Small-x resummation for muon colliders

Marco Bonvini

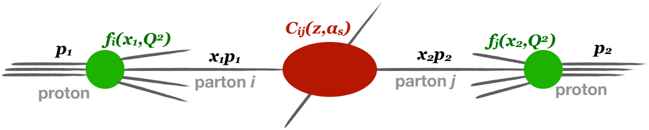
INFN, Sezione di Roma

Resummation, Evolution, Factorization (REF 2025)

16 Oct 2025, Milano, Italy

Based on a work in progress with Stefano Frixione and Giovanni Stagnitto

Theoretical predictions with hadrons in the initial state



Collinear factorization theorem: $\tau = \frac{Q^2}{s}$ $\quad y = Y - \frac{1}{2}\log\frac{x_1}{x_2}$

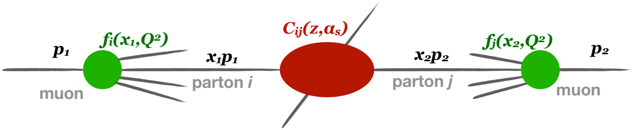
$$\frac{d\sigma}{dQ^2 dY dp_t \dots} = \sum_{i,j}\int_\tau^1 dx_1 \int_\tau^1 dx_2\, f_i(x_1,Q^2) f_j(x_2,Q^2)\, C_{ij}\left(\frac{\tau}{x_1x_2}, y, p_t, \dots, \alpha_s, \alpha\right)$$

$$Q^2 \frac{d}{dQ^2} f_i(x,Q^2) = \sum_j \int_x^1 \frac{dz}{z} P_{ij}(z, \alpha_s(Q^2), \alpha(Q^2))\, f_j\left(\frac{x}{z}, Q^2\right) \qquad \leftarrow \text{DGLAP evolution}$$

- coefficient functions $C_{ij}(x,y,p_t,\dots,\alpha_s)$ (observable-dependent, perturbative)
- splitting functions $P_{ij}(x,\alpha_s,\alpha)$ (universal, perturbative)
- proton's parton distribution functions (PDFs) $f_i(x,Q^2)$ (universal, nonperturbative)

Proton's PDFs $f_i(x,Q_0^2)$ at a reference scale Q_0 are fitted from data

Theoretical predictions with leptons in the initial state



Collinear factorization theorem: $\tau = \frac{Q^2}{s}$ $\quad y = Y - \frac{1}{2}\log\frac{x_1}{x_2}$

$$\frac{d\sigma}{dQ^2 dY dp_t \dots} = \sum_{i,j} \int_\tau^1 dx_1 \int_\tau^1 dx_2 \, f_i(x_1, Q^2) f_j(x_2, Q^2) \, C_{ij}\left(\frac{\tau}{x_1 x_2}, y, p_t, \dots, \alpha_s, \alpha\right)$$

$$Q^2 \frac{d}{dQ^2} f_i(x, Q^2) = \sum_j \int_x^1 \frac{dz}{z} P_{ij}\left(z, \alpha_s(Q^2), \alpha(Q^2)\right) f_j\left(\frac{x}{z}, Q^2\right)$$ ← DGLAP evolution

- coefficient functions $C_{ij}(x, y, p_t, \dots, \alpha_s)$ (observable-dependent, perturbative)
- splitting functions $P_{ij}(x, \alpha_s, \alpha)$ (universal, perturbative)
- muon's parton distribution functions (PDFs) $f_i(x, Q^2)$ (universal, perturbative)

Muon's PDFs $f_i(x, m_\mu^2)$ can be computed perturbatively!
It describes (and resums) initial-state radiation in a convenient framework

Muon's initial-state radiation

In the $\overline{\text{MS}}$ scheme, the muon's PDFs are

$$f_\mu(x, Q^2) = \delta(1 - x)$$

$$f_\gamma(x, Q^2) = 0$$

$$f_i(x, Q^2) = 0 \qquad\qquad i = \bar{\mu}, e^-, e^+, q, \bar{q}, g$$

Muon's initial-state radiation

In the $\overline{\text{MS}}$ scheme, the muon's PDFs are [Frixione 1909.03886]

$$f_\mu(x,Q^2) = \delta(1-x) + \frac{\alpha}{2\pi}\left[\frac{1+x^2}{1-x}\left(\log\frac{Q^2}{m_\mu^2(1-x)^2} - 1\right)\right]_+$$

$$f_\gamma(x,Q^2) = 0 + \frac{\alpha}{2\pi}\frac{1+(1-x)^2}{x}\left(\log\frac{Q^2}{m_\mu^2 x^2} - 1\right)$$

$$f_i(x,Q^2) = 0 + 0 \qquad i = \bar\mu, e^-, e^+, q, \bar q, g$$

Muon's initial-state radiation

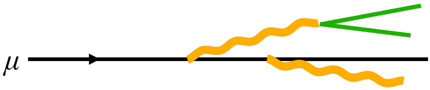
In the $\overline{\text{MS}}$ scheme, the muon's PDFs are [Frixione 1909.03886]

$$f_\mu(x,Q^2) = \delta(1-x) + \frac{\alpha}{2\pi}\left[\frac{1+x^2}{1-x}\left(\log\frac{Q^2}{m_\mu^2(1-x)^2} - 1\right)\right]_+ + \ldots$$

$$f_\gamma(x,Q^2) = 0 + \frac{\alpha}{2\pi}\frac{1+(1-x)^2}{x}\left(\log\frac{Q^2}{m_\mu^2 x^2} - 1\right) + \ldots$$

$$f_i(x,Q^2) = 0 + 0 + \ldots \qquad i = \bar\mu, e^-, e^+, q, \bar q, g$$

Now computed also to NNLO in QED [Stahlhofen 2508.16964] [Schnubel,Szafron 2509.09618]



These represent the initial condition (typically at $Q \sim m_\mu$) for the evolution

Evolution from the initial scale

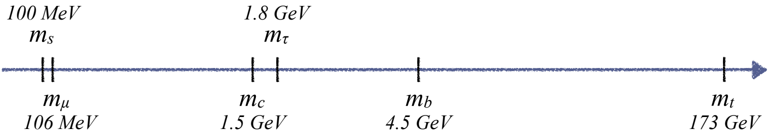
DGLAP evolution

$$Q^2 \frac{d}{dQ^2} f_i(x, Q^2) = \sum_j \int_x^1 \frac{dz}{z} P_{ij}\big(z, \alpha_s(Q^2), \alpha(Q^2)\big) f_j\left(\frac{x}{z}, Q^2\right)$$

Splitting functions $P_{ij}(z, \alpha_s(Q^2), \alpha(Q^2))$ known up to

- NNLO (α_s^3) and partial N^{3}LO (α_s^4) in QCD
- NLO (α^2) in QED
- NLO ($\alpha_s \alpha$) in mixed QED-QCD

Evolution starts from the muon scale ($m_\mu \sim 106$MeV)



The strong coupling α_s at low scales

Evolution from $Q \sim 100$MeV to $Q \sim 1$GeV is in a non-perturbative regime of QCD

How to deal with this problem?

Step 1:
Extend the running of α_s to lower scales using known analytic extensions which add a non-perturbative contribution

The simplest realization is

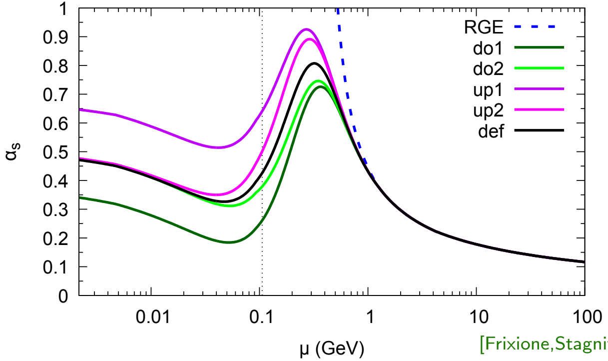
$$\alpha_s(Q^2) = \frac{1}{\beta_0 \log\frac{Q^2}{\Lambda^2}} \quad \to \quad \frac{1}{\beta_0}\left[\frac{1}{\log\frac{Q^2}{\Lambda^2}} + \frac{1}{1-\frac{Q^2}{\Lambda^2}}\right]$$

which removes the Landau pole, and gives a monotonic behaviour which tends to $\alpha_s(0) = 1/\beta_0$

Taking into account some constraints from event shapes and structure functions, one can modify the non-perturbative term to obtain a form which better agrees with data [Webber 9805484]

This can be extended to higher order running, including flavour thresholds, and adding variations as a measure of the uncertainty [Frixione,Stagnitto 2309.07516]

The strong coupling α_s at low scales



[Frixione,Stagnitto 2309.07516]

Step 2:
Use perturbative computations with these values of α_s (and cross fingers!)

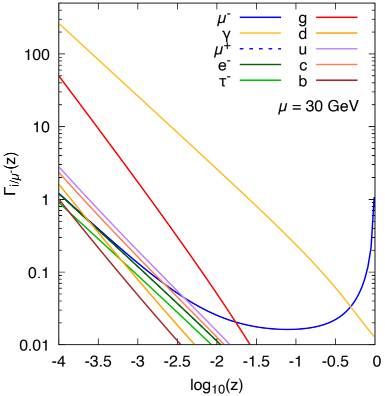
Note that α_s reaches values as high as 0.7-0.9...

A rather different approach has been adopted in [Garosi,Marzocca,Trifinopoulos 2303.16964] and [Han,Ma,Xie 2103.09844], where QCD is switched off below some scale, introducing an IR sensitivity

The PDFs for a muon collider

After evolving the muon's PDFs as described, using LO QED+QCD evolution, this plot is obtained at $Q = 30\text{GeV}$

The dominant PDFs at medium/small x are the photon and the gluon.



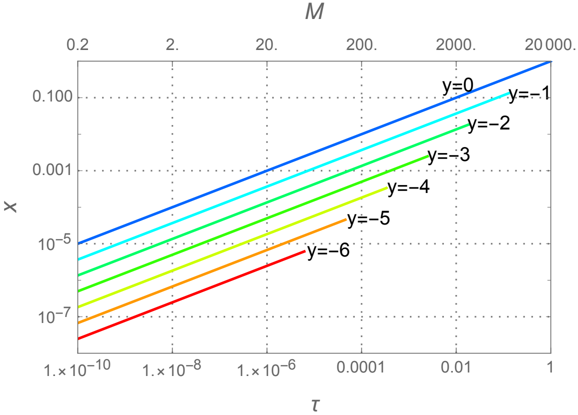
[Frixione,Stagnitto 2309.07516]

A precise determination of the gluon PDF in the muon requires the resummation of small-x logarithms

[MB,Frixione,Stagnitto (work in progress)]

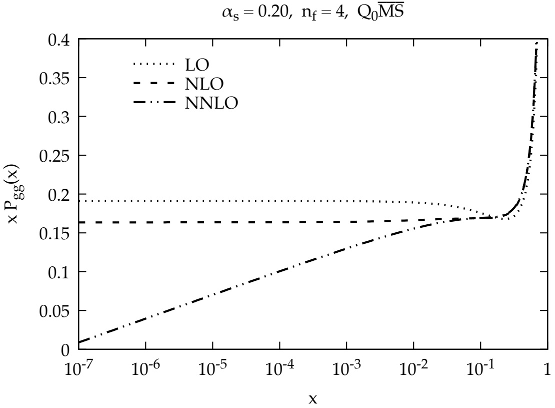
Kinematic reach of a 20 TeV muon collider

Typical values of x contributing at different invariant masses and rapidities



For example, Higgs production at a $\mathbf{20}$ TeV muon collider needs x as small as $x \sim \mathbf{4 \cdot 10^{-5}}$, with typical values in the range $x \sim \mathbf{10^{-4} \div 10^{-3}}$

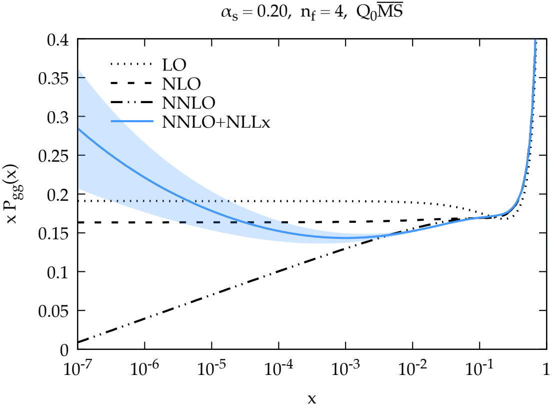
$P_{gg}(x, \alpha_s)$ splitting function



Logarithms start to grow for $x \lesssim 10^{-2}$ $\rightarrow$ **perturbative instability**

Small-x logarithms in gluon-gluon splitting function

$\boldsymbol{P_{gg}(x, \alpha_s)}$ splitting function



Logarithms start to grow for $x \lesssim 10^{-2}$ → **perturbative instability**

Resummation obtained with my HELL public code [MB,Marzani,Peraro EPJC 76(2016)11] [MB,Marzani,Muselli JHEP 12(2017)117] [MB,Marzani JHEP 06(2018)145]

following works of [Altarelli,Ball,Forte] [Catani,Ciafaloni,Colferai,Hautmann,Salam,Stasto] [Thorne,White]

Small-x resummation at large α_s

Problem: HELL 3.0 can only reach values of $\boldsymbol{\alpha_s}$ as high as $\boldsymbol{\alpha_s \sim 0.3}$
But now we need to reach $\boldsymbol{\alpha_s \sim 0.8}$!!

First part of the solution:
Improving various parts of the code, both numerical and conceptual aspects

But there still is a problem:
One ingredient, γ_{qg}, is not really resummed to all orders. The first coefficients of its expansion have been computed long ago [Catani,Hautmann 9405388]

$$\gamma_{qg}(\alpha_s, N) = \frac{\alpha_s}{3\pi} T_R \left\{ 1 + \frac{5}{3}\frac{\bar{\alpha}_s}{N} + \frac{14}{9}\left(\frac{\bar{\alpha}_s}{N}\right)^2 + \left[\frac{82}{81} + 2\,\zeta_3\right]\left(\frac{\bar{\alpha}_s}{N}\right)^3 + \left[\frac{122}{243} + \frac{25}{6}\zeta_3\right]\left(\frac{\bar{\alpha}_s}{N}\right)^4 + \left[\frac{146}{729} + \frac{14}{3}\zeta_3 + 2\,\zeta_5\right]\left(\frac{\bar{\alpha}_s}{N}\right)^5 + \ldots \right\}$$

More coefficients have been later computed numerically [Altarelli,Ball,Forte 0802.0032]

The HELL implementation is based on a finite number of these coefficients $\rightarrow$ so it's not really all-order ☹

The resummation of γ_{qg}

$\gamma_{qg}(\alpha_s, N)$ can be extracted from the equation for the factorization of the quark Green function [Catani,Hautmann 9405388]

$$G^{(0)}_{qg}(\alpha_s, N, \epsilon) = G_{qg}(\alpha_s, N, \epsilon)\,\Gamma_{gg}(\alpha_s, N, \epsilon) + \Gamma_{qg}(\alpha_s, N, \epsilon)$$

with $(S_\epsilon = e^{-\epsilon\psi(1)}/4\pi)$

$$\Gamma_{gg}(\alpha_s, N, \epsilon) = \exp\left(\frac{1}{\epsilon}\int_0^{\alpha_s S_\epsilon}\frac{d\alpha}{\alpha}\,\gamma_{gg}(\alpha, N)\right)$$

$$\Gamma_{qg}(\alpha_s, N, \epsilon) = \frac{1}{\epsilon}\int_0^{\alpha_s S_\epsilon}\frac{d\alpha}{\alpha}\,\gamma_{qg}(\alpha, N)\Gamma_{gg}(\alpha/S_\epsilon)$$

$$G^{(0)}_{qg}(\alpha_s, N, \epsilon) = \frac{\alpha_s S_\epsilon}{3\pi\epsilon} T_R \sum_{k=0}^{\infty}\left(\frac{\bar\alpha_s S_\epsilon}{N}\right)^k \sum_{j=-k}^{\infty} d_{kj}\epsilon^j$$

where d_{kj} are complicated coefficients known recursively.

Requiring that $G_{qg}(\alpha_s, N, \epsilon)$ is finite for $\epsilon \to 0$ it is possible to solve the equation order by order both for $G_{qg}(\alpha_s, N, \epsilon)$ and for $\gamma_{qg}(\alpha_s, N)$.

But this is an order-by-order extraction, not a resummation!

All-order expression for γ_{qg}

The rational part of γ_{qg} was actually known to all orders [Catani,Hautmann 9405388]

$$\gamma_{qg}(\alpha_s, N) = \frac{\alpha_s}{3\pi} T_R \frac{1}{4}\left[3\exp\frac{2\bar{\alpha}_s}{N} + \exp\frac{2\bar{\alpha}_s}{3N}\right] + \text{terms with } \zeta_k$$

All-order expression for γ_{qg}

The rational part of γ_{qg} was actually known to all orders [Catani,Hautmann 9405388]

$$\gamma_{qg}(\alpha_s, N) = \frac{\alpha_s}{3\pi} T_R \frac{1}{4}\left[3\exp\frac{2\bar\alpha_s}{N} + \exp\frac{2\bar\alpha_s}{3N}\right] + \text{terms with } \zeta_k$$

We have been able to find a complete closed form for γ_{qg}

$$\gamma_{qg}(\alpha_s, N) = \frac{\alpha_s}{3\pi} T_R\, h_{qg}(\gamma_{gg}(\alpha_s, N))$$

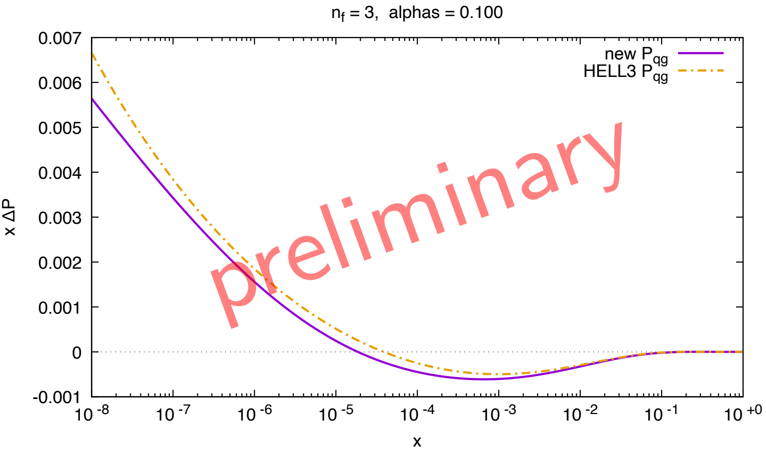
$$h_{qg}(M) = \frac{M\chi(M)}{4}\left[3\exp\Big(2MF(M)\Big) + \exp\left(\frac{2}{3}MF(M)\right)\right]$$

$$F(M) = \text{a function that you will see once we will publish our paper ☺}$$

$$\chi(M) = 2\psi(1) - \psi(M) - \psi(1-M) \qquad \text{(BFKL kernel)}$$

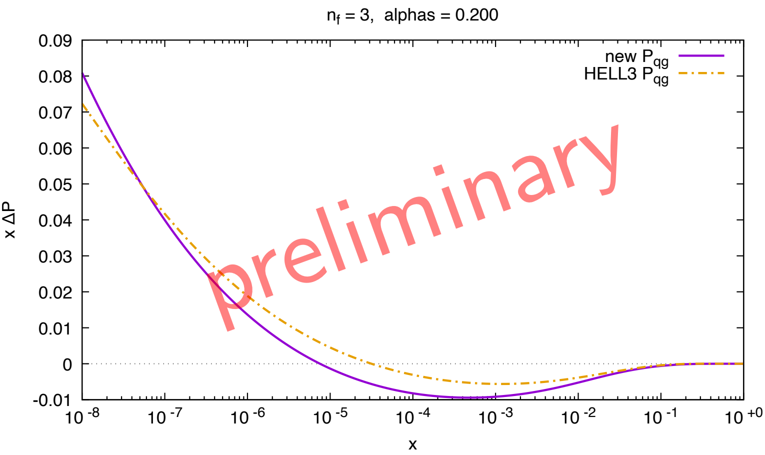
We also have analytic results for the coefficients, and a closed all-order form for G_{qg} at $\mathcal{O}(\epsilon^0)$

How does the resummed P_{qg} look like



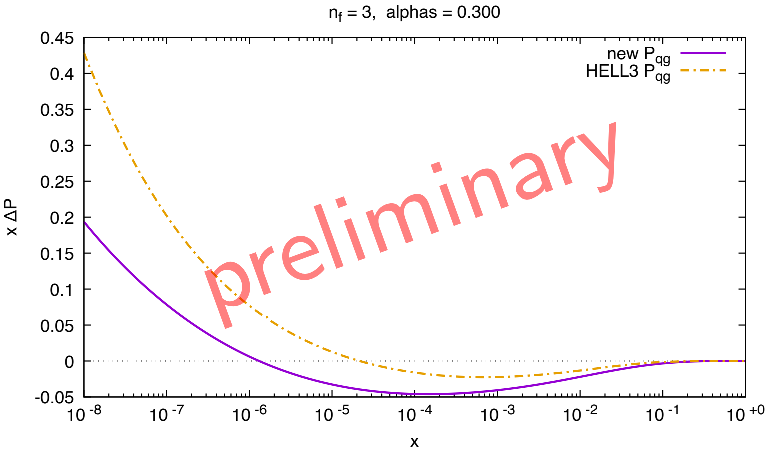
Previous HELL 3.0 implementation is good at small α_s, but it gets worse and worse as α_s increases

How does the resummed P_{qg} look like



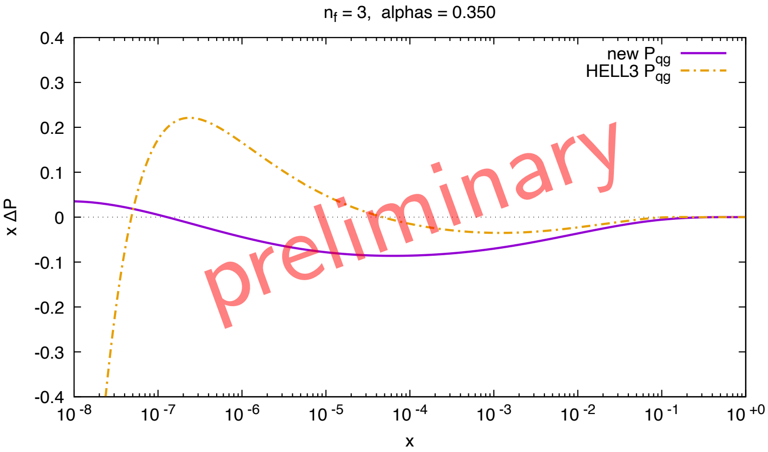
Previous HELL 3.0 implementation is good at small α_s, but it gets worse and worse as α_s increases

How does the resummed P_{qg} look like



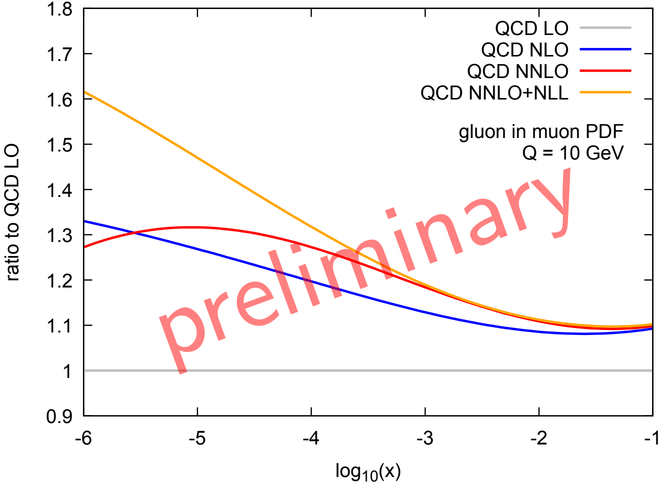
Previous HELL 3.0 implementation is good at small α_s, but it gets worse and worse as α_s increases

How does the resummed P_{qg} look like



Previous HELL 3.0 implementation is good at small α_s, but it gets worse and worse as α_s increases

Preliminary results on the effect of small-x resummation on the gluon PDF in the muon (no resummed matching conditions so far) [MB,Frixione,Stagnitto (work in progress)]

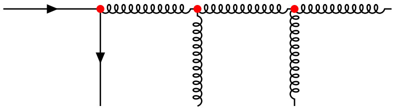


Do we need to worry about small-x logs in QED evolution?

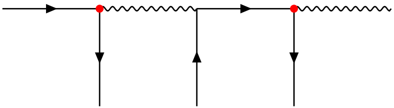
There are small-x logs in QED splitting functions, appearing already at $\mathcal{O}(\alpha)$ in $P_{\gamma f}$
Are they enhanced?

Yes, but much less than in QCD

QCD: Single-log enhancement, due to non-abelian nature of strong interactions



QED: Half-log enhancement (one extra power of the log every two powers of α), due to the abelian nature of EM interactions



Given also that $\alpha \ll \alpha_s$, their resummation is definitely not needed

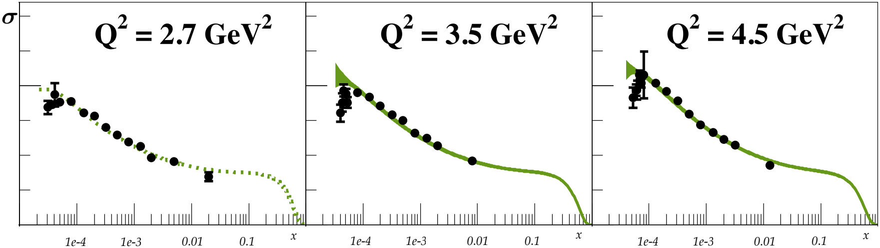
Conclusions

- ISR for a muon collider can be efficiently described through PDFs in a collinear factorisation framework
- These PDFs are perturbative, and they are computed from an initial condition at the muon scale and evolved upwards with DGLAP
- In doing so, evolution passes through low scales where QCD is non-perturbative
- Analytic coupling allows to describe this region ...
- ... but still α_s gets large ($\alpha_s \sim 0.8$) thus requiring a good control on perturbative ingredients
- At small x the photon and gluon PDFs dominate, calling for the resummation of small-x logarithms
- New analytic all-order results for the resummation of γ_{qg} ☺
- Now resummed results with HELL (new v4) can reach high values of α_s
- Sizable impact on muon PDFs

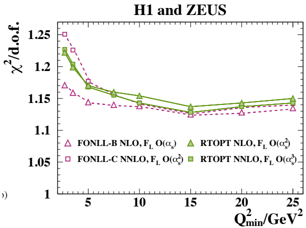
Backup slides

Deep-inelastic scattering (DIS) data from HERA extend down to $x \sim 3 \times 10^{-5}$ in the "perturbative region" $Q^2 > 2\,\text{GeV}^2$

Tension between HERA data at low Q^2 and low x with fixed-order theory



Also leads to a deterioration of the χ^2 of PDF fits when including low-Q^2 data



The first PDF fits with small-x resummation

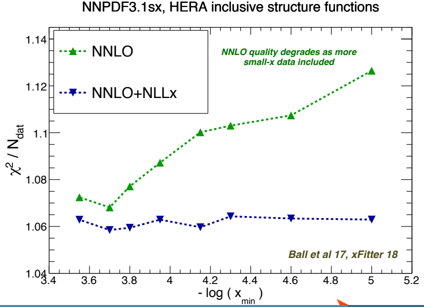
Small-x resummation available from the HELL code

NNPDF3.1sx [1710.05935]	xFitter [1802.00064, see also 1902.11125]
NeuralNet parametrization of PDFs	polynomial paramterization
MonteCarlo uncertainty	Hessian uncertainty
charm PDF is fitted	charm PDF perturbatively generated
DIS+tevatron+LHC ($\sim$ **4000** datapoints)	only HERA data ($\sim$ **1200** datapoints)
NLO, NLO+NLLx, NNLO, NNLO+NLLx	NNLO, NNLO+NLLx

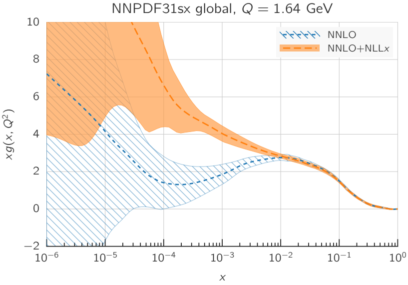
The quality of the fit improves substantially including small-x resummation

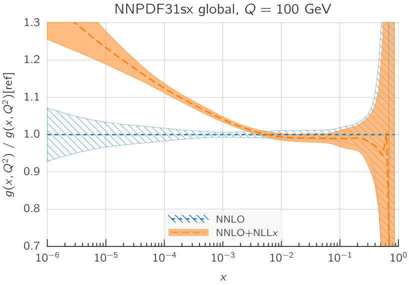
$\chi^2/N_{\rm dat}$	NNLO	NNLO+NLLx
xFitter	**1.23**	**1.17**
NNPDF3.1sx	**1.130**	**1.100**
		smaller!

Stable upon inclusion of low-x data $\rightarrow$



Impact of small-x resummation on PDFs: the gluon



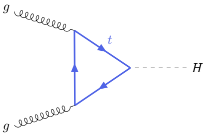


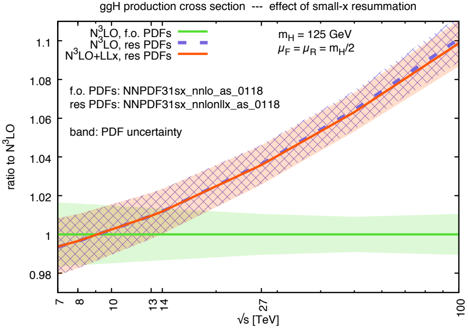
Dramatic effect of resummation on the gluon PDF at $x \lesssim 10^{-3}$

Significant impact expected for LHC and future high-energy collider phenomenology

At colliders $x_{\min} = Q^2/s \to$ small-x resummation more relevant at **low invariant masses** and at **higher collider energies**

$gg \to H$ inclusive cross section
[MB EPJC 78(2018)10]



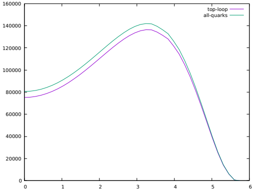


ggH cross section at FCC-hh $\sim \mathbf{10\%}$ larger than fixed order!

At LHC $+1\%$ effect

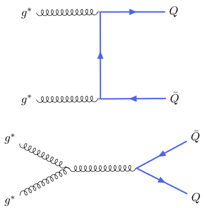
Larger effect expected at differential level in certain kinematic regions

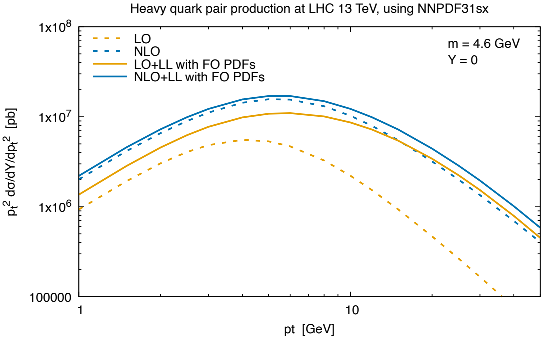
Preliminary parton-level results for fully differential Higgs production [Bernardini,MB,Silvetti (work in progress)]



Fully differential small-x resummation: Heavy-quark pair production at LHC

Fully differential heavy-quark pair production at small x [MB,Silvetti EPJC 83(2023)4]





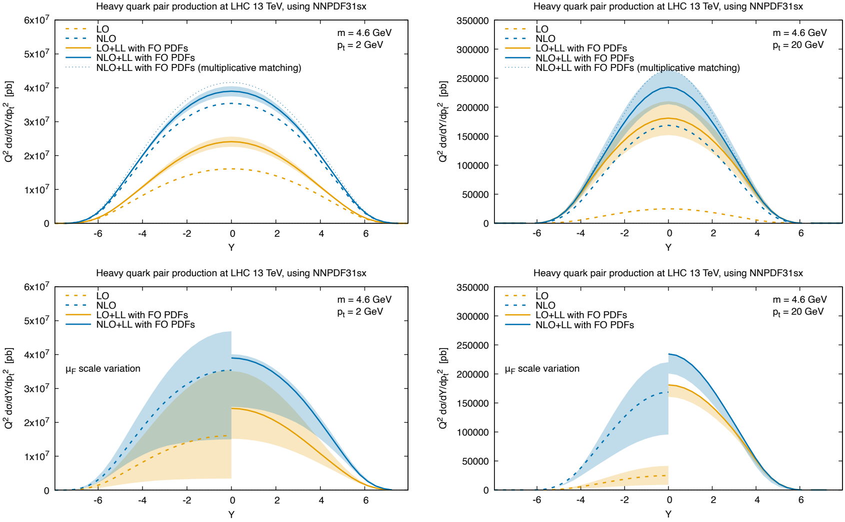
At large p_t a larger perturbative instability, cured by resummation of small-x logs

Induced by kinematic constraint $\quad x\, e^{2|y|} \leq \dfrac{1}{1+\frac{p_t^2}{m^2}} \quad$ in $\quad C(x, y, p_t, \alpha_s)$

$b\bar{b}$ and $c\bar{c}$ sensitive to very small $x \to$ can constrain PDFs! [Gauld,Rojo 1610.09373]

Heavy-quark pair production at LHC

$p_t = 2$ GeV $\qquad\qquad$ $p_t = 20$ GeV



How small can x be at pp colliders?

$y = Y - \frac{1}{2}\log\frac{x_1}{x_2}$

$$\frac{d\sigma}{dQ^2\,dY\,dp_t\dots} = \sum_{i,j=g,q}\int_\tau^1 dx_1 \int_\tau^1 dx_2\, f_i(x_1,Q^2)\, f_j(x_2,Q^2)\, C_{ij}\left(\frac{\tau}{x_1x_2}, y, p_t, \dots, \alpha_s\right)$$

The longitudinal variables $x_1, x_2,\ x = \dfrac{\tau}{x_1x_2}$ can get as small as $\tau = \dfrac{Q^2}{s}$

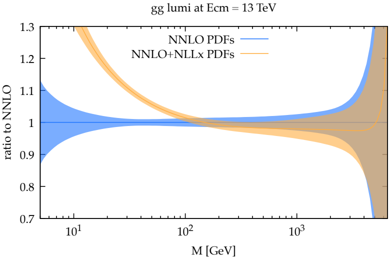
τ	Higgs	low mass Drell-Yan	$b\bar{b}$	$c\bar{c}$
LHC (13 TeV)	10^{-4}	$\sim 10^{-6}$	$\sim 10^{-6}$	$\sim 10^{-7}$
FCC-hh (100 TeV)	10^{-6}	$\sim 10^{-8}$	$\sim 10^{-8}$	$\sim 10^{-9}$

FCC-hh probes two orders of magnitude smaller x

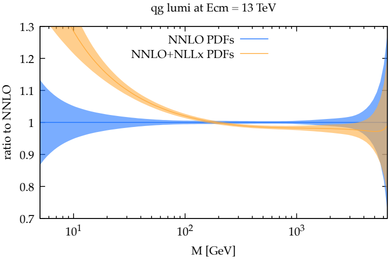
High-energy (small-x) logarithms $\log\dfrac{1}{x}$ become more relevant at **low invariant masses** and at **higher collider energies**

Parton luminosities at LHC

gg:

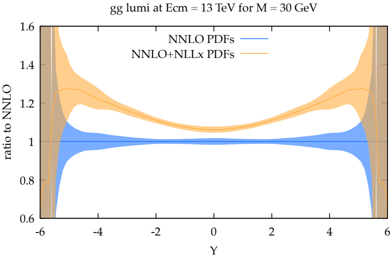


qg:

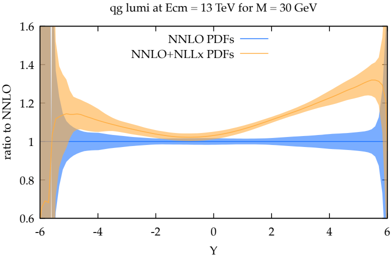


Difference more pronounced in differential distributions at large rapidity

gg:

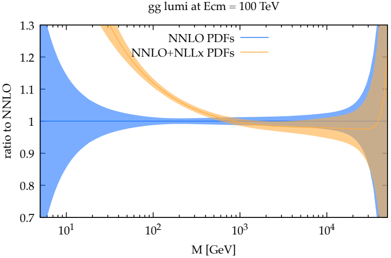


qg:

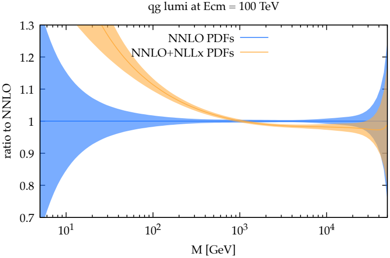


Parton luminosities at FCC-hh

gg:

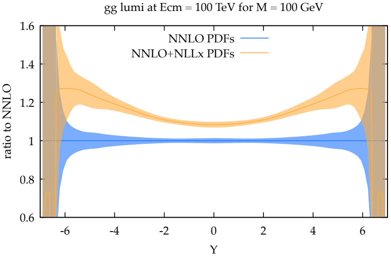


qg:

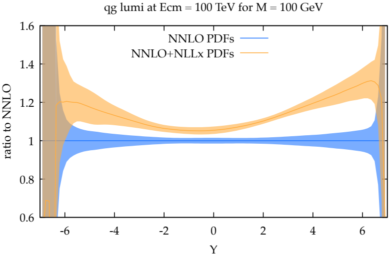


Large effects also at the EW scale, especially at large rapidities

gg:



qg:



Why is the effect of resummation mostly driven by the PDFs?

Let's consider again the collinear factorization formula

$$\frac{d\sigma}{dQ^2 dY\ldots} = \int_\tau^1 \frac{dz}{z} \int d\hat{y}\, f_i\left(\sqrt{\frac{\tau}{z}} e^{\hat{y}}, Q^2\right) f_j\left(\sqrt{\frac{\tau}{z}} e^{-\hat{y}}, Q^2\right) \frac{dC_{ij}}{dy\ldots}(z, Y-\hat{y}, \ldots, \alpha_s)$$

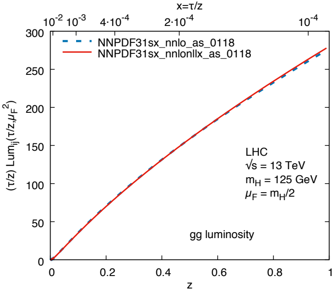
The small z integration region, where logs in C are large, is weighted by the PDFs at large momentum fractions $x = \sqrt{\frac{\tau}{z}} e^{\pm\hat{y}}$

Since PDFs die fast at large x, especially the gluon, the small-z region is suppressed!

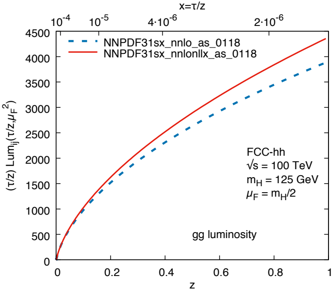
Rather, the large z region is enhanced by the gluon-gluon luminosity

In that region, the difference between fixed-order and resummed PDFs is large

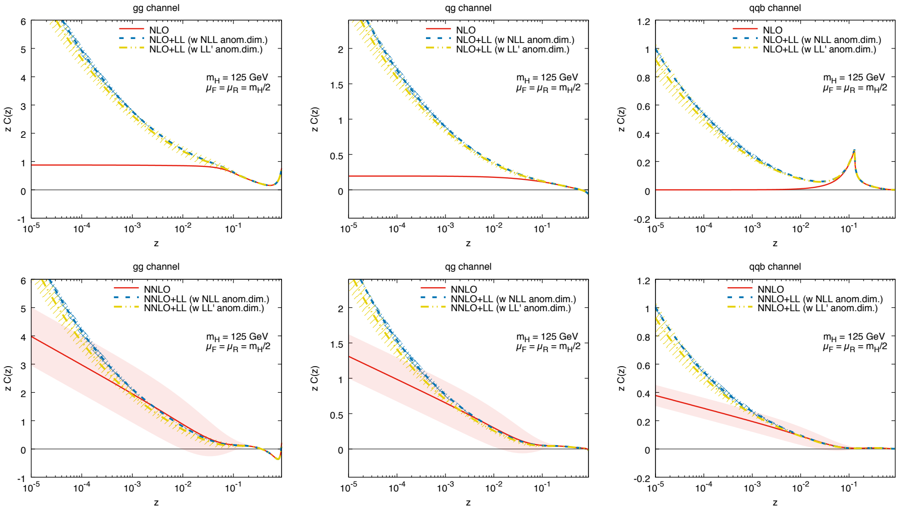
LHC:



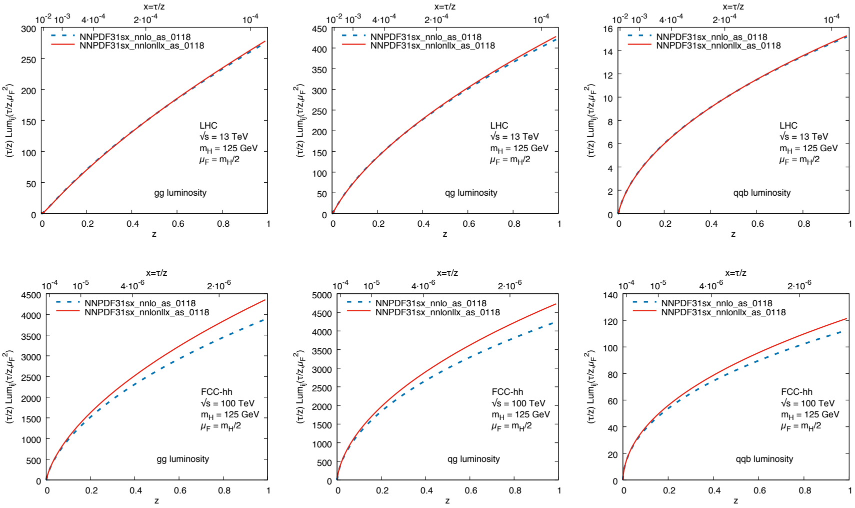
FCC:



Higgs production: parton-level results



Parton luminosities for ggH



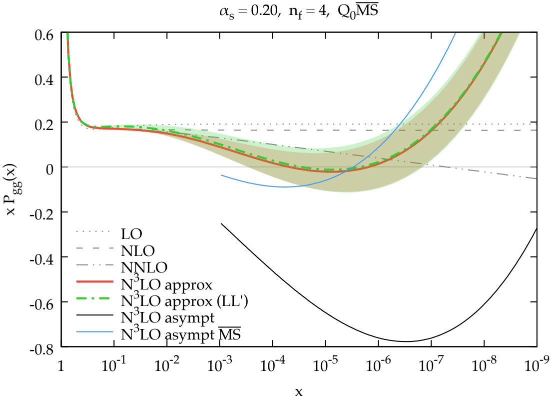
$$\sigma(\tau, Q^2) = \sigma_0(Q^2) \sum_{i,j=g,q} \int_\tau^1 \frac{dz}{z}\, C_{ij}(z, \alpha_s)\, \mathscr{L}_{ij}\left(\frac{\tau}{z}, \mu_\mathrm{F}^2\right) \qquad \tau = \frac{m_H^2}{s}$$

Towards N^{3}LO evolution

Recent impressive progress towards N^{3}LO splitting functions

[Davies,Vogt,Ruijl,Ueda,Vermaseren 1610.07477] [Moch,Ruijl,Ueda,Vermaseren,Vogt 1707.08315]

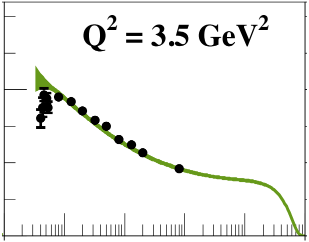
At small x, approximate predictions from NLLx resummation [MB,Marzani 1805.06460]



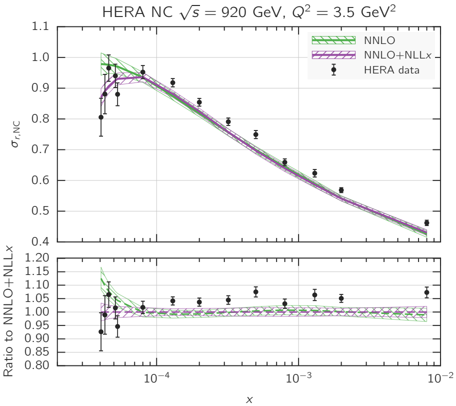
Large uncertainties from subleading logs

N^{3}LO splitting functions are much more unstable at small x $\to$ need resummation!

Fit results: description of the HERA data



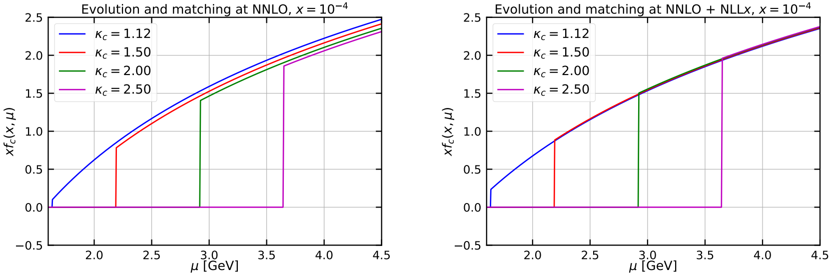
→



The better description mostly comes from a larger resummed F_L

$$\sigma_{r,\mathrm{NC}} = F_2(x_{\mathsf{Bj}}, Q^2) - \frac{y^2}{1+(1-y)^2} F_L(x_{\mathsf{Bj}}, Q^2) \qquad\qquad y = \frac{Q^2}{x_{\mathsf{Bj}} s}$$

Matching conditions at the charm threshold



The perturbatively generated charm PDF is much less dependent on the matching scale when small-x resummation is included!