Heavy-Quark Mass Effects in Energy-Energy Correlation

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Milano, 15 October 2025

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Energy-Energy Correlation (EEC)

The Energy-Energy Correlation (EEC) gives the distribution of the angular distance χ between jets produced in (high energy) e^+e^- annihilation to hadrons.

Definition:

$$\frac{d\Sigma}{d\cos\chi} \equiv \sum_{N=2}^{\infty} \sum_{i,j=1}^{N} \int \frac{E_i\, E_j}{Q^2}\, \delta\left(\cos\chi - \cos\theta_{i,j}\right) d\sigma_N \left(e^+e^- \mapsto h_i + h_j + X\right)$$

[Basham et al., 1978]

> Q (Hard Scale) $=$ Center Of Mass (COM) energy;
>
> $\theta_{i,j} =$ angle between space directions (3-momenta) of hadrons h_i and h_j.

Consider all particle pairs (i, j), weighted by their energy product $E_i\, E_j$, at a given angular separation χ.

In perturbation theory (hadrons $\mapsto$ partons):

$$e^+e^- \to q\bar{q},\, q\bar{q}g,\, \ldots$$

- We consider the **2-jet region**, where the final state consists of two, back-to-back jets. [Collins and Soper, '81; Kodaira and Trentadue, '82]

At lowest order:

$$\frac{1}{\sigma_{tot}} \frac{d\Sigma}{d\cos\chi} = \frac{1}{2}\,\delta(1+\cos\chi) + \frac{1}{2}\,\delta(1-\cos\chi).$$

Two spikes of the same strength in the 2-jet limit ($\chi = \pi$) and in the forward limit ($\chi = 0$), at the kinematical boundary ($0 \le \chi \le \pi$).

The 2-jet region is affected by double (Sudakov) logarithms, while the forward region is affected by single logarithms of hard collinear nature (no soft enhancement).

Correct Normalization

By integrating over all correlation angles χ, δ-function constraints disappears, so

$$\int_0^\pi \frac{d\Sigma}{d\cos\chi}\, d\cos\chi \;=\; \int d\sigma \;=\; \sigma_{tot},$$

since:

$$\sum_{i,j=1}^{N} E_i\, E_j \;=\; \left(\sum_{i=1}^{N} E_i\right)^2 \;=\; Q^2.$$

Remark: it was crucial to include also the diagonal terms $i = j$.

Infrared Safety

The EEC is an **infrared**, i.e. **soft** and **collinear**, safe observable.

[Sterman and Weinberg, '77]

1. **Soft Safety:** If the energy of a parton i (a gluon) of an (i, j) pair goes to zero ($E_i \to 0^+$), its contribution to the EEC vanishes due to the energy weighting factor:

$$\frac{E_i\, E_j}{Q^2} \to 0^+ \qquad \text{for} \quad E_i \to 0^+$$

2. **Collinear Safety:** The EEC is insensitive to (i.e. it does not change under) the collinear splitting of a parton $i \mapsto a + b$. Since energy is conserved ($E_i = E_a + E_b$):

$$E_i \mapsto x\, E_i + (1 - x) E_i \qquad (0 \le x \le 1)$$

The weight is unchanged:

$$\frac{x\, E_i\, E_j}{Q^2} + \frac{(1 - x)\, E_i\, E_j}{Q^2} = \frac{E_i\, E_j}{Q^2}.$$

Note that collinear safety is compatible only with linearity in E_i and E_j, while soft safety is compatible with $(E_i\, E_j)^a$, $a > 0$.

Conclusion

An absolute prediction of the EEC in perturbative QCD (pQCD) is possible — no infrared cutoff needed.
One assumes local parton-hadron duality (soft hadronization mechanism) and replaces hadrons with partons.
As is usually the case, perturbative calculations suffer from "infrared problems" in all-order resummation, because of Landau-pole effects.

Two-Jet Region & Sudakov Logarithms

In the two-jet region, it's convenient to use the (unitary) variable:

$$y \equiv \frac{1 + \cos\chi}{2} \ll 1$$

Perturbative expansion of the partially-integrated EEC distribution (or event fraction) contains large double logarithms of infrared origin (soft and/or collinear), the Sudakov logs

$$\alpha_S^n \log^k \left(\frac{1}{y} \right),$$

with:

$$n = 1, 2, 3, \cdots; \qquad k = 1, 2, 3, \cdots, 2\,n.$$

[Dokshitzer et al., '78]

Up to two logs per loop (i.e. per power of α_S).

Problem: Hard process, $Q \gg \Lambda_{QCD}$,

$$\alpha_S \equiv \alpha_S(Q) \ll 1.$$

But:

$$\log\left(\frac{1}{y}\right) \gg 1,$$

may imply:

$$\alpha_S \log\left(\frac{1}{y}\right) \sim 1.$$

Ordinary (i.e. truncated) perturbative expansion spoiled!

Problem formally similar to the secular problem in celestial mechanics (back to Poincare' and Lindstedt) —anaharmonic perturbations of classical oscillators.

In 2-jet region, the differential EEC is conveniently written as:

$$\frac{1}{\sigma_{tot}} \frac{d\Sigma}{dy} = \frac{1}{2} C(\alpha_S)\, \mathcal{S}(y, \alpha_S) + \mathrm{Rem}(y, \alpha_S)$$

[CTTW, '93]

Objects involved:

1. $C\,(\alpha_S) \,=\, 1 \,+\, c_1\,\alpha_S \,+\, c_2\,\alpha_S^2 \,+\, \ldots$
   a short-distance, process-dependent **Coefficient function**, to be computed in ordinary perturbation theory;
2. $\mathcal{S}\,(y, \alpha_S) \,=\, 1 \,+\, \sum_{n=1}^{\infty} \sum_{k=1}^{2n} c_{n,k}\,\alpha_S^n\,\log^k(y)$
   a **Sudakov Form Factor**, i.e. a process-independent, long-distance dominated form factor.
   It factorizes (large) infrared logarithms and is computed using **Resummed Perturbation Theory**.
3. $\mathrm{Rem}\,(y, \alpha_S) \,=\, \sum_{n=1}^{\infty} \alpha_S^n\,\mathrm{Rem}_n(y),$
   a process-dependent, short-distance dominated **Remainder Function**, to be computed in ordinary perturbation theory, important away from the 2-jet region,

$$\lim_{y \to 0^+} \int_0^y \mathrm{Rem}\,\left(y', \alpha_S\right)\, dy' \,=\, 0.$$

Sudakov Form Factor

The Sudakov form factor has a (generalized) exponential structure in **impact parameter (b) space**:

$$\mathcal{S}(y) = \int_0^\infty d(Qb)\,\frac{Qb}{2}\,J_0\left(bQ\sqrt{y}\right)\,S(b),$$

where

$$S(b) = e^{-w(b)}$$

is the form factor in b-space, having an exponential structure.

[Curci and Greco, Parisi and Petronzio, '80]

The exponent, $w(b)$, called the radiator, is the (effective) one-soft-gluon distribution in impact-parameter space b.

In practise, the Sudakov form factor can be (easily) resummed to all orders because it exponentiates in b-space.

Exponentiation in QCD

The exponentiation of Sudakov form factors is a consequence of:

1. **Factorization of matrix elements** for multiple soft gluon emission, $n = 2, 3, 4, \cdots$, approximate in QCD ($M \equiv M_1$):

$$M_n(k_1, k_2, \ldots, k_n) \simeq \frac{1}{n!} M(k_1)\, M(k_2) \cdots M(k_n);$$

2. **Factorization of kinematical constraints** via (2-dimensional) Fourier transform:

$$\delta^{(2)}\left(\sum_{j=1}^{n} \vec{k}_\perp^{(j)}\right) = \int \frac{d^2 b}{(2\pi)^2} \prod_{j=1}^{n} e^{i\vec{b}\cdot\vec{k}_\perp^{(j)}}.$$

Note on QCD vs. QED

- In QED, soft matrix elements exactly factorize (eikonal identity). It describes a free (photon) field coupled to a classical source.
- In QCD, this is more complex because soft gluons are self-interacting (they carry color charge). However, gluon correlation effects cancel.

Radiator $w(b)$ in the standard (massless) case

$$w(b) = \int_0^{Q^2} \frac{dk_\perp^2}{k_\perp^2} \left\{ A\left[\alpha_S\left(k_\perp^2\right)\right] \log\left(\frac{Q^2}{k_\perp^2}\right) + B\left[\alpha_S\left(k_\perp^2\right)\right] \right\} \left[1 - J_0\left(bk_\perp\right)\right]$$

- $J_0(x)$ is the Bessel function of first kind of order zero;
- The function $A\left(\alpha_S\right) = \sum_{n=1}^{\infty} A_n\, \alpha_S^n$ describes (leading) double-logarithmic effects, produced by both soft and collinear gluons;
- The function $B\left(\alpha_S\right) = \sum_{n=1}^{\infty} B_n\, \alpha_S^n$ describes (subleading) single-logarithmic effects, produced by hard collinear gluons, or soft ones (in higher orders);
- $k_\perp$ is soft-gluon transverse momentum.

Remarks:

1. "Resummability" stronger property than being infrared safe;
2. Explicitly checked against explicit Feynman diagram computations up to $\mathcal{O}\left(\alpha_S^3\right)$ included.

A Brief Selection of EEC References

Literature on EEC – involving fixed-order and resummed perturbative calculations, nonperturbative models, as well as experimental measurements at different energies — is huge. A selection follows.

Fixed-Order Calculations

- Richards, Stirling & Ellis, 1982
- Del Duca et al., 2016
- Dixon et al., 2018
- Moult & Zhu, 2018
- Ebert et al., 2021
- Duhr et al., 2022

Resummed Computations

- Collins & Soper, '81
- Kodaira & Trentadue, '82
- de Florian & Grazzini, '04
- Tulipant et al., '17
- Kardos et al., '18
- Ferrera & U.A., '24

Relevance of Mass Effects

1. At a center-of-mass energy $Q = 30$ GeV (Desy):

$$\left(\frac{2m_b}{Q}\right)^2 \sim 0.1 \sim \alpha_S(Q).$$

   Beauty mass effects are of the same order as *first-order* QCD corrections.

2. At the Z^0 peak, $Q = M_Z \approx 91.2$ GeV (LEP and SLD):

$$\left(\frac{2m_b}{Q}\right)^2 \sim 0.01 \sim \alpha_S^2(Q).$$

   Beauty mass effects are of the same order as *second-order* QCD corrections.

Note on Flavor-Inclusive Case

There is an additional suppression factor in the flavor-inclusive case:

- $\simeq 0.09$ for γ-exchange
- $\simeq 0.22$ for Z^0-exchange

Radiator $W(b)$ in the Massive Case $(0 < m \ll Q)$

(New) Radiator in the massive case:

$$W(b) = \int_{m^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2} \left\{ A[\alpha_S(k_\perp^2)] \log\left(\frac{Q^2}{k_\perp^2}\right) + B[\alpha_S(k_\perp^2)] \right\} \left[1 - J_0(bk_\perp)\right]$$
$$+ \int_0^{m^2} \frac{dk_\perp^2}{k_\perp^2} \left\{ A[\alpha_S(k_\perp^2)] \log\left(\frac{Q^2}{m^2}\right) + D[\alpha_S(k_\perp^2)] \right\} \left[1 - J_0(bk_\perp)\right]$$

[G. Ferrera & U.A., 2024]

It occurs the *new function*

$$D\left(\alpha_S\right) = \sum_{n=1}^{\infty} D_n\, \alpha_S^n,$$

describing soft emissions off the heavy quark legs (in addition to previous $A\left(\alpha_S\right)$ and $B\left(\alpha_S\right)$ functions).

Remarks on Massive Resummation Formula

- **Consistency with massless resummation.**
  In the massless limit ($m \mapsto 0^+$), the first integral extends down to 0 while the second vanishes, simply recovering the standard massless resummation formula.
- **Mass corrections** effectively:
  - Introduce a cutoff at $k_\perp \approx m$ for the massless evolution.
  - Add a new contribution for $k_\perp < m$, where the collinear logarithm is "frozen":
  $$\log\left(\frac{Q^2}{k_\perp^2}\right) \mapsto \log\left(\frac{Q^2}{m^2}\right);$$
  - Introduce a new function $D(\alpha_S)(\neq B(\alpha_S))$ for subleading effects coming from soft, large-angle emissions.
    On the contrary, leading effects are controlled by the unique function $A(\alpha_S)$;

Further Remarks

- **Dead Cone Effect.**
  All mass effects (except for the D term) can be obtained from the massless case by imposing a minimum emission angle:

  $$\theta \gtrsim \frac{2m}{Q}.$$

  This is the well-known dead-cone effect dating back to *classical electrodynamics.*
- For beauty quarks ($m_b \gg \Lambda_{QCD}$), and to some extent for charm quarks (borderline), the large-$k_\perp$ integral in the massive form factor is not affected by the Landau pole, while the low-$k_\perp$ integral is;
- The small-$k_\perp$ component is at most linear in $\log(Q^2/m^2)$ (same result obtained long ago in Wilson-line analysis of massive quark form factor [Korchemsky and Radyushkin, 1987]).

Explicit Form Factor in b-space

By integrating as usual on $k_\perp$ (step approximation), the Sudakov form factor in impact-parameter (or b-)space is naturally split into a short-distance part and a long-distance one:

$$S(b) = \begin{cases} S_S(b) & \text{if} \quad b \le b_0/m; \\ S_L(b) & \text{if} \quad b > b_0/m; \end{cases}$$

where $b_0 \equiv 2\, e^{-\gamma_E} = 1.123\cdots$.

- **Short-distance form factor**:

$$S_S(b) \equiv \exp\left[L\, g_1(\lambda) + g_2(\lambda) + \alpha_S\, g_3(\lambda) + \ldots\right];$$

  where $\lambda \equiv \beta_0\, \alpha_S(Q)\, L$ with $L \equiv \log\left(Q^2 b^2/b_0^2\right)$;

- **Long-distance form factor**:

$$S_L(b) \equiv \bar{S}_S \exp\left\{\log\left(\frac{Q^2}{m^2}\right) F\left[\rho, \alpha_S(m)\right] + H\left[\rho, \alpha_S(m)\right]\right\}.$$

We have defined the (matching) constant:

$$\bar{S}_S \equiv S_S\left(b \mapsto \frac{b_0}{m}\right)$$

and the variable:

$$\rho \equiv \beta_0\,\alpha_S(m)\,\log\left(\frac{m^2 b^2}{b_0^2}\right).$$

The leading function F has a standard perturbative expansion:

$$F\left(\rho, \alpha_S\right) \equiv \sum_{n=0}^{\infty} \alpha_S^n\, f_{n+1}(\rho);$$

as well as the subleading function:

$$H\left(\rho, \alpha_S\right) \equiv \sum_{n=0}^{\infty} \alpha_S^n\, h_{n+2}(\rho).$$

NLL Short-Distance Form Factor

The logarithmic approximations are defined as usual:

- Leading-Log (**LL**) approximation: keep only g_1;
- Next-to-Leading-Log (**NLL**) approximation: keep g_1 *and* g_2.

And so on.

The functions for the short-distance form factor $S_S(b)$ are the same as in the massless case:

$$
\begin{aligned}
g_1(\lambda) &= +\frac{A_1}{\beta_0}\,\frac{\lambda + \log(1-\lambda)}{\lambda};\\
g_2(\lambda) &= -\frac{A_2}{\beta_0^2}\left[\frac{\lambda}{1-\lambda} + \log(1-\lambda)\right] + \frac{B_1}{\beta_0}\log(1-\lambda) + \\
&\quad + \frac{A_1\,\beta_1}{\beta_0^3}\left[\frac{\lambda + \log(1-\lambda)}{1-\lambda} + \frac{1}{2}\log^2(1-\lambda)\right].
\end{aligned}
$$

The LL coefficients reads:

$$A_1 = \frac{C_F}{\pi}.$$

The NLL coefficient read:

$$A_2 = \frac{C_F}{\pi^2}\left[C_A\left(\frac{67}{36} - \frac{\pi^2}{12}\right) - \frac{5}{18}\,n_f\right]$$

and:

$$B_1 = -\frac{3}{2}\frac{C_F}{\pi}.$$

[Kodaira and Trentadue, '82]

As usual, $C_F = 4/3$, $C_A = 3$ and, at the Z^0 peak, $n_f = 5$.

Key Difference

The only difference with respect to the massless case is the occurrence of a cutoff $b \approx 1/m$ on large b values.

NLL Long-Distance Form Factor

For the long-distance factor $S_L(b)$, the LL functions read:

$$f_1(\rho) = \frac{A_1}{\beta_0} \log(1 - \rho)$$

and:

$$h_1(\rho) = 0.$$

The NLL functions read:

$$f_2(\rho) = -\frac{A_2}{\beta_0} \frac{\rho}{1 - \rho} + \frac{A_1 \beta_1}{\beta_0^2} \frac{\rho + \log(1 - \rho)}{1 - \rho}$$

and:

$$h_2(\rho) = \frac{D_1}{\beta_0} \log(1 - \rho);$$

with $D_1 = -C_F/\pi$.

Remark: the function $h_2(\rho)$ rescales the hard scale

$$Q \;\mapsto\; c_1\, Q,$$

with:

$$c_1 \;\equiv\; e^{D_1/(2A_1)} \;\approx\; 0.606,$$

a general fact already known to the russian school [Dokshitzer et al.]. That implies the dead-cone (dc) opening angle becomes larger:

$$\vartheta_{dc} \;=\; \frac{2m}{Q} \mapsto \frac{\vartheta_{dc}}{c_1}$$

The dead-cone angle increases almost by a factor of two.

Plots in Impact Parameter (b-)Space

- **Continuous** curves are the plots of the **massive** form factors;
- **Dashed** lines represent the **massless** cases — for comparison.

- Leading-Logarithmic, **LL**, approximation: **Red**;
- Next-to-Leading-Logarithmic, **NLL**, appr.: **Blue**;
- Next-to-Next-to-Leading-Logarithmic, **NNLL**, appr.: **Green**.

Massive form factors decay **slower** at **large** (transverse) distances, $b > b_{CR}$, than massless ones. The behavior changes indeed from a double-logarithmic to a single-logarithmic one. That is because the dead-cone effect (acting already at leading, double-log level) suppresses small-angle radiation.

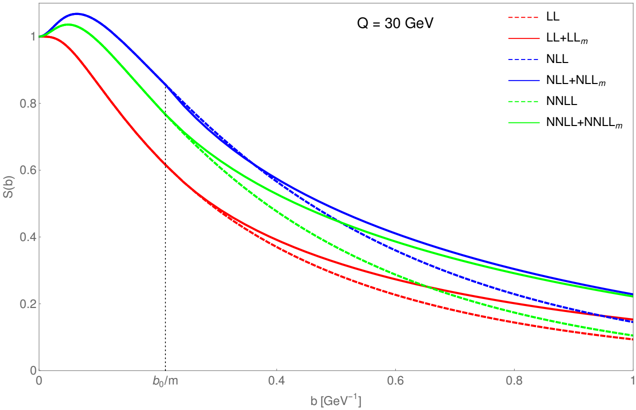


Figure: Sudakov form factor $S(b)$ in impact-paramter (b-)space at COM energy $Q = 30\,\mathrm{GeV}$. The (dotted black) vertical line sets the transition from the small-distance (effectively massless) domain to the long-distance one. Only in the latter, the massive-massless splitting occurs.

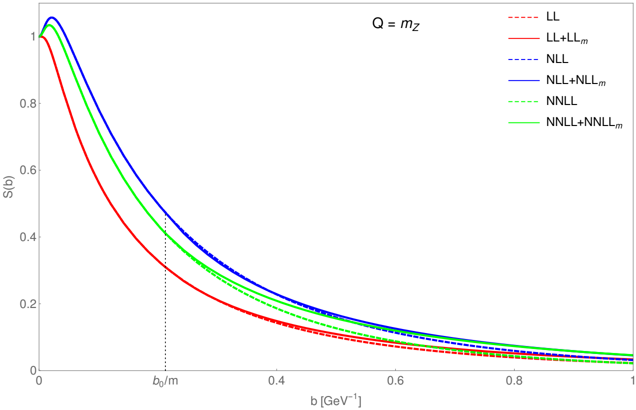


Figure: Sudakov form factor $S(b)$ in impact-paramter (b)-space at COM energy $Q = m_Z \simeq 91.2\,\mathrm{GeV}$. The vertical line sets the transition from the small-distance domain to the long-distance one. Because of the larger hard scale, mass effects are much smaller than in previous plot.

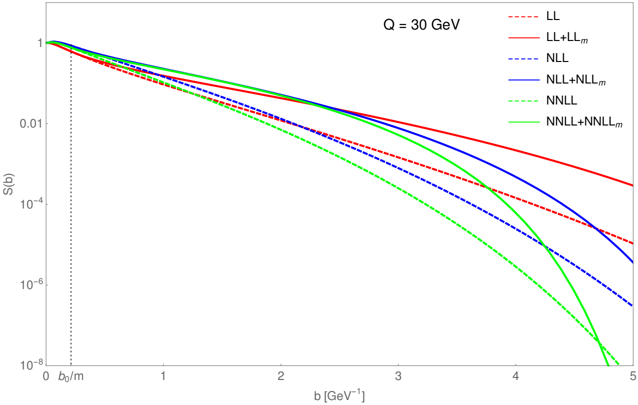


Figure: Sudakov form factor $S(b)$ for large values of the impact paramter $b : 0 \mapsto 5\,\mathrm{GeV}^{-1} \simeq 1\,\mathrm{Fermi}$ (the typical size of a hadron), at COM energy $Q = 30\,\mathrm{GeV}$. The vertical scale is logarithmic.

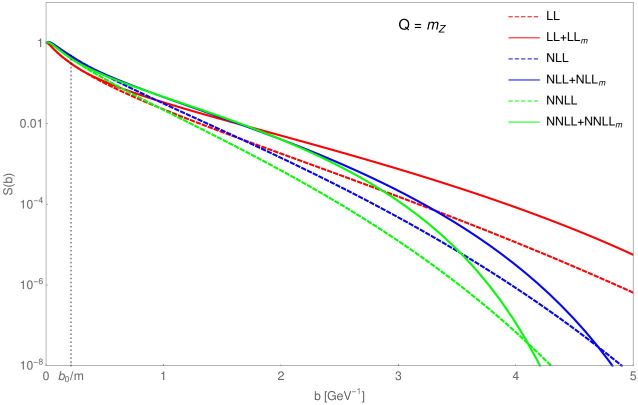
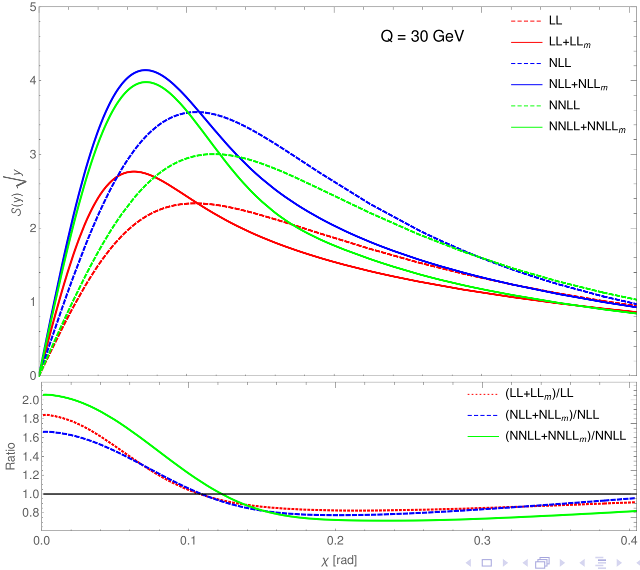
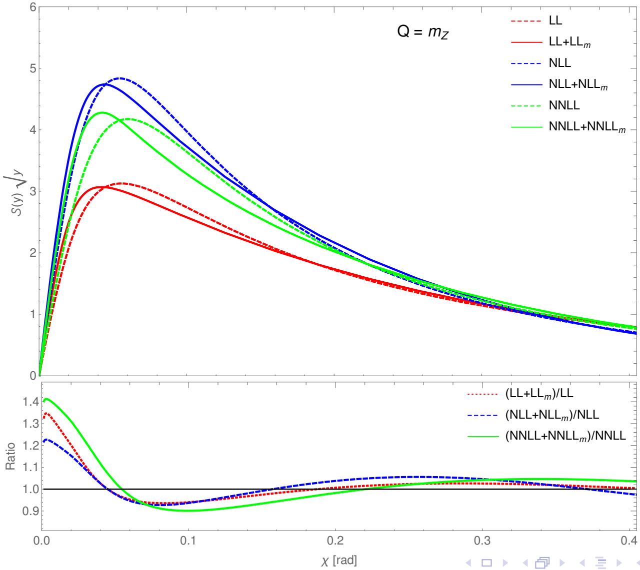


Figure: Sudakov form factor $S(b)$ for large values of the impact paramter $b : 0 \mapsto 5\,\mathrm{GeV}^{-1}$, at COM energy $Q = m_Z \simeq 91.2\,\mathrm{GeV}$. The vertical scale is logarithmic.

Plots in Physical Angle (χ-)Space

- By including mass effects, in general the EEC peak becomes **higher** and moves to **smaller angles** χ (roughly, less radiation emitted because of dead cone).
- At a hard scale $Q = 30$ GeV, beauty mass effects ($m_b \approx 5.28$ GeV) are pretty large (see fig.1).
- At the Z^0 peak ($Q = M_{Z^0} \approx 91.2$ GeV), mass effects are reduced by an order of magnitude, as expected (see fig.2).
- In the flavor-inclusive case, there is a reduction factor:
    - $\approx 1/11 \approx 0.09$ for γ-exchange;
    - ≈ 0.22 for Z^0-exchange.


Q = 30 GeV
LL
LL+LL$_m$
NLL
NLL+NLL$_m$
NNLL
NNLL+NNLL$_m$
$S(y)\sqrt{y}$
0
1
2
3
4
5
(LL+LL$_m$)/LL
(NLL+NLL$_m$)/NLL
(NNLL+NNLL$_m$)/NNLL
Ratio
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.0
0.1
0.2
0.3
0.4
χ [rad]


Q = m_Z
LL
LL+LL$_m$
NLL
NLL+NLL$_m$
NNLL
NNLL+NNLL$_m$
$S(y)\sqrt{y}$
(LL+LL$_m$)/LL
(NLL+NLL$_m$)/NLL
(NNLL+NNLL$_m$)/NNLL
Ratio
χ [rad]

Conclusions and Outlook

We have presented a *simple generalization*, to the massive case, of the (standard) resummation formula for the Energy-Energy Correlation in the 2-jet limit. We assumed the heavy quark to be ultra-relativistic: $0 < m \ll Q$.

The heavy-quark mass effects are perturbative, short-distance effects, not involving any nonperturbative model dependence.

Further work involves different directions:

- Current phenomenological analyses can be improved by implementing the new massive form factor;
- The massive resummation formula can be generalized to the case in which the heavy-quark mass m is not assumed anymore to be much smaller than the hard scale Q.

- Mass effects in the Coefficient Function,

$$C\left(\alpha_S\right) \;\mapsto\; C\left(\alpha_S;\, m/Q\right),$$

as well as in the Remainder function,

$$\mathrm{Rem}\left(y, \alpha_S\right) \;\mapsto\; \mathrm{Rem}\left(y, \alpha_S;\, m/Q\right),$$

can be computed (they are expected to be smaller than the Sudakov effects).

One should implement the "old" numerical calculations of the Energy-Energy Correlation in the massive case to $\mathcal{O}(\alpha_S)$

[Csikor, Ali & Barreiro, 1984].

- One can generalize the present treatment of heavy-quark mass effects to other $k_\perp$-like shape variables (e.g., jet broadening) — a relatively straightforward thing.