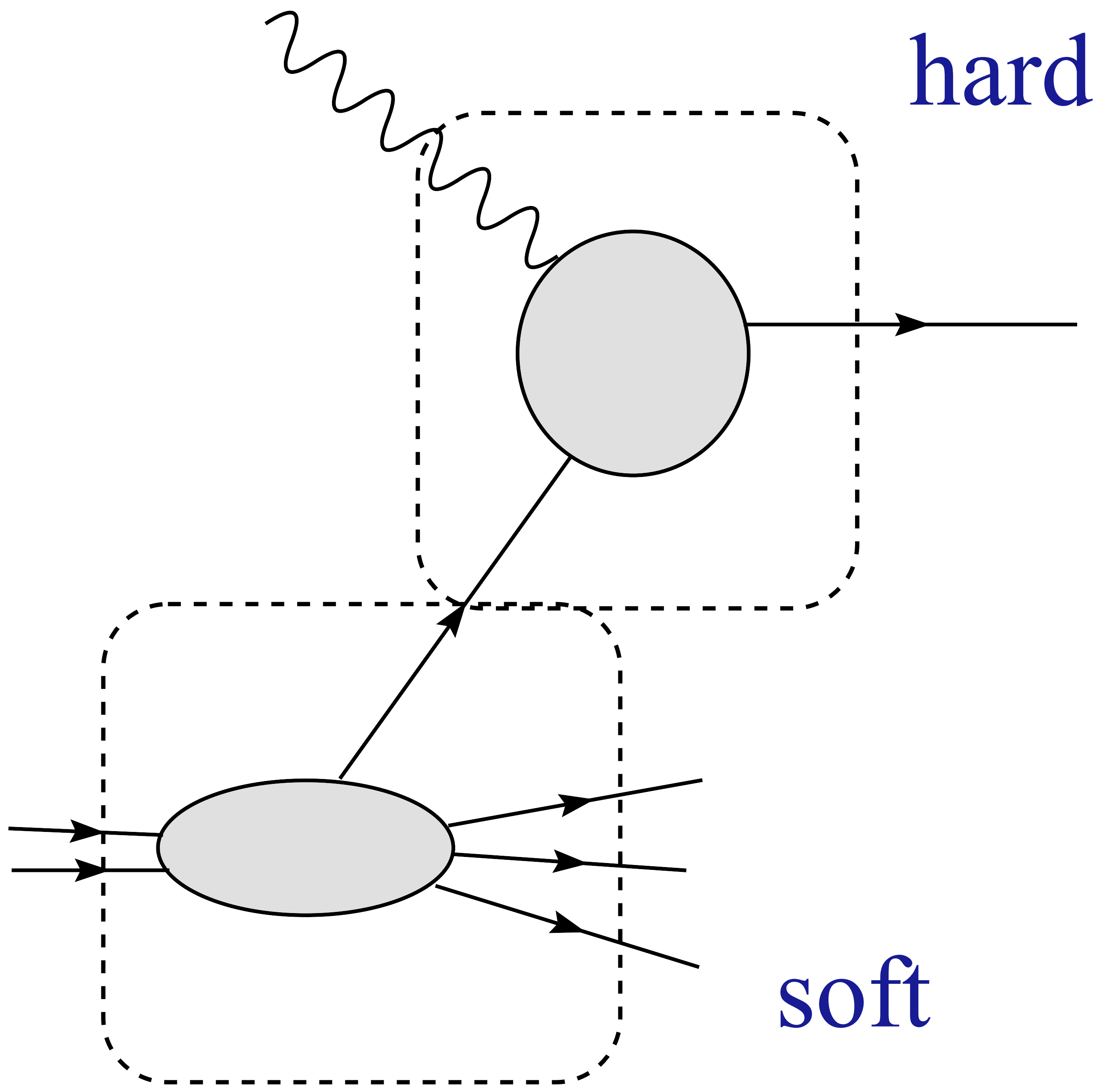

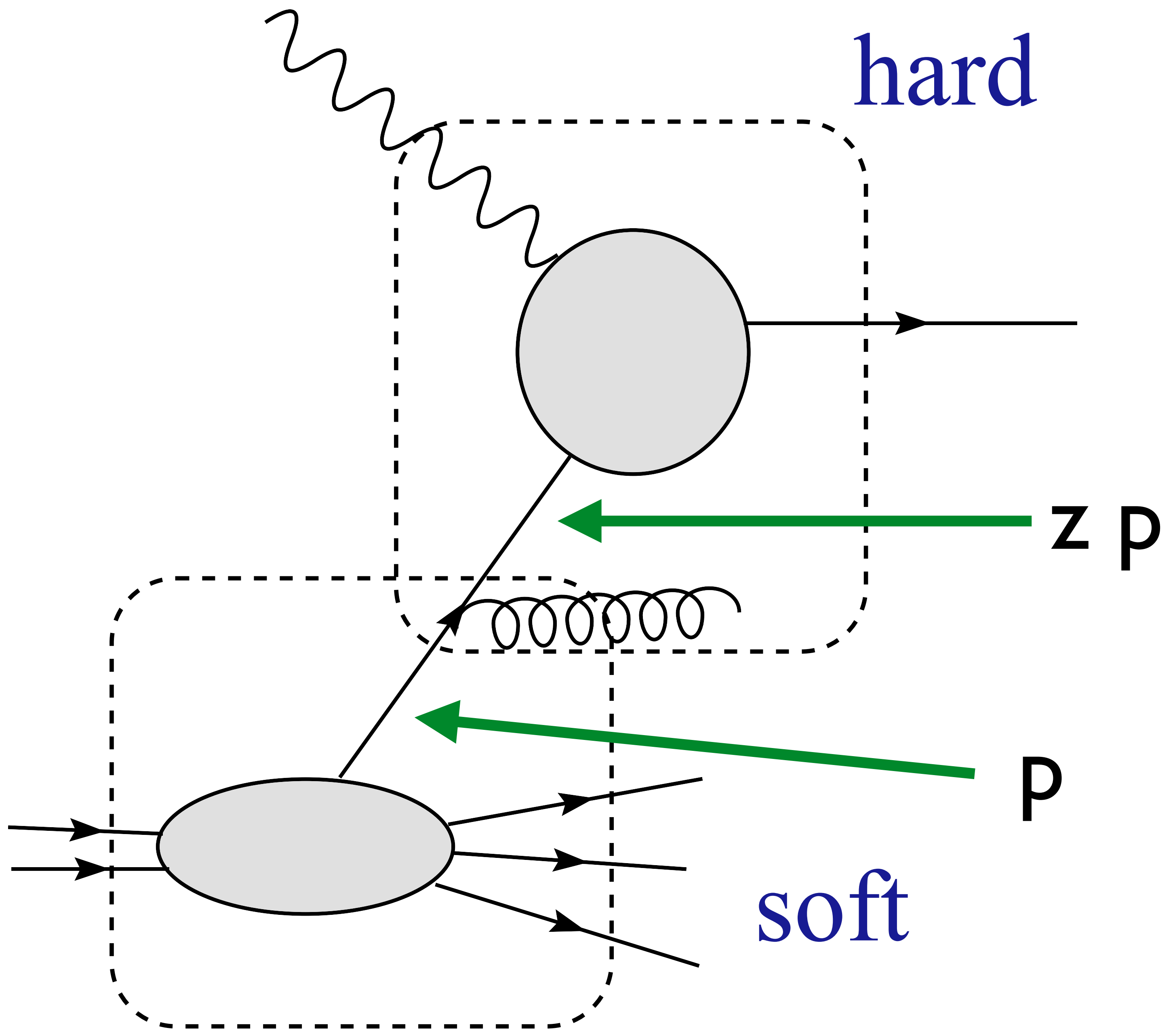

and result is divergent … $k_T \to 0$

$$\left|\mathcal{M}_{\gamma^* q\to q\, g}\right|^2 \simeq \frac{2}{k_T^2} 4\pi\alpha_S \boxed{\hat{P}_{qq}(z)} \left|\mathcal{M}_{\gamma^* q\to q}\right|^2$$

prob. of a quark with momentum fraction z

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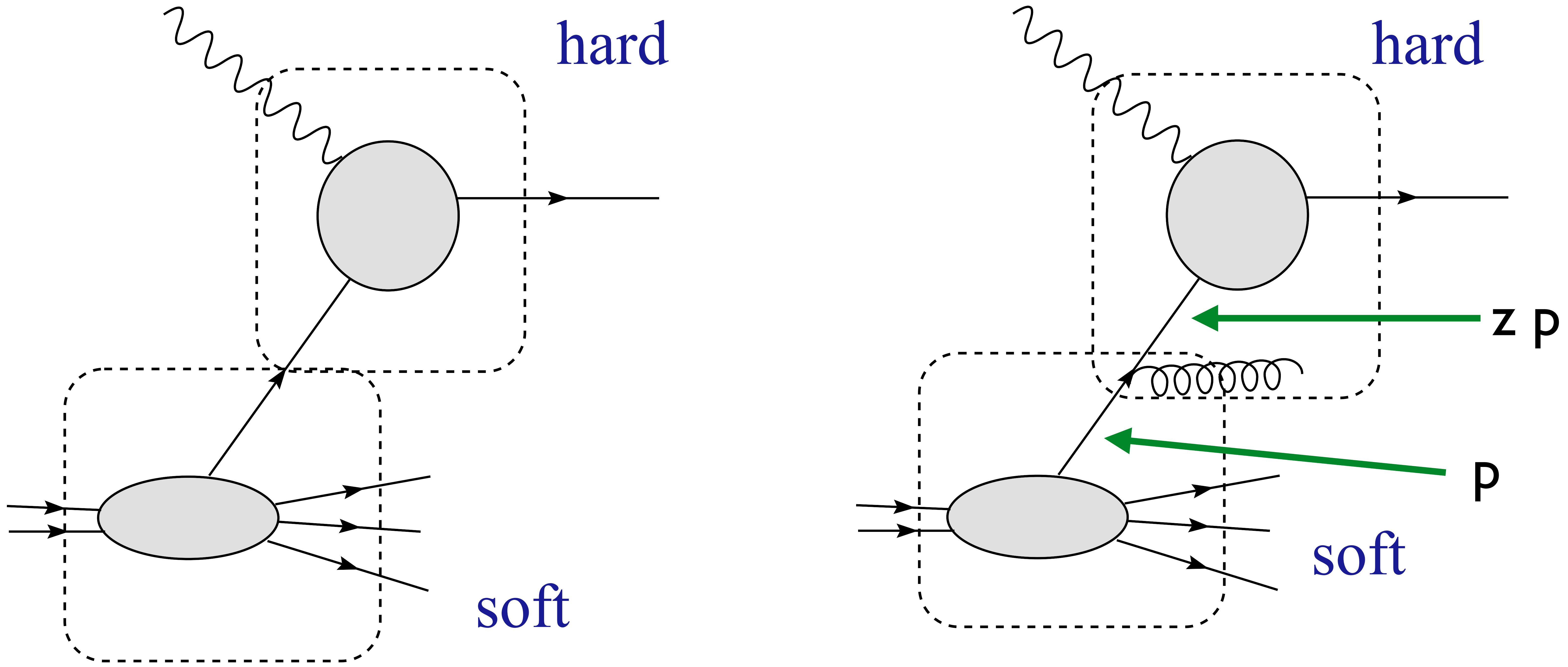

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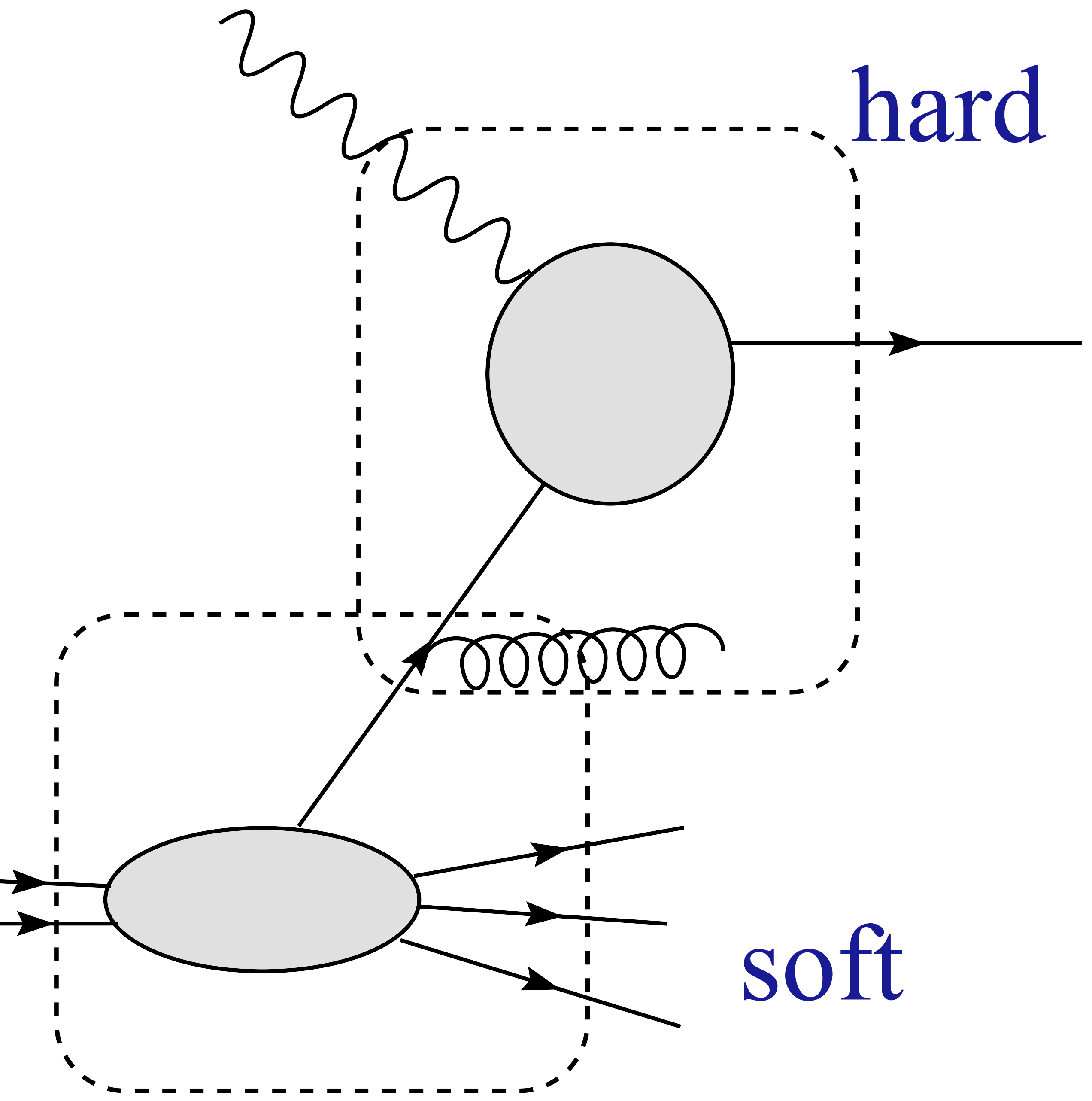
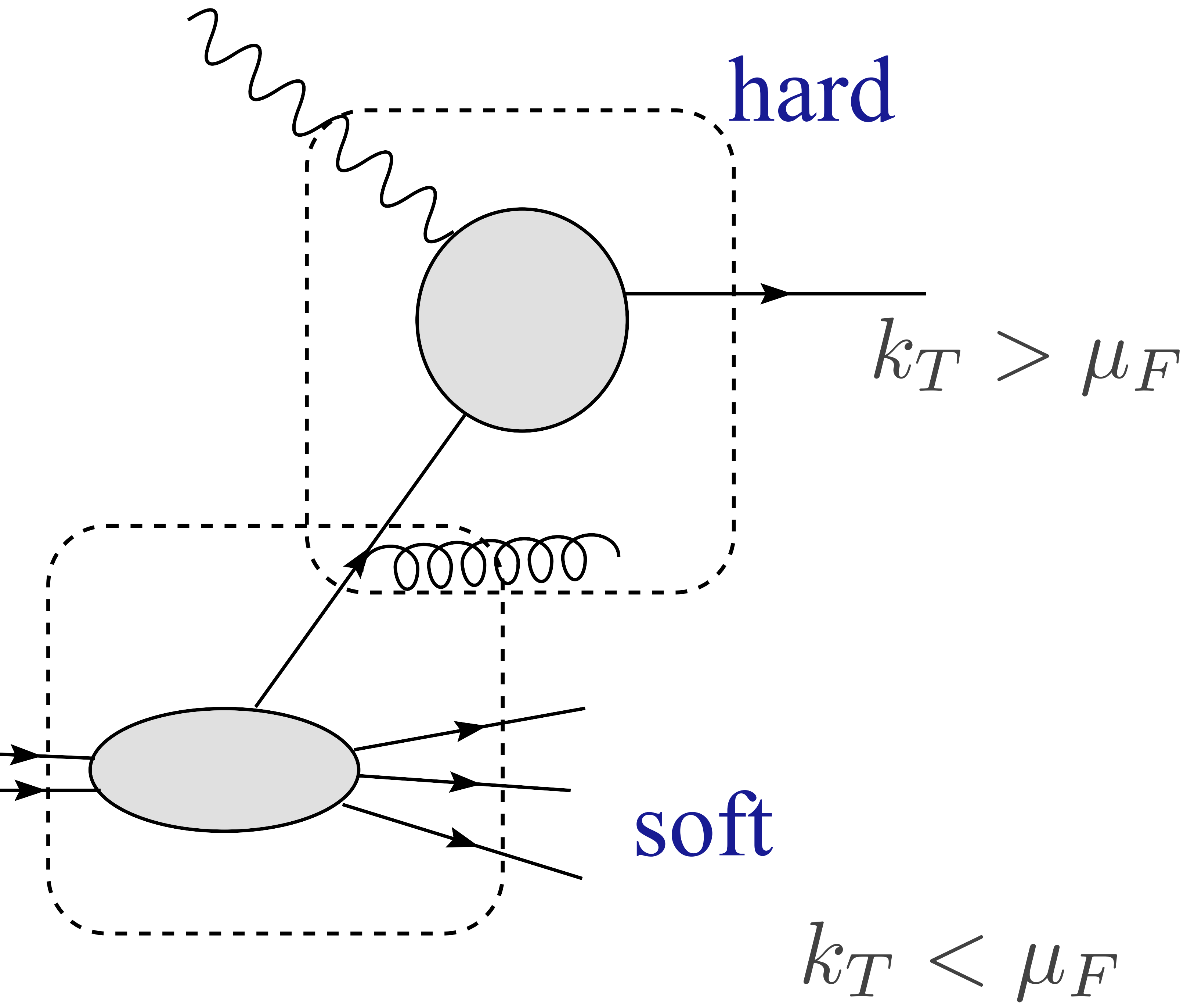
Regularize the divergence with a cut-off $\int_{\mu_0^2}^{Q^2} \frac{dk_T^2}{k_T^2} \sim \log\left(\frac{Q^2}{\mu_0^2}\right)$

$\mu_0^2 \lesssim k_T^2 < Q^2$

prob. of a quark with momentum fraction z



$$F_2^{cor}(x,Q^2) = \sum_q e_q^2\, x\, \frac{\alpha_s}{2\pi} \log \underbrace{\left(\frac{Q^2}{\mu_0^2}\right)} \int_x^1 \frac{dy}{y} P_{qq}(y)\, q\left(\frac{x}{y}\right) + \text{finite}$$



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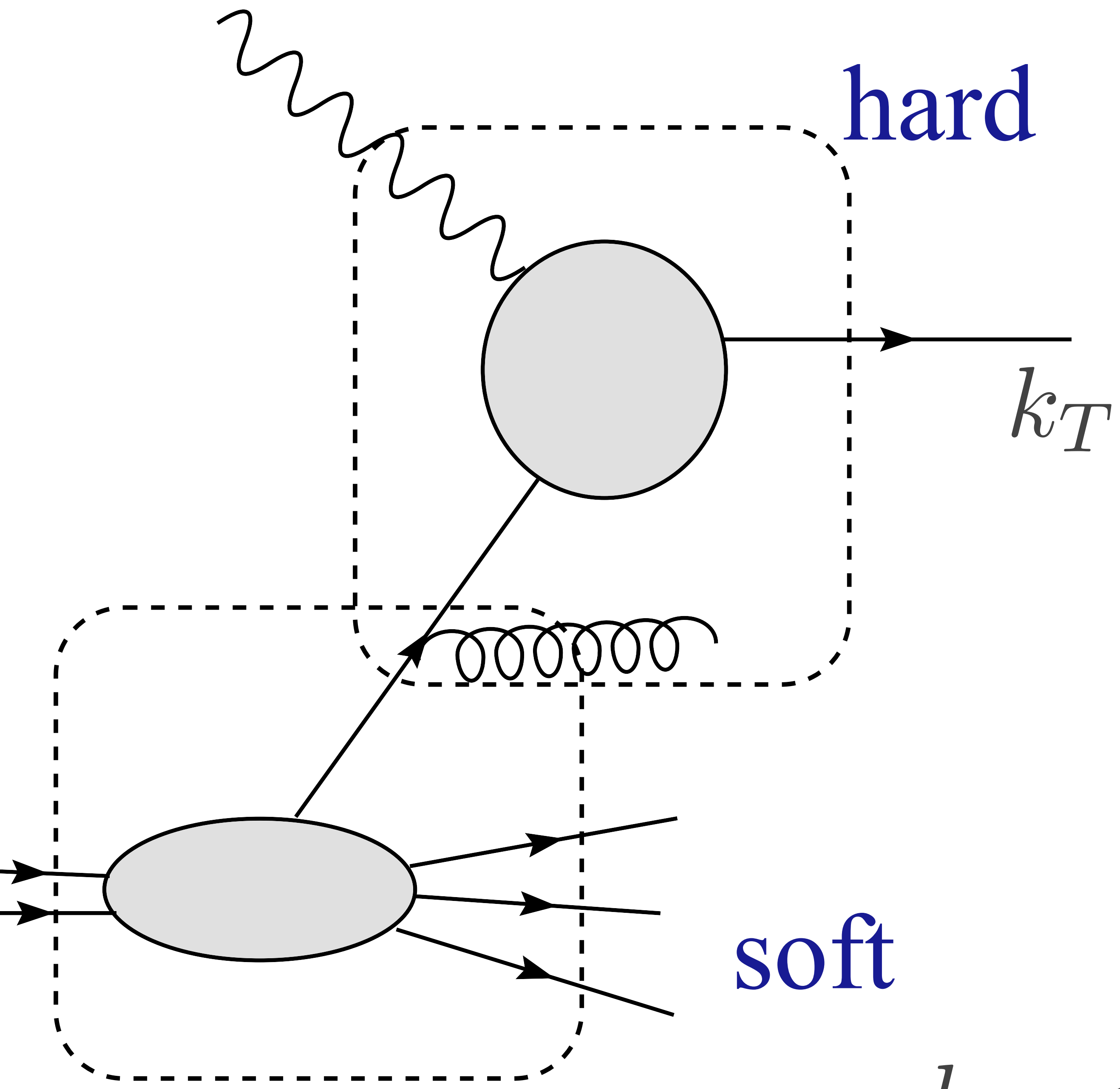
Introduce new factorization scale



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$$\log\left(\frac{Q^2}{\mu_0^2}\right) = \log\left(\frac{\mu_F^2}{\mu_0^2}\right) + \log\left(\frac{Q^2}{\mu_F^2}\right)$$

soft (and divergent) to PDF

Hard (and finite)

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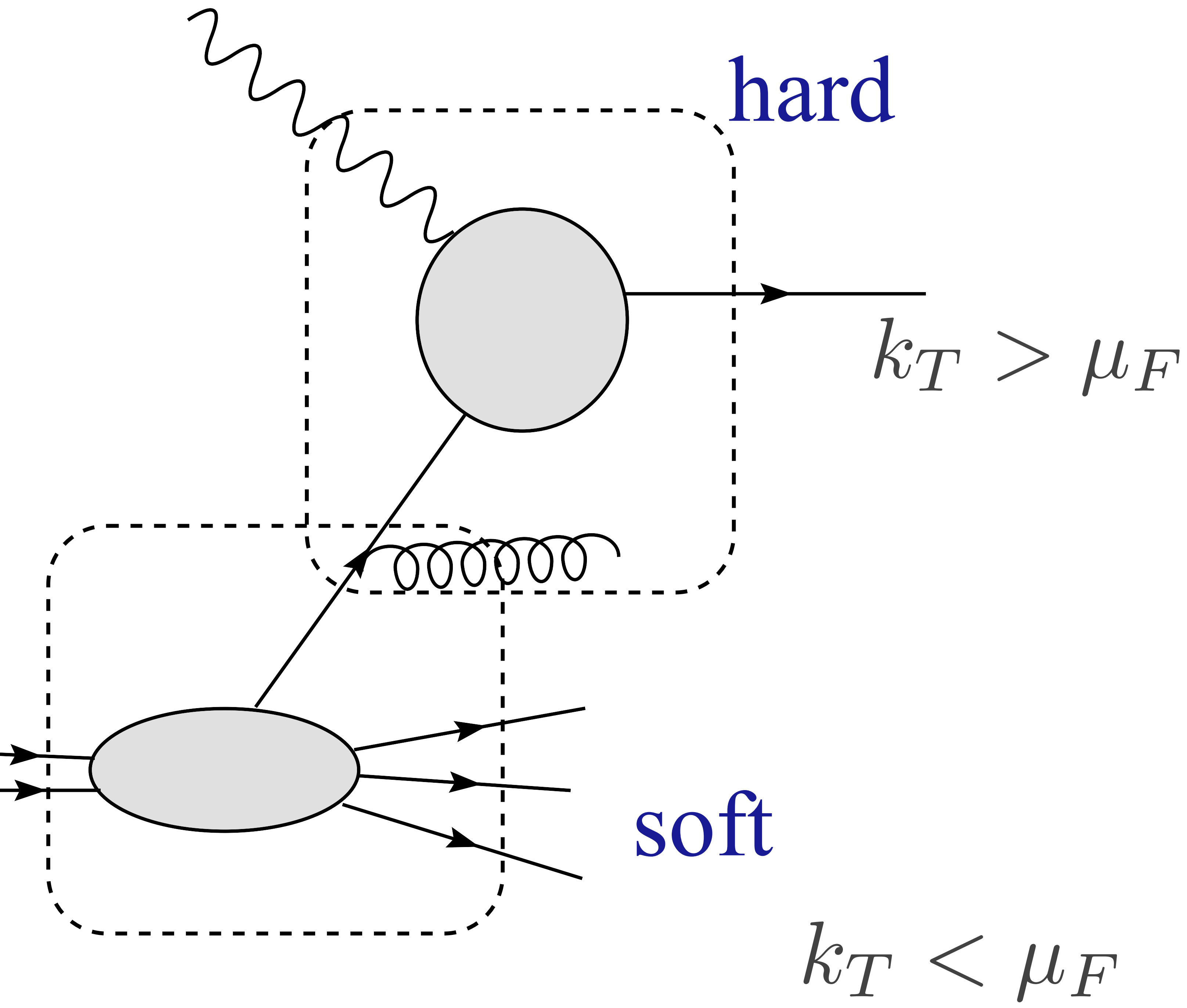

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Introduce new factorization scale

Factorization IR equivalent to UV renormalization (DR and fact. scheme)

$$q(x,\mu_F^2) = q(x) + \frac{\alpha_s}{2\pi} \log\left(\frac{\mu_F^2}{\mu_0^2}\right) \int_x^1 \frac{dy}{y} P_{qq}(y)\, q\left(\frac{x}{y}\right)$$

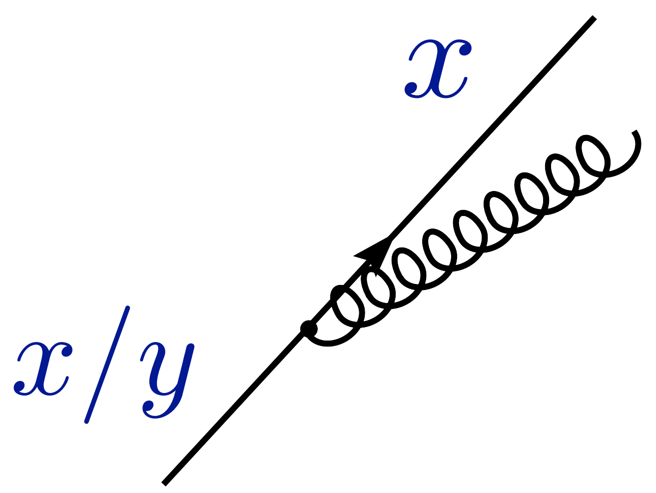
Cross section (structure function) finite in terms of UNIVERSAL factorized PDFs

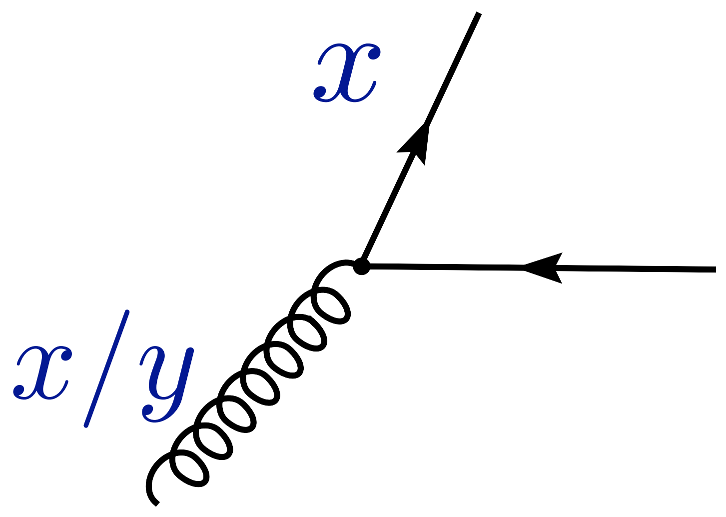
Altarelli-Parisi equation : SPLITTING FUNCTIONS (RGE like: resummation of collinear logs)

Increase “resolution” scale: resolve more details of “partonic structure”

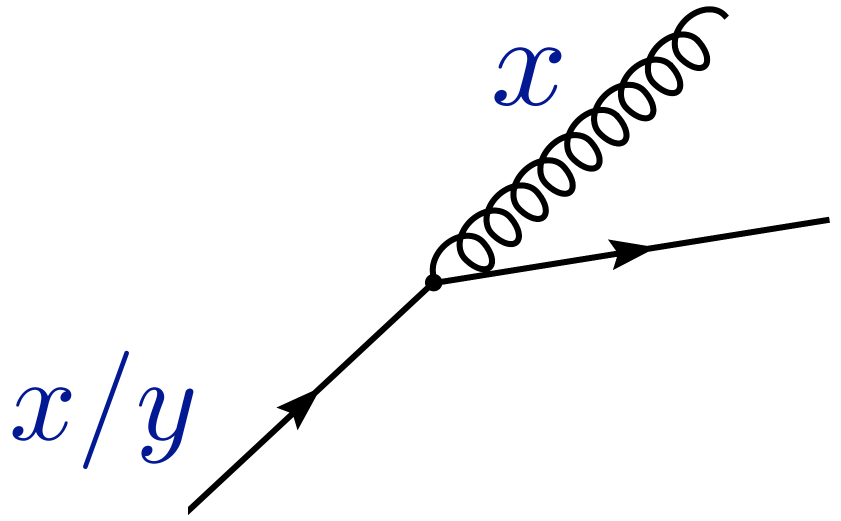
$$\frac{\partial q(x,\mu_F^2)}{\partial \log(\mu_F^2)} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dy}{y} P_{qq}(y)\, q\left(\frac{x}{y},\mu_F^2\right) + \frac{\alpha_s}{2\pi}\int_x^1 \frac{dy}{y} P_{qg}(y)\, g\left(\frac{x}{y},\mu_F^2\right)$$

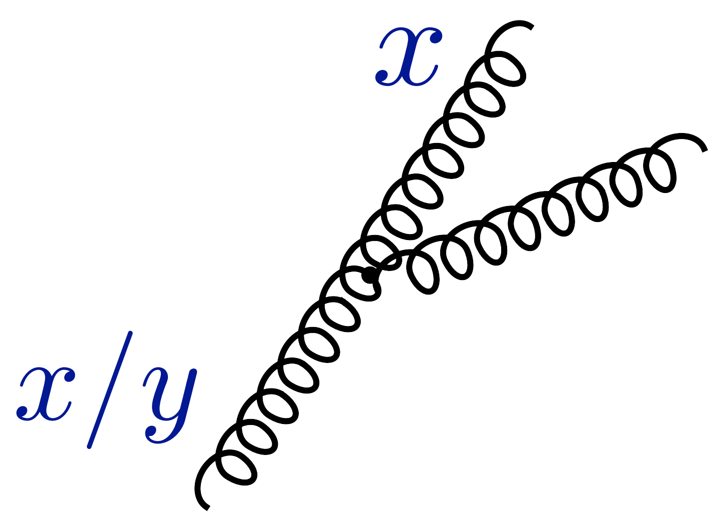
Probabilistic interpretation





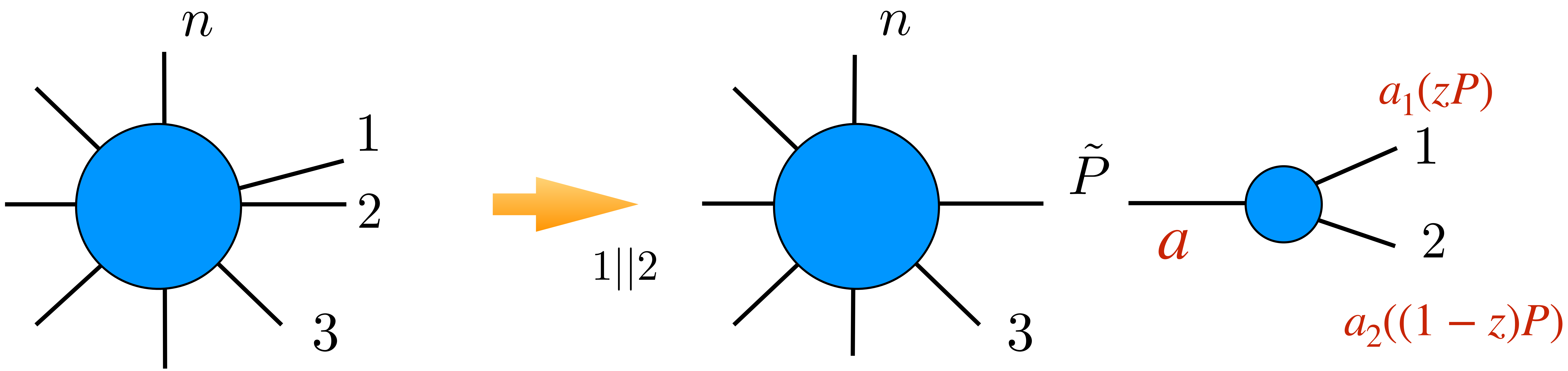
$$\frac{\partial g(x,\mu_F^2)}{\partial \log(\mu_F^2)} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dy}{y} P_{gq}(y) \sum_q q\left(\frac{x}{y},\mu_F^2\right) + \frac{\alpha_s}{2\pi}\int_x^1 \frac{dy}{y} P_{gg}(y)\, g\left(\frac{x}{y},\mu_F^2\right)$$





Origin of pdf factorization: Strict factorization at the amplitude level

n particle amplitude



1-*2* collinear

n-1 particle amplitude

Collinear limit

Kinematical details $k_\perp \to 0$

$$p_1^\mu = zp^\mu + k_\perp^\mu - \frac{k_\perp^2}{z}\frac{n^\mu}{2p\cdot n} \qquad p_2^\mu = (1-z)p^\mu - k_\perp^\mu - \frac{k_\perp^2}{1-z}\frac{n^\mu}{2p\cdot n} \qquad s_{12} \equiv 2p_1\cdot p_2 = -\frac{k_\perp^2}{z(1-z)}$$

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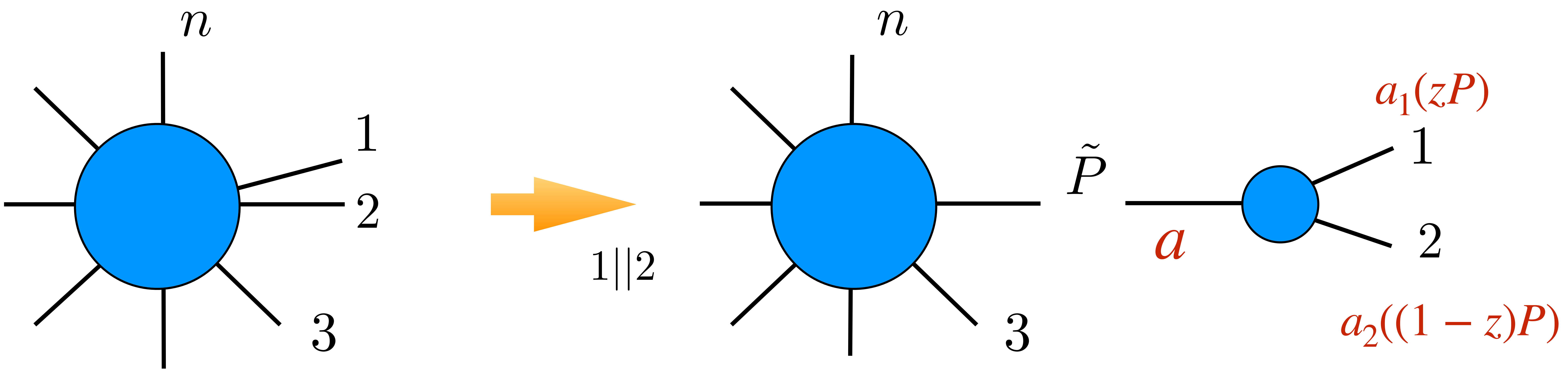

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scattering amplitude in color+spin space (vector)

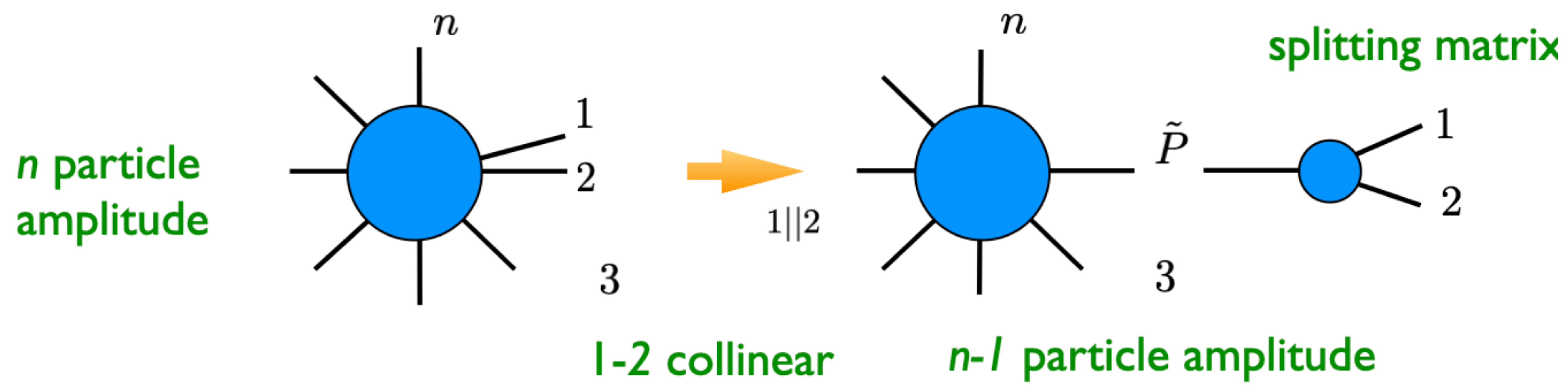
$$\mathcal{M}^{c_1,c_2,\ldots;s_1,s_2,\ldots}(p_1,p_2,\ldots) \equiv \Big(\langle c_1,c_2,\ldots| \otimes \langle s_1,s_2,\ldots|\Big)\,|\mathcal{M}(p_1,p_2,\ldots)\rangle$$

Collinear limit

$$|\mathcal{M}^{(0)}(p_1,p_2,\ldots,p_n)\rangle \simeq \boldsymbol{Sp}^{(0)}(p_1,p_2;\widetilde{P})\;|\mathcal{M}^{(0)}(\widetilde{P},\ldots,p_n)\rangle$$

matrix in color+spin space: no dependence on color of non-collinear partons (identity in that space)

splitting amplitudes



$$Sp^{(0)\,(c_1,c_2;\,c)}(p_1,p_2;\widetilde{P}) \equiv \langle c_1,c_2|\ \boldsymbol{Sp}^{(0)}(p_1,p_2;\widetilde{P})\ |c\rangle$$

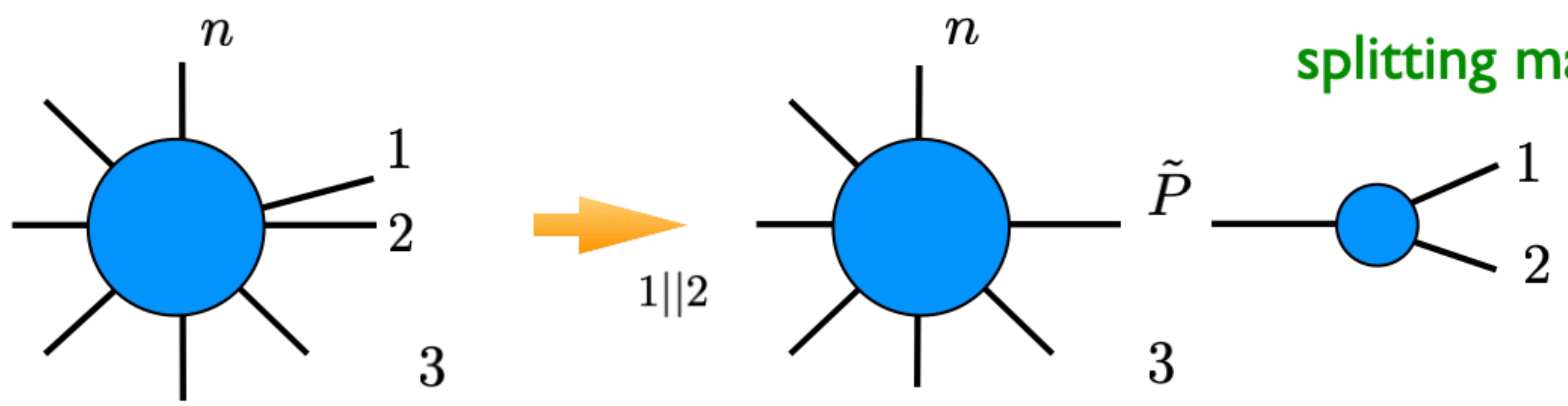
"AP kernel" $\mathbf{P}^{(0,R)} = \left(\boldsymbol{Sp}^{(0,R)}\right)^{\dagger} \boldsymbol{Sp}^{(0,R)}$

"almost" the AP kernel at this order
not true at higher orders (just a part)

$$\left|\mathscr{M}_{a_1,a_2,\ldots}\left(p_1,p_2,p_3\ldots\right)\right|^2 \simeq \frac{2}{s_{12}} 4\pi\mu^{2\epsilon}\alpha_S\,\hat{P}\left(z;\epsilon\right)\left|\mathscr{M}_{a,\ldots}\left(P,p_3\ldots\right)\right|^2$$

splitting amplitudes

n particle amplitude



splitting matrix

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PDFs (parton model) factorization direct result from strict factorization of amplitudes

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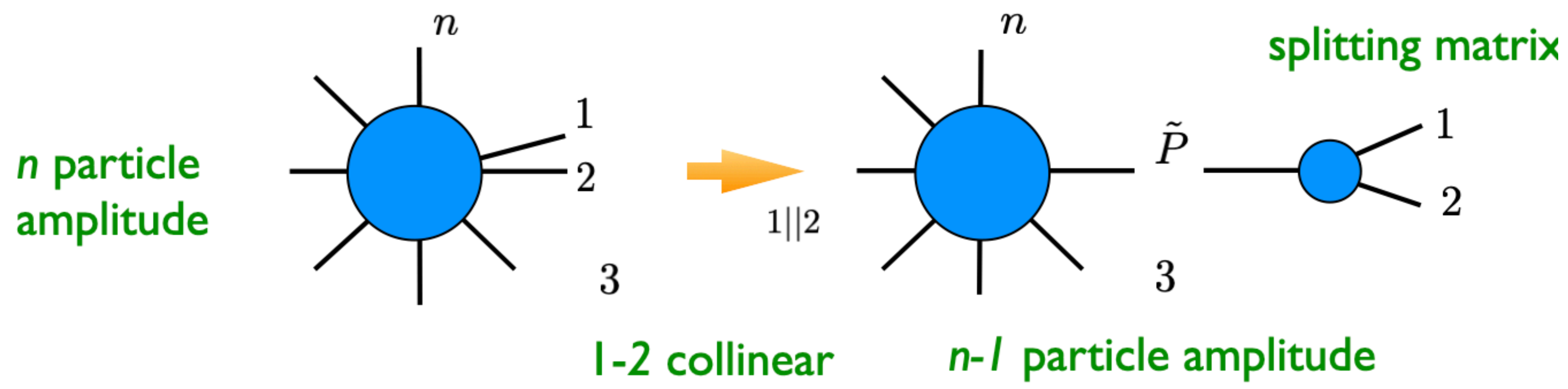

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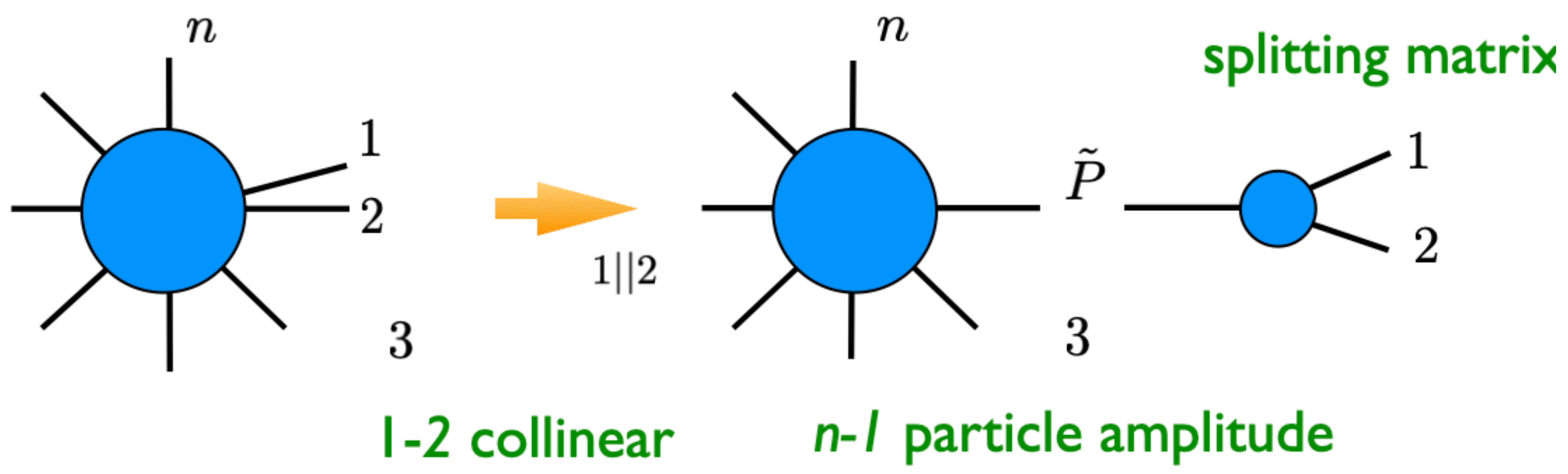
The key point for factorization is that splitting functions do not depend on non-collinear partons

splitting amplitudes



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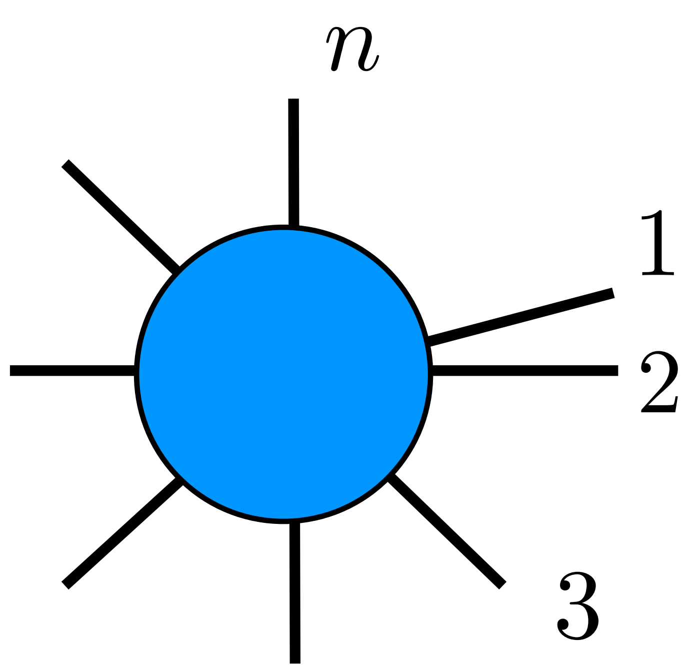
Not true at higher orders…

- Factorization at the lowest order can be understood in terms of different scales



when partons 1 and 2 become almost collinear
there are two very different scale regimes

- invariant mass s_{12} is much smaller than s_{1j} , s_{2j} and s_{ij} with $i, j = 3,4,...n$

Factorization : interactions between collinear partons take place at large space-time distances while interactions between non-collinear and collinear with non-collinear take place a small distances

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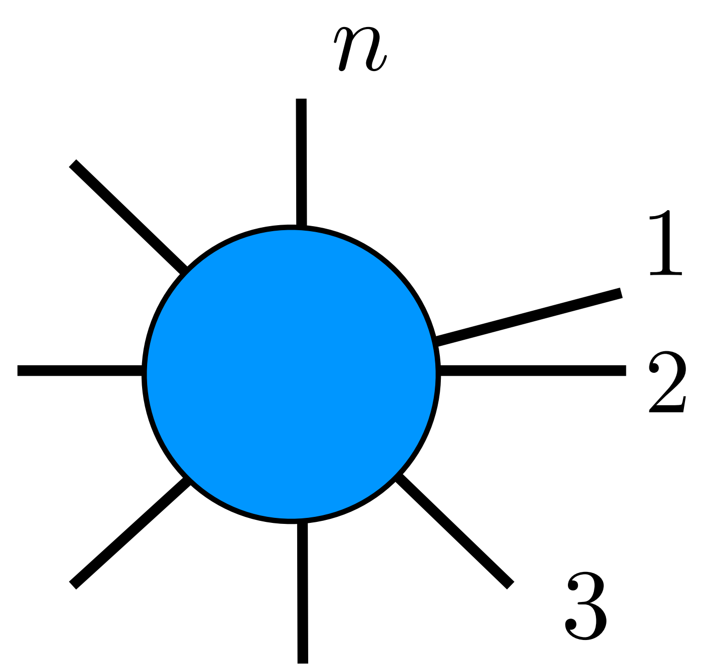

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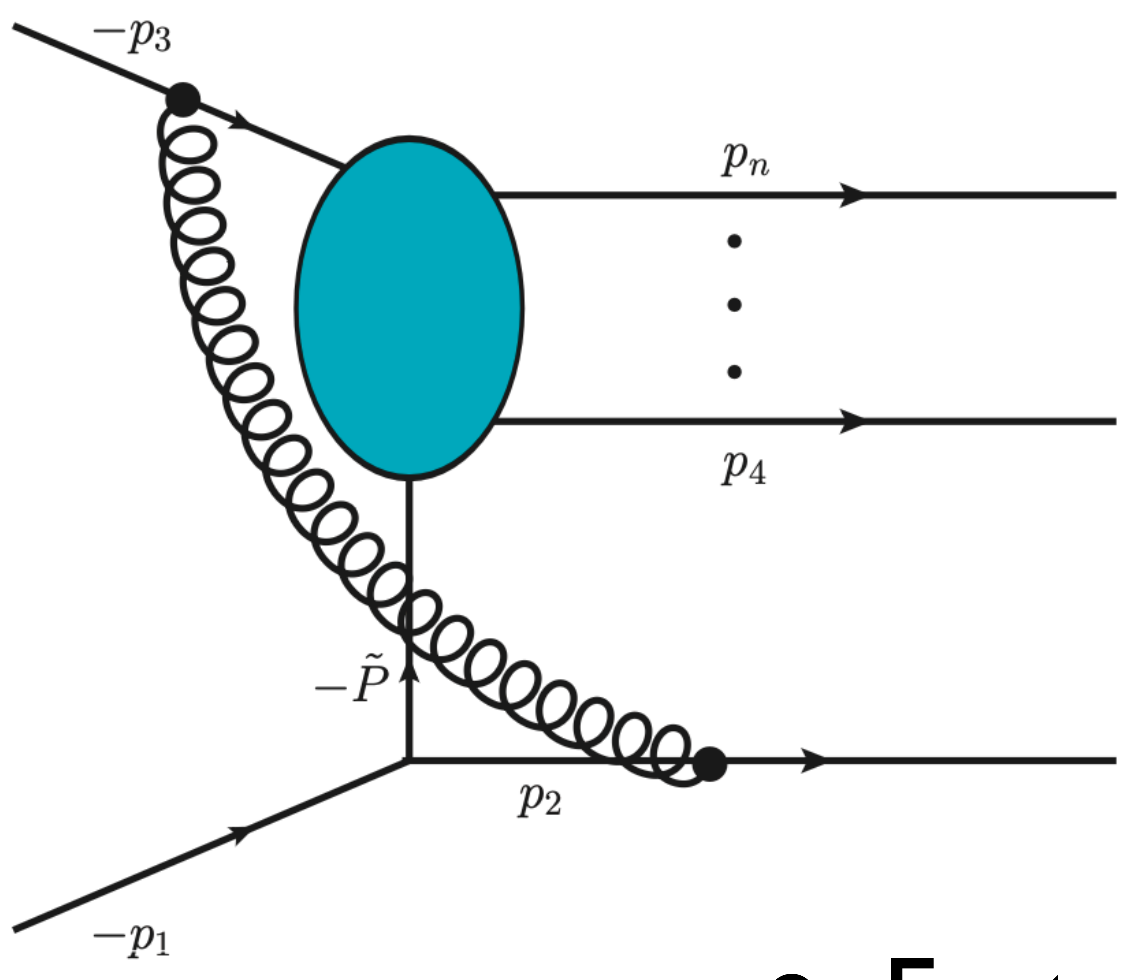


- The situation can be different when there are loops : a soft wide-angle gluon can produce pairwise interactions between collinear and non-collinear partons (proportional to $T_i \cdot T_j$) and **DO** spoil factorization

- Factorization can be recovered (in some cases) due to color coherence and causality

‣ **Collinear limit at one loop** (TIME LIKE - final state singularity)

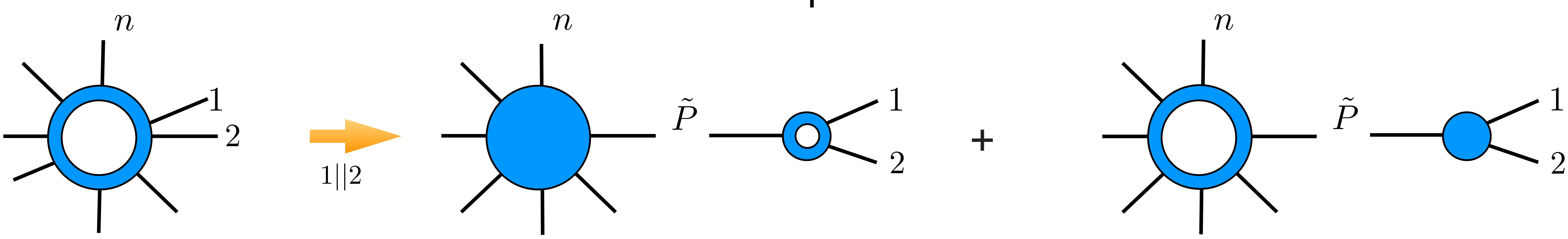
$$|\mathcal{M}^{(1)}(p_1,p_2,\ldots,p_n)\rangle \simeq \boldsymbol{Sp}^{(1)}(p_1,p_2;\widetilde{P})\ |\mathcal{M}^{(0)}(\widetilde{P},\ldots,p_n)\rangle + \boldsymbol{Sp}^{(0)}(p_1,p_2;\widetilde{P})\ |\mathcal{M}^{(1)}(\widetilde{P},\ldots,p_n)\rangle$$

1 loop splitting

Born reduced amplitude

splitting LO

1 loop reduced amplitude



Splitting amplitudes in the time-like (final state) region are known up to 3 loops !

$\boldsymbol{Sp}^{(2)}\left(p_1, p_2; \widetilde{P}\right)$ Bern, Dixon, Kosower (2004)
Badger, Glover (2004)

$\boldsymbol{Sp}^{(3)}\left(p_1, p_2; \widetilde{P}\right)$ Guan, Herzog, Ma, Mistlberger, Suresh (2024)

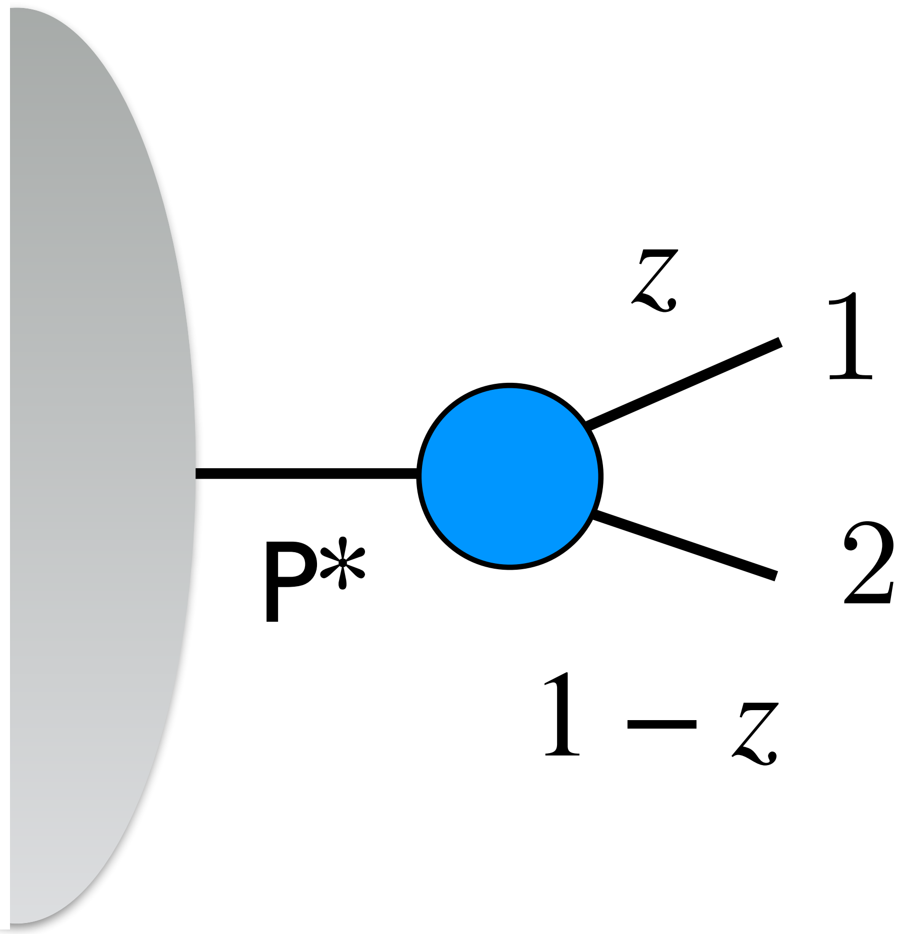
Not the AP kernel (just one configuration)

Timelike (final state)

$E_1 > 0$

$E_2 > 0$

$s_{12} > 0$



crossing

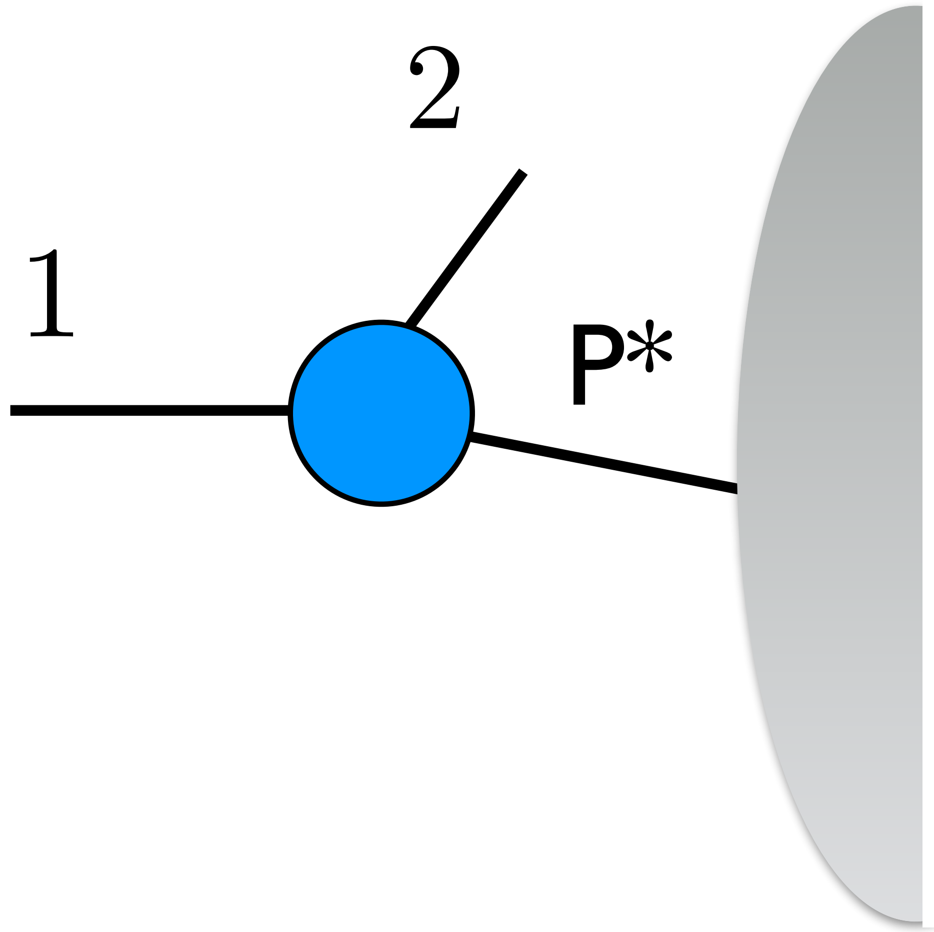
$$p_1 \to -p_1$$

$$z \to \frac{1}{z} \quad > 0$$

$$1 - z \to 1 - \frac{1}{z} \quad < 0$$

$$s_{12} \to -s_{12}$$

Spacelike (initial state)



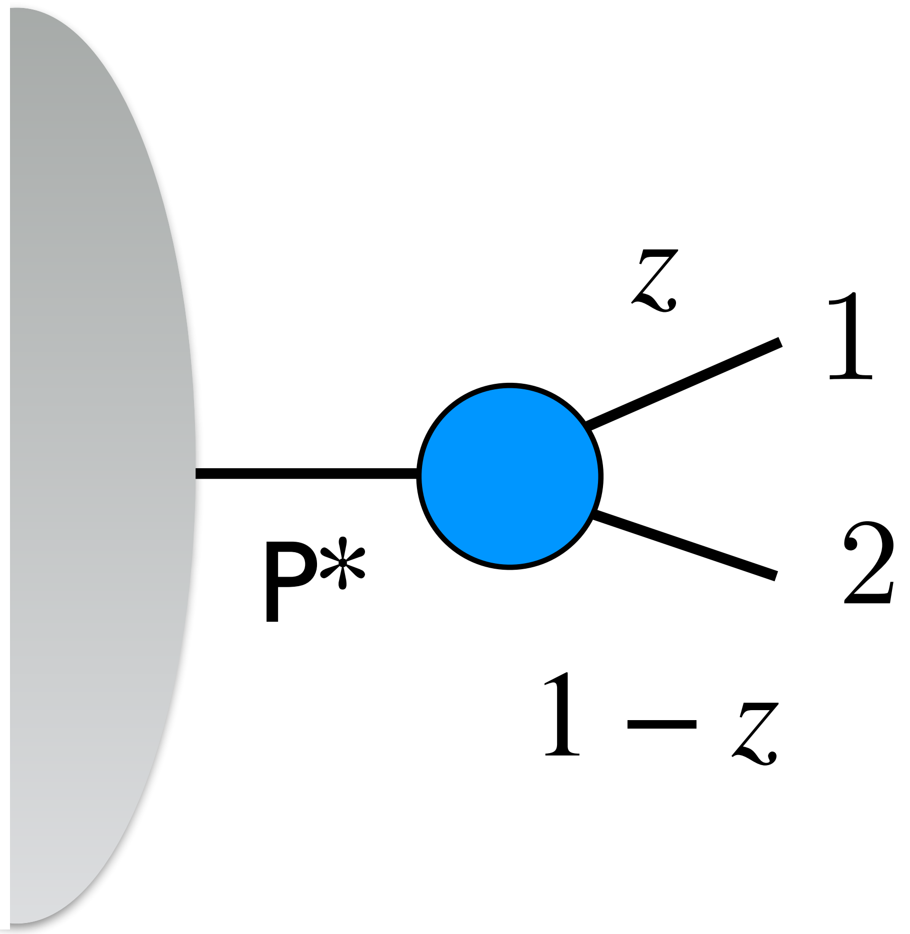
$E_1 < 0$

$E_2 > 0$

$s_{12} < 0$

Timelike (final state)

$E_1 > 0$
$E_2 > 0$
$s_{12} > 0$



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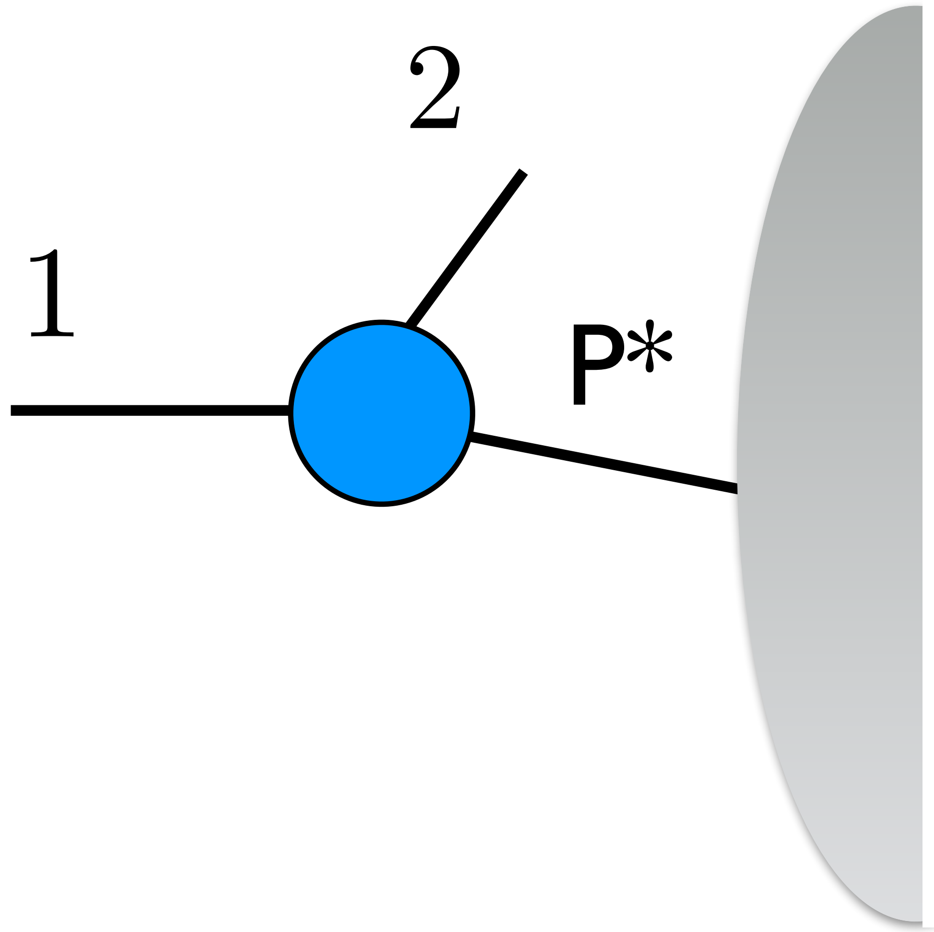


$E_1 < 0$
$E_2 > 0$
$s_{12} < 0$

Drell-Yan-Levy relation
for AP kernels

$$P_{ij}^{(T),(0)}(z) = z\mathcal{AC}\left[\quad P_{ji}^{(S),(0)}(x = \frac{1}{z})\right]$$

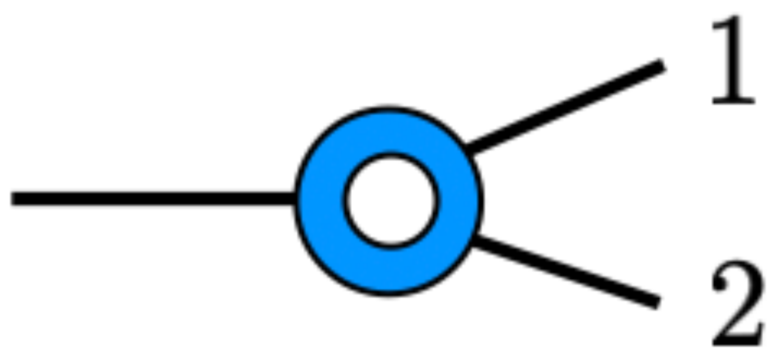
Drell, Levy, Yan (1969)

Signs of $z_1, z_2 \,.\, s_{12}$ are relevant : Splitting functions include transcendental functions (logs and Polylogs)

$$\ln^2(1-x) \quad \longrightarrow \quad \ln^2(x-1) - \pi^2$$

- Well known feature for timelike (fragmentation functions) and space like (parton distribution) evolution kernels : not a problem as long as they are process independent (same for DIS and pp, in SL case)

TL



$$\boldsymbol{Sp}^{(1)}(p_1,p_2;\widetilde{P}) = \boldsymbol{Sp}_H^{(1)}(p_1,p_2;\widetilde{P}) + I_C(p_1,p_2;\widetilde{P})\ \boldsymbol{Sp}^{(0)}(p_1,p_2;\widetilde{P})$$

rational
and finite

transcendental
and all divergences

usual prescription

Bern, Del Duca, Schmidt (1998)
Bern, Del Duca, Kilgore, Schmidt (1999)
Kosower, Uwer (1999)

$$\begin{aligned} I_C(p_1,p_2;\widetilde{P}) \quad &= \quad g_S^2\, c_\Gamma \left(\frac{-s_{12}-i0}{\mu^2}\right)^{-\epsilon} \\ &\times \quad \left\{ \frac{1}{\epsilon^2}\left(C_{12}-C_1-C_2\right) + \frac{1}{\epsilon}\left(\gamma_{12}-\gamma_1-\gamma_2+b_0\right) \right. \\ &- \quad \left. \frac{1}{\epsilon}\left[\left(C_{12}+C_1-C_2\right) f(\epsilon;z_1) + \left(C_{12}+C_2-C_1\right) f(\epsilon;z_2)\right]\right\} \end{aligned}$$

usual Casimir and color factors

$d = 4 - 2\epsilon$

$$f(\epsilon;1/x) \equiv \frac{1}{\epsilon}\left[\,{}_2F_1(1,-\epsilon;1-\epsilon;1-x) - 1\right] = \ln x - \epsilon\left[\mathrm{Li}_2(1-x) + \sum_{k=1}^{+\infty} \epsilon^k\, \mathrm{Li}_{k+2}(1-x)\right]$$

need i0 prescription… not trivial.. we obtain it from explicit calculation of SL and TL collinear limit

The general expression for all TL and SL collinear limits (assume $z_1 > 0$ for simplicity)

$$\boldsymbol{I}_C(p_1, p_2 \boxed{; p_3, \dots, p_n)} = g_S^2\, c_\Gamma \left(\frac{-s_{12} - i0}{\mu^2} \right)^{-\epsilon} \times \left\{ \frac{1}{\epsilon^2} \left(C_{12} - C_1 - C_2 \right) + \frac{1}{\epsilon} \left(\gamma_{12} - \gamma_1 - \gamma_2 + b_0 \right) - \frac{1}{\epsilon} \left[\left(C_{12} + C_1 - C_2 \right) f(\epsilon; z_1) - \boxed{2 \sum_{j=3}^{n} \boldsymbol{T}_2 \cdot \boldsymbol{T}_j\, f(\epsilon; z_2 - i0 s_{j2})} \right] \right\}$$

dependence on NON-COLLINEAR partons

The factorization-breaking term (pole and notice non-trivial "kinematical" dependence on i0 prescription)

$$\boldsymbol{\delta}(p_1, p_2; p_3, \dots, p_n) = + \frac{2}{\epsilon} \sum_{j=3}^{n} \boldsymbol{T}_2 \cdot \boldsymbol{T}_j\, f(\epsilon; z_2 - i0 s_{j2})$$

branch-cut singularity on $z_2 < 0$

when $z_2 = 1 - z \to 1 - \frac{1}{z} < 0$ function evaluated above or below branch depending on kinematics of non-collinear partons (sign of s_{j2} depends parton incoming/outgoing)

Why not appearing in simpler TL case? : z_2 is positive (no need for prescription) and color conservation

$$\boldsymbol{\delta}\left(p_1,p_2;p_3,\ldots,p_n\right) = +\frac{2}{\epsilon}\sum_{j=3}^{n}\boldsymbol{T}_2\cdot\boldsymbol{T}_j\, f\Big(\epsilon;z_2-i0s_{j2}\Big) = +\frac{2}{\epsilon}T_2\; f\left(\epsilon;z_2\right)\sum_{j=3}^{n}\boldsymbol{T}_j$$

color conservation $\sum_{k=1}^{n}\boldsymbol{T}_k = 0$ $\Rightarrow$ $\sum_{j=3}^{n}\boldsymbol{T}_j = -(\boldsymbol{T}_1+\boldsymbol{T}_2)$

$$(\boldsymbol{T}_1+\boldsymbol{T}_2)^2-\boldsymbol{T}_1^2-\boldsymbol{T}_2^2 = 2\,\boldsymbol{T}_1\cdot\boldsymbol{T}_2$$

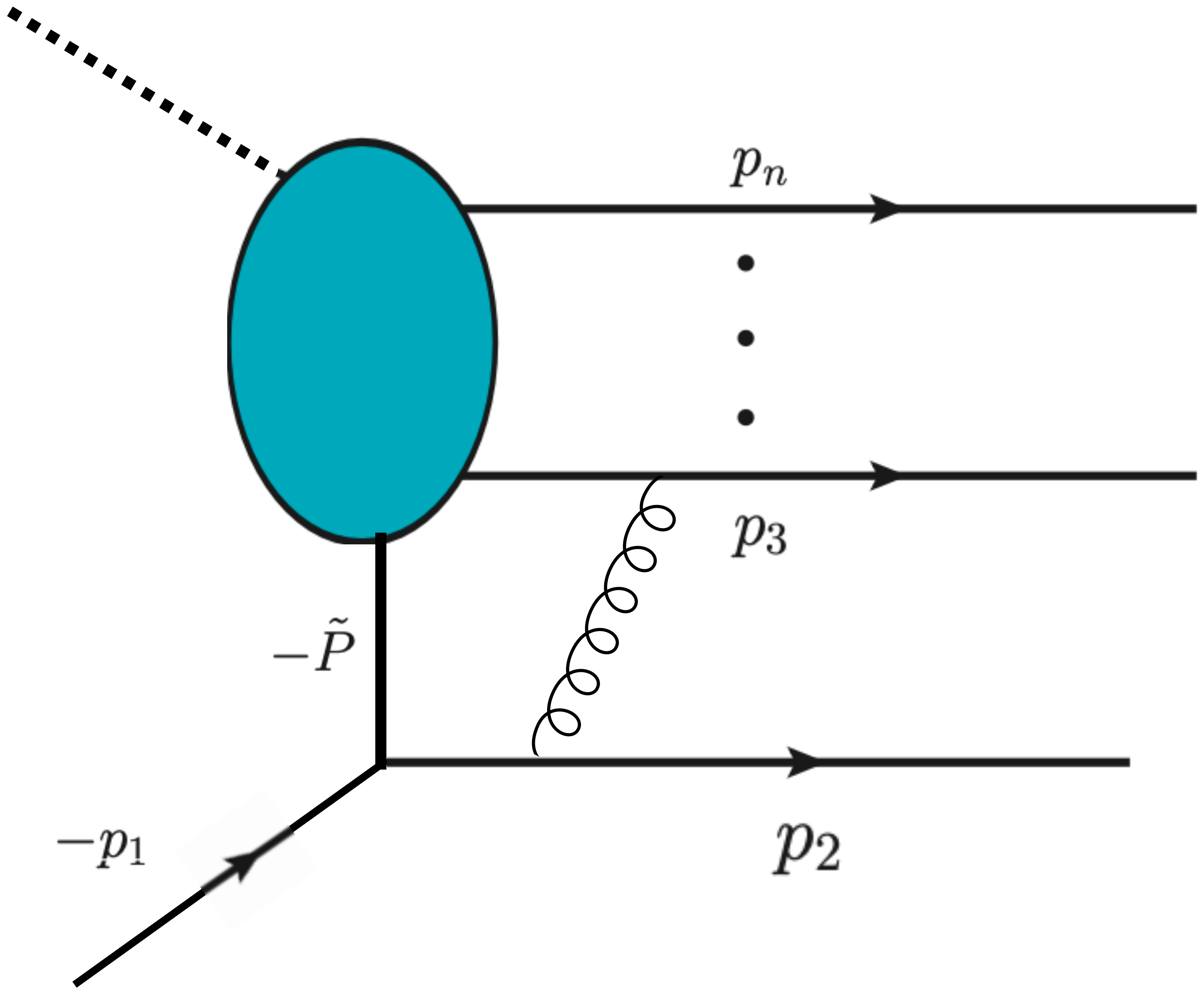
casimir factors $\boldsymbol{T}_k^2 = C_k$

TL Case factorizes
$$\boldsymbol{\delta}\left(p_1,p_2;p_3,\ldots,p_n\right) = -\frac{1}{\epsilon}\left(C_{12}+C_2-C_1\right)f\left(\epsilon;z_2\right)$$

Color Coherence : *non-collinear partons act coherently as a single parton, whose color charge is equal to the total charge of the non-collinear partons. Owing to colour conservation, this color charge is equal (modulo the overall sign) to the color charge of the parent (off-shell P*=1+2) parton*

- in general, DIS has only one parton in initial state $s_{j2} > 0 \; (j \geq 3)$

$$\boldsymbol{\delta}\left(p_1, p_2; p_3, \ldots, p_n\right) = +\frac{2}{\epsilon} \sum_{j=3}^{n} \boldsymbol{T}_2 \cdot \boldsymbol{T}_j\, f\left(\epsilon; z_2 - i0 s_{j2}\right) = +\frac{2}{\epsilon} T_2\, f\left(\epsilon; z_2 - i0\right) \sum_{j=3}^{n} \boldsymbol{T}_j$$



- There are interactions between 2 and the non-collinear partons that are separately not factorized $\boldsymbol{T}_2 \cdot \boldsymbol{T}_j$

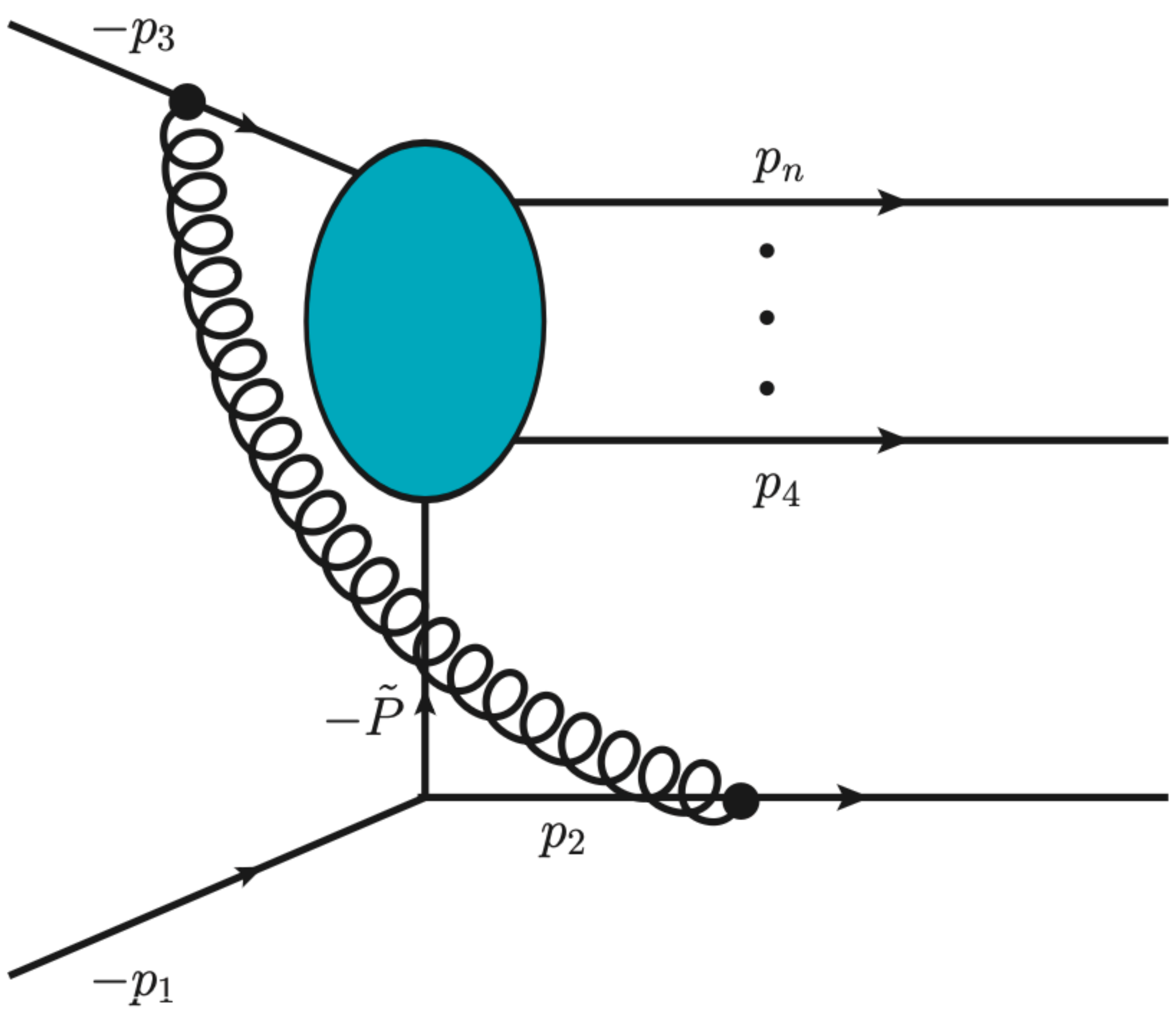
- But, since all non-collinear partons are in the final state s_{j2} are all positive and imaginary parts combine coherently to *mimic* single effective interaction with parent parton P

$$\boldsymbol{\delta}(p_1, p_2; p_3, \ldots, p_n) = -\frac{1}{\epsilon}\left(C_{12} + C_2 - C_1\right) f(\epsilon; z_2 - i0)\,, \qquad s_{j2} > 0 \; (j \geq 3)$$

Effective strict factorization in DIS

- in pp collisions there is different partonic environment: two partons in initial state

$$\boldsymbol{\delta}\left(p_1, p_2; p_3, \ldots, p_n\right) = +\frac{2}{\epsilon} \sum_{j=3}^{n} \boldsymbol{T}_2 \cdot \boldsymbol{T}_j\, f\left(\epsilon; z_2 - i0 s_{j2}\right) \qquad s_{23} < 0 < s_{j2}\ (j \geq 4)$$



can not factorize the color structure because of parton 3 with different sign contribution

IS collinear splitting in pp collisions with $n \geq 4$ partons involves color correlations with non-collinear and explicit factorization breaking

imaginary part π …

$$\boldsymbol{\delta}(p_1, p_2; p_3, \ldots, p_n) = -\frac{1}{\epsilon}\left(C_{12} + C_2 - C_1\right) f(\epsilon; z_2 - i0) + \frac{i}{\epsilon}\, 4\, \boldsymbol{T}_2 \cdot \boldsymbol{T}_3\, f_I(\epsilon; z_2)$$

antihermitian contribution

- But one loop amplitudes with collinear divergences enter in every NNLO calculation at the LHC
- Does it mean that NNLO calculations are wrong and there is no factorization?
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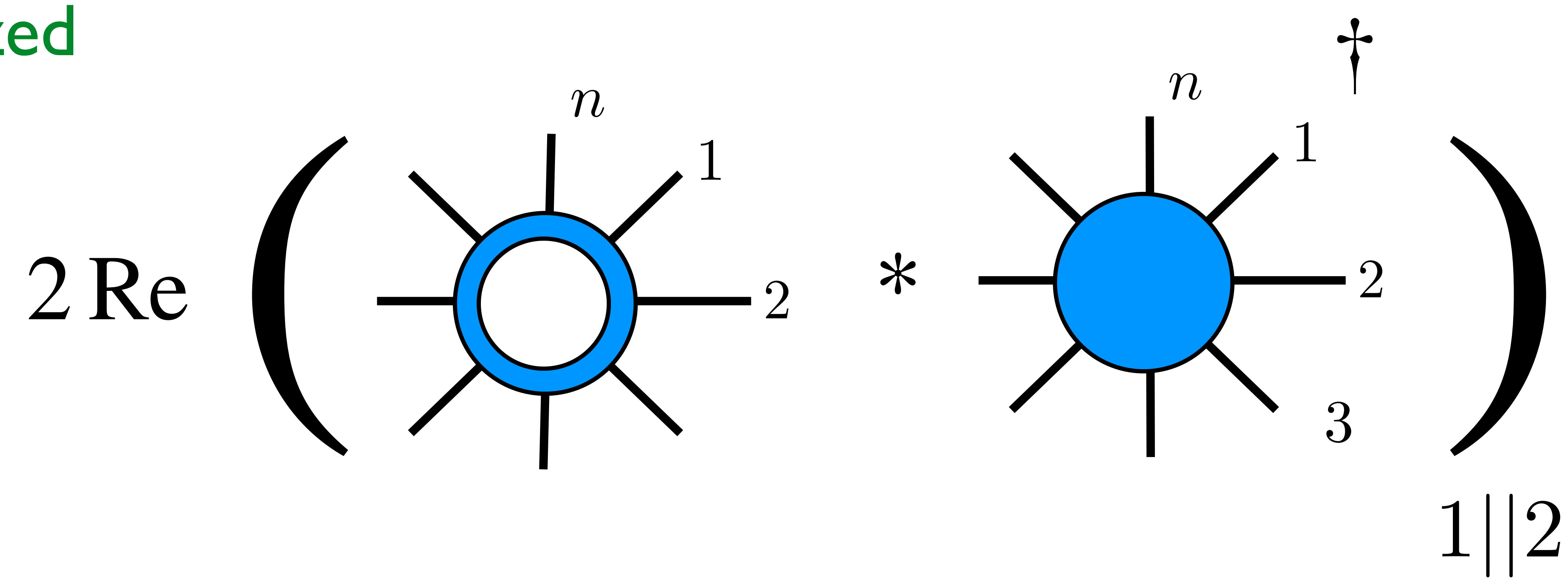
- But one loop amplitudes with collinear divergences enter in every NNLO calculation at the LHC
- Does it mean that NNLO calculations are wrong and there is no factorization?
- Breaking of factorization appears at the amplitude level…but cross section involves squared amplitudes
- The collinear limit relevant at NNLO (interference between 1 loop and Born) is

$$\langle \mathcal{M}^{(0,R)} | \mathcal{M}^{(1,R)} \rangle + \text{ c.c. } \simeq \left[\langle \overline{\mathcal{M}}^{(0,R)} | \; \mathbf{P}^{(0,R)} \, | \overline{\mathcal{M}}^{(1,R)} \rangle + \text{ c.c. } \right] + \langle \overline{\mathcal{M}}^{(0,R)} | \, \mathbf{P}^{(1,R)} \, | \overline{\mathcal{M}}^{(0,R)} \rangle$$

lowest order strictly factorized

$$\mathbf{P}^{(1,R)} = \left(\boldsymbol{Sp}^{(0,R)} \right)^{\dagger} \boldsymbol{Sp}^{(1,R)} \; + \; \text{h.c.}$$



is factorized because factorization breaking term in the splitting amplitude is *antihermitian*

In summary (one-loop)

- 1 loop splitting amplitudes violate strict collinear factorization, but effect is antihermitian

- cancels at the level of cross sections (squared amplitudes) $\Rightarrow$ NNLO calculations at LHC are safe

  - splitting amplitudes used for subtraction methods

- What about higher orders? : we will see that there are hermitian and antihermitian contributions

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- collinear limit up to two loops

$$
\begin{aligned}
|\mathcal{M}^{(0,R)}\rangle &\simeq \boldsymbol{Sp}^{(0,R)}\,|\overline{\mathcal{M}}^{(0,R)}\rangle\,, \\
|\mathcal{M}^{(1,R)}\rangle &\simeq \boldsymbol{Sp}^{(1,R)}\,|\overline{\mathcal{M}}^{(0,R)}\rangle + \boldsymbol{Sp}^{(0,R)}\,|\overline{\mathcal{M}}^{(1,R)}\rangle\,, \\
|\mathcal{M}^{(2,R)}\rangle &\simeq \boldsymbol{Sp}^{(2,R)}\,|\overline{\mathcal{M}}^{(0,R)}\rangle + \boldsymbol{Sp}^{(1,R)}\,|\overline{\mathcal{M}}^{(1,R)}\rangle + \boldsymbol{Sp}^{(0,R)}\,|\overline{\mathcal{M}}^{(2,R)}\rangle
\end{aligned}
$$

renormalized quantities

- The two-loop splitting matrix can be decomposed as in terms of the known 1-loop results…

$$\boldsymbol{Sp}^{(2,R)} = \widetilde{\boldsymbol{I}}_C^{(2)}(\epsilon)\ \boldsymbol{Sp}^{(0,R)} + \widetilde{\boldsymbol{I}}_C^{(1)}(\epsilon)\ \boldsymbol{Sp}^{(1,R)} + \widetilde{\boldsymbol{Sp}}^{(2)\,\mathrm{fin.}}$$

2 loop

1 loop, both break fact.

finite (not computed here)

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- New structure appears: involves three-part color correlations (double and single poles)

$$\widetilde{\boldsymbol{\Delta}}_C^{(2)}(\epsilon) = \left(\frac{\alpha_S(\mu^2)}{2\pi}\right)^2 \left(\frac{-s_{12}}{\mu^2}\right)^{-2\epsilon} \pi\, f_{abc} \sum_{i=1,2} \sum_{\substack{j,k=3 \\ j\neq k}}^{n} T_i^a T_j^b T_k^c\, \Theta(-z_i)\, \mathrm{sign}(s_{ij})\, \Theta(-s_{jk})$$
$$\times \ln\left(-\frac{s_{j\widetilde{P}}\, s_{k\widetilde{P}}\, z_1 z_2}{s_{jk}\, s_{12}} - i0\right) \left[-\frac{1}{2\,\epsilon^2} + \frac{1}{\epsilon} \ln\left(\frac{-z_i}{1-z_i}\right)\right] .$$

cancels for TL (z>0)

requires SL + non-collinear parton in initial and final state : cancels for DIS

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cancels for TL (z>0)

requires SL + non-collinear parton in initial and final state : cancels for DIS

- Contains both hermitian (depends on momenta) and non-hermitian contributions (only on signs)
- Finite part (not-computed) also contains factorization breaking terms with same color correlations

- Now we take the square and see what remains!

iteration of 1-loop, factorized (antihermitian)

$$\mathbf{P}^{(2,R)} = \widetilde{I}_P^{(1)}(\epsilon)\,\mathbf{P}^{(1,R)} + \widetilde{I}_P^{(2)}(\epsilon)\,\mathbf{P}^{(0,R)} + \left(\boldsymbol{Sp}_H^{(1,R)}\right)^\dagger \boldsymbol{Sp}_H^{(1,R)}$$
$$+ \left(\boldsymbol{Sp}^{(0,R)}\right)^\dagger \widetilde{\mathbf{\Delta}}_P^{(2)}(\epsilon)\,\boldsymbol{Sp}^{(0,R)} + \left[\left(\boldsymbol{Sp}^{(0,R)}\right)^\dagger \widetilde{\boldsymbol{Sp}}^{(2)\,\mathrm{fin.}} + \mathrm{h.c.}\right]$$

divergent, factorization breaking (from hermitian part)

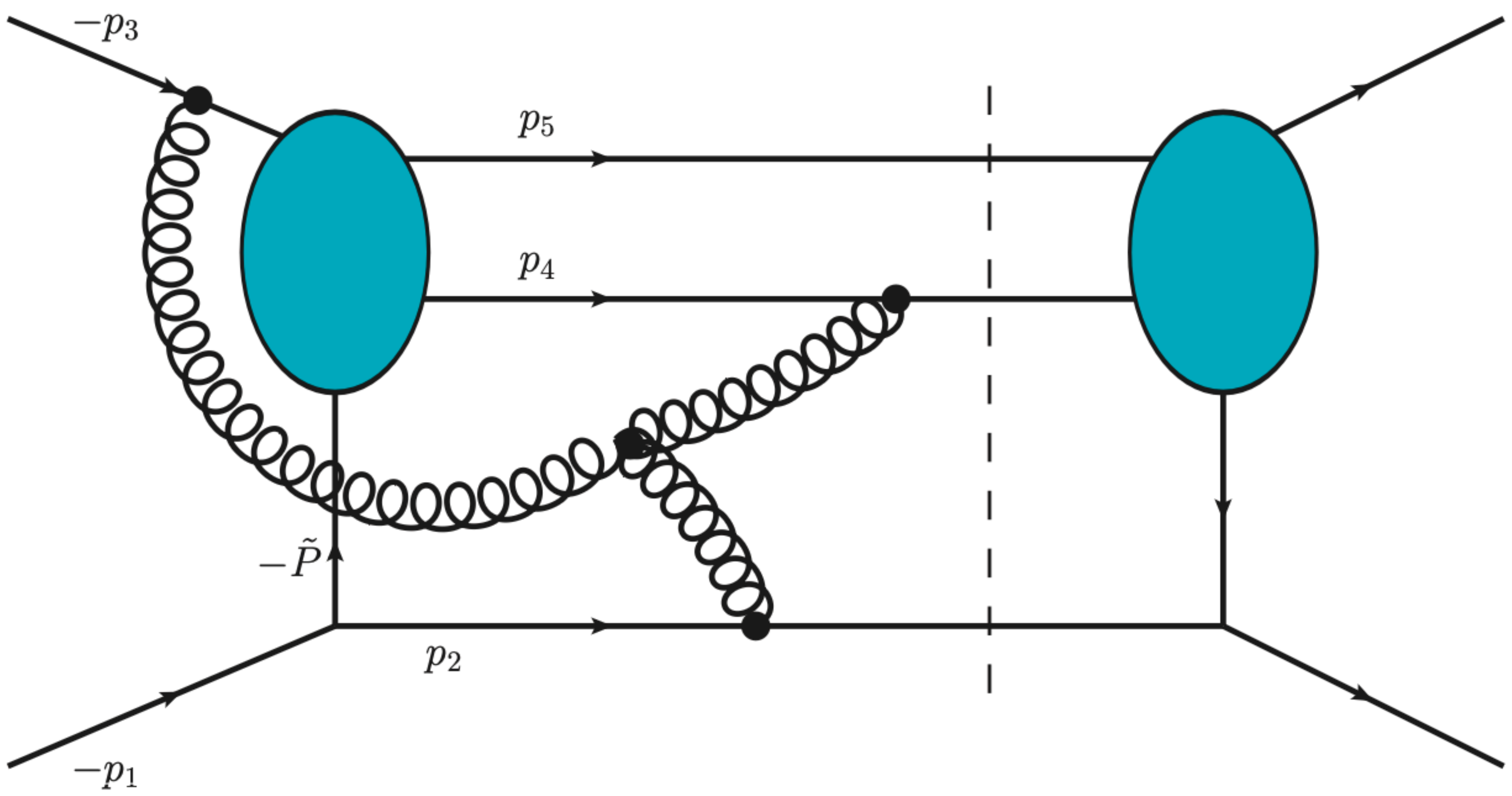
finite, not computed but factorization breaking

- Non abelian, only for SL, vanishing in DIS

$$\widetilde{\mathbf{\Delta}}_P^{(2)}(\epsilon) = \left(\frac{\alpha_S(\mu^2)}{2\pi}\right)^2 \left(\frac{-s_{12}}{\mu^2}\right)^{-2\epsilon} 2\,\pi\, f_{abc} \sum_{i=1,2} \sum_{\substack{j,k=3\\ j\neq k}}^{n} T_i^a\, T_j^b\, T_k^c\, \Theta(-z_i)\,\mathrm{sign}(s_{ij})\,\Theta(-s_{jk})$$
$$\times \ln\left(\frac{s_{j\widetilde{P}}\, s_{k\widetilde{P}}\, z_1 z_2}{s_{jk}\, s_{12}}\right)\left[-\frac{1}{2\,\epsilon^2} + \frac{1}{\epsilon}\ln\left(\frac{-z_i}{1-z_i}\right)\right].$$

- Because of color conservation it requires 5 QCD partons: 2 collinear, 1 extra incoming, 2 extra final

- Simplest case, two loop (times Born) for parton + parton → 3 partons (2 jet at LHC at N3LO)



with 1(IS) and 2 (FS) collinear
parton 2 with very small q_T

$$\widetilde{\mathbf{\Delta}}_P^{(2)}(\epsilon) = \frac{\alpha_S^2(\mu^2)}{\pi} \left(\frac{-s_{12}}{\mu^2}\right)^{-2\epsilon} f_{abc}\, T_2^a\, T_3^b\, T_4^c \ln\left(\frac{s_{34}\ s_{5\widetilde{P}}}{s_{35}\ s_{4\widetilde{P}}}\right) \left[-\frac{1}{2\,\epsilon^2} + \frac{1}{\epsilon} \ln\left(-\frac{z_2}{z_1}\right)\right]$$

- There is a clear factorization term, non-vanishing, in the splitting function
- Involves correlations between collinear and non-collinear partons including kinematical dependence
- Multiple collinear factorization also discussed

Catani, DdeF, Rodrigo (2012) Cieri, Dhani, Rodrigo (2024)
Duhr, Venkata, Zhang (2025)

One more step needed to get the contribution to the N3LO cross section : $\langle \mathscr{M} \,|\, \mathscr{M} \rangle$

expectation value

the N3LO contribution involves

Born level amplitude

$$\langle \overline{\mathcal{M}} | \left(\boldsymbol{Sp}^{(0,R)} \right)^{\dagger} \, \widetilde{\boldsymbol{\Delta}}_{P}^{(2)}(\epsilon) \, \boldsymbol{Sp}^{(0,R)} \; | \overline{\mathcal{M}} \rangle = \langle \overline{\mathcal{M}}^{(0,R)} | \left(\boldsymbol{Sp}^{(0,R)} \right)^{\dagger} \, \widetilde{\boldsymbol{\Delta}}_{P}^{(2)}(\epsilon) \, \boldsymbol{Sp}^{(0,R)} \, | \overline{\mathcal{M}}^{(0,R)} \rangle$$

In a color basis where $\mathbf{T}_i \cdot \mathbf{T}_j$ is real the pure QCD amplitude is also real and $\Big| \overline{\mathscr{M}}^{(0)} \rangle \langle \overline{\mathscr{M}}^{(0)} \Big|$ symmetric

Forshaw, Seymour, Siódmok (2012) In the same basis $\widetilde{\Delta}_{p}^{(2)}(\epsilon)$ is antisymmetric

symmetric antisymmetric

using that $\langle \mathscr{M}^{(0)} \big| A \big| \mathscr{M}^{(0)} \rangle = \mathrm{Tr}\left[\Big| \mathscr{M}^{(0)} \rangle \langle \mathscr{M}^{(0)} \Big| A \right] = 0$

factorization breaking terms (IR part) vanish in pure QCD for cross-section

- Formally involves only the IR part of the 2-loop contribution, finite part is not known in QCD

Henn, Ma, Xu, Yan, Zhang, Zhu (2024) compute full result (including finite part) in $\mathcal{N} = 4$ Super-Yang-Mills

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- The "vanishing" only occurs at N3LO, at higher order one finds

2 loop vanishing

$$\langle \overline{\mathcal{M}} | \left(\boldsymbol{Sp}^{(0,R)}\right)^{\dagger} \widetilde{\boldsymbol{\Delta}}_P^{(2)}(\epsilon)\, \boldsymbol{Sp}^{(0,R)}\, |\overline{\mathcal{M}}\rangle = \langle \overline{\mathcal{M}}^{(0,R)} | \left(\boldsymbol{Sp}^{(0,R)}\right)^{\dagger} \widetilde{\boldsymbol{\Delta}}_P^{(2)}(\epsilon)\, \boldsymbol{Sp}^{(0,R)}\, |\overline{\mathcal{M}}^{(0,R)}\rangle$$
$$+ \left[\langle \overline{\mathcal{M}}^{(1,R)} | \left(\boldsymbol{Sp}^{(0,R)}\right)^{\dagger} \widetilde{\boldsymbol{\Delta}}_P^{(2)}(\epsilon)\, \boldsymbol{Sp}^{(0,R)}\, |\overline{\mathcal{M}}^{(0,R)}\rangle + \text{c.c.}\right] + \text{higher orders}$$

$\mathbf{P}^{(3,R)}$ 3 loop result involves 1-loop amplitude with extra phases

In summary (two-loops and more)

- 2loop splitting amplitudes violate strict collinear factorization, effect is antihermitian and hermitian

- effect survives at the level of (squared) splitting matrix

  ➡ appear at N3LO parton + parton → 3 partons (2 jet at LHC

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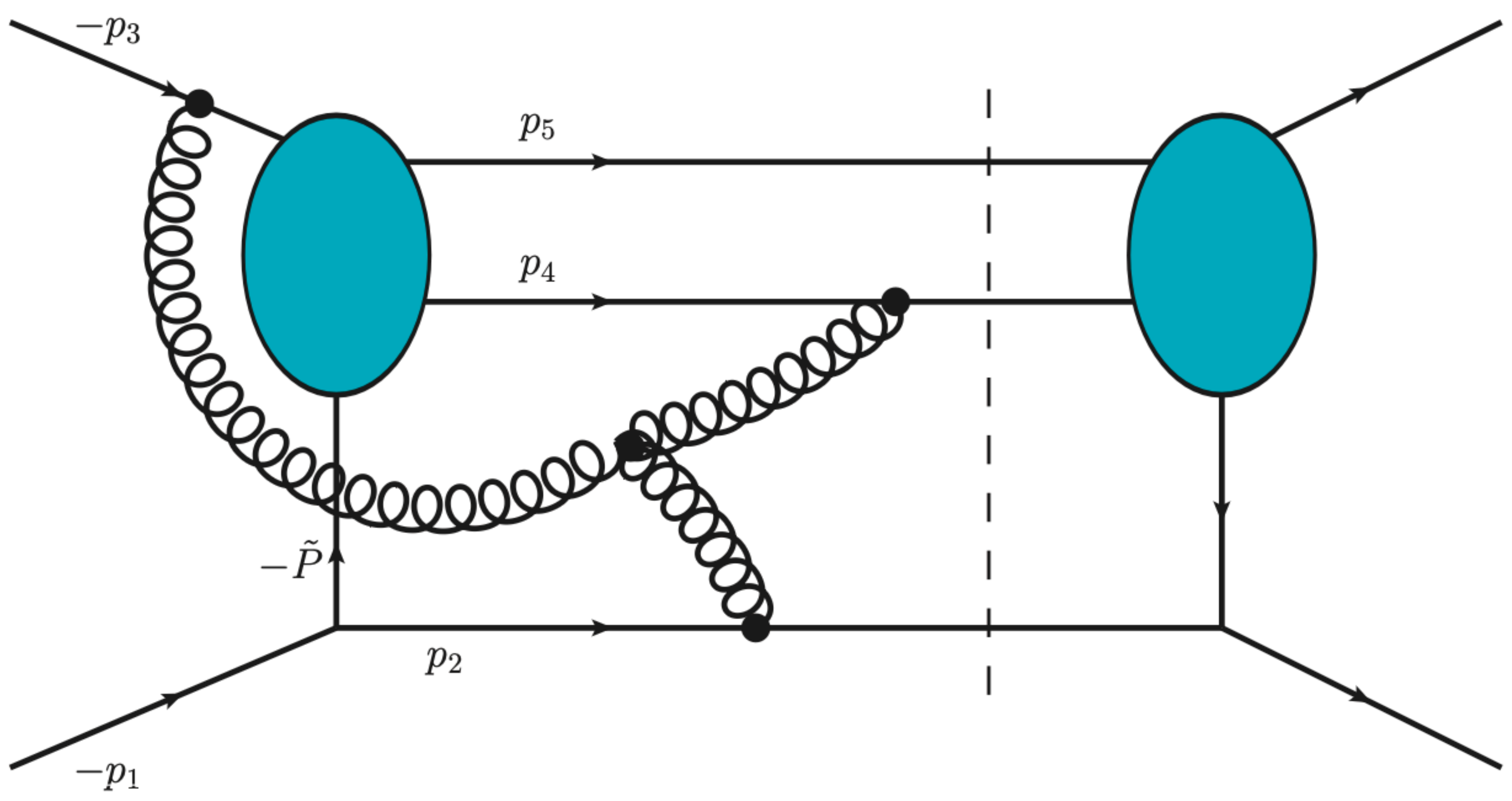
- in any case, strict factorization guarantees mass singularities cancellation but not the other way around

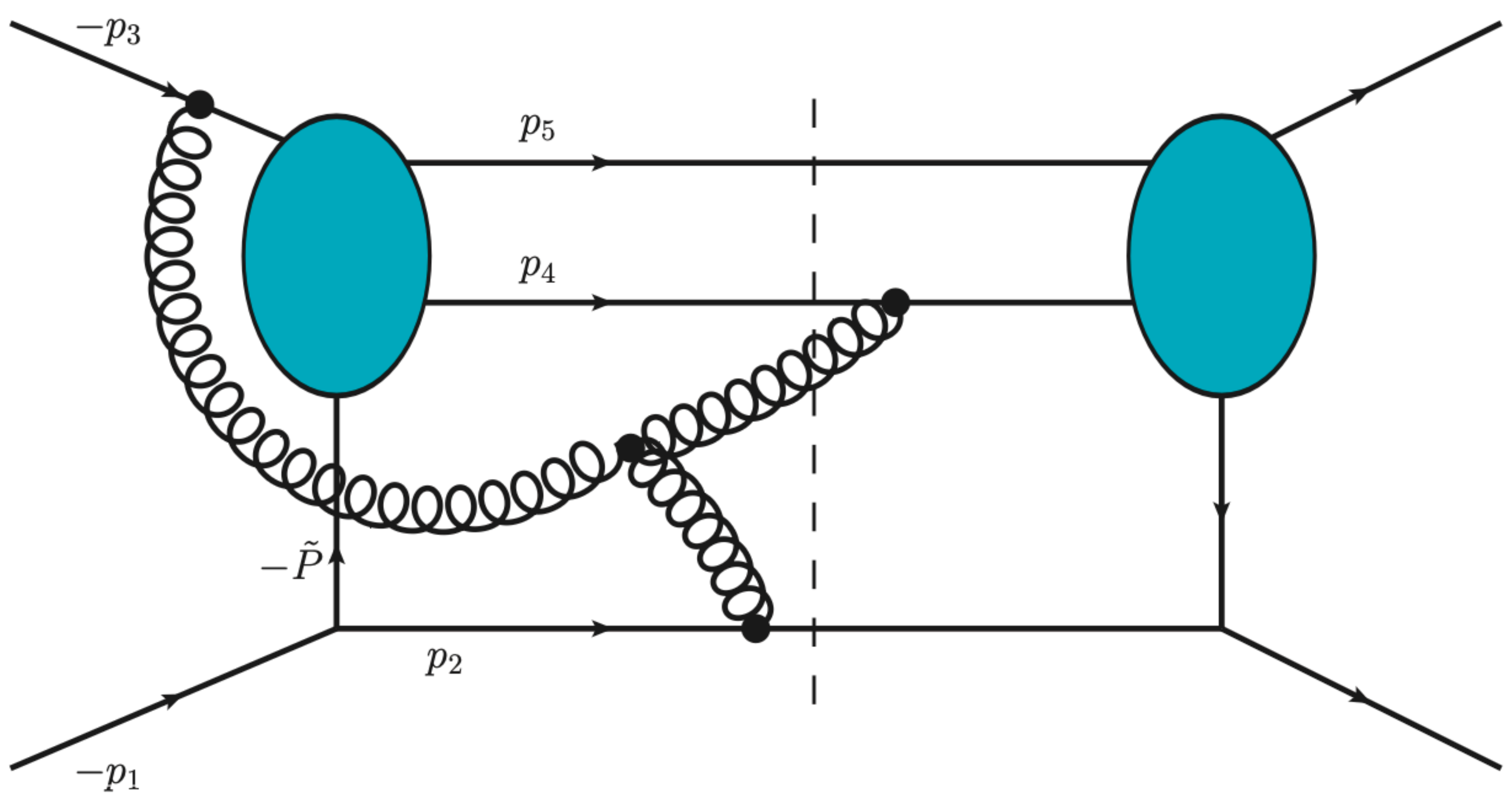
  still mass factorization can be the result of cancellation between different configurations

- Many configurations contribute at N3LO, this one has similar structure

I loop $\text{parton parton} \to \text{parton parton} + 1\,\text{collinear parton} + 1\,\text{soft parton}$



original 2 loop



same diagram with different cut
same color structure and kinematics
in the soft limit of real gluon

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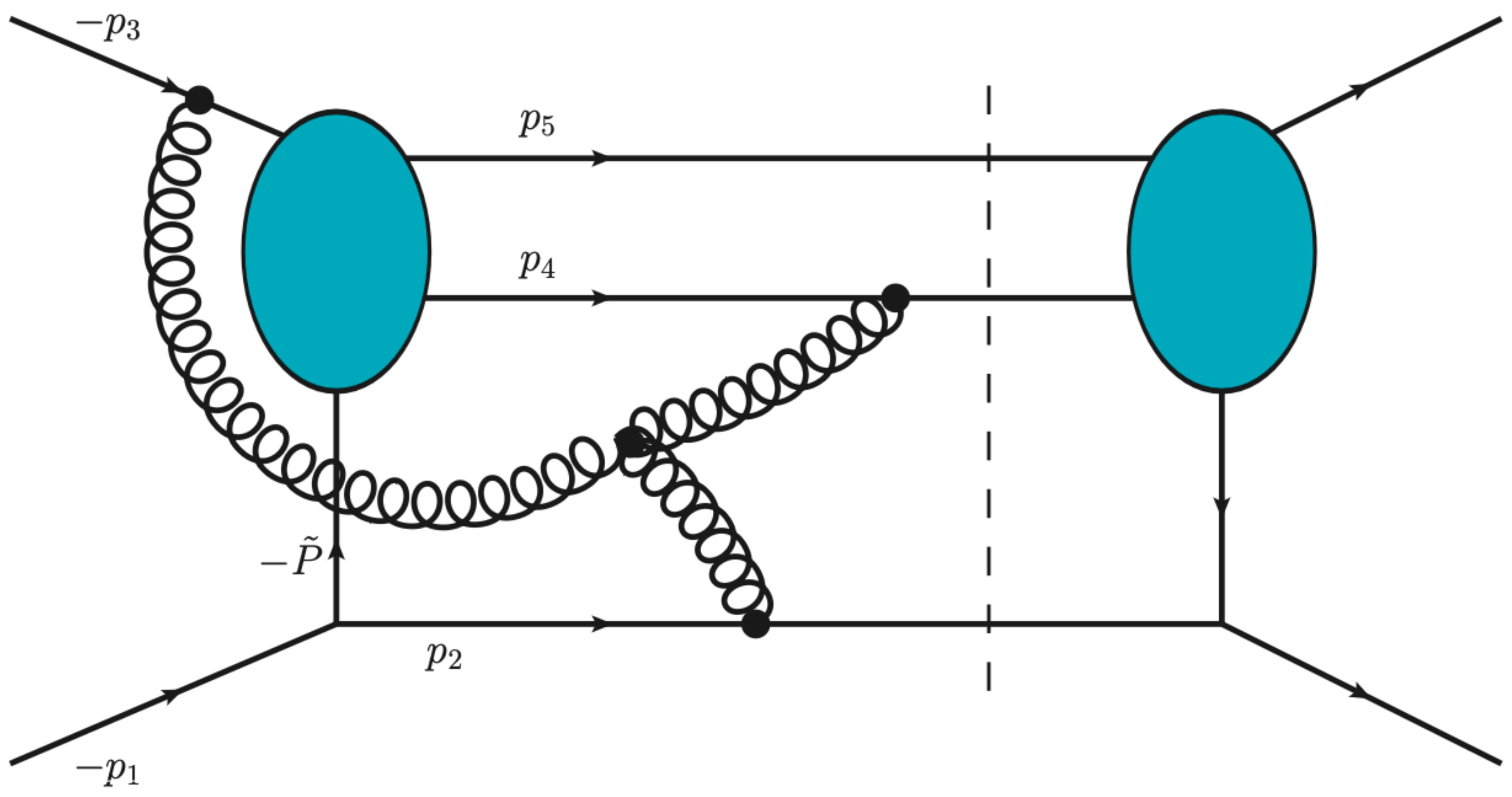

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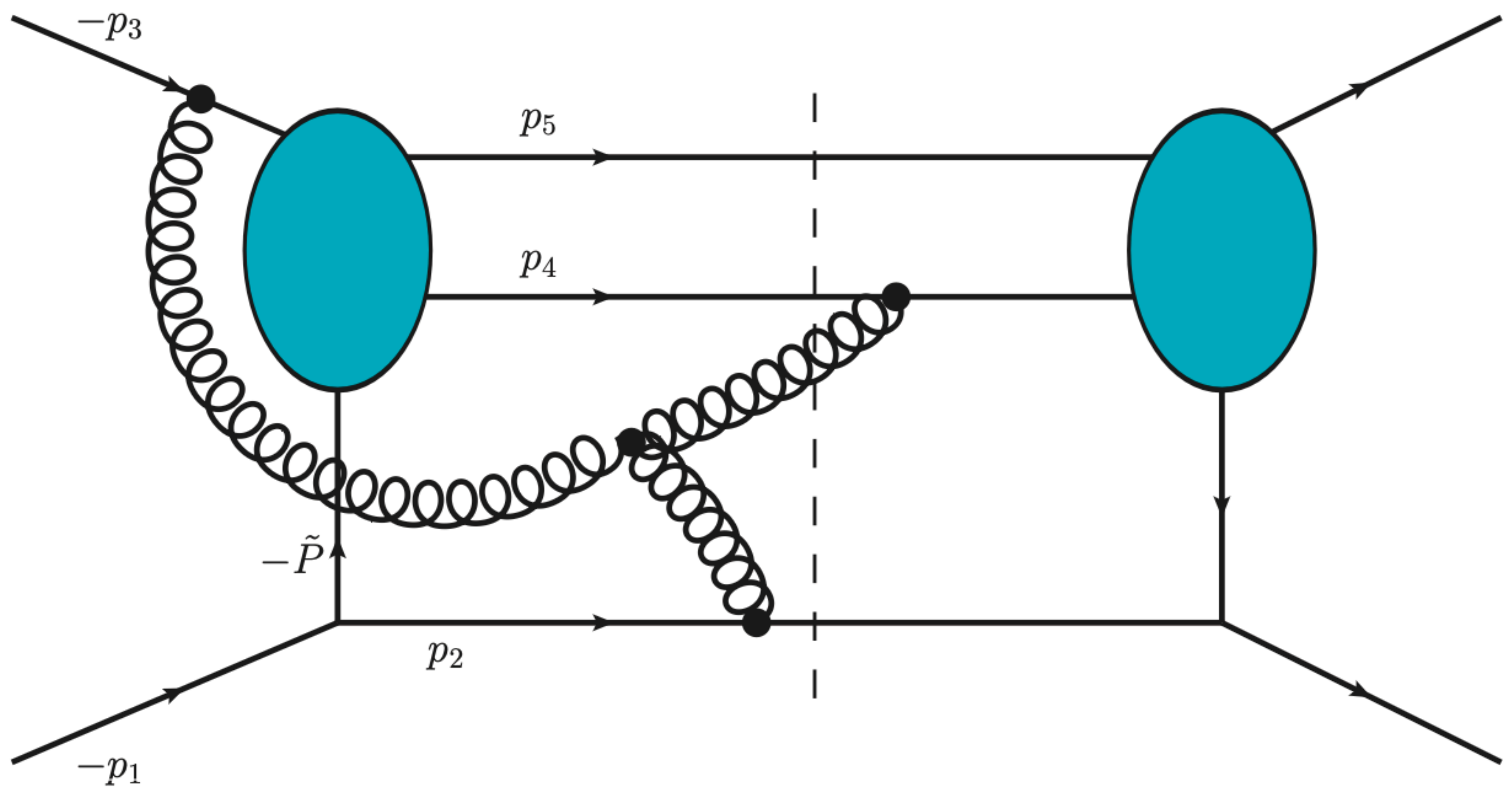

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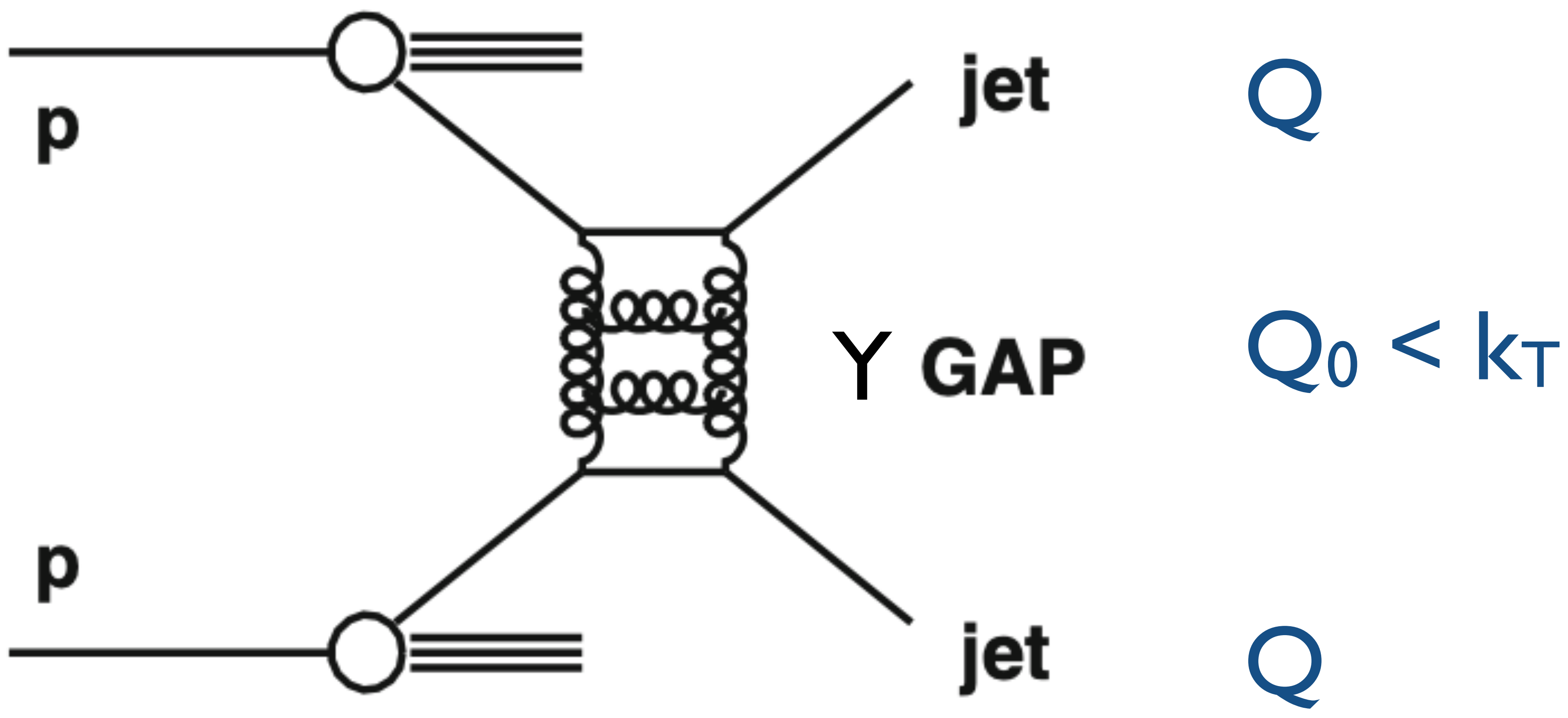
- soft and collinear factorization recently discussed

Cieri, Dhani, Rodrigo (2024)

Proof of a cancellation requieres a dedicated computation!

book-keeping of terms very complicated for subtraction method

- Even if there is cancellation, effects remain in observables : super-leading logs in large rapidity gaps generated by breaking of strict collinear factorization at higher orders



assuming that collinear radiation logs can be absorbed into parton distributions (evolution) expects $\alpha_s^n \ln^n(Q/Q_0)$

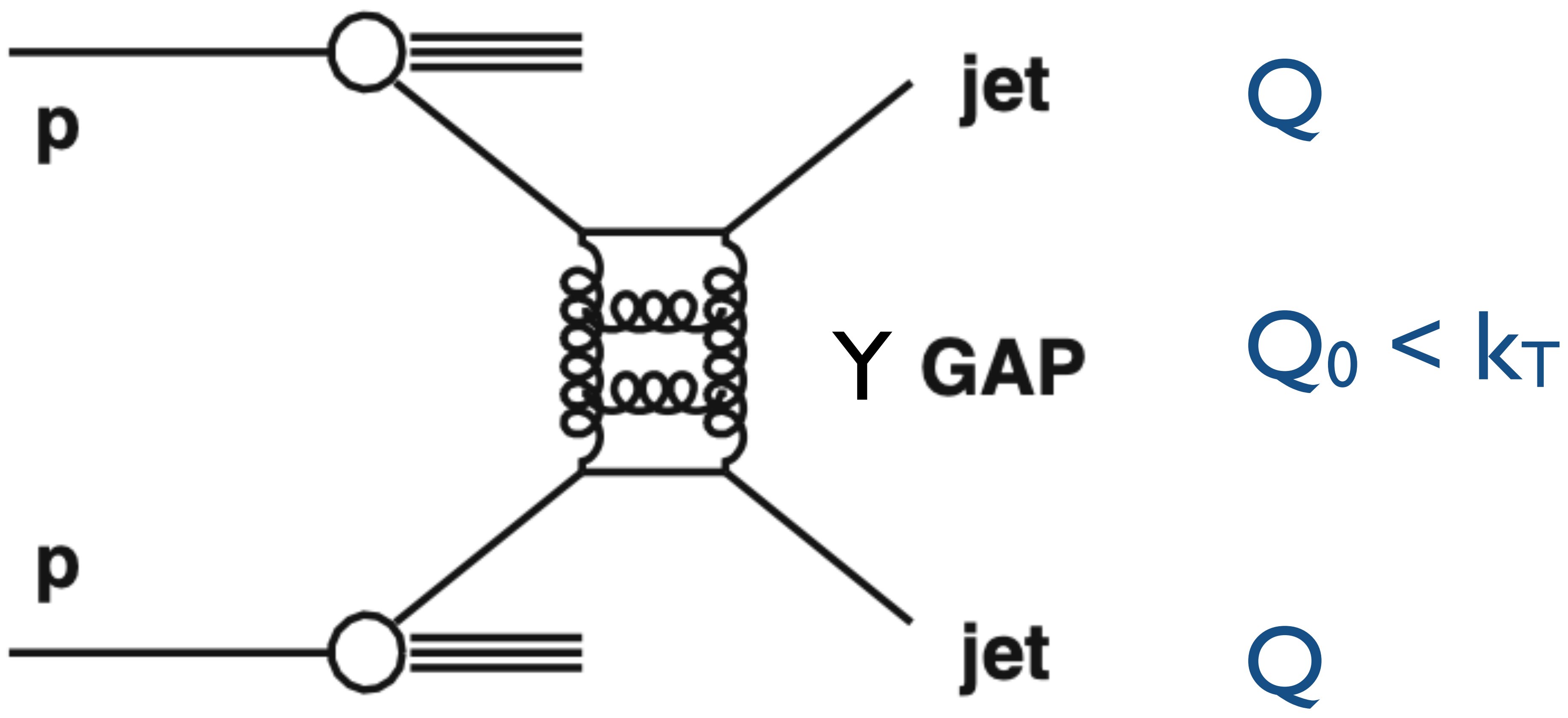
but larger logs appear starting at 4th order

$$\sigma_{1,gg} = -\sigma_0 \left(\frac{2\alpha_s}{\pi}\right)^4 \ln^5\left(\frac{Q}{Q_0}\right) \pi^2 Y \frac{3N^2+4}{80}$$

$$\alpha_s^n L^{2n-3} \pi^2 Y$$

Dasgupta, Salam (2001)
Forshaw, Kyrieleis, Seymour (2006)

- Even if there is cancellation, effects remain in observables : super-leading logs in large rapidity gaps generated by breaking of strict collinear factorization at higher orders



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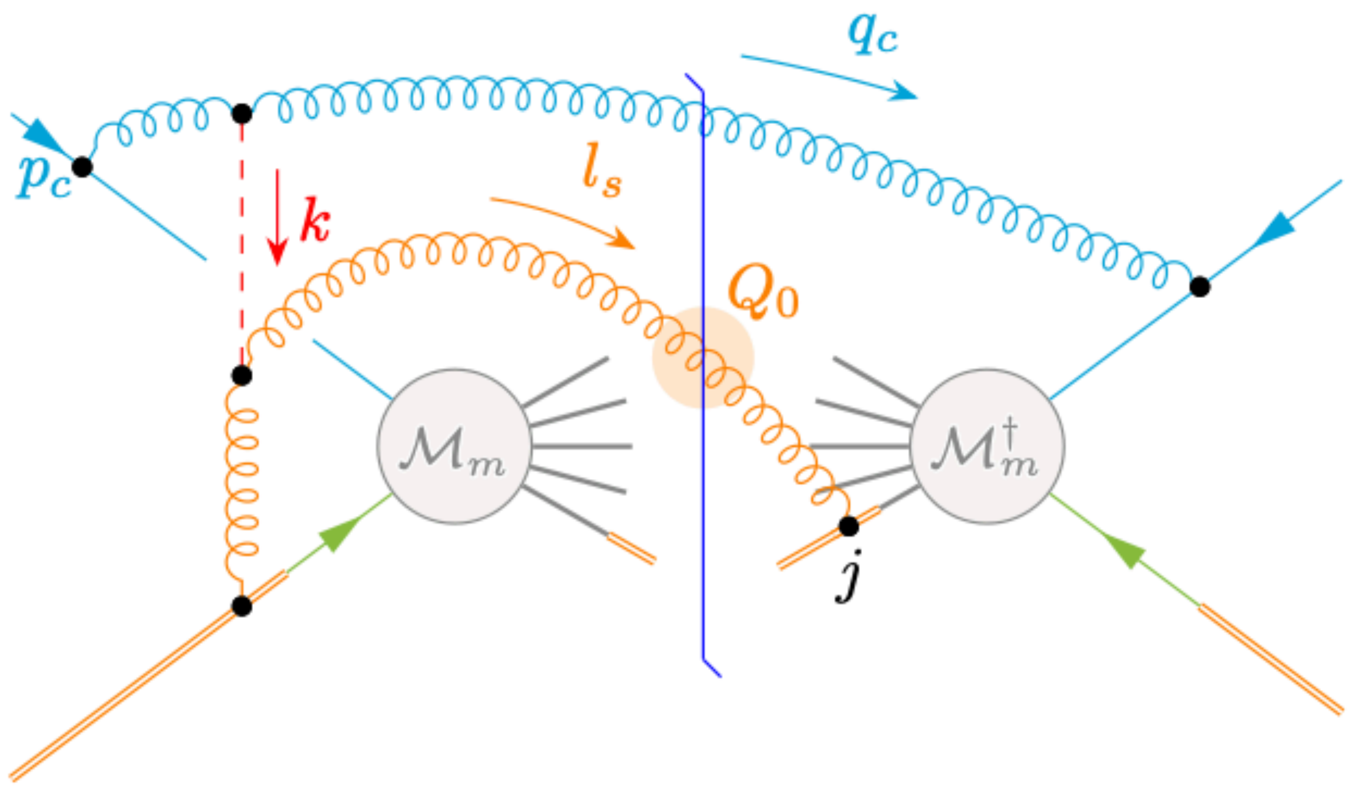
$$\alpha_s^n L^{2n-3} \pi^2 Y$$

Dasgupta, Salam (2001)
Forshaw, Kyrieleis, Seymour (2006)

- double Logs generated by exchange of Coulomb/Glauber gluons : how to reconcile this with the single logarithmic evolution implied by PDF factorization?

Becher, Hager, Jaskiewicz, Neubert, Schwienbacher (2024)

- Find intricate mechanism to cancel double log by Glauber gluons: interplay of space-like collinear and soft gluon emission restores factorization for non-global large rapidly gap cross section (jet-veto)



Conclusions

- Strict collinear factorization is breaking in the SL region
- 1 loop effects are "imaginary" and disappear in cross-section
- 2 loop effects include real contributions, requiere 5 QCD partons

  vanish if pure QCD Born amplitude
- 3 loop effects are there : N4LO
- There must be an intricate cancellation with collinear+soft to restore PDF factorization
- Even in that case, there are remnants of "partial" cancellation : super leading logs

Without Factorization


Without Factorization
With Factorization

THANKS


Backup

splitting amplitudes

$$Sp^{(0)\,(c_1,c_2;c)}(p_1,p_2;\widetilde{P}) \equiv \langle c_1,c_2|\,\boldsymbol{Sp}^{(0)}(p_1,p_2;\widetilde{P})\,|c\rangle$$

$q \to q_1 g_2$

$$Sp^{(0)\,(\alpha_1,a_2;\alpha)}_{q_1 g_2}(p_1,p_2;\widetilde{P}) = \mu^\epsilon\, g_S\, t^{a_2}_{\alpha_1\alpha}\,\frac{1}{s_{12}}\,\overline{u}(p_1)\,\not{\varepsilon}(p_2)\,u(\widetilde{P}) \quad ,$$

$\bar{q} \to \bar{q}_1 g_2$

$$Sp^{(0)\,(\alpha_1,a_2;\alpha)}_{\bar{q}_1 g_2}(p_1,p_2;\widetilde{P}) = \mu^\epsilon\, g_S\,\left(-t^{a_2}_{\alpha\,\alpha_1}\right)\,\frac{1}{s_{12}}\,\overline{v}(\widetilde{P})\,\not{\varepsilon}(p_2)\,v(p_1) \quad ,$$

$g \to q_1 \bar{q}_2$

$$Sp^{(0)\,(\alpha_1,\alpha_2;a)}_{q_1\bar{q}_2}(p_1,p_2;\widetilde{P}) = \mu^\epsilon\, g_S\, t^{a}_{\alpha_1\alpha_2}\,\frac{1}{s_{12}}\,\overline{u}(p_1)\,\not{\varepsilon}^*(\widetilde{P})\,v(p_2) \quad ,$$

$g \to g_1 g_2$

$$Sp^{(0)\,(a_1,a_2;a)}_{g_1 g_2}(p_1,p_2;\widetilde{P}) = \mu^\epsilon\, g_S\, i\, f_{a_1 a_2\, a}\,\frac{2}{s_{12}}$$
$$\times\left[\varepsilon(p_1)\cdot\varepsilon(p_2)\; p_1\cdot\varepsilon^*(\widetilde{P}) + \varepsilon(p_2)\cdot\varepsilon^*(\widetilde{P})\; p_2\cdot\varepsilon(p_1) - \varepsilon(p_1)\cdot\varepsilon^*(\widetilde{P})\; p_1\cdot\varepsilon(p_2)\right]$$

Altarelli-Parisi Splitting functions in d-dimensions

with spin correlations
(factorization is not complete at
the squared matrix elements due to spin)

$$\hat{P}_{qg}^{ss'}(z,k_\perp;\epsilon) = \hat{P}_{\bar{q}g}^{ss'}(z,k_\perp;\epsilon) = \delta_{ss'}\, C_F \left[\frac{1+z^2}{1-z} - \epsilon(1-z)\right]$$

$$\hat{P}_{gq}^{ss'}(z,k_\perp;\epsilon) = \hat{P}_{g\bar{q}}^{ss'}(z,k_\perp;\epsilon) = \delta_{ss'}\, C_F \left[\frac{1+(1-z)^2}{z} - \epsilon z\right]$$

$$\hat{P}_{q\bar{q}}^{\mu\nu}(z,k_\perp;\epsilon) = \hat{P}_{\bar{q}q}^{\mu\nu}(z,k_\perp;\epsilon) = T_R \left[-g^{\mu\nu} + 4z(1-z)\frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2}\right]$$

$$\hat{P}_{gg}^{\mu\nu}(z,k_\perp;\epsilon) = 2C_A \left[-g^{\mu\nu}\left(\frac{z}{1-z} + \frac{1-z}{z}\right) - 2(1-\epsilon)z(1-z)\frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2}\right]$$

$$\langle \hat{P}_{qg}(z;\epsilon)\rangle = \langle \hat{P}_{\bar{q}g}(z;\epsilon)\rangle = C_F \left[\frac{1+z^2}{1-z} - \epsilon(1-z)\right]$$

$$\langle \hat{P}_{gq}(z;\epsilon)\rangle = \langle \hat{P}_{g\bar{q}}(z;\epsilon)\rangle = C_F \left[\frac{1+(1-z)^2}{z} - \epsilon z\right]$$

$$\langle \hat{P}_{q\bar{q}}(z;\epsilon)\rangle = \langle \hat{P}_{\bar{q}q}(z;\epsilon)\rangle = T_R \left[1 - \frac{2z(1-z)}{1-\epsilon}\right]$$

$$\langle \hat{P}_{gg}(z;\epsilon)\rangle = 2C_A \left[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)\right]$$

after average of polarizations of parton a
usual Splitting functions

$$\left|\mathcal{M}_{a_1,a_2,\ldots}(p_1,p_2,p_3\ldots)\right|^2 \simeq \frac{2}{s_{12}} 4\pi\mu^{2\epsilon}\alpha_S\, \hat{P}(z;\epsilon)\left|\mathcal{M}_{a,\ldots}(P,p_3\ldots)\right|^2$$

Universal factorization

PDFs (parton model) factorization direct result from collinear factorization (roughly speaking)

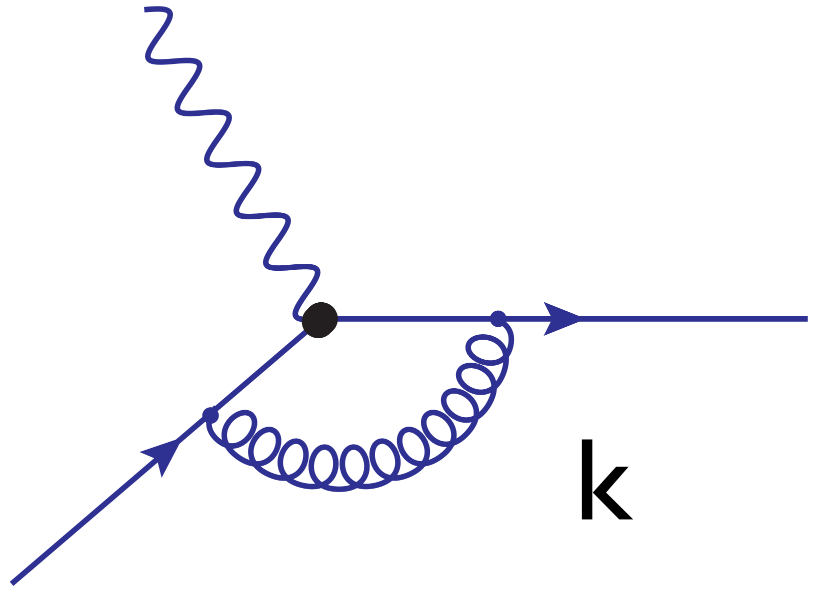
Parton model with QCD corrections: problems…

- Real and virtual contributions : separately divergent

UV

$$\int_0^\infty dk$$

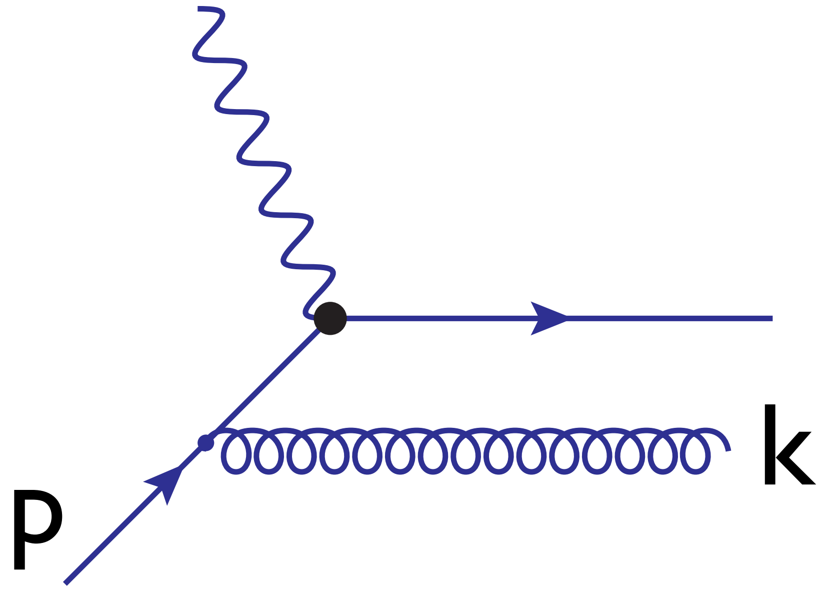
IR



1 loop

+

$$\int dk$$



$$\frac{-1}{(p-k)^2} = \frac{1}{2\,p\cdot k} = \frac{1}{2E_q E_g(1-\cos\theta)}$$

dimensional regularization $\rightarrow$ $\dfrac{1}{\epsilon^2}$

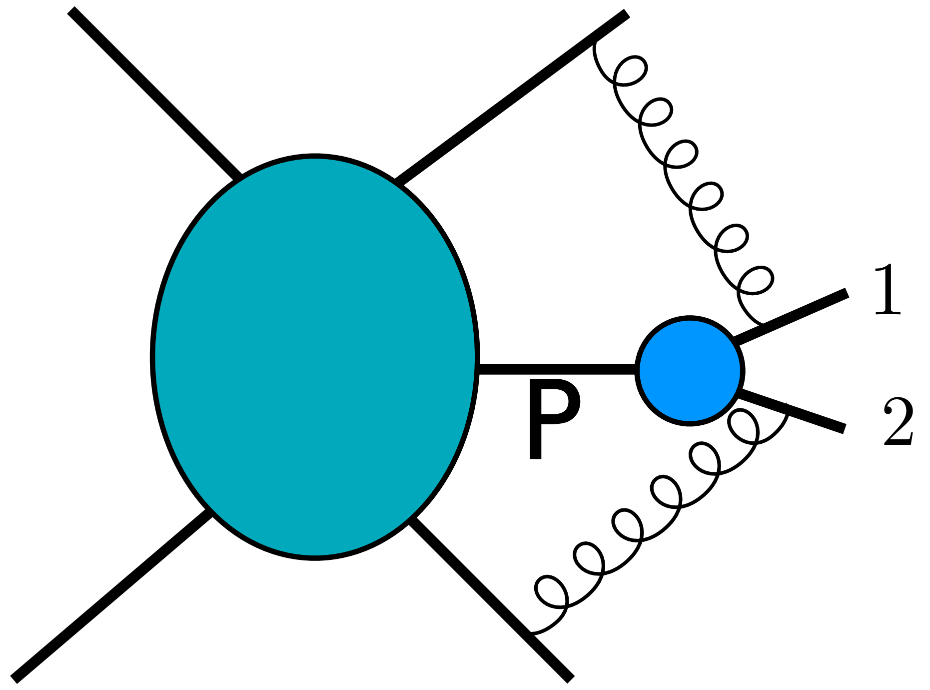
$\epsilon = d - 4$

1 extra parton

IR in soft/collinear configurations

soft divergences cancel but collinear divergence remains in partonic cross section…

factorization of mass singularities KLN theorem



$t \to +\infty$

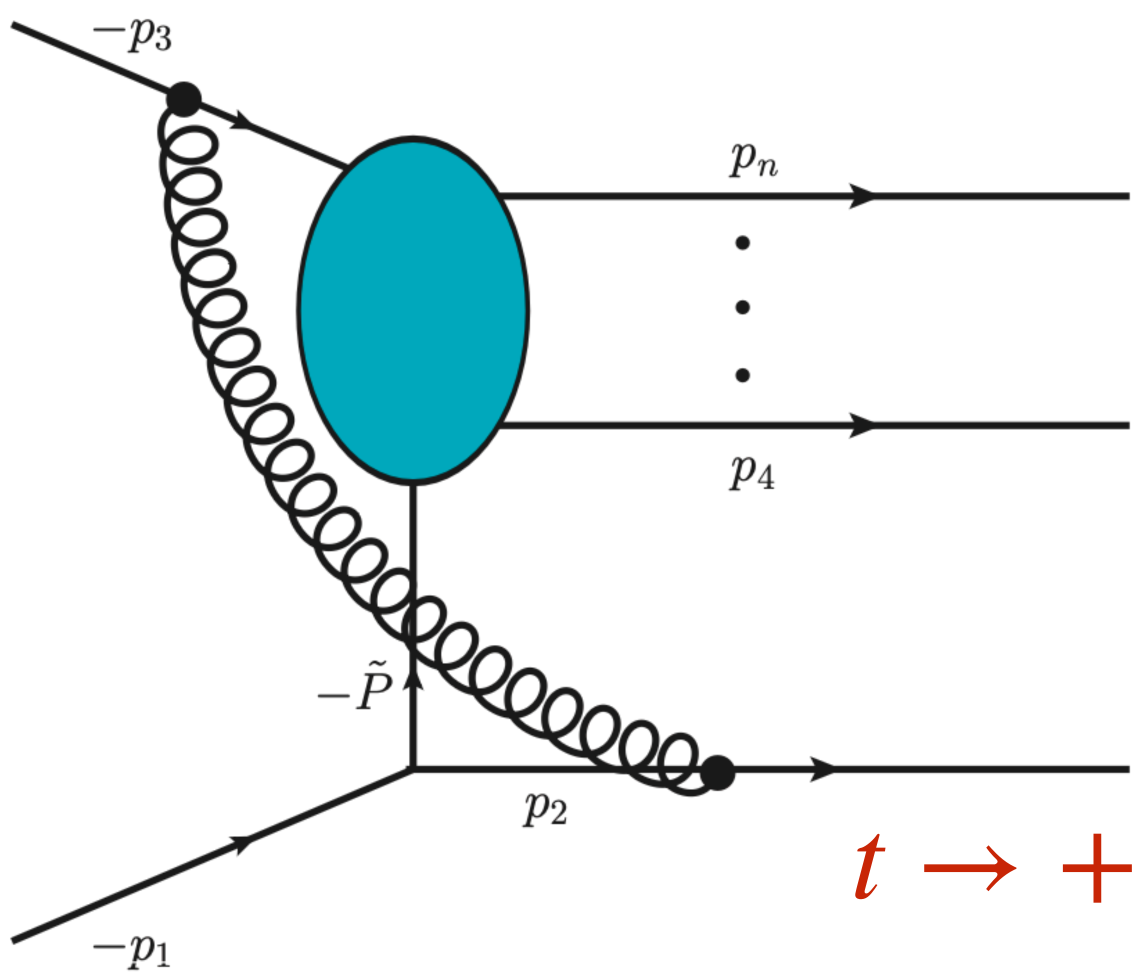
why color coherence in TL and DIS and NOT in PP ?

$-\frac{i}{\epsilon}\pi T_j \cdot T_2$

the absorptive contribution when a slightly off-shell gluon (non-abelian Coulomb or Glauber gluon) is exchanged and depends in the "energy sign" (both initial or both final state) : causal origin! $t \to \pm\infty$

Color coherence still valid in loops for TL, everything happens at $t \to +\infty$, no distinction between large space and time

$t \to -\infty$



$t \to +\infty$

Color coherence not valid in loops for SL, collinear process involves IS and FS , gauge theories generate interactions between $t \to -\infty$ and $t \to +\infty$ (distinction between large space and time)

- given that the non-factorizable term appears in the single pole, there is a simple way to compute it

we know the IR structure of one and two-loop amplitudes

$$|\mathcal{M}^{(1,R)}\rangle = \boldsymbol{I}_M^{(1)}(\epsilon)\,|\mathcal{M}^{(0,R)}\rangle + |\mathcal{M}^{(1)\,\mathrm{fin.}}\rangle$$

$$\boldsymbol{I}_M^{(1)}(\epsilon) = \frac{\alpha_S(\mu^2)}{2\pi}\,\frac{1}{2}\left\{ -\sum_{i=1}^{n}\left(\frac{1}{\epsilon^2}C_i + \frac{1}{\epsilon}\gamma_i\right) - \frac{1}{\epsilon}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \boldsymbol{T}_i\cdot\boldsymbol{T}_j\,\ln\left(\frac{-s_{ij}-i0}{\mu^2}\right)\right\}$$

- Formulae is valid for both the original amplitude with n partons and the reduced with n-1

$$\boldsymbol{Sp}^{(1,R)}\,|\overline{\mathcal{M}}^{(0,R)}\rangle \simeq |\mathcal{M}^{(1,R)}\rangle - \boldsymbol{Sp}^{(0,R)}\,|\overline{\mathcal{M}}^{(1,R)}\rangle$$

n-1   *n*   *n-1*

the IR divergente part is

$$\boldsymbol{Sp}^{(1,R)} \simeq \boldsymbol{I}_M^{(1)}(\epsilon)\,\boldsymbol{Sp}^{(0,R)} - \boldsymbol{Sp}^{(0,R)}\,\boldsymbol{I}_{\overline{M}}^{(1)}(\epsilon) + \mathcal{O}(\epsilon^0)$$

just apply collinear limit

$$\ln\left(\frac{-s_{ji}-i0}{\mu^2}\right) \simeq \ln(z_i - i0 s_{ji}) + \ln\left(\frac{-s_{j\widetilde{P}}-i0}{\mu^2}\right)$$

$$\boxed{\boldsymbol{Sp}^{(1,R)} \simeq -\frac{2}{\epsilon}\sum_{\substack{i\in C\\ j\in NC}} \boldsymbol{T}_j\cdot\boldsymbol{T}_i\,\ln(z_i - i0 s_{ji})\;\boldsymbol{Sp}^{(0,R)} + \text{factorizable terms}}$$

very useful for 2-loops!

- look only at the IR structure (poles) at the two loop order

$$|\mathcal{M}^{(2,R)}\rangle = \boldsymbol{I}_M^{(2)}(\epsilon)\,|\mathcal{M}^{(0,R)}\rangle + \boldsymbol{I}_M^{(1)}(\epsilon)\,|\mathcal{M}^{(1,R)}\rangle + \,|\mathcal{M}^{(2)\,\text{fin.}}\rangle$$

$$\boldsymbol{I}_M^{(1)}(\epsilon) = \frac{\alpha_S(\mu^2)}{2\pi}\,\frac{1}{2}\left\{ -\sum_{i=1}^{n}\left(\frac{1}{\epsilon^2}C_i + \frac{1}{\epsilon}\gamma_i\right) - \frac{1}{\epsilon}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \boldsymbol{T}_i\cdot\boldsymbol{T}_j\,\ln\left(\frac{-s_{ij}-i0}{\mu^2}\right)\right\}$$

$$K = \left(\frac{67}{18} - \frac{\pi^2}{6}\right)C_A - \frac{5}{9}N_f$$

$$\begin{aligned}\boldsymbol{I}_M^{(2)}(\epsilon) &= -\frac{1}{2}\left[\boldsymbol{I}_M^{(1)}(\epsilon)\right]^2 + \frac{\alpha_S(\mu^2)}{2\pi}\left\{+\frac{1}{\epsilon}b_0\left[\boldsymbol{I}_M^{(1)}(2\epsilon) - \boldsymbol{I}_M^{(1)}(\epsilon)\right] + K\,\boldsymbol{I}_M^{(1)}(2\epsilon)\right\}\\ &+ \left(\frac{\alpha_S(\mu^2)}{2\pi}\right)^2\frac{1}{\epsilon}\sum_{i=1}^{n} H_i^{(2)}\ .\end{aligned}$$

- Formulae is valid for both the original amplitude with *n* partons and the reduced with *n-1* and we know exactly the one-loop splitting amplitude $\mathbf{Sp}^{(1,R)}$

$$|\mathcal{M}^{(2,R)}\rangle \simeq \boldsymbol{Sp}^{(2,R)}\,|\overline{\mathcal{M}}^{(0,R)}\rangle + \boldsymbol{Sp}^{(1,R)}\,|\overline{\mathcal{M}}^{(1,R)}\rangle + \boldsymbol{Sp}^{(0,R)}\,|\overline{\mathcal{M}}^{(2,R)}\rangle$$

just apply collinear limit $\ln\left(\frac{-s_{ji}-i0}{\mu^2}\right) \simeq \ln(z_i - i0s_{ji}) + \ln\left(\frac{-s_{j\widetilde{P}}-i0}{\mu^2}\right)$

- The two-loop splitting matrix can be decomposed as in terms of the known 1-loop results…

$$\boldsymbol{Sp}^{(2,R)} = \widetilde{\boldsymbol{I}}_C^{(2)}(\epsilon)\ \boldsymbol{Sp}^{(0,R)} + \widetilde{\boldsymbol{I}}_C^{(1)}(\epsilon)\ \boldsymbol{Sp}^{(1,R)} + \widetilde{\boldsymbol{Sp}}^{(2)\,\text{fin.}}$$

2 loop

1 loop, both break fact.

finite (not computed here)

+ new 2-loop operator

$$\widetilde{\boldsymbol{I}}_C^{(2)}(\epsilon) = -\frac{1}{2}\left[\widetilde{\boldsymbol{I}}_C^{(1)}(\epsilon)\right]^2 + \frac{\alpha_S(\mu^2)}{2\pi}\left\{\frac{1}{\epsilon}\, b_0\left[\widetilde{\boldsymbol{I}}_C^{(1)}(2\epsilon) - \widetilde{\boldsymbol{I}}_C^{(1)}(\epsilon)\right] + K\,\widetilde{\boldsymbol{I}}_C^{(1)}(2\epsilon)\right.$$
$$\left. + \frac{\alpha_S(\mu^2)}{2\pi}\left(\frac{-s_{12}-i0}{\mu^2}\right)^{-2\epsilon}\frac{1}{\epsilon}\left(H_1^{(2)} + H_2^{(2)} - H_{12}^{(2)}\right)\right\} + \widetilde{\boldsymbol{\Delta}}_C^{(2)}(\epsilon)\ ,$$

several terms violating factorization involving two-parton color correlations $\mathbf{T}_i \cdot \mathbf{T}_j$ from 1-loop

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several terms violating factorization involving two-parton color correlations $\mathbf{T}_i \cdot \mathbf{T}_j$ from 1-loop

- New structure appears: involves three-part color correlations (double and single poles)

$$\widetilde{\boldsymbol{\Delta}}_C^{(2)}(\epsilon) = \left(\frac{\alpha_S(\mu^2)}{2\pi}\right)^2\left(\frac{-s_{12}}{\mu^2}\right)^{-2\epsilon}\pi\, f_{abc}\sum_{i=1,2}\sum_{\substack{j,k=3\\ j\neq k}}^{n} T_i^a T_j^b T_k^c\, \Theta(-z_i)\,\text{sign}(s_{ij})\,\Theta(-s_{jk})$$
$$\times\ \ln\left(-\frac{s_{j\widetilde{P}}\, s_{k\widetilde{P}}\, z_1 z_2}{s_{jk}\, s_{12}} - i0\right)\left[-\frac{1}{2\,\epsilon^2} + \frac{1}{\epsilon}\ln\left(\frac{-z_i}{1-z_i}\right)\right]\ .$$

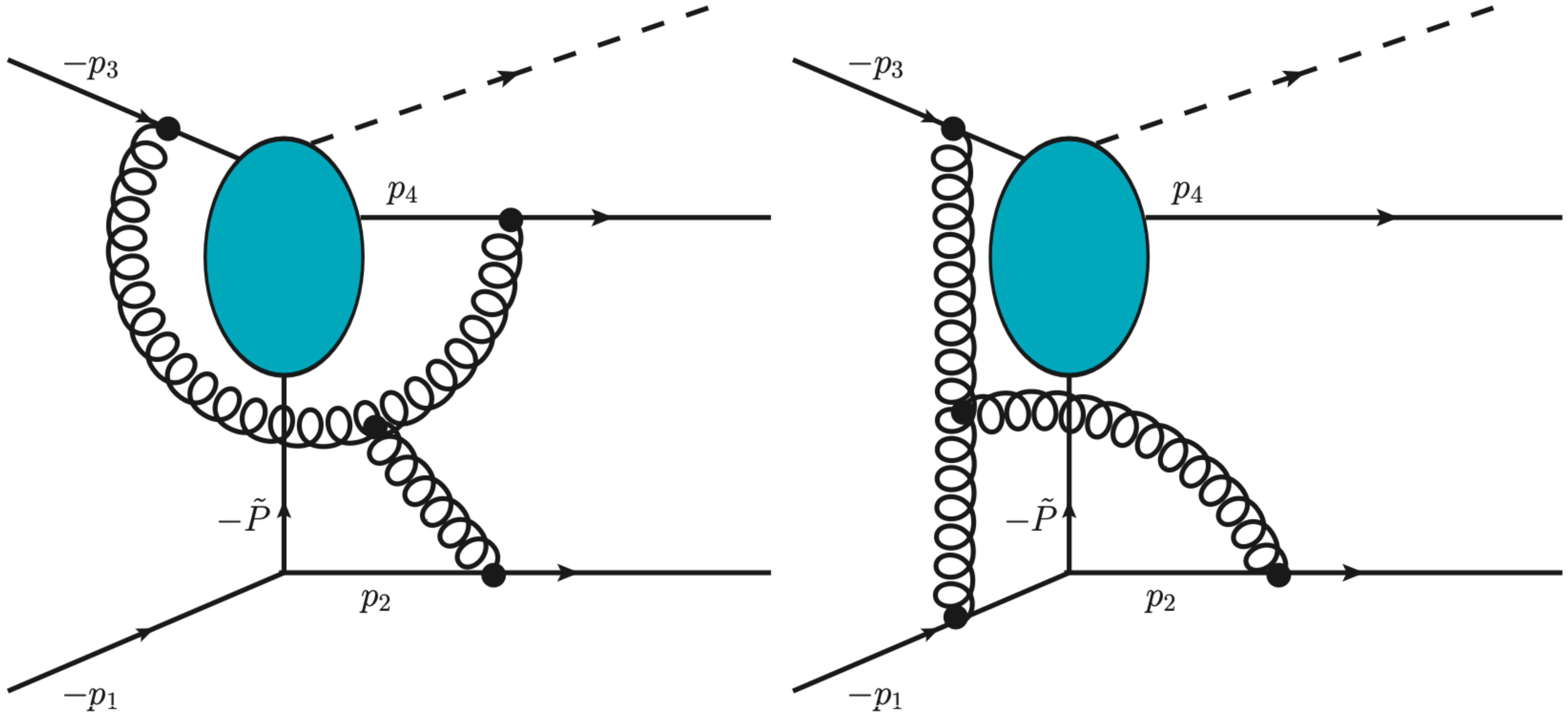
cancels for TL (z>0)

requires SL + non-collinear parton in initial and final state : cancels for DIS

$\widetilde{\Delta}_C^{(2)}(\epsilon)$ typically contributes to the SL collinear limit in hadron–hadron hard-scattering processes with at least 4 QCD partons

$1||2$

$z_2 < 0 \ , \quad s_{34} < 0 \ , \quad s_{23} < 0$



$$\widetilde{\boldsymbol{\Delta}}_C^{(2)}(\epsilon) \simeq \frac{\alpha_S^2(\mu^2)}{2\pi} \left(\frac{-s_{12}}{\mu^2}\right)^{-2\epsilon} f_{abc}\, T_1^a\, T_2^b\, T_3^c \ln\left(-\frac{s_{31}\, s_{42}}{s_{34}\, s_{12}} - i0\right) \left[-\frac{1}{2\,\epsilon^2} + \frac{1}{\epsilon} \ln\left(-\frac{z_2}{z_1}\right)\right]$$

- Factorization breaking depends also on the momenta (not only on sign of the energies as 1-loop)
- Contains both hermitian (depends on momenta) and non-hermitian contributions (only on signs)
- Finite part (not-computed) also contains factorization breaking terms with same color correlations

- Now we take the square and see what remains!

iteration of 1-loop, factorized (antihermitian)

$$\mathbf{P}^{(2,R)} = \widetilde{I}_P^{(1)}(\epsilon)\,\mathbf{P}^{(1,R)} + \widetilde{I}_P^{(2)}(\epsilon)\,\mathbf{P}^{(0,R)} + \left(\boldsymbol{Sp}_H^{(1,R)}\right)^\dagger \boldsymbol{Sp}_H^{(1,R)} + \left(\boldsymbol{Sp}^{(0,R)}\right)^\dagger \widetilde{\boldsymbol{\Delta}}_P^{(2)}(\epsilon)\,\boldsymbol{Sp}^{(0,R)} + \left[\left(\boldsymbol{Sp}^{(0,R)}\right)^\dagger \widetilde{\boldsymbol{Sp}}^{(2)\,\mathrm{fin.}} + \mathrm{h.c.}\right]$$

divergent, factorization breaking (from hermitian part)

finite, not computed but factorization breaking

- Non abelian, only for SL, vanishing in DIS

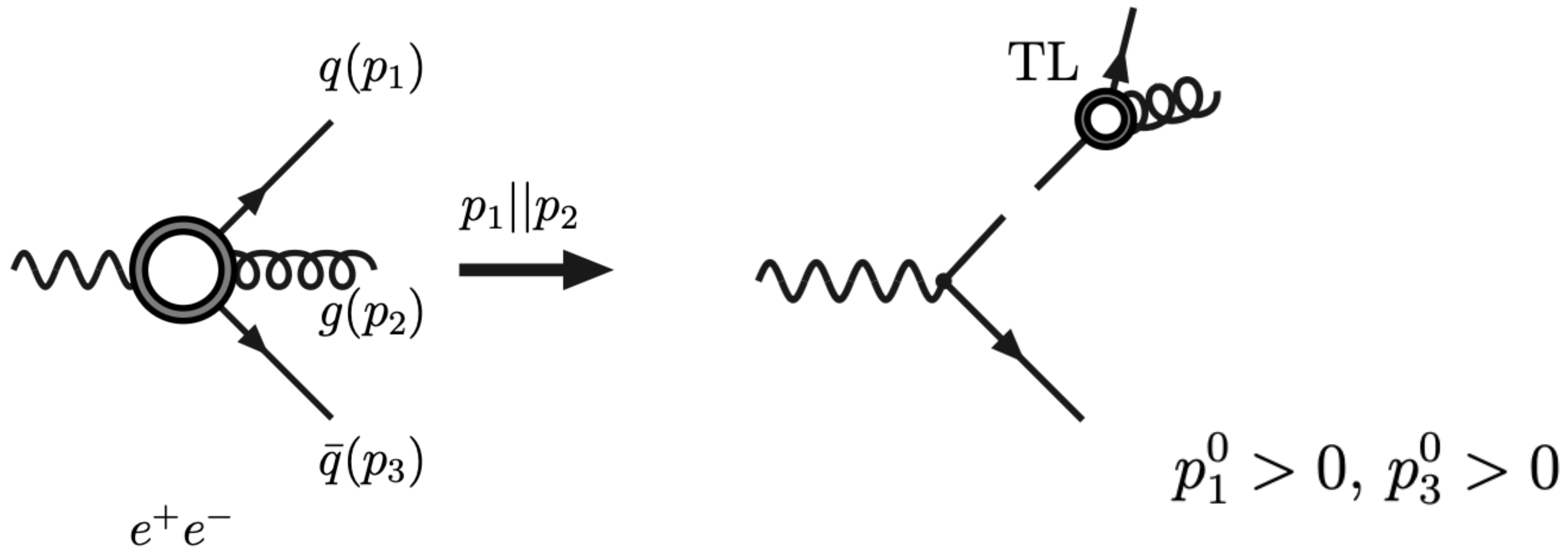
$$\widetilde{\boldsymbol{\Delta}}_P^{(2)}(\epsilon) = \left(\frac{\alpha_S(\mu^2)}{2\pi}\right)^2 \left(\frac{-s_{12}}{\mu^2}\right)^{-2\epsilon} 2\,\pi\, f_{abc} \sum_{i=1,2} \sum_{\substack{j,k=3\\ j\neq k}}^{n} T_i^a\, T_j^b\, T_k^c\, \Theta(-z_i)\,\mathrm{sign}(s_{ij})\,\Theta(-s_{jk}) \times \ln\left(\frac{s_{j\widetilde{P}}\, s_{k\widetilde{P}}\, z_1 z_2}{s_{jk}\, s_{12}}\right) \left[-\frac{1}{2\,\epsilon^2} + \frac{1}{\epsilon}\ln\left(\frac{-z_i}{1-z_i}\right)\right] .$$

- Because of color conservation it requires 5 QCD partons: 2 collinear, 1 extra incoming, 2 extra final

Simplest case: only 3 QCD partons (color sum closed)

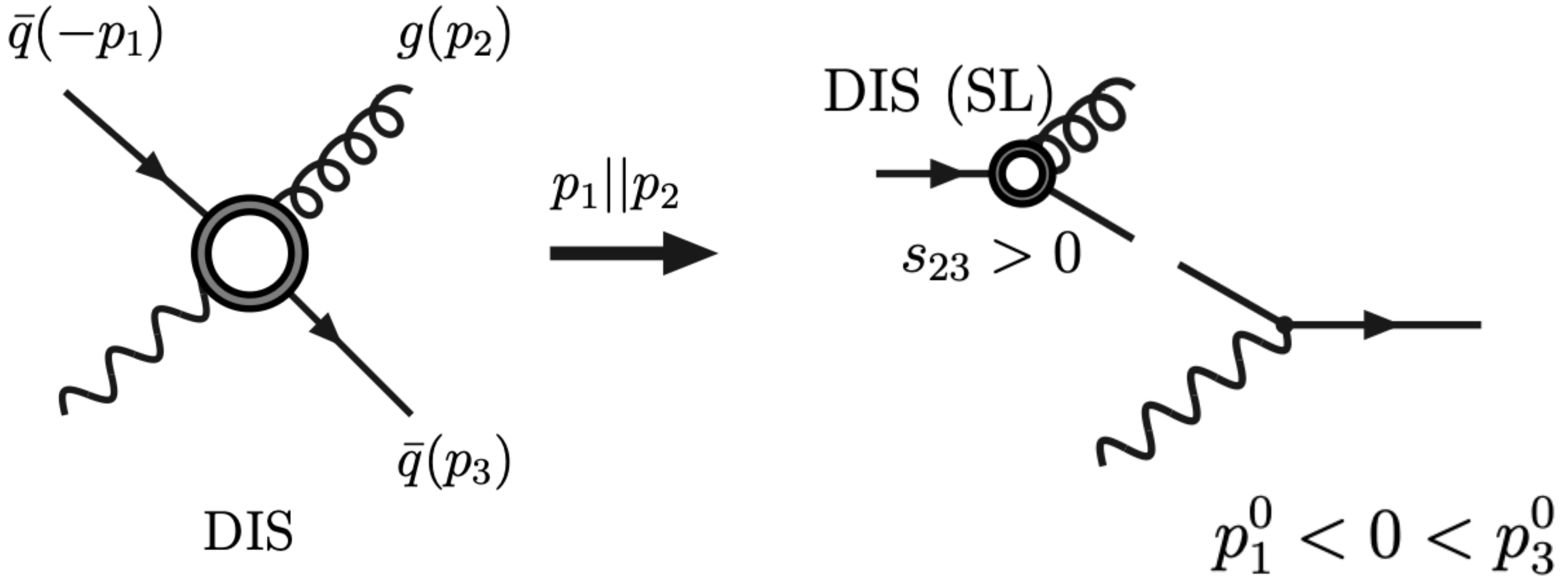
$$\boldsymbol{\delta}(p_1,p_2;p_3) = -\frac{1}{\epsilon}\Big(C_{12}+C_2-C_1\Big)\, f(\epsilon; z_2 - i0 s_{23})$$

just pole structure
differ only in imaginary part



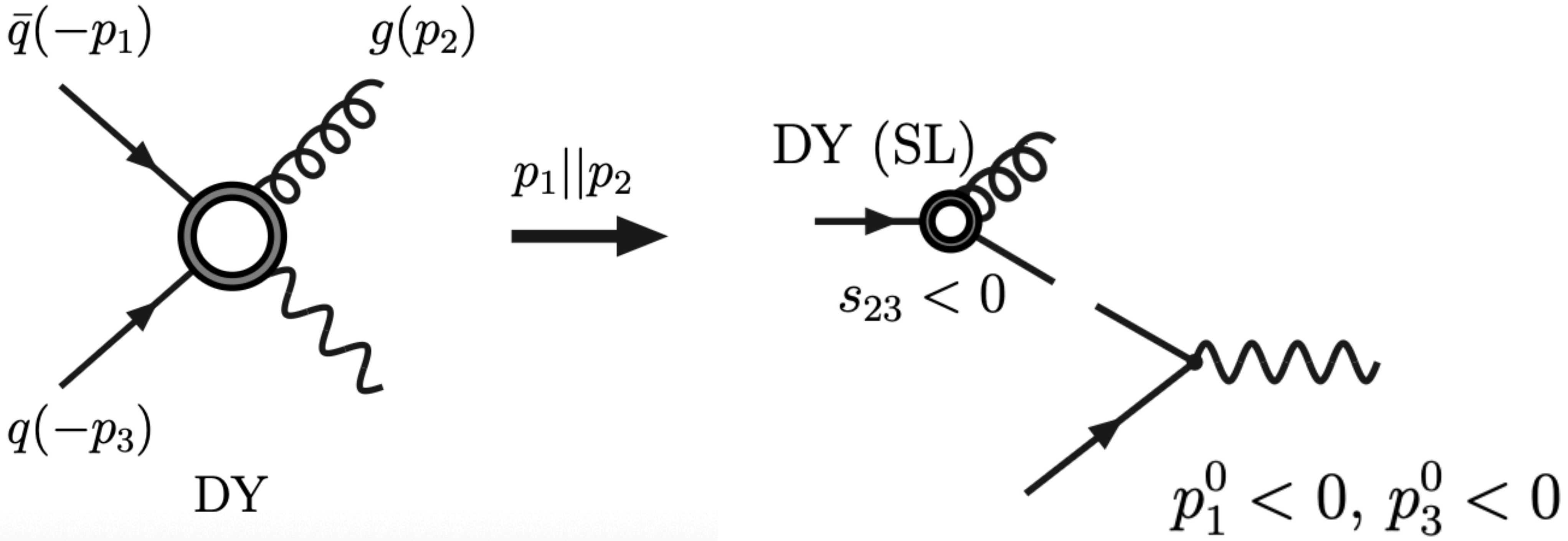
$$\boldsymbol{\delta}^{(e^+e^-)}(p_1,p_2;p_3) = -\frac{1}{\epsilon}\, C_A\, f(\epsilon; z_2)$$

$$-\frac{C_A}{\epsilon}\log(z_2) + \cdots$$



$$\boldsymbol{\delta}^{(\mathrm{DIS})}(p_1,p_2;p_3) = -\frac{1}{\epsilon}\, C_A\, f(\epsilon; z_2 - i0)$$

$$-\frac{C_A}{\epsilon}\left[\log(|z_2|) + i\pi\right] + \cdots$$



$$\boldsymbol{\delta}^{(\mathrm{DY})}(p_1,p_2;p_3) = -\frac{1}{\epsilon}\, C_A\, f(\epsilon; z_2 + i0)$$

$$-\frac{C_A}{\epsilon}\left[\log(|z_2|) - i\pi\right] + \cdots$$

- in any case, strict factorization guarantees mass singularities cancellation but not the other way around

still mass factorization can be the result of cancellation between different configurations

- Back to the simplest example $pp \to 2\text{jet}$ at N3LO $\quad$ $\text{parton parton} \to \text{parton parton} \quad \text{at LO}$

- Many configurations contribute, some of them are

3 loop $\text{parton parton} \to \text{parton parton}$

2 loop $\text{parton parton} \to \text{parton parton} + \text{soft parton}$

"wrong" kinematics can cancel
only $\delta(1-z)$ contributions

1 loop $\text{parton parton} \to \text{parton parton} + 2 \text{ collinear partons}$

triple collinear, either factorizes or
involves two color correlations

Born $\text{parton parton} \to \text{parton parton} + 3 \text{ collinear partons}$

quadruple collinear, Born level.
strict factorization

1 loop $\text{parton parton} \to \text{parton parton} + 1 \text{ collinear parton} + 1 \text{ soft parton}$

this one could make it!